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Engineering Economy

Chapter 1: Introduction to Engineering Economy

The purpose of this book is to develop and illustrate the principles and methodology required to answer the basic economic question of any design: Do its benefits exceed its cost?

Engineering economy…

involves the systematic evaluation of the economic merits of proposed solutions to engineering problems.

Solutions to engineering problems must

- promote the well-being and survival of an organization,
- embody creative and innovative technology and ideas,
- permit identification and scrutiny of their estimated outcomes, and
- translate profitability to the "bottom line" through a valid and acceptable measure of merit.

Engineering economic analysis can play a role in many types of situations.

- Choosing the best design for a high-efficiency gas furnace.
- Selecting the most suitable robot for a welding operation on an automotive assembly line.
- Making a recommendation about whether jet airplanes for an overnight delivery service should be purchased or leased.
- Determining the optimal staffing plan for a computer help desk.

There are seven fundamental principles of engineering economy.

- Develop the alternatives
- Focus on the differences
- Use a consistent viewpoint
- Use a common unit of measure
- Consider all relevant criteria
- Make uncertainty explicit
- Revisit your decisions

Engineering economic analysis procedure

- Problem definition
- Development of alternatives
- Development of prospective outcomes
- Selection of a decision criterion
- Analysis and comparison of alternatives.
- Selection of the preferred alternative.
- Performance monitoring and postevaluation of results.

Electronic spreadsheets are a powerful addition to the analysis arsenal.

- Most engineering economy problems can be formulated and solved using a spreadsheet.
- Large problems can be quickly solved.
- Proper formulation allows key parameters to be changed.
- Graphical output is easily generated.

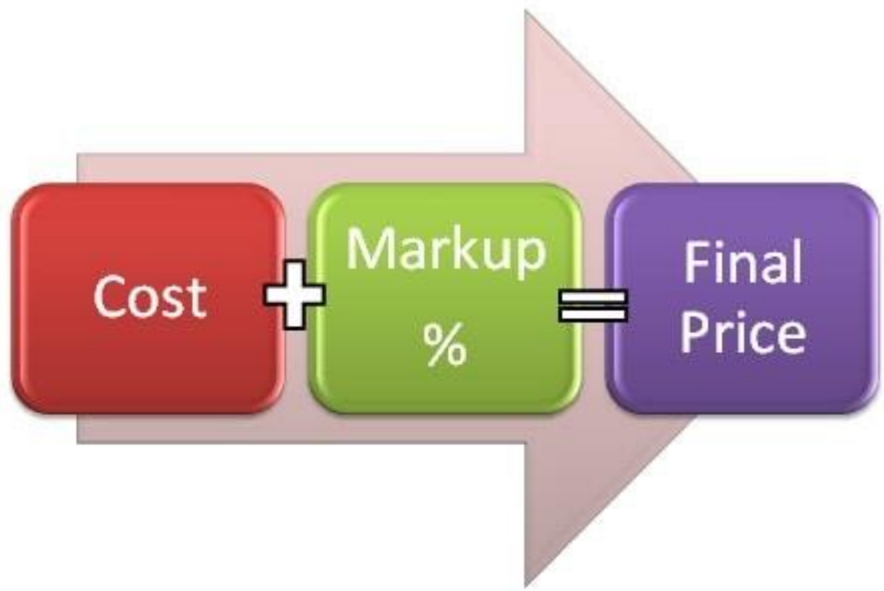
CHAPTER 2
Cost Concepts and Design Economics

Cost Terminology

What is Cost?

In business and accounting, cost is the monetary value spent by a company to produce a product.

It does not include profit mark-up



Fixed, Variable, and Incremental Costs

- Fixed costs:
- Are those costs that don't change based on production levels.
- Unaffected by changes in activity level over a feasible range of operations for the capacity or capability available (unaffected by changes in activity level).

Examples:
- Insurance and Taxes on facilities
- administrative salaries
- License fees
- Interest cost on borrowed capital
- Rent.

Fixed, Variable, and Incremental Costs

- Variable costs:

Are costs that vary with output.

- Change according to the quantity of a good or service being produced (vary in total with the quantity of output (or similar measure of activity)).

Examples:

- Costs of material and labor used in a product

or service

- Sales tax
- Hauling
- Packaging
- Fuel costs

Fixed and Variable Costs Example

EXAMPLE 2-1

In connection with surfacing a new highway, a contractor has a choice of two sites on which to set up the asphalt-mixing plant equipment. The contractor estimates that it will cost \$2.75 per cubic yard mile (yd^3-mile) to haul the asphalt-paving material from the mixing plant to the job location. Factors relating to the two mixing sites are as follows (production costs at each site are the same):

Cost Factor	Site *A*	Site *B*
Average hauling distance	4 miles	3 miles
Monthly rental of site	\$2,000	\$7,000
Cost to set up and remove equipment	\$15,000	\$50,000
Hauling expense	\$2.75/yd^3-mile	\$2.75/yd^3-mile
Flagperson	Not required	\$150/day

The job requires 50,000 cubic yards of mixed-asphalt-paving material. It is estimated that four months (17 weeks of five working days per week) will be required for the job.

Compare the two sites in terms of their fixed, variable, and total costs. Assume that the cost of the return trip is negligible.
Which is the better site?

For the selected site, how many cubic yards of paving material does the contractor have to deliver before starting to make a profit if paid \$12 per cubic yard delivered to the job location?

Solution

Cost	Fixed	Variable	Site *A*		Site *B*	
Rent	√		=	\$8,000	=	\$28,000
Setup/removal	√		=	15,000	=	50,000
Flagperson	√		=	0	5(17)(\$150) =	12,750
Hauling		√	4(50,000)(\$2.75) =	550,000	3(50,000)(\$2.75) =	412,500
			Total:	\$573,000		\$503,250

Site B, which has the larger fixed costs, has the smaller total cost for the job. Note that the extra fixed costs of Site B are being "traded off" for reduced variable costs at this site.

The contractor will begin to make a profit at the point where total revenue equals total cost as a function of the cubic yards of asphalt pavement mix delivered. Based on Site B, we have

3(\$2.75) = \$8.25 in variable cost per yd3 delivered,

Total cost = total revenue

\$90,750 + \$8.25x = \$12x

x = 24,200 yd3 delivered.

Therefore, by using Site B, the contractor will begin to make a profit on the job after delivering 24,200 cubic yards of material.

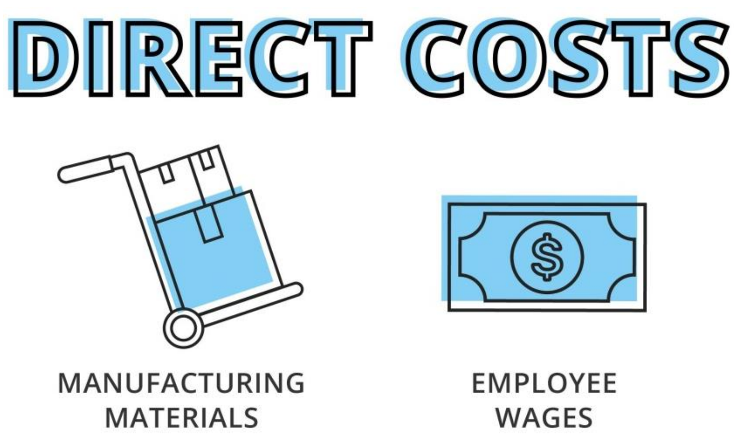
Direct, Indirect, and Standard Costs

- **Direct costs**

Are costs that can be reasonably measured and allocated to a specific output or work activity.

Examples:

Labor and material costs directly associated with a product, service, or construction activity



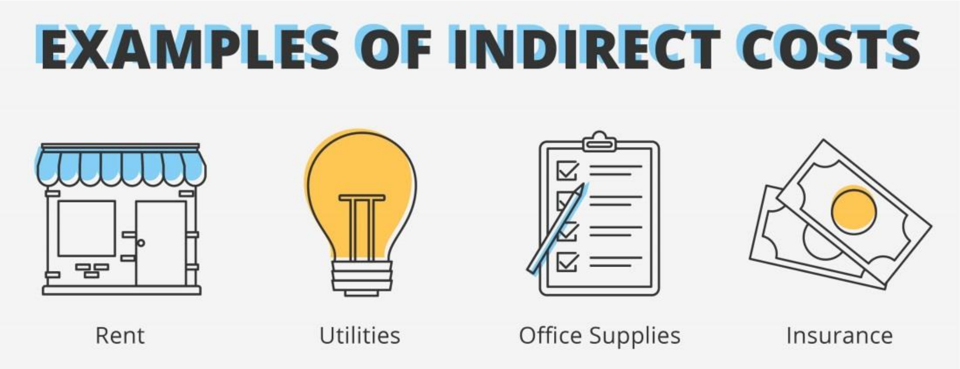
Direct, Indirect, and Standard Costs

- **Indirect costs:**

Are costs that are difficult to allocate to a specific output or work activity (also overhead or burden).

Examples:

- Costs of common tools
- General supplies
- Equipment maintenance in a plant



Direct, Indirect, and Standard Costs

- **Standard costs:**

Are planned costs per unit of output.

- Standard costs play an important role in cost control and other management functions (cost per unit of output, established in advance of production or service delivery)

Cash Cost versus Book Cost

- Cash Cost:

A cost that involves payment of cash (and results in a cash flow)

- Book Cost:

a cost that does not involve a cash transaction but is reflected in the accounting system, and it is the future expenses incurred for the alternatives being analyzed.

(equipment, machines, Depreciation)

Sunk Cost

- Is one that has occurred in the past and has no relevance to estimates of future costs and revenues related to an alternative course of action.

- Independent of future costs
- Incurred in past
- Cannot be recovered

Examples:

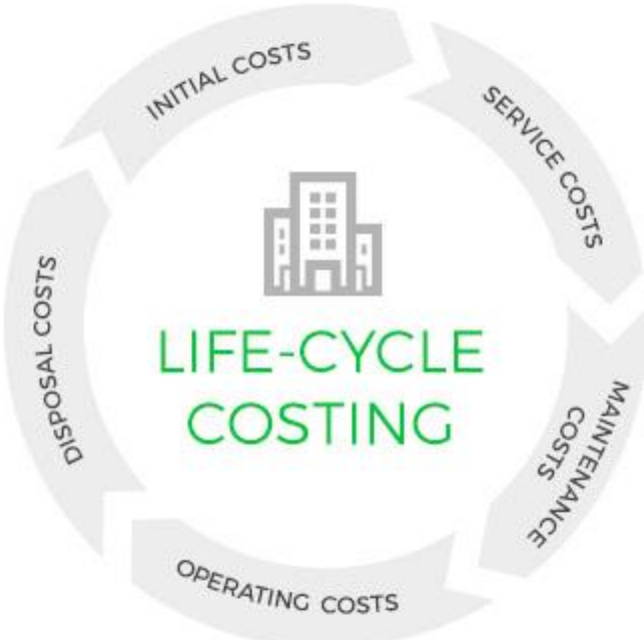
- Earnest money on a house
- Money spent on a passport

Example (Sunk Cost)

The concept of sunk cost is illustrated in this simple example. Suppose that Joe College finds a motorcycle he likes and pays \$40 as a down payment, which will be applied to the \$1,300 purchase price but must be forfeited if he decides not to take the cycle. Over the weekend, Joe finds another motorcycle he considers equally desirable for a purchase price of \$1,230. For the purpose of deciding which cycle to purchase, the \$40 is a sunk cost and thus would not enter into the decision, except that it lowers the remaining cost of the first cycle. The decision is between paying an additional \$1,260 (\$1,300 – \$40) for the first motorcycle versus \$1,230 for the second motorcycle.

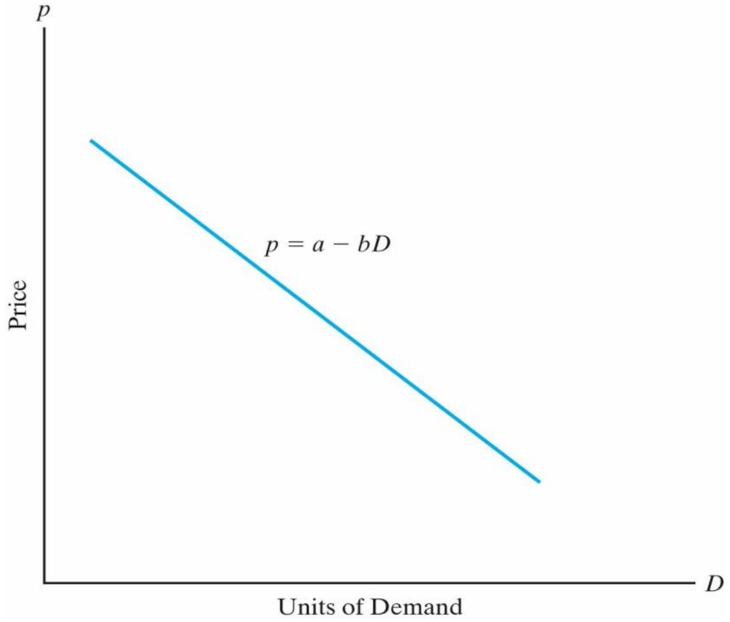
Life-Cycle Cost

- It refers to a summation of all the costs related to a product, structure, system, or service during its life span.



The general price-demand relationship

The demand for a product or service is directly related to its price according to p=a-bD where p is price, D is demand, and a and b are constants that depend on the particular product or service.



The general price-demand relationship is ?

Total revenue depends on price and demand.

Total revenue is the product of the selling price per unit, *p*, and the number of units sold, *D*.

$$TR = pD = (a - bD)D = aD - bD^2$$

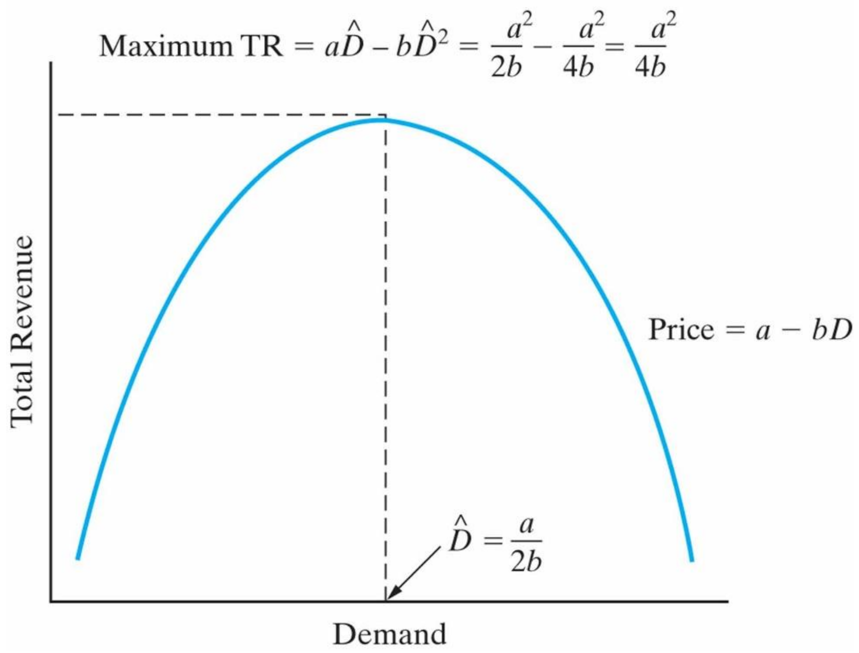
$$\text{for} \quad 0 \le D \le \frac{a}{b} \quad \text{and} \quad a > 0, \quad b > 0$$

Calculus can help determine the demand that maximizes revenue.

$$\frac{dTR}{dD} = a - 2bD = 0$$

Solving, the optimal demand is

$$\hat{D} = \frac{a}{2b}$$



We can also find maximum profit…

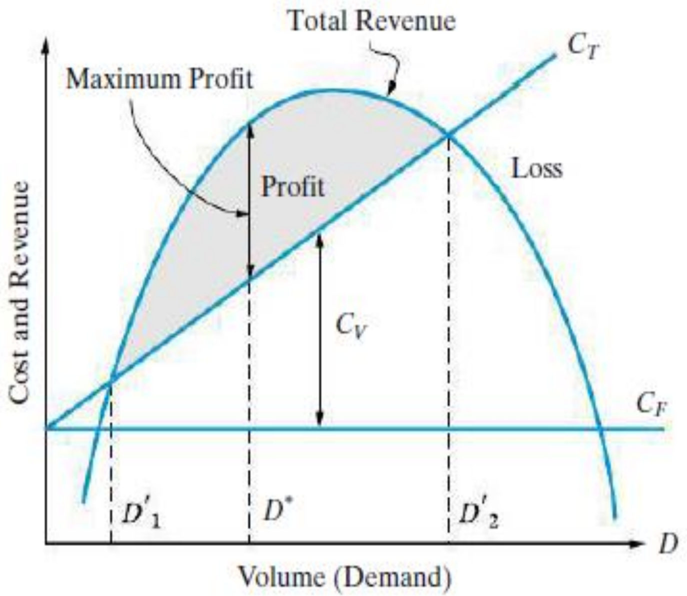
at any demand D, **total cost** is $CT = CF + CV$, where CF and CV denote fixed and variable costs, respectively. For the linear relationship assumed here, $CV = cv \cdot D$, where cv is the variable cost per unit.

Profit is revenue minus cost, so

Profit (loss) = total revenue − total costs
$= (aD - bD^2) - (CF + cvD)$

$= -bD^2 + (a - cv)D - CF$

for $0 \le D \le a/b$ and $a > 0$, $b > 0$.



Differentiating, we can find the value of ***D*** that maximizes profit.

$$\frac{d(\text{profit})}{dD} = a - c_v - 2bD = 0 \qquad D^* = \frac{a - c_v}{2b}$$

we can find revenue/cost breakeven.

To ensure that we have *maximized* profit (rather than minimized it), the sign of the second derivative must be negative. Checking this, we find that

$$\frac{d^2(\text{profit})}{dD^2} = -2b,$$

Breakeven is found when **total revenue = total cost**. Solving, we find the demand at which this occurs.

Total revenue = total cost (breakeven point)

$$aD - bD^2 \quad = \quad CF + cvD$$

$$-bD^2 + (a - cv)D - CF = 0$$

$$D' = \frac{-(a - c_v) \pm \sqrt{(a - c_v)^2 - 4(-b)(-C_F)}}{2(-b)}$$

we can solve for the breakeven points D'_1 and D'_2

This will ensure that D'_1 and D'_2 have real positive, unequal values

Opportunity Cost

- Is the monetary advantage foregone due to limited resources.

- The cost of the best rejected opportunity.

Example:

- Consider a student who could earn $20,000 for working during a year but chooses instead to go to school for a year and spend $5,000 to do so.

The opportunity cost of going to school for that year is $25,000: $5,000 cash outlay and $20,000 for income foregone.

EXAMPLE 2-4

A company produces an electronic timing switch that is used in consumer and commercial products. The fixed cost (*CF*) is $73,000 per month, and the variable cost (*cv*) is $83 per unit. The selling price per unit is p = $180 – 0.02(*D*), based on Equation (2-1). For this situation,

(a) determine the optimal volume for this product and confirm that a profit occurs (instead of a loss) at this demand.

(b) find the volumes at which breakeven occurs; that is, what is the range of profitable demand? Solve by hand and by spreadsheet.

Solution

$$\text{(a)}\quad D^* = \frac{a - c_v}{2b} = \frac{\$180 - \$83}{2(0.02)} = 2{,}425 \text{ units per month}$$

Is $(a - cv) > 0$?

($180 – $83) = $97, which is greater than 0.

And is (total revenue – total cost) > 0 for $D* = 2{,}425$ units per month?

$[\$180(2{,}425) - 0.02(2{,}425)^2] - [\$73{,}000 + \$83(2{,}425)] = \$44{,}612$

A demand of $D* = 2{,}425$ units per month results in a maximum profit of $44,612 per month. Notice that the second derivative is negative (−0.04).

Given the quadratic equation $ax^2 + bx + c = 0$, the roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$D' = \frac{-97 \pm [(97)^2 - 4(-0.02)(-73{,}000)]^{0.5}}{2(-0.02)}$$

$$D'_1 = \frac{-97 + 59.74}{-0.04} = 932 \text{ units per month}$$

$$D'_2 = \frac{-97 - 59.74}{-0.04} = 3{,}918 \text{ units per month.}$$

Thus, the range of profitable demand is 932–3,918 units per month.

EXAMPLE 2-5

An engineering consulting firm measures its output in a standard service hour unit, which is a function of the personnel grade levels in the professional staff. The variable cost (cv) is $62 per standard service hour. The charge-out rate [i.e., selling price (p)] is $85.56 per hour. The maximum output of the firm is 160,000 hours per year, and its fixed cost (CF) is $2,024,000 per year. For this firm,

(a) what is the breakeven point in standard service hours and in percentage of total capacity?

(b) what is the percentage reduction in the breakeven point (sensitivity) if fixed costs are reduced 10%; if variable cost per hour is reduced 10%; and if the selling price per unit is increased by 10%?

Solution (a)

$$\text{Total revenue} = \text{total cost} \qquad \text{(breakeven point)}$$

$$pD' = C_F + c_v D'$$

$$D' = \frac{C_F}{(p - c_v)},$$

and

$$D' = \frac{\$2{,}024{,}000}{(\$85.56 - \$62)} = 85{,}908 \text{ hours per year}$$

$$D' = \frac{85{,}908}{160{,}000} = 0.537,$$

or 53.7% of capacity.

Solution (b)

(b) A 10% reduction in C_F gives

$$D' = \frac{0.9(\$2{,}024{,}000)}{(\$85.56 - \$62)} = 77{,}318 \text{ hours per year}$$

and

$$\frac{85{,}908 - 77{,}318}{85{,}908} = 0.10,$$

or a 10% reduction in D'.

A 10% reduction in c_v gives

$$D' = \frac{\$2{,}024{,}000}{[\$85.56 - 0.9(\$62)]} = 68{,}011 \text{ hours per year}$$

and

$$\frac{85{,}908 - 68{,}011}{85{,}908} = 0.208,$$

or a 20.8% reduction in D'.

A 10% increase in p gives

$$D' = \frac{\$2{,}024{,}000}{[1.1(\$85.56) - \$62]} = 63{,}021 \text{ hours per year}$$

and

$$\frac{85{,}908 - 63{,}021}{85{,}908} = 0.266,$$

or a 26.6% reduction in D'.

Engineers must consider cost in the design of products, processes and services.

- “Cost-driven design optimization” is critical in today’s competitive business environment.
- In our brief examination we examine discrete and continuous problems that consider a single primary cost driver.

Two main tasks are involved in cost-driven design optimization.

1. Determine the optimal value for a certain alternative's design variable.
2. Select the best alternative, each with its own unique value for the design variable.

Cost models are developed around the design variable, X.

Optimizing a design with respect to cost is a four-step process.

- Identify the design variable that is the primary cost driver.
- Express the cost model in terms of the design variable.
- For continuous cost functions, differentiate to find the optimal value. For discrete functions, calculate cost over a range of values of the design variable.
- Solve the equation in step 3 for a continuous function. For discrete, the optimum value has the minimum cost value found in step 3.

Here is a simplified cost function.

$$\text{Cost} = aX + \frac{b}{X} + k$$

where,

a is a parameter that represents the directly varying cost(s),

b is a parameter that represents the indirectly varying cost(s),

k is a parameter that represents the fixed cost(s), and

X represents the design variable in question.

“Present economy studies” can ignore the time value of money.

- Alternatives are being compared over one year or less.
- When revenues and other economic benefits vary among alternatives, choose the alternative that maximizes overall profitability of defect-free output.
- When revenues and other economic benefits are not present or are constant among alternatives, choose the alternative that minimizes total cost per defect-free unit.

EXAMPLE 2-6

The cost of operating a jet-powered commercial (passenger-carrying) airplane varies as the three-halves (3/2) power of its velocity; specifically, $CO = knv^{3/2}$, where n is the trip length in miles, k is a constant of proportionality, and v is velocity in miles per hour. It is known that at 400 miles per hour, the *average* cost of operation is $300 per mile. The company that owns the aircraft wants to minimize the cost of operation, but that cost must be balanced against the cost of the passengers' time (CC), which has been set at $300,000 per hour.

(a) At what velocity should the trip be planned to minimize the total cost, which is the sum of the cost of operating the airplane and the cost of passengers' time?

(b) How do you know that your answer for the problem in Part (a) minimize sthe total cost?

Solution

(a) The equation for total cost (C_T) is

$$C_T = C_O + C_C = knv^{3/2} + (\$300{,}000 \text{ per hour})\left(\frac{n}{v}\right),$$

where n/v has time (hours) as its unit.

Now we solve for the value of k:

$$\frac{C_O}{n} = kv^{3/2}$$

$$\frac{\$300}{\text{mile}} = k\left(400\frac{\text{miles}}{\text{hour}}\right)^{3/2}$$

$$k = \frac{\$300/\text{mile}}{\left(400\frac{\text{miles}}{\text{hour}}\right)^{3/2}}$$

$$k = \frac{\$300/\text{mile}}{8000\left(\frac{\text{miles}^{3/2}}{\text{hour}^{3/2}}\right)}$$

$$k = \$0.0375\frac{\text{hours}^{3/2}}{\text{miles}^{5/2}}.$$

Thus,

$$C_T = \left(\$0.0375\frac{\text{hours}^{3/2}}{\text{miles}^{5/2}}\right)(n \text{ miles})\left(v\frac{\text{miles}}{\text{hour}}\right)^{3/2} + \left(\frac{\$300{,}000}{\text{hour}}\right)\left(\frac{n \text{ miles}}{v\dfrac{\text{miles}}{\text{hour}}}\right)$$

$$C_T = \$0.0375nv^{3/2} + \$300{,}000\left(\frac{n}{v}\right).$$

Next, the first derivative is taken:

$$\frac{dC_T}{dv} = \frac{3}{2}(\$0.0375)nv^{1/2} - \frac{\$300{,}000n}{v^2} = 0.$$

So,

$$0.05625v^{1/2} - \frac{300{,}000}{v^2} = 0$$

$$0.05625v^{5/2} - 300{,}000 = 0$$

$$v^{5/2} = \frac{300{,}000}{0.05625} = 5{,}333{,}333$$

$$v^* = (5{,}333{,}333)^{0.4} = 490.68 \text{ mph}.$$

(b) Finally, we check the second derivative to confirm a minimum cost solution:

$$\frac{d^2C_T}{dv^2} = \frac{0.028125}{v^{1/2}} + \frac{600{,}000}{v^3} \qquad \text{for } v > 0, \text{ and therefore, } \frac{d^2C_T}{dv^2} > 0.$$

The company concludes that $v = 490.68$ mph minimizes the total cost of this particular airplane's flight.

The Time Value of Money

Introduction

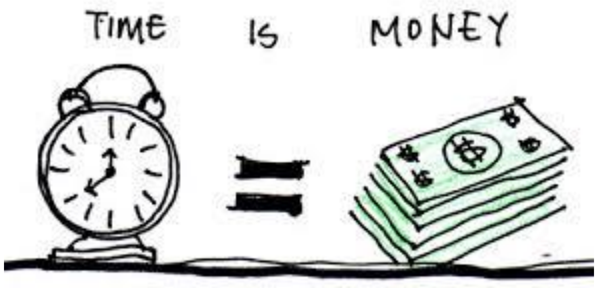
Basic Concepts

The objective of Chapter 4 is to explain time value of money calculations and to illustrate economic equivalence

- The term *capital* refers to wealth in the form of money or property that can be used to produce more wealth.

- The majority of engineering economy studies involve commitment of capital for extended periods of time, so the effect of time must be considered.
- It is recognized that a dollar today is worth more than a dollar one or more years from now because of the interest (or profit) it can earn. Therefore, money has a *time value*.

Money has a time value



Return to capital in the form of interest and profit is an essential ingredient of engineering economy studies.

Interest is a fee that is charged for the use of someone else's money.

The size of the fee will depend upon:

1. The total amount of money borrowed
2. The length of time over which it is borrowed.

Example: An engineer wishes to borrow $20,000 in order to start his own business. A bank will lend him the money provided he agrees to repay $920 per month for two years. How much interest is he being charged.

Whenever money is borrowed or invested, one party acts as the lender and another party as the borrower.

The lender is the owner of the money, and the borrower pays interest to the lender for the use of the lender’s money.

Example: Savings account

Interest Rate

If a given amount of money is borrowed for a specified period of time (typically, one year), a certain percentage of the money is charged as interest. This percentage is called the interest rate.

Example:

a. A student deposits $1,000 in a savings account that pays interest at the rate of 6% per year. How much money will the student have after one year?

b. An investor makes a loan of $5,000, to be repaid in one lump sum at the end of one year. What annual interest rate corresponds to a lump-sum payment of $5,425?

Why Consider Return to Capital?

1. Interest and profit pay the providers of capital for forgoing its use during the time the capital is being used.

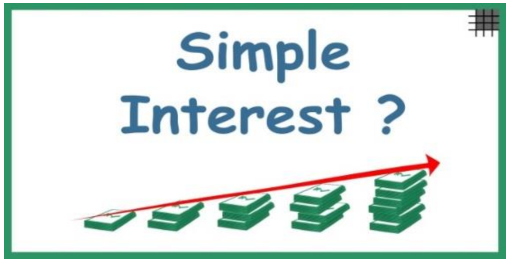
2. Interest and profit are payments for the *risk* the investor takes in permitting another person, or an organization, to use his or her capital.

3. Any project or venture must provide a sufficient return to be financially attractive to the suppliers of money or property.

Simple Interest

When the total interest earned or charged is linearly proportional to the initial amount of the loan (principal), the interest rate, and the number of interest periods for which the principal is committed, the interest and interest rate are said to be *simple (*not used frequently in modern commercial practice) .

Simple interest is defined as a fixed percentage of the principal (the amount of money borrowed), multiplied by the life of the loan



$$\underline{I} = (P)(N)(i),$$

Where

$\underline{I}$ = total amount of simple interest , earned or paid

P = principal amount lent or borrowed;

N = life of the loan ,number of interest periods, (e.g., years);

i = interest rate per interest period (expressed as a decimal)

It is understood that N and i refer to the same unit of time

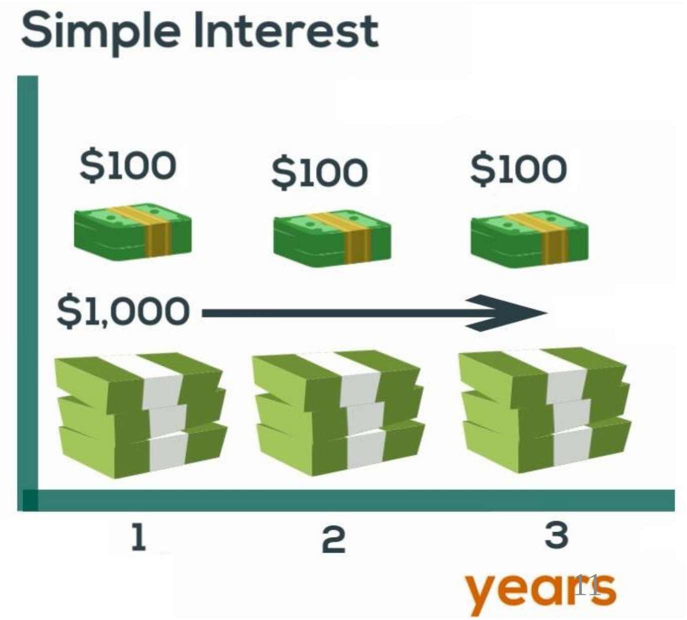
The total amount repaid at the end of N interest periods is $P + \underline{I}$

Example

If $1,000 were loaned for three years at a simple interest rate of 10% per year, find

1. The total interest earned

2. The total amount owed at the end of three years



Example

- If $5,000 were loaned for five years at a simple interest rate of 7% per year, the interest earned would be

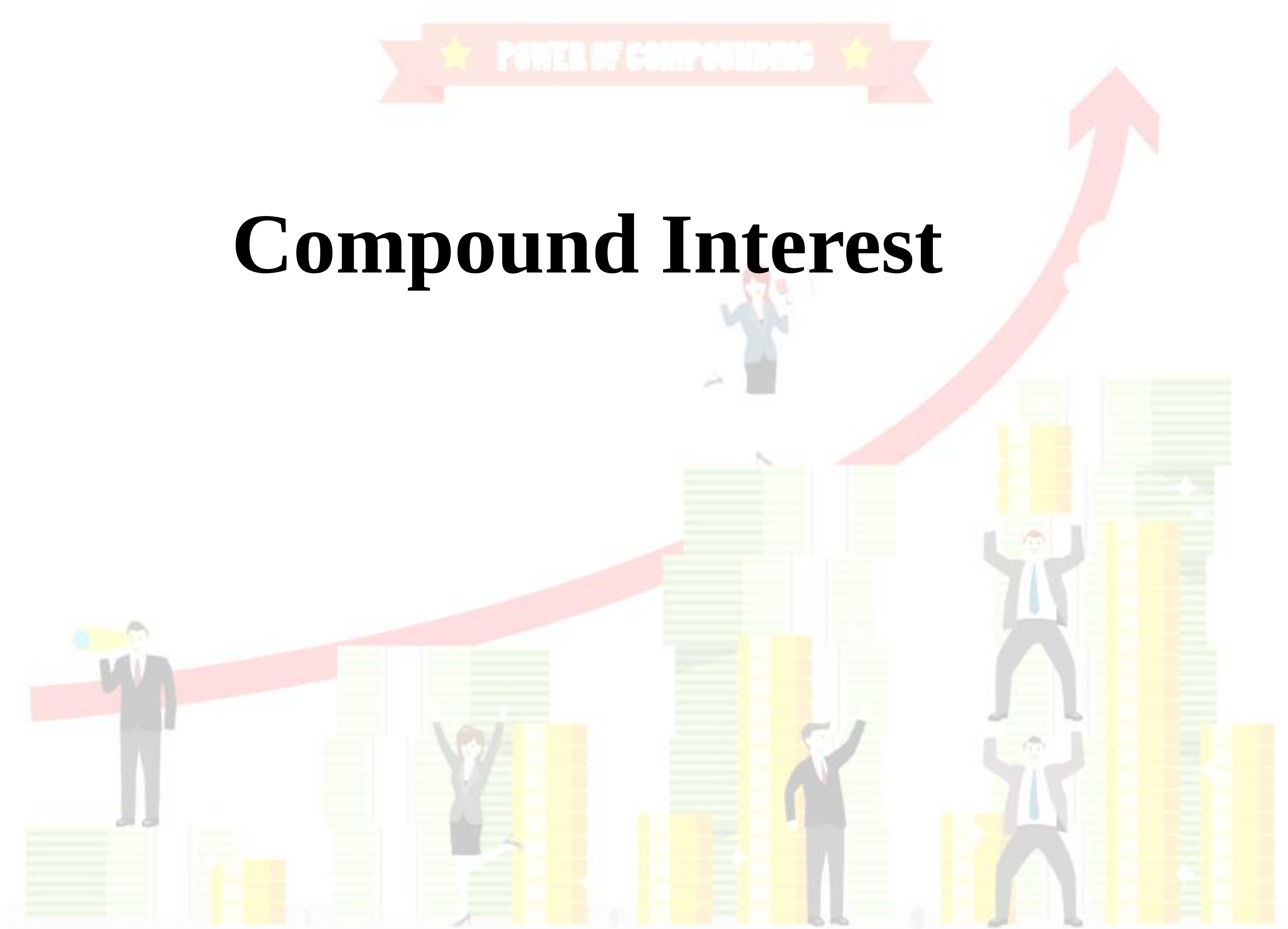
$$\underline{\mathrm{I}} = \$5,000 \quad \text{x} \quad 5 \quad \text{x} \quad 0.07 = \$1,750$$

So, the total amount repaid at the end of five years would be the original amount ($5,000) plus the interest ($1,750), or $6,750.

Example

A student borrows \$3,000 from his uncle in order to finish school. His uncle agrees to charge him simple interest at the rate of 5½% per year. Suppose the student waits two years and then repays the entire loan. How much will he have to repay?

Compound Interest



When interest is *compounded*, the total time period is subdivided into several *interest periods* (e.g., one year, three months, one month). Interest is credited at the end of each interest period and is allowed to accumulate from one interest period to the next.

During a given interest period, the current interest is determined as a percentage of the total amount owed (i.e., the principal plus the previously accumulated interest). Thus, for the first interest period, the interest is determined as

$$I_1 = iP$$

And the total amount accumulated is

$$F_1 = P + I_1 = P + iP = P(1 + i)$$

For the second interest period, the interest is determined as

$$I_2 = iF_1 = i(1 + i)P$$

And the total amount accumulated is

$$F_2 = P + I_1 + I_2 = P + iP + i(1 + i)P = P(1 + i)^2$$

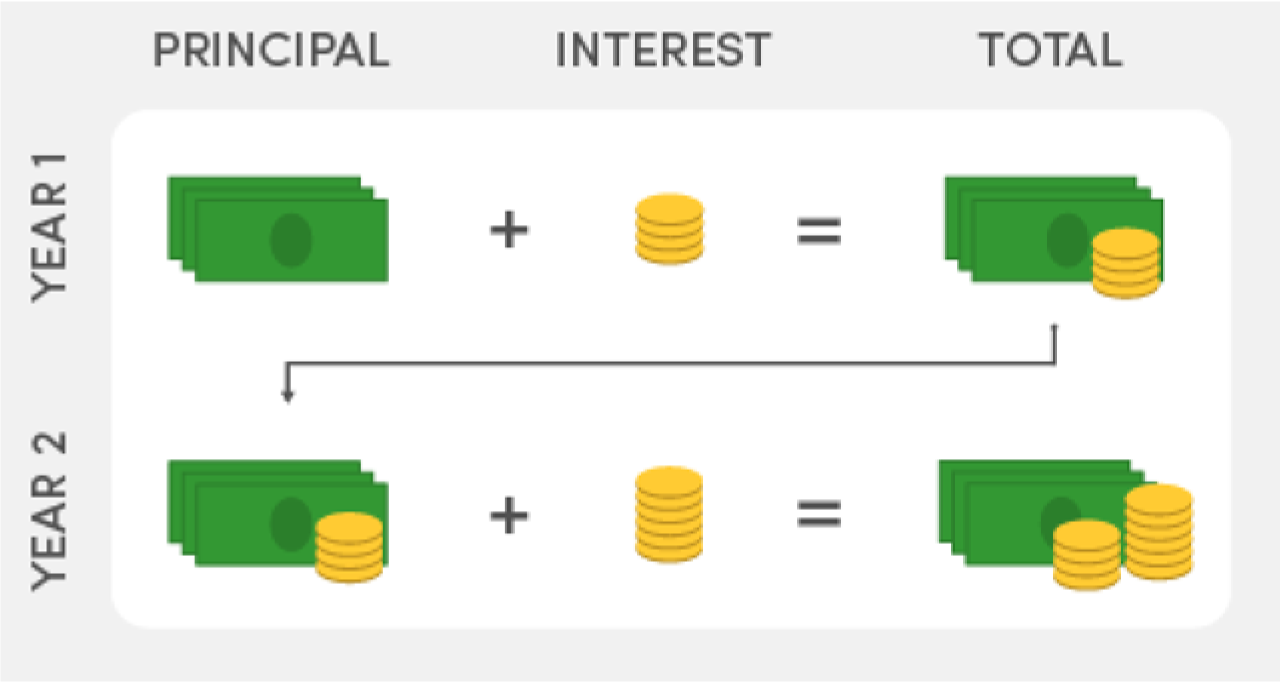
For the third interest period,

$$I_3 = i(1 + i)^2 P \qquad\qquad F_3 = P(1 + i)^3$$

In general, if there are N interest periods

$$F = P(1 + i)^N$$

Which is the so-called law of compound interest. Notice that F, the total amount of money accumulated, increase exponentially with N, the time measured in interest periods.



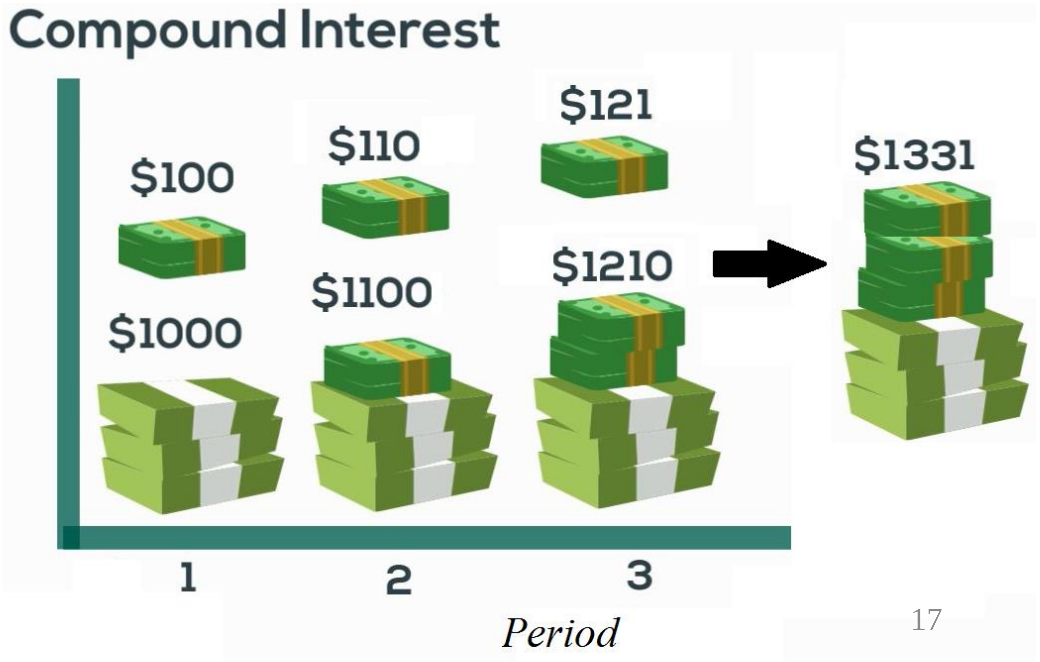
Example

If \$1,000 were loaned for three periods at an interest rate of 10% compounded each period, find

1. The total interest earned =F-P=1331-1000=331

2. The total amount owed at the end of three Periods

F=1000(1+.1)^3=1331



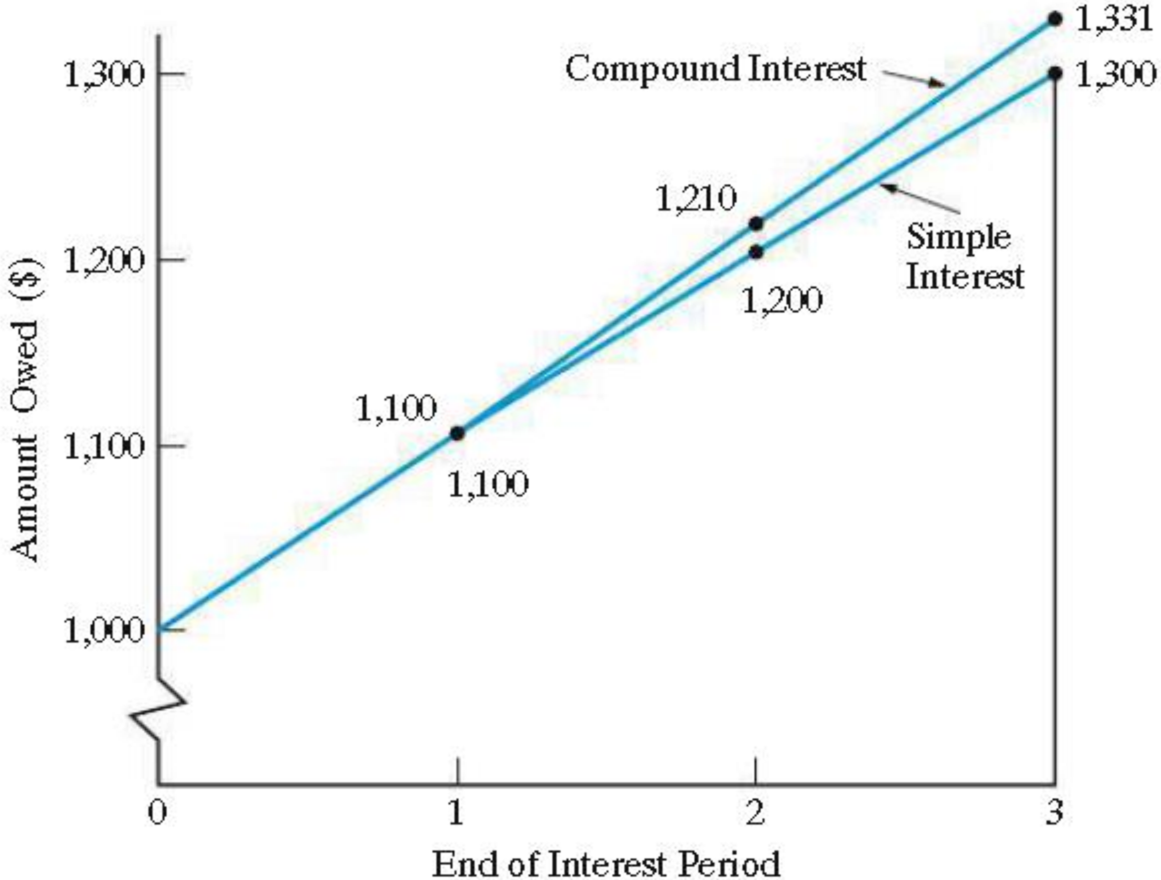
- Compound interest reflects both the remaining principal and any accumulated interest. For $1,000 at 10%…

Period	(1) Amount owed at beginning of period	(2)=(1)x10% Interest amount for period	(3)=(1)+(2) Amount owed at end of period
1	$1,000	$100	$1,100
2	$1,100	$110	$1,210
3	$1,210	$121	$1,331

Compound interest is commonly used in personal and professional financial transactions.

Simple interest does consider the time value of money but does not involve compounding of interest. Compound interest is much more common in practice than simple interest

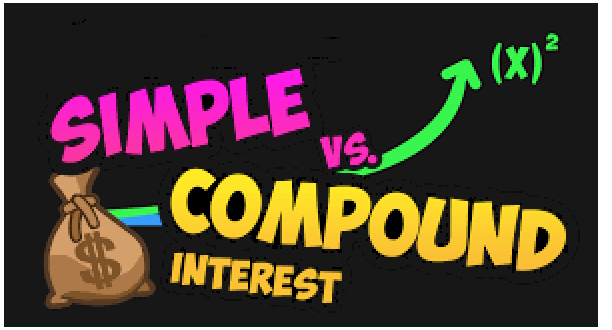
[Compound]



[Simple]

Illustration of Simple versus Compound Interest

Example



A student deposits \$5,000 in a savings account that pays interest at the rate of 6% per year, compounded annually. If all of the money is allowed to accumulate, how much will the student have after 12 years? Compare this with the amount that would have accumulated if simple interest had been paid.

The Concept of Equivalence

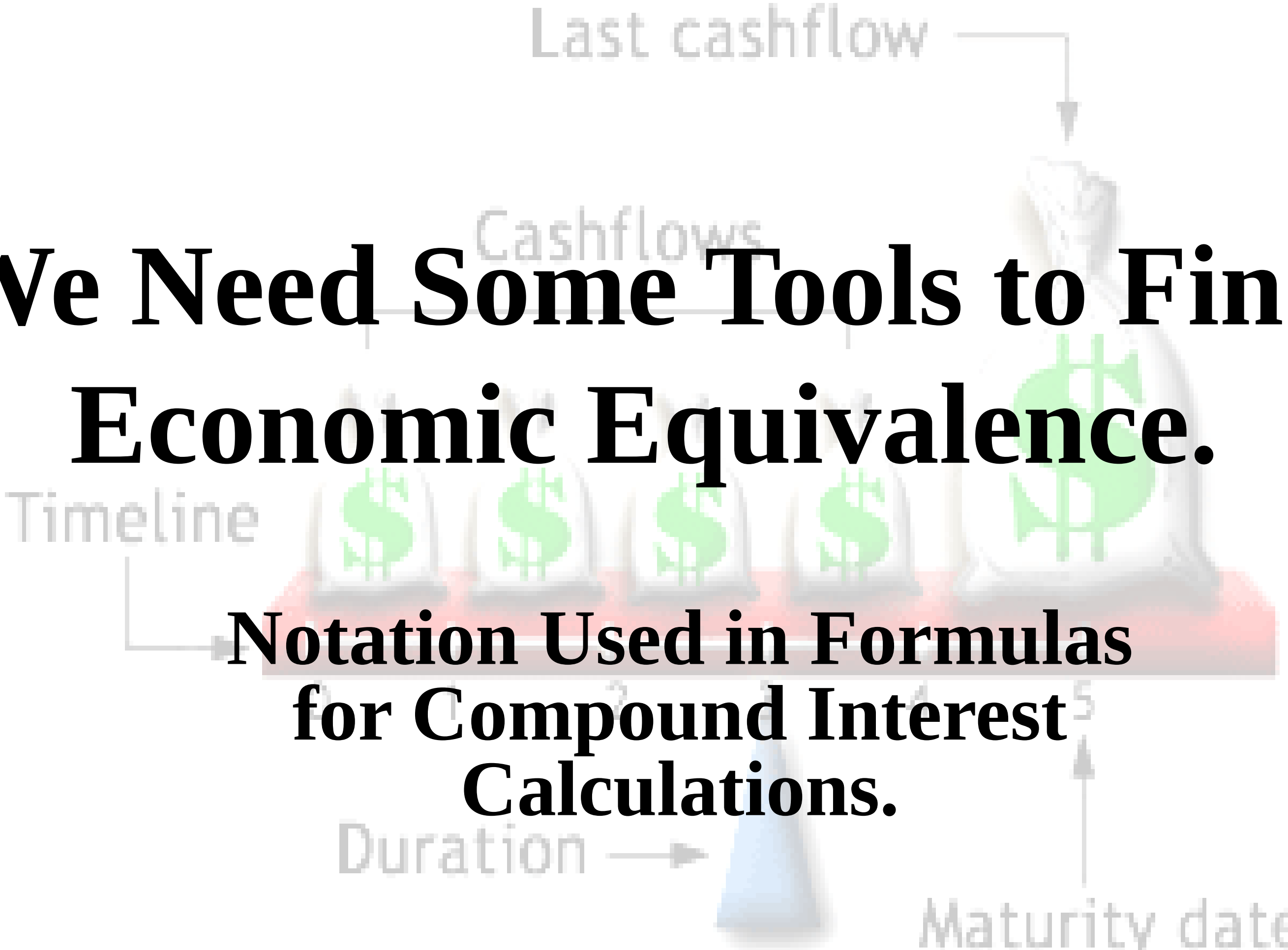
Economic equivalence allows us to compare alternatives on a common basis.

- Each alternative can be reduced to an *equivalent basis* dependent on
  - interest rate,
  - amount of money involved, and
  - timing of monetary receipts or expenses.

- Using these elements, we can "move" cash flows so that we can compare them at particular points in time.

We Need Some Tools to Find Economic Equivalence.

Notation Used in Formulas for Compound Interest Calculations.



The following notation is utilized in formulas for compound interest calculations:

i = effective interest rate per interest period;

N = number of compounding (interest) periods;

P = present sum of money; the *equivalent* value of one or more cash flows at a reference point in time called the present;

F = future sum of money; the *equivalent* value of one or more cash flows a a reference point in time called the future;

A = end-of-period cash flows (or *equivalent* end-of-period values) in a uniform series continuing for a specified number of periods, starting at the end of the first period and continuing through the last period.

Cash Flows

A *cash flow* is the difference between total cash receipts (*inflows*) and total cash disbursements (*outflows*) for a given period of time.

Cash flows are very important in engineering economics because they form the basis for evaluating projects, equipment, and investment alternatives.

The easiest way to visualize a cash flow is through a *cash flow diagram*, in which the individual cash flows are represented as vertical arrows along a horizontal time scale.

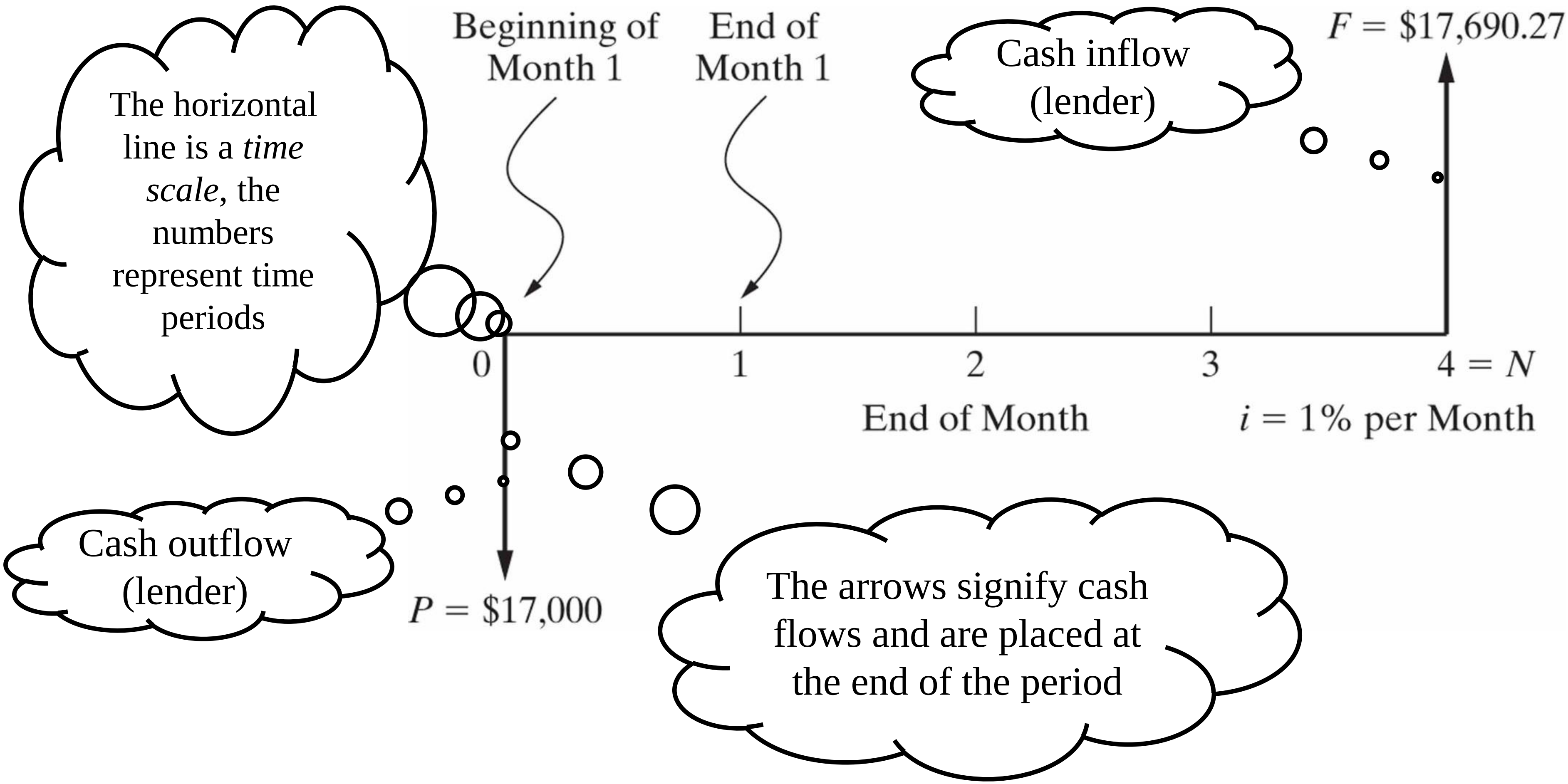
Positive cash flows are represented by upward-pointing arrows, and negative cash flows by downward-pointing arrows; the length of an arrow is proportional to the magnitude of the corresponding cash flow. Each cash flow is assumed to occur at the *end* of the respective time period.

- The use of cash-flow (time) diagrams or tables is strongly recommended for situations in which the analyst needs to clarify or visualize what is involved when flows of money occur at various times.

- The difference between total cash inflows (receipts) and cash outflows (expenditures) for a specified period of time (e.g., one year) is the net cash flow for the period.

"Hold all my calls, Barbara. I'm going to find out how cashflow works."

In a lender-borrower situation, an inflow for the one is an outflow for the other. Hence, the cash flow diagram for the lender will be the mirror image in the timeline of the cash flow diagram for the borrower.



The horizontal line is a *time scale*, the numbers represent time periods
Beginning of Month 1
End of Month 1
Cash inflow (lender)
$F = \$17,690.27$
0
1
2
3
$4 = N$
End of Month
$i = 1\%$ per Month
Cash outflow (lender)
$P = \$17,000$
The arrows signify cash flows and are placed at the end of the period

Example: Cash-Flow Diagramming

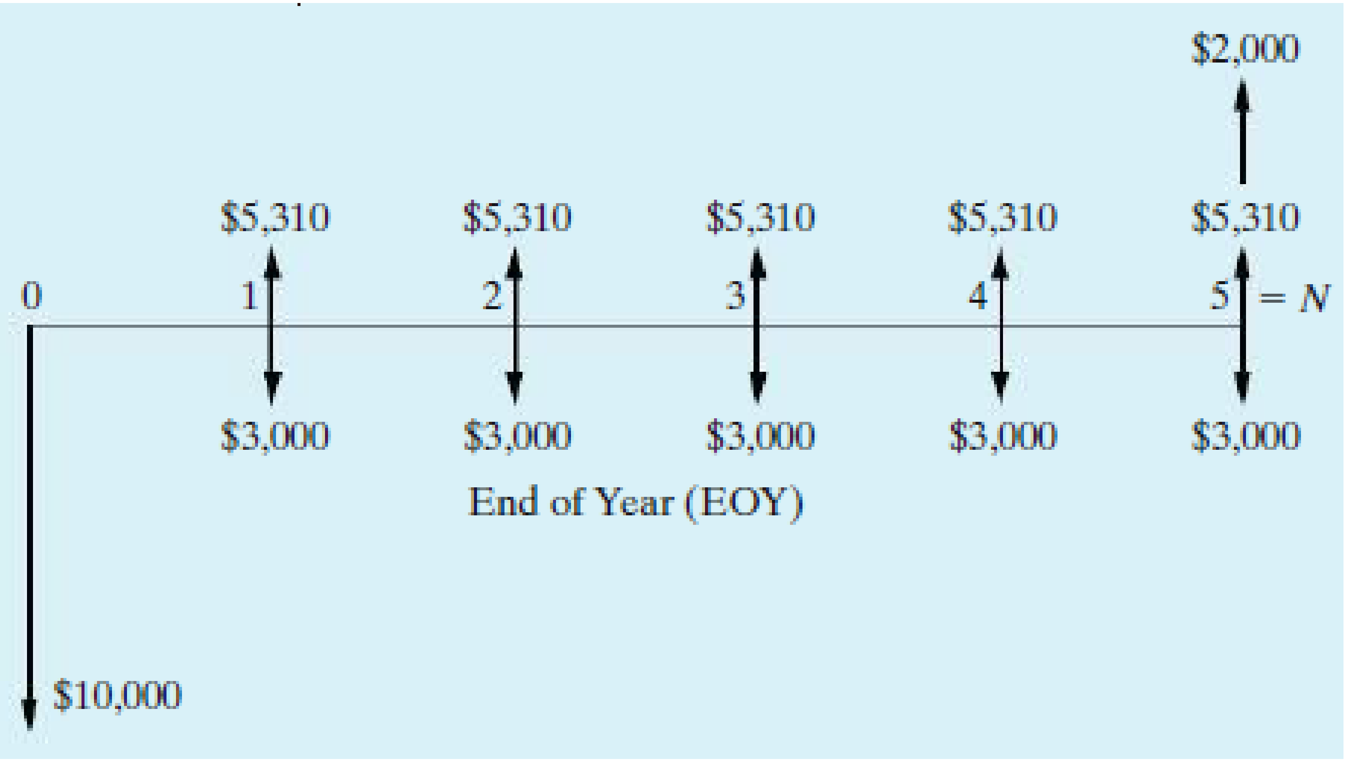
"Well you can't upset me.
Timpson - what's the bad news you've got?"

Before evaluating the economic merits of a proposed investment, the XYZ Corporation insists that its engineers develop a cash-flow diagram of the proposal. An investment of $10,000 can be made that will produce uniform annual revenue of $5,310 for five years and then have a market (recovery) value of $2,000 at the end of year (EOY) five. Annual expenses will be $3,000 at the end of each year for operating and maintaining the project. Draw a cash-flow diagram for the five-year life of the project. Use the ,99viewpoint.??

?}9

Solution

- As shown in the figure below, the initial investment of $10,000 and annual expenses of $3,000 are cash outflows, while annual revenues and the market value are cash inflows.



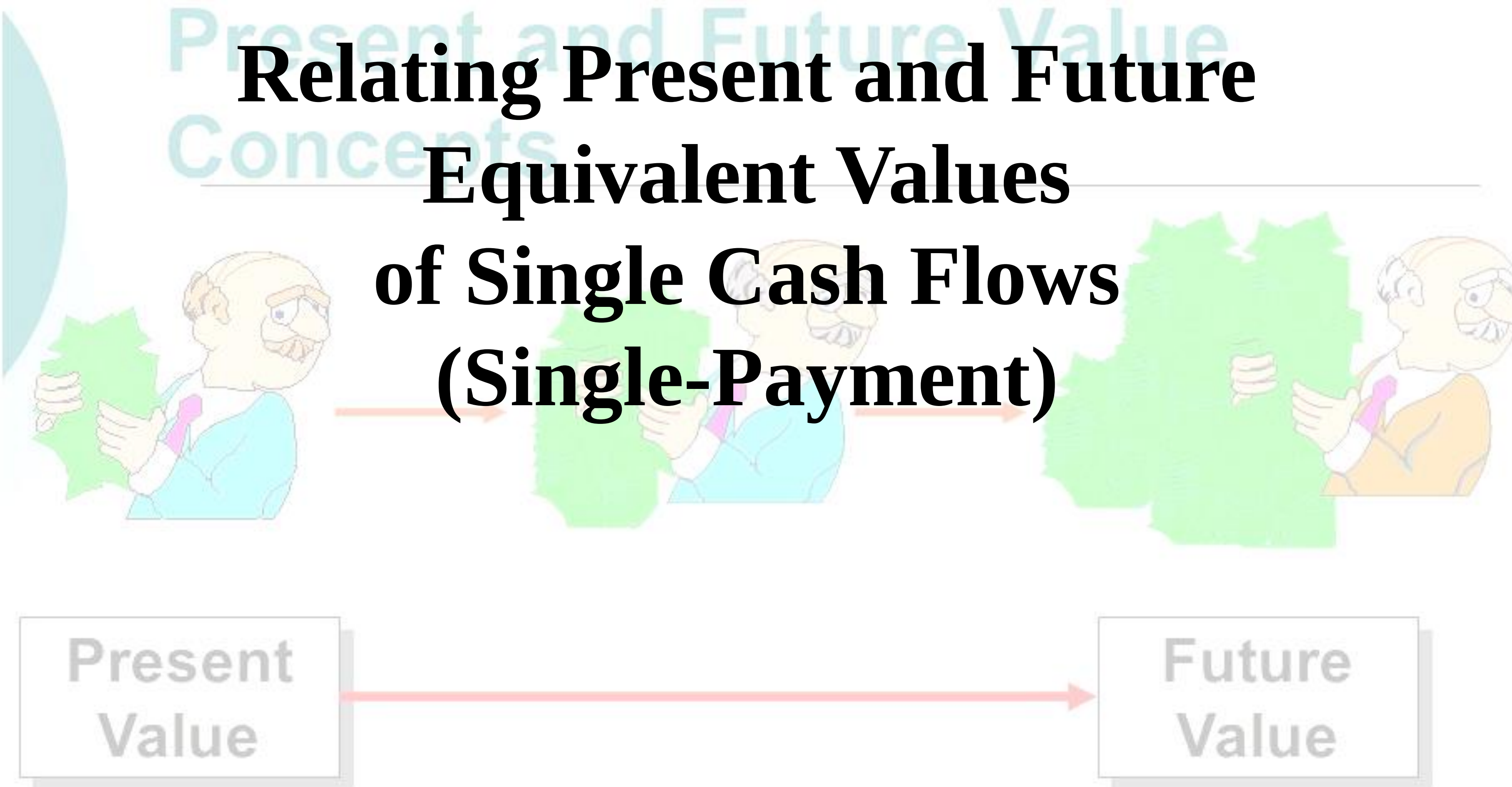
Test Yourself

Suppose that you have a savings plan covering the next ten years, according to which you put aside $600 today, $500 at the end of every other year for the next five years, and $400 at the end of each year for the remaining five years. As part of this plan, you expected to withdraw $300 at the end of every year for the first 3 years, and $350 at the end of every other year thereafter.

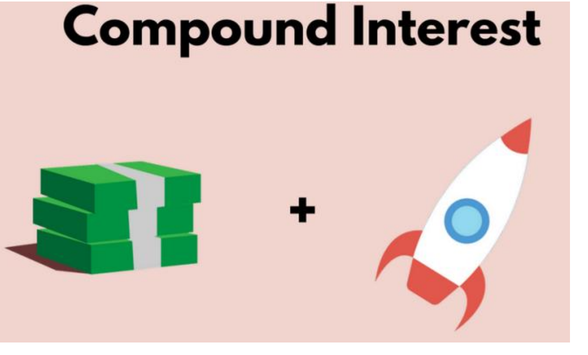
a. Tabulate *your* cash flows

b. Draw *your* cash flow diagram.

Relating Present and Future Equivalent Values of Single Cash Flows (Single-Payment)



- We can apply compound interest formulas to "move" cash flows along the cash flow diagram.



Single-Payment, Compound-Amount Factor
(Finding *F* when Given *P*)

Suppose that a given sum of money, P, earns interest at a rate *i*, compounded annually. We have already seen that the total amount of money, F, which will have accumulated from an investment of P dollars after N years is given by:

$$F = P(1 + i)^N$$

The ratio

$$F/P = (1 + i)^N$$

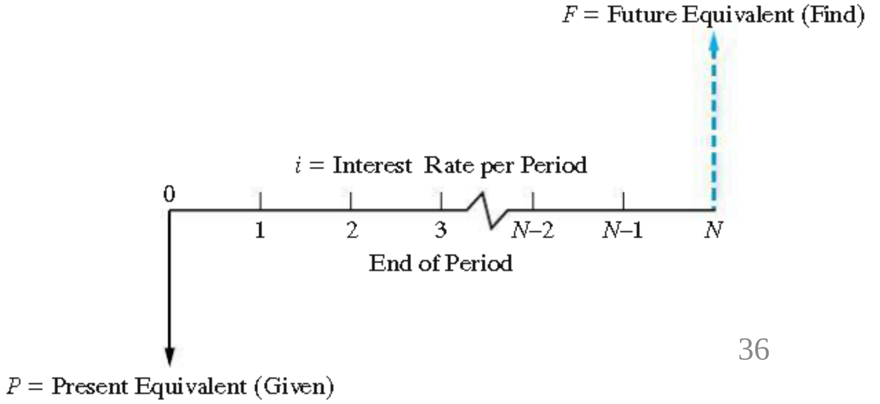
Is called the *single-payment, compound-amount* factor. Numerical values of this factor may be calculated from previous equation or obtained from compound interest tables such as those shown in Appendix *C*.

A fuller notation, *(F/P, i%, N)*, is helpful when setting up the solution to a compound interest problem.

$(1+i)^N = (F/P, i\%, N)$

$F = P(1 + i)^N$

$F = P(F/P, i\%, N)$



Example: Future Equivalent of a Present Sum

Suppose that you borrow $8,000 now, promising to repay the loan principal plus accumulated interest in four years at i = 10% per year. How much would you repay at the end of four years?

Solution

In general, we see that $F = P(1+i)^N$, and the total amount to be repaid is **$11,713**.

Example - Revisited

Suppose that you borrow \$8,000 now, promising to repay the loan principal plus accumulated interest in four years at i = 10% per year. How much would you repay at the end of four years?

TABLE C-13 Discrete Compounding; $i = 10\%$

	Single Payment		Uniform Series				Uniform Gradient		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor	
	To Find F Given P	To Find P Given F	To Find F Given A	To Find P Given A	To Find A Given F	To Find A Given P	To Find P Given G	To Find A Given G	
N	F/P	P/F	F/A	P/A	A/F	A/P	P/G	A/G	N
1	1.1000	0.9091	1.0000	0.9091	1.0000	1.1000	0.000	0.0000	1
2	1.2100	0.8264	2.1000	1.7355	0.4762	0.5762	0.826	0.4762	2
3	1.3310	0.7513	3.3100	2.4869	0.3021	0.4021	2.329	0.9366	3
4	1.4641	0.6830	4.6410	3.1699	0.2155	0.3155	4.378	1.3812	4
5	1.6105	0.6209	6.1051	3.7908	0.1638	0.2638	6.862	1.8101	5
6	1.7716	0.5645	7.7156	4.3553	0.1296	0.2296	9.684	2.2236	6
7	1.9487	0.5132	9.4872	4.8684	0.1054	0.2054	12.763	2.6216	7
8	2.1436	0.4665	11.4359	5.3349	0.0874	0.1874	16.029	3.0045	8
9	2.3579	0.4241	13.5795	5.7590	0.0736	0.1736	19.422	3.3724	9
10	2.5937	0.3855	15.9374	6.1446	0.0627	0.1627	22.891	3.7255	10
11	2.8531	0.3505	18.5312	6.4951	0.0540	0.1540	26.396	4.0641	11
12	3.1384	0.3186	21.3843	6.8137	0.0468	0.1468	29.901	4.3884	12
13	3.4523	0.2897	24.5227	7.1034	0.0408	0.1408	33.377	4.6988	13
14	3.7975	0.2633	27.9750	7.3667	0.0357	0.1357	36.801	4.9955	14
15	4.1772	0.2394	31.7725	7.6061	0.0315	0.1315	40.152	5.2789	15
16	4.5950	0.2176	35.9497	7.8237	0.0278	0.1278	43.416	5.5493	16
17	5.0545	0.1978	40.5447	8.0216	0.0247	0.1247	46.582	5.8071	17
18	5.5599	0.1799	45.5992	8.2014	0.0219	0.1219	49.640	6.0526	18
19	6.1159	0.1635	51.1591	8.3649	0.0195	0.1195	52.583	6.2861	19

Example: $2,500 at time zero is equivalent to how much after six years if the interest rate is 8% per year?

TABLE C-11 Discrete Compounding; $i = 8\%$

	Single Payment		Uniform Series				Uniform Gradient		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor	
N	To Find F Given P F/P	To Find P Given F P/F	To Find F Given A F/A	To Find P Given A P/A	To Find A Given F A/F	To Find A Given P A/P	To Find P Given G P/G	To Find A Given G A/G	N
1	1.0800	0.9259	1.0000	0.9259	1.0000	1.0800	0.000	0.0000	1
2	1.1664	0.8573	2.0800	1.7833	0.4808	0.5608	0.857	0.4808	2
3	1.2597	0.7938	3.2464	2.5771	0.3080	0.3880	2.445	0.9487	3
4	1.3605	0.7350	4.5061	3.3121	0.2219	0.3019	4.650	1.4040	4
5	1.4693	0.6806	5.8666	3.9927	0.1705	0.2505	7.372	1.8465	5
6	1.5869	0.6302	7.3359	4.6229	0.1363	0.2163	10.523	2.2763	6
7	1.7138	0.5835	8.9228	5.2064	0.1121	0.1921	14.024	2.6937	7
8	1.8509	0.5403	10.6366	5.7466	0.0940	0.1740	17.806	3.0985	8
9	1.9990	0.5002	12.4876	6.2469	0.0801	0.1601	21.808	3.4910	9
10	2.1589	0.4632	14.4866	6.7101	0.0690	0.1490	25.977	3.8713	10
11	2.3316	0.4289	16.6455	7.1390	0.0601	0.1401	30.266	4.2395	11
12	2.5182	0.3971	18.9771	7.5361	0.0527	0.1327	34.634	4.5957	12
13	2.7196	0.3677	21.4953	7.9038	0.0465	0.1265	39.046	4.9402	13
14	2.9372	0.3405	24.2149	8.2442	0.0413	0.1213	43.472	5.2731	14
15	3.1722	0.3152	27.1521	8.5595	0.0368	0.1168	47.886	5.5945	15
16	3.4259	0.2919	30.3243	8.8514	0.0330	0.1130	52.264	5.9046	16
17	3.7000	0.2703	33.7502	9.1216	0.0296	0.1096	56.588	6.2037	17
18	3.9960	0.2502	37.4502	9.3719	0.0267	0.1067	60.843	6.4920	18
19	4.3157	0.2317	41.4463	9.6036	0.0241	0.1041	65.013	6.7697	19
20	4.6610	0.2145	45.7620	9.8181	0.0219	0.1019	69.090	7.0369	20
21	5.0338	0.1987	50.4229	10.0168	0.0198	0.0998	73.063	7.2940	21
22	5.4365	0.1839	55.4568	10.2007	0.0180	0.0980	76.926	7.5412	22
23	5.8715	0.1703	60.8933	10.3711	0.0164	0.0964	80.673	7.7786	23
24	6.3412	0.1577	66.7648	10.5288	0.0150	0.0950	84.300	8.0066	24
25	6.8485	0.1460	73.1059	10.6748	0.0137	0.0937	87.804	8.2254	25
30	10.0627	0.0994	113.2832	11.2578	0.0088	0.0888	103.456	9.1897	30
35	14.7853	0.0676	172.3168	11.6546	0.0058	0.0858	116.092	9.9611	35
40	21.7245	0.0460	259.0565	11.9246	0.0039	0.0839	126.042	10.5699	40
45	31.9204	0.0313	386.5056	12.1084	0.0026	0.0826	133.733	11.0447	45
50	46.9016	0.0213	573.7702	12.2335	0.0017	0.0817	139.593	11.4107	50
60	101.2571	0.0099	1253.2133	12.3766	0.0008	0.0808	147.300	11.9015	60
80	471.9548	0.0021	5886.9354	12.4735	0.0002	0.0802	153.800	12.3301	80
100	2199.7613	0.0005	27484.5157	12.4943	^{a}	0.0800	155.611	12.4545	100
∞				12.5000		0.0800			∞

^{a} Less than 0.0001.

Examples

1. A student deposits $1,000 in a savings account that pays interest at the rate of 6% per year, compounded annually. If all of the money is allowed to accumulate, how much money will the student have after 12 years?

2. $2,500 at time zero is equivalent to how much after six years if the interest rate is 8% per year?

Solution

$$F = \$2,500(F/P, 8\%, 6) = \$2,500(1.5869) = \$3,967$$

Single-Payment, Present-Worth Factor
(Finding *P* when Given *F*)

- From Equation $F = P(1 + i)^N$. Solving this for P gives the relationship

$$P = F\left(\frac{1}{1+i}\right)^N = \boldsymbol{F(1+i)^{-N}}$$

- The quantity $(1 + i)^{-N}$ is called the *single-payment, present-worth factor*. Numerical values for this factor are given in the third column of the tables in Appendix C for a wide range of values of i and N.

$$(1+i)^{-N} = (P/F, i\%, N)$$

$$P = F\,(P/F, i\%, N)$$

Example: Present Equivalent of a Future Amount of Money

An investor (owner) has an option to purchase a tract of land that will be worth $10,000 in six years. If the value of the land increases at 8% each year, how much should the investor be willing to pay now for this property?

Solution

The purchase price can be determined from Equation (4-5) and Table C-11 in Appendix C as follows:

$P = \$10{,}000(P/F, 8\%, 6)$

$P = \$10{,}000(0.6302)$

$= \$6{,}302.$

Example: $3,000 at the end of year seven is equivalent to how much today (time zero) if the interest rate is 6% per year?

TABLE C-9 Discrete Compounding; i = 6%

	Single Payment		Uniform Series				Uniform Gradient		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor	
N	To Find F Given P F/P	To Find P Given F P/F	To Find F Given A F/A	To Find P Given A P/A	To Find A Given F A/F	To Find A Given P A/P	To Find P Given G P/G	To Find A Given G A/G	N
1	1.0600	0.9434	1.0000	0.9434	1.0000	1.0600	0.000	0.0000	1
2	1.1236	0.8900	2.0600	1.8334	0.4854	0.5454	0.890	0.4854	2
3	1.1910	0.8396	3.1836	2.6730	0.3141	0.3741	2.569	0.9612	3
4	1.2625	0.7921	4.3746	3.4651	0.2286	0.2886	4.946	1.4272	4
5	1.3382	0.7473	5.6371	4.2124	0.1774	0.2374	7.935	1.8836	5
6	1.4185	0.7050	6.9753	4.9173	0.1434	0.2034	11.459	2.3304	6
7	1.5036	0.6651	8.3938	5.5824	0.1191	0.1791	15.450	2.7676	7
8	1.5938	0.6274	9.8975	6.2098	0.1010	0.1610	19.842	3.1952	8
9	1.6895	0.5919	11.4913	6.8017	0.0870	0.1470	24.577	3.6133	9
10	1.7908	0.5584	13.1808	7.3601	0.0759	0.1359	29.602	4.0220	10
11	1.8983	0.5268	14.9716	7.8869	0.0668	0.1268	34.870	4.4213	11
12	2.0122	0.4970	16.8699	8.3838	0.0593	0.1193	40.337	4.8113	12
13	2.1329	0.4688	18.8821	8.8527	0.0530	0.1130	45.963	5.1920	13
14	2.2609	0.4423	21.0151	9.2950	0.0476	0.1076	51.713	5.5635	14
15	2.3966	0.4173	23.2760	9.7122	0.0430	0.1030	57.555	5.9260	15
16	2.5404	0.3936	25.6725	10.1059	0.0390	0.0990	63.459	6.2794	16
17	2.6928	0.3714	28.2129	10.4773	0.0354	0.0954	69.401	6.6240	17
18	2.8543	0.3503	30.9057	10.8276	0.0324	0.0924	75.357	6.9597	18
19	3.0256	0.3305	33.7600	11.1581	0.0296	0.0896	81.306	7.2867	19
20	3.2071	0.3118	36.7856	11.4699	0.0272	0.0872	87.230	7.6051	20
21	3.3996	0.2942	39.9927	11.7641	0.0250	0.0850	93.114	7.9151	21
22	3.6035	0.2775	43.3923	12.0416	0.0230	0.0830	98.941	8.2166	22
23	3.8197	0.2618	46.9958	12.3034	0.0213	0.0813	104.701	8.5099	23
24	4.0489	0.2470	50.8156	12.5504	0.0197	0.0797	110.381	8.7951	24
25	4.2919	0.2330	54.8645	12.7834	0.0182	0.0782	115.973	9.0722	25
30	5.7435	0.1741	79.0582	13.7648	0.0126	0.0726	142.359	10.3422	30
35	7.6861	0.1301	111.4348	14.4982	0.0090	0.0690	165.743	11.4319	35
40	10.2857	0.0972	154.7620	15.0463	0.0065	0.0665	185.957	12.3590	40
45	13.7646	0.0727	212.7435	15.4558	0.0047	0.0647	203.110	13.1413	45
50	18.4202	0.0543	290.3359	15.7619	0.0034	0.0634	217.457	13.7964	50
60	32.9877	0.0303	533.1282	16.1614	0.0019	0.0619	239.043	14.7909	60
80	105.7960	0.0095	1746.5999	16.5091	0.0006	0.0606	262.549	15.9033	80

Examples

1. A certain sum of money will be deposited in a savings account that pays interest at the rate of 6% per year, compounded annually. If all of the money is allowed to accumulate, how much must be deposited initially so that $5,000 will have accumulated after 10 years?

2. $3,000 at the end of year seven is equivalent to how much today (time zero) if the interest rate is 6% per year?

Solution

$$P = \$3,000(P/F, 6\%, 7) = \$3,000(0.6651) = \$1,995$$

Test Yourself

Betty will need $12,000 in five years to pay for a major overhaul on her tractor engine. She has found an investment that will provide a 5% return on her invested funds. How much does Betty need to invest today so she will have her overhaul funds in five years?

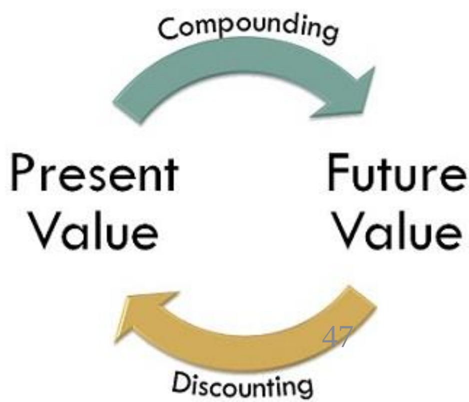
Important Rules

Rule A. Cash flows cannot be added or subtracted unless they occur at the same point in time.

Rule B. To move a cash flow forward in time by one time unit, multiply the magnitude of the cash flow by (1 + *i*), where *i* is the interest rate that reflects the time value of money. (Compounding)

Rule C. To move a cash flow backward in time by one time unit, divide the magnitude of the cash flow by (1 + *i*).

(Discounting)



Unknown Interest Rate
(Finding the Interest Rate Given *P*, *F*, and *N*)

There are situations in which we know two sums of money (*P* and *F*) and how much time separates them (*N*), but we don't know the interest rate (*i*) that makes them equivalent.

$$\boldsymbol{i = \sqrt[N]{F/P} - 1}$$

Example: The Inflating Price of Gasoline

The average price of gasoline in 2005 was $2.31 per gallon. In 1993, the average price was $1.07. What was the average annual rate of increase in the price of gasoline over this 12-year period?

Solution

With respect to the year 1993, the year 2005 is in the future. Thus, $P = \$1.07$, $F = \$2.31$, and $N = 12$. Using Equation (4-6), we find $i = \sqrt[12]{2.31/1.07} - 1 = 0.0662$ or 6.62% per year.

Example:

If we want to turn \$500 into \$1,000 over a period of 10 years, at what interest rate would we have to invest it?

Unknown Number of Years (Finding *N* when Given *P*, *F*, and *i*)

Sometimes we are interested in finding the amount of time needed for a present sum to grow into a future sum at a specified interest rate.

$$F = P(1 + i)^N$$

$$(1 + i)^N = (F/P)$$

Using logarithms,

$$N \log(1 + i) = \log\left({}^{F}/_{P}\right)$$

$$\boldsymbol{N = \frac{\log\left({}^{F}/_{P}\right)}{log(1 + i)}}$$

Example: When Will Gasoline Cost $5.00 per Gallon?

In the previous example, the average price of gasoline was given as $2.31 in 2005. We computed the average annual rate of increase in the price of gasoline to be 6.62%. If we assume that the price of gasoline will continue to inflate at this rate, how long will it be before we are paying $5.00 per gallon?

Solution

We have $P = \$2.31$, $F = \$5.00$, and $i = 6.62\%$ per year. Using Equation (4-7), we find

$$N = \frac{\log(\$5.00/\$2.31)}{\log(1 + 0.0662)} = \frac{\log(2.1645)}{\log(1.0662)} = 12.05 \text{ years.}$$

So, if gasoline prices continue to increase at the same rate, we can expect to be paying $5.00 per gallon in 2017.

Example:

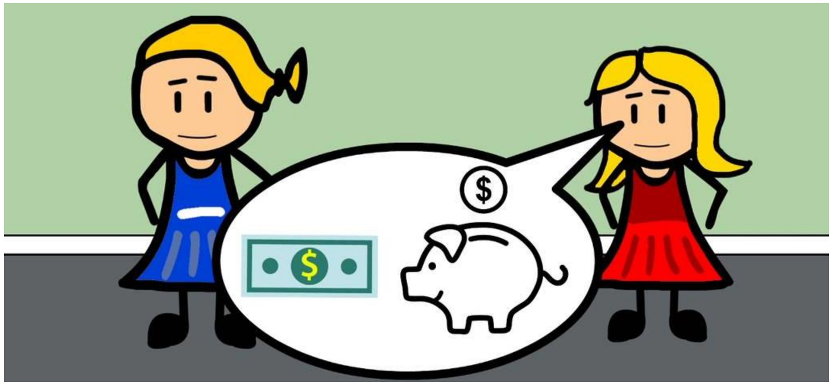
How long would it take for $500 invested today at 15% interest per year to be worth $1,000?

Example

How many years will be required for a given sum of money to triple, if it is deposited in a bank account that pays 6% per year, compounded annually?

Examples

Example 1: What's the FV of an initial $100 after 3 years if i = 10%?



Example 2: What's the PV of $100 due in 3 years if i = 10%?

Example 3: If sales grow at 8% per year, how long before sales double?

Example 4: Your cousin want to buy a fancy watch with $425 , instead you suggest that she buy an inexpensive watch with $25 and invest the difference (i.e. $400) for 40 years with an interest of 9% per year how much she will get after 40 years ?

Example 5: How long dose it take for $1,000 to quadruple in value when the interest rate is 8%?

Example 6: A person invested 100,000 JD with a 9% interest for five years. How many did he have at the end of the fifth year?

Example 7: How much will be the population of Jordan after fifty years if the number now is ten million and the increase rate is 3.5%

Example 8: A person borrowed 50 thousand JD for six years at an annual interest rate of 15%. If he paid 50 thousand JD at the end of the second year .How much he must paid at the end of the sixth year?

Example 9: You deposited $100,000 in a bank giving annual interest rate of 8%. At the end of the first year you deposited $10,000 and at the end of the fourth year you withdrew $20,000 from the balance. At the end of the fifth year you withdrew another $20,000. At the end of the seventh year, you deposited $10,000 in your balance. How much will be in your balance at the end of the twelfth year?

TABLE C-8 Discrete Compounding; $i = 5\%$

	Single Payment		Uniform Series				Uniform Gradient		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor	
	To Find F Given P	To Find P Given F	To Find F Given A	To Find P Given A	To Find A Given F	To Find A Given P	To Find P Given G	To Find A Given G	
N	F/P	P/F	F/A	P/A	A/F	A/P	P/G	A/G	N
1	1.0500	0.9524	1.0000	0.9524	1.0000	1.0500	0.000	0.0000	1
2	1.1025	0.9070	2.0500	1.8594	0.4878	0.5378	0.907	0.4878	2
3	1.1576	0.8638	3.1525	2.7232	0.3172	0.3672	2.635	0.9675	3
4	1.2155	0.8227	4.3101	3.5460	0.2320	0.2820	5.103	1.4391	4
5	1.2763	0.7835	5.5256	4.3295	0.1810	0.2310	8.237	1.9025	5
6	1.3401	0.7462	6.8019	5.0757	0.1470	0.1970	11.968	2.3579	6
7	1.4071	0.7107	8.1420	5.7864	0.1228	0.1728	16.232	2.8052	7
8	1.4775	0.6768	9.5491	6.4632	0.1047	0.1547	20.970	3.2445	8
9	1.5513	0.6446	11.0266	7.1078	0.0907	0.1407	26.127	3.6758	9
10	1.6289	0.6139	12.5779	7.7217	0.0795	0.1295	31.652	4.0991	10
11	1.7103	0.5847	14.2068	8.3064	0.0704	0.1204	37.499	4.5144	11
12	1.7959	0.5568	15.9171	8.8633	0.0628	0.1128	43.624	4.9219	12
13	1.8856	0.5303	17.7130	9.3936	0.0565	0.1065	49.988	5.3215	13
14	1.9799	0.5051	19.5986	9.8986	0.0510	0.1010	56.554	5.7133	14
15	2.0789	0.4810	21.5786	10.3797	0.0463	0.0963	63.288	6.0973	15
16	2.1829	0.4581	23.6575	10.8378	0.0423	0.0923	70.160	6.4736	16
17	2.2920	0.4363	25.8404	11.2741	0.0387	0.0887	77.141	6.8423	17
18	2.4066	0.4155	28.1324	11.6896	0.0355	0.0855	84.204	7.2034	18
19	2.5270	0.3957	30.5390	12.0853	0.0327	0.0827	91.328	7.5569	19
20	2.6533	0.3769	33.0660	12.4622	0.0302	0.0802	98.488	7.9030	20
21	2.7860	0.3589	35.7193	12.8212	0.0280	0.0780	105.667	8.2416	21
22	2.9253	0.3418	38.5052	13.1630	0.0260	0.0760	112.846	8.5730	22
23	3.0715	0.3256	41.4305	13.4886	0.0241	0.0741	120.009	8.8971	23
24	3.2251	0.3101	44.5020	13.7986	0.0225	0.0725	127.140	9.2140	24
25	3.3864	0.2953	47.7271	14.0939	0.0210	0.0710	134.228	9.5238	25
30	4.3219	0.2314	66.4388	15.3725	0.0151	0.0651	168.623	10.9691	30
35	5.5160	0.1813	90.3203	16.3742	0.0111	0.0611	200.581	12.2498	35
40	7.0400	0.1420	120.7998	17.1591	0.0083	0.0583	229.545	13.3775	40
45	8.9850	0.1113	159.7002	17.7741	0.0063	0.0563	255.315	14.3644	45
50	11.4674	0.0872	209.3480	18.2559	0.0048	0.0548	277.915	15.2233	50
60	18.6792	0.0535	353.5837	18.9293	0.0028	0.0528	314.343	16.6062	60
80	49.5614	0.0202	971.2288	19.5965	0.0010	0.0510	359.646	18.3526	80
100	131.5013	0.0076	2610.0252	19.8479	0.0004	0.0504	381.749	19.2337	100
∞				20.0000		0.0500			∞

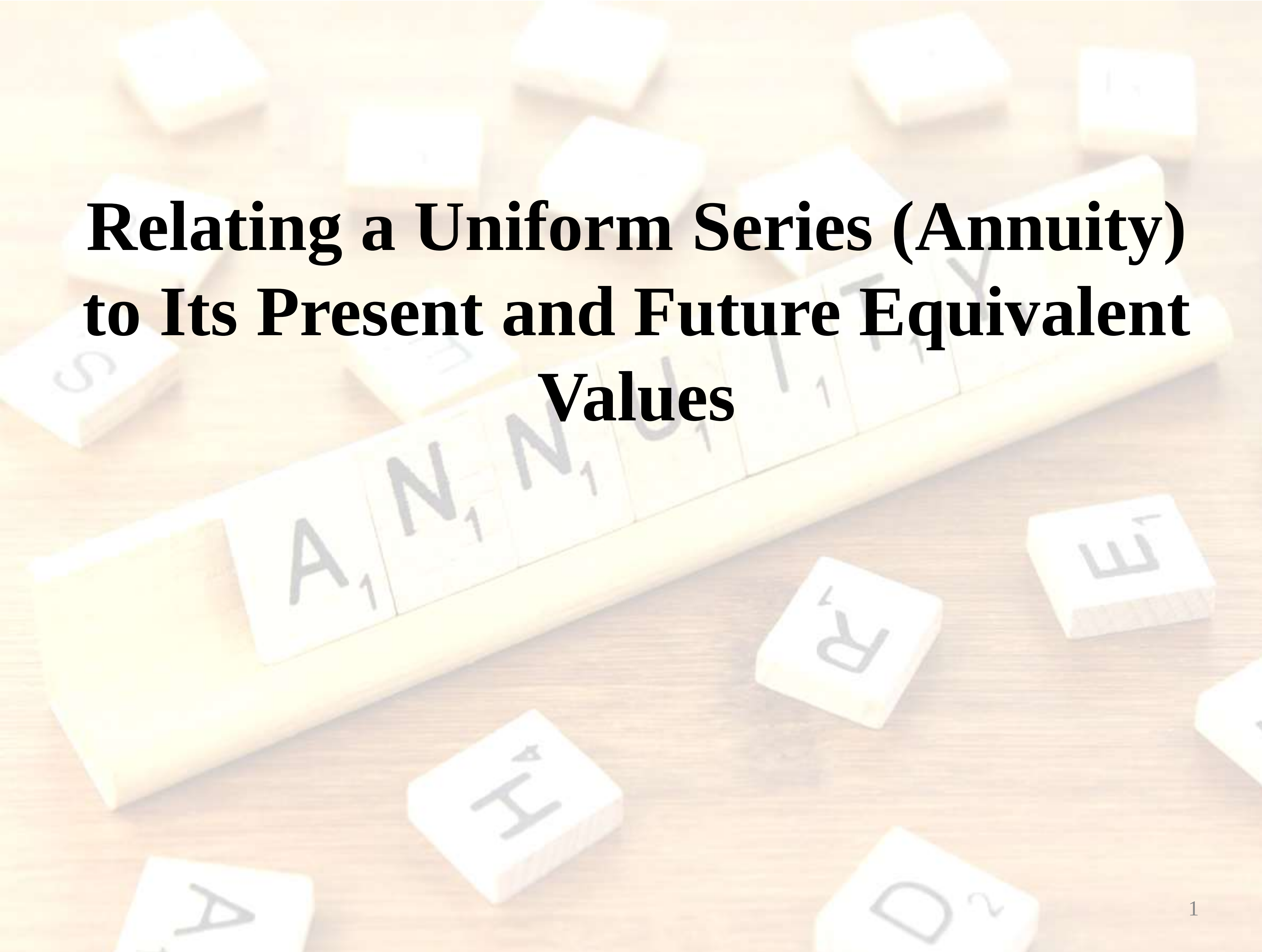
TABLE C-12 Discrete Compounding; $i = 9\%$

	Single Payment		Uniform Series				Uniform Gradient		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor	
	To Find F Given P	To Find P Given F	To Find F Given A	To Find P Given A	To Find A Given F	To Find A Given P	To Find P Given G	To Find A Given G	
N	F/P	P/F	F/A	P/A	A/F	A/P	P/G	A/G	N
1	1.0900	0.9174	1.0000	0.9174	1.0000	1.0900	0.000	0.0000	1
2	1.1881	0.8417	2.0900	1.7591	0.4785	0.5685	0.842	0.4785	2
3	1.2950	0.7722	3.2781	2.5313	0.3051	0.3951	2.386	0.9426	3
4	1.4116	0.7084	4.5731	3.2397	0.2187	0.3087	4.511	1.3925	4
5	1.5386	0.6499	5.9847	3.8897	0.1671	0.2571	7.111	1.8282	5
6	1.6771	0.5963	7.5233	4.4859	0.1329	0.2229	10.092	2.2498	6
7	1.8280	0.5470	9.2004	5.0330	0.1087	0.1987	13.375	2.6574	7
8	1.9926	0.5019	11.0285	5.5348	0.0907	0.1807	16.888	3.0512	8
9	2.1719	0.4604	13.0210	5.9952	0.0768	0.1668	20.571	3.4312	9
10	2.3674	0.4224	15.1929	6.4177	0.0658	0.1558	24.373	3.7978	10
11	2.5804	0.3875	17.5603	6.8052	0.0569	0.1469	28.248	4.1510	11
12	2.8127	0.3555	20.1407	7.1607	0.0497	0.1397	32.159	4.4910	12
13	3.0658	0.3262	22.9534	7.4869	0.0436	0.1336	36.073	4.8182	13
14	3.3417	0.2992	26.0192	7.7862	0.0384	0.1284	39.963	5.1326	14
15	3.6425	0.2745	29.3609	8.0607	0.0341	0.1241	43.807	5.4346	15
16	3.9703	0.2519	33.0034	8.3126	0.0303	0.1203	47.585	5.7245	16
17	4.3276	0.2311	36.9737	8.5436	0.0270	0.1170	51.282	6.0024	17
18	4.7171	0.2120	41.3013	8.7556	0.0242	0.1142	54.886	6.2687	18
19	5.1417	0.1945	46.0185	8.9501	0.0217	0.1117	58.387	6.5236	19
20	5.6044	0.1784	51.1601	9.1285	0.0195	0.1095	61.777	6.7674	20
21	6.1088	0.1637	56.7645	9.2922	0.0176	0.1076	65.051	7.0006	21
22	6.6586	0.1502	62.8733	9.4424	0.0159	0.1059	68.205	7.2232	22
23	7.2579	0.1378	69.5319	9.5802	0.0144	0.1044	71.236	7.4357	23
24	7.9111	0.1264	76.7898	9.7066	0.0130	0.1030	74.143	7.6384	24
25	8.6231	0.1160	84.7009	9.8226	0.0118	0.1018	76.927	7.8316	25
30	13.2677	0.0754	136.3075	10.2737	0.0073	0.0973	89.028	8.6657	30
35	20.4140	0.0490	215.7108	10.5668	0.0046	0.0946	98.359	9.3083	35
40	31.4094	0.0318	337.8824	10.7574	0.0030	0.0930	105.376	9.7957	40
45	48.3273	0.0207	525.8587	10.8812	0.0019	0.0919	110.556	10.1603	45
50	74.3575	0.0134	815.0836	10.9617	0.0012	0.0912	114.325	10.4295	50
60	176.0313	0.0057	1944.7921	11.0480	0.0005	0.0905	118.968	10.7683	60
80	986.5517	0.0010	10950.5741	11.0998	0.0001	0.0901	122.431	11.0299	80
100	5529.0408	0.0002	61422.6755	11.1091	[a]	0.0900	123.234	11.0930	100
∞				11.1111		0.0900			∞

[a] Less than 0.0001.

TABLE C-15 Discrete Compounding; $i = 15\%$

	Single Payment		Uniform Series				Uniform Gradient		
	Compound Amount Factor	Present Worth Factor	Compound Amount Factor	Present Worth Factor	Sinking Fund Factor	Capital Recovery Factor	Gradient Present Worth Factor	Gradient Uniform Series Factor	
	To Find F Given P	To Find P Given F	To Find F Given A	To Find P Given A	To Find A Given F	To Find A Given P	To Find P Given G	To Find A Given G	
N	F/P	P/F	F/A	P/A	A/F	A/P	P/G	A/G	N
1	1.1500	0.8696	1.0000	0.8696	1.0000	1.1500	0.000	0.0000	1
2	1.3225	0.7561	2.1500	1.6257	0.4651	0.6151	0.756	0.4651	2
3	1.5209	0.6575	3.4725	2.2832	0.2880	0.4380	2.071	0.9071	3
4	1.7490	0.5718	4.9934	2.8550	0.2003	0.3503	3.786	1.3263	4
5	2.0114	0.4972	6.7424	3.3522	0.1483	0.2983	5.775	1.7228	5
6	2.3131	0.4323	8.7537	3.7845	0.1142	0.2642	7.937	2.0972	6
7	2.6600	0.3759	11.0668	4.1604	0.0904	0.2404	10.192	2.4498	7
8	3.0590	0.3269	13.7268	4.4873	0.0729	0.2229	12.481	2.7813	8
9	3.5179	0.2843	16.7858	4.7716	0.0596	0.2096	14.755	3.0922	9
10	4.0456	0.2472	20.3037	5.0188	0.0493	0.1993	16.980	3.3832	10
11	4.6524	0.2149	24.3493	5.2337	0.0411	0.1911	19.129	3.6549	11
12	5.3503	0.1869	29.0017	5.4206	0.0345	0.1845	21.185	3.9082	12
13	6.1528	0.1625	34.3519	5.5831	0.0291	0.1791	23.135	4.1438	13
14	7.0757	0.1413	40.5047	5.7245	0.0247	0.1747	24.973	4.3624	14
15	8.1371	0.1229	47.5804	5.8474	0.0210	0.1710	26.693	4.5650	15
16	9.3576	0.1069	55.7175	5.9542	0.0179	0.1679	28.296	4.7522	16
17	10.7613	0.0929	65.0751	6.0472	0.0154	0.1654	29.783	4.9251	17
18	12.3755	0.0808	75.8364	6.1280	0.0132	0.1632	31.157	5.0843	18
19	14.2318	0.0703	88.2118	6.1982	0.0113	0.1613	32.421	5.2307	19
20	16.3665	0.0611	102.4436	6.2593	0.0098	0.1598	33.582	5.3651	20
21	18.8215	0.0531	118.8101	6.3125	0.0084	0.1584	34.645	5.4883	21
22	21.6447	0.0462	137.6316	6.3587	0.0073	0.1573	35.615	5.6010	22
23	24.8915	0.0402	159.2764	6.3988	0.0063	0.1563	36.499	5.7040	23
24	28.6252	0.0349	184.1678	6.4338	0.0054	0.1554	37.302	5.7979	24
25	32.9190	0.0304	212.7930	6.4641	0.0047	0.1547	38.031	5.8834	25
30	66.2118	0.0151	434.7451	6.5660	0.0023	0.1523	40.753	6.2066	30
35	133.1755	0.0075	881.1702	6.6166	0.0011	0.1511	42.359	6.4019	35
40	267.8635	0.0037	1779.0903	6.6418	0.0006	0.1506	43.283	6.5168	40
45	538.7693	0.0019	3585.1285	6.6543	0.0003	0.1503	43.805	6.5830	45
50	1083.6574	0.0009	7217.7163	6.6605	0.0001	0.1501	44.096	6.6205	50
60	4383.9987	0.0002	29219.9916	6.6651	α	0.1500	44.343	6.6530	60
80	71750.8794	α	478332.5293	6.6666	α	0.1500	44.436	6.6656	80
100	1174313.4507	α	7828749.6713	6.6667	α	0.1500	44.444	6.6666	100
∞				6.6667		0.1500			∞



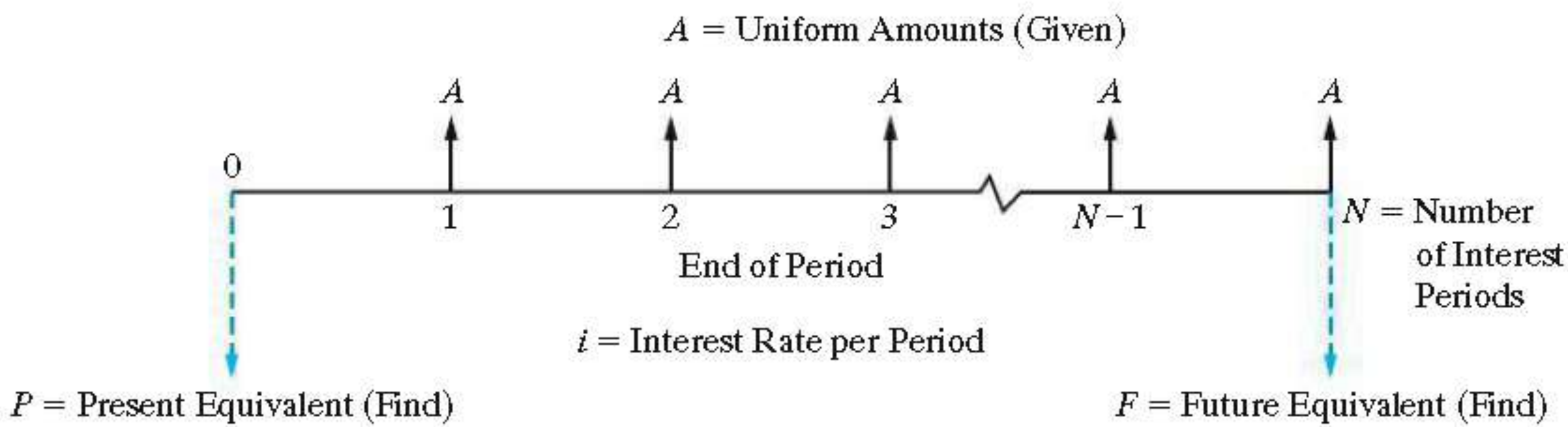
Relating a Uniform Series (Annuity) to Its Present and Future Equivalent Values

The formulas and tables to be presented are derived such that *A* occurs at the end of each period, and thus,

1. *P* (present equivalent value) occurs one interest period before the first *A* (uniform amount),

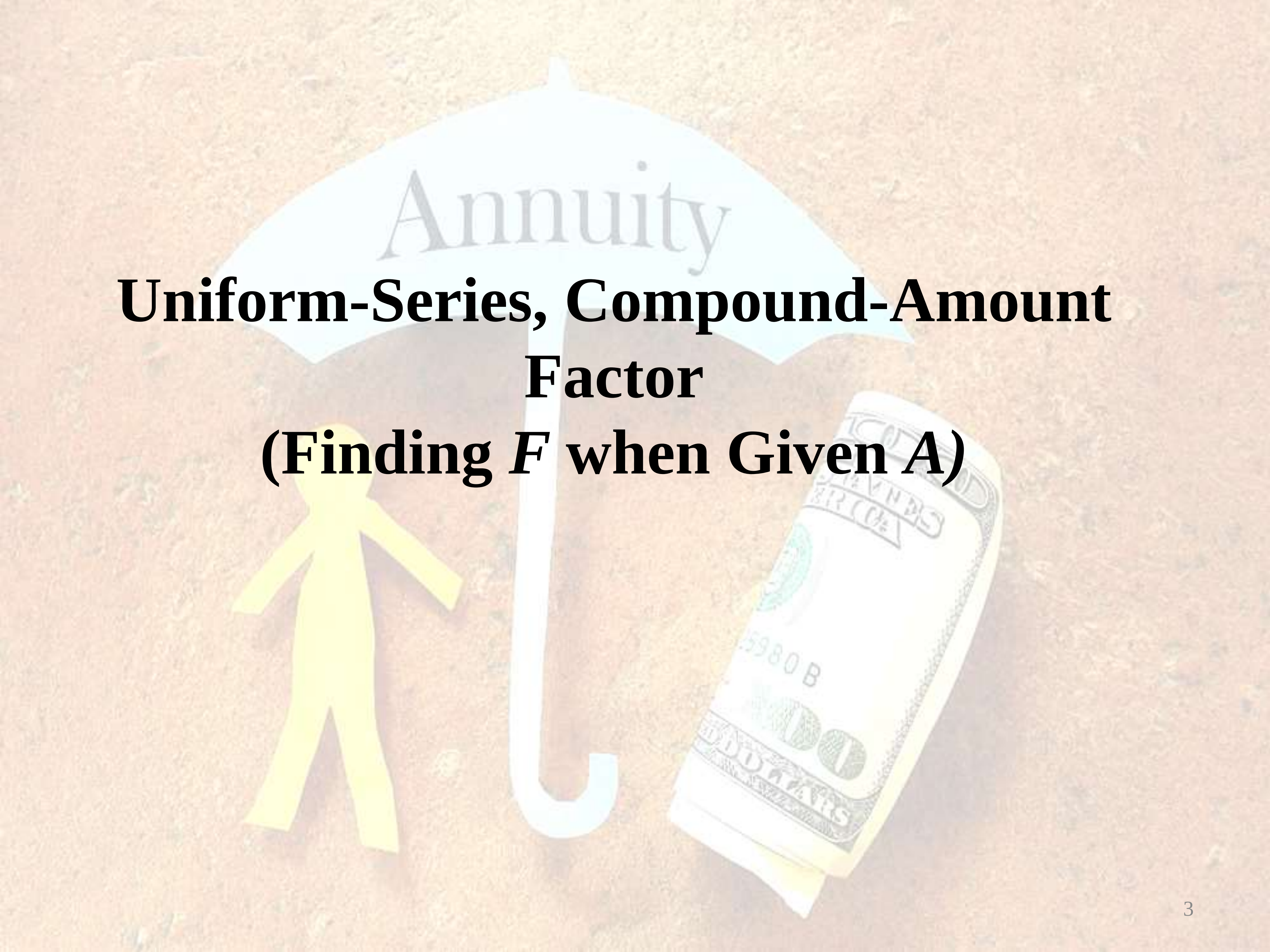
2. *F* (future equivalent value) occurs at the same time as the last *A*, and *N* periods after *P*, and

3. *A* (annual equivalent value) occurs at the end of periods 1 through *N*, inclusive.

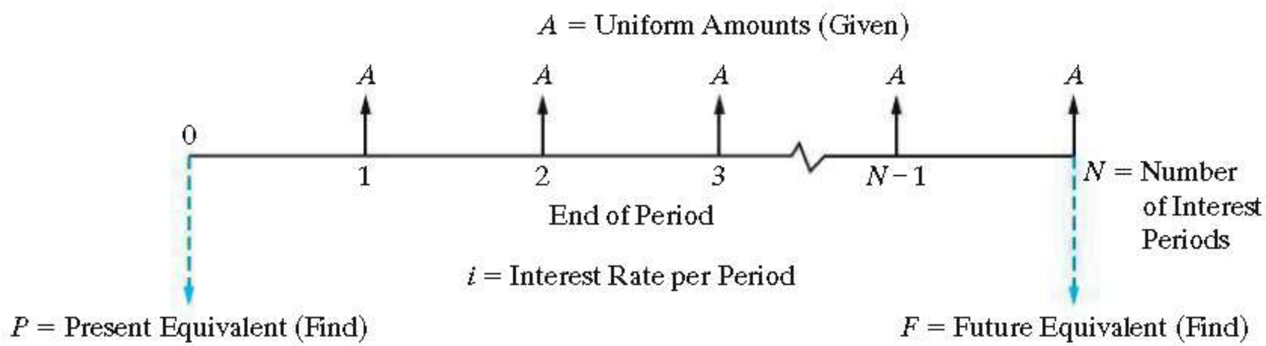


General Cash-Flow Diagram Relating Uniform Series (Ordinary Annuity) to Its Present Equivalent and Future Equivalent Values

Uniform-Series, Compound-Amount Factor
(Finding *F* when Given *A*)



If a cash flow in the amount of *A* dollars occurs at the end of each period for *N* periods and *i*% is the interest (profit or growth) rate per period, the future equivalent value, *F*, at the end of the *N*th period is obtained by summing the future equivalents of each of the cash flows.



$$F = A + A(i+1) + A\,(i+1)^2 + A\,(i+1)^3 + \;... + A\,(i+1)^{N-1} \quad \ldots (1)$$

Multiply both sides by (*i*+1)

$$F(i+1) = A(i+1) + A\,(i+1)^2 + A\,(i+1)^3 + A\,(i+1)^4 + \cdots + A\,(i+1)^N \quad \ldots (2)$$

Subtract equation (1) from equation (2)

$$F(1+i) - F = -A + A(1+i)^N$$

$$F(1+i-1) = A[(1+i)^N - 1]$$

$$Fi = A[(1+i)^N - 1]$$

$$F = A\left[\frac{(1+i)^N - 1}{i}\right]$$

The quantity $\left[\frac{(1+i)^N-1}{i}\right]$ is called the *uniform series compound amount factor*

$$\boldsymbol{F = A\left[\frac{(1+i)^N - 1}{i}\right] = ({}^{F}/_{A}\ , i\%, N)}$$

Example:

How much will you have in 40 years if you save \$3,000 each year and your account earns 8% interest each year?

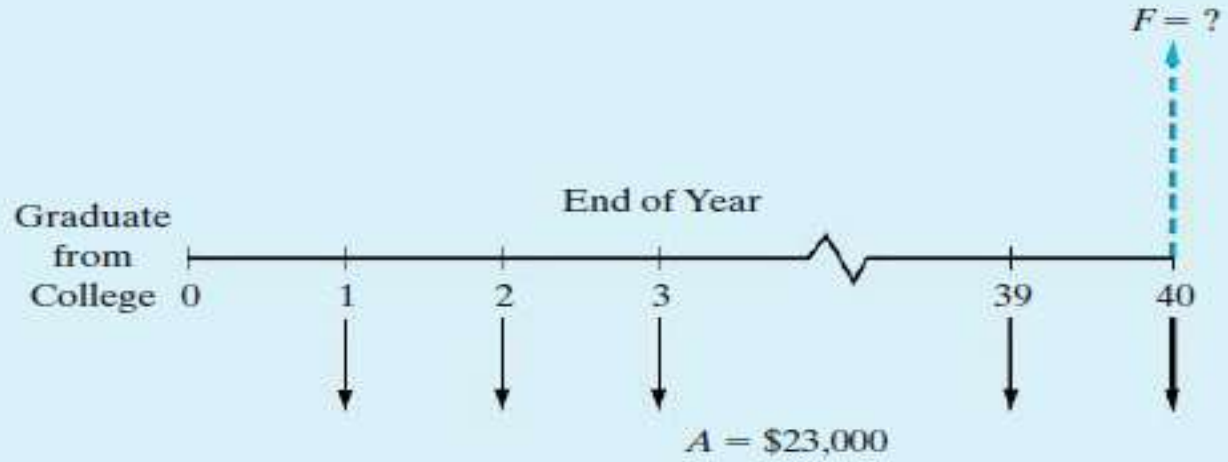
Solution

$$F = \$3,000(F/A, 8\%, 40) = \$3,000(259.0565) = \$777,170$$

Example 4-7: Future Value of a College Degree

A recent government study reported that a college degree is worth an extra \$23,000 per year in income (A) compared to what a high-school graduate makes. If the interest rate (i) is 6% per year and you work for 40 years (N), what is the future compound amount (F) of this extra income?

Solution: The viewpoint we will use to solve this problem is that of "lending" the \$23,000 of extra annual income to a savings account (or some other investment vehicle). The future equivalent is the amount that can be withdrawn after the 40th deposit is made.



Notice that the future equivalent occurs at the *same time* as the last deposit of \$23,000.

$$F = \$23{,}000(F/A,\ 6\%,\ 40)$$
$$= \$23{,}000(154.762)$$
$$= \$3{,}559{,}526$$

The bottom line is "Get your college degree!"

EXAMPLE 4-8

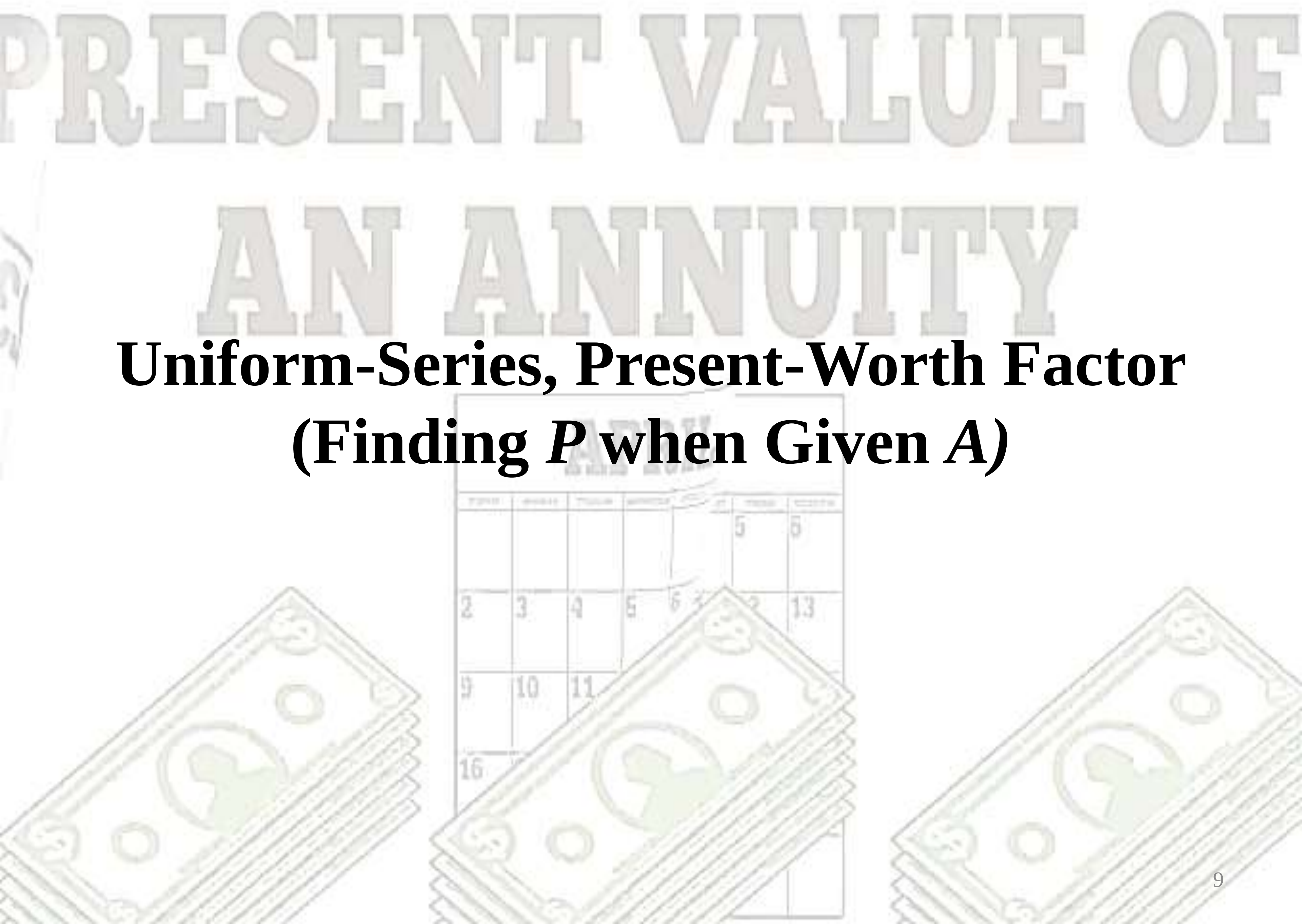
To illustrate further the amazing effects of compound interest, we consider the credibility of this statement: “If you are 20 years of age and save \$1.00 each day for the rest of your life, you can become a millionaire.” Let’s assume that you live to age 80 and that the annual interest rate is 10% (i = 10%). Under these specific conditions, we compute the future compound amount (F) to be

F = \$365/year (F/A, 10%, 60 years)

= \$365 (3,034.81)

= \$1,107,706.

Thus, the statement is true for the assumptions given! The moral is to start saving early and let the “magic” of compounding work on your behalf!



Uniform-Series, Present-Worth Factor (Finding *P* when Given *A*)

From Equation , $F = P(1 + i)^N$. Substituting for F into Equation $F = A\left[\frac{(1+i)^N-1}{i}\right]$,

$$P(1+i)^N = A\left[\frac{(1+i)^N - 1}{i}\right]$$

Dividing both sides by $(1+i)^N$, we get

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

The quantity $\left[\frac{(1+i)^N-1}{i(1+i)^N}\right]$ is called the *uniform series present worth factor*

$$\boldsymbol{P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right] = A(P/A\ ,i\%, N)}$$

Example:

How much would be needed today to provide an annual amount of \$50,000 each year for 20 years, at 9% interest each year?

Solution

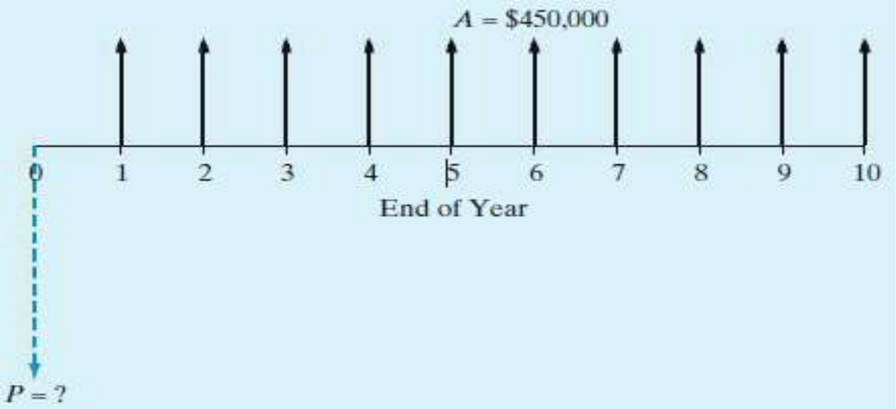
$$P = \$50,000(P/A, 9\%, N) = \$50,000(9.1285) = \$456,427$$

EXAMPLE 4-9

A micro-brewery is considering the installation of a newly designed boiler system that burns the dried, spent malt and barley grains from the brewing process. The boiler will produce process steam that powers the majority of the brewery's energy operations, saving $450,000 per year over the boiler's expected life of 10 years. If the interest rate is 12% per year, how much money can the brewery afford to invest in the new boiler system?

Solution

In the cash flow diagram below, notice that the affordable amount (i.e., the present equivalent, *P*) occurs one time period (year) before the first end-of-year cash flow of $450,000.



The increase in annual cash flow is $450,000, and it continues for 10 years at 12% annual interest. The upper limit on what the brewery can afford to spend on the new boiler is:

P = $450,000 ($P/A$, 12%, 10)
= $450,000 (5.6502)
= $2,542,590.

Example :

An engineer who is planning his retirement has decided that he will have to withdraw $10,000 from his savings account at the end of each year. How much money must the engineer have in the bank at the start of his retirement, if his money earns 6% per year, compounded annually, and he is planning a 12-year retirement (i.e., 12 annual withdrawals)?

Example 4-10 : How Much Is a Lifetime Oil Change Offer Worth?

"Make your best deal with us on a new automobile and we'll change your oil for free for as long as you own the car!" If you purchase a car from this dealership, you expect to have four free oil changes per year during the five years you keep the car. Each oil change would normally cost you $30. If you save your money in a mutual fund earning 2% per quarter, how much are the oil changes worth to you at the time you buy the car?

Uniform-Series, Sinking-Fund Factor (Finding *A* when Given *F*)

Taking equation

$$F = A\left[\frac{(1+i)^N - 1}{i}\right]$$

And solving for A, we find that

$$A = F\left[\frac{i}{(1+i)^N - 1}\right]$$

The quantity $\left[\frac{i}{(1+i)^N-1}\right]$ is called the *sinking fund factor (the reciprocal of the uniform-series, compound-amount factor).*

$$\boldsymbol{A = F\left[\frac{i}{(1+i)^N-1}\right] = F\left(A/F\ , i\%, N\right)}$$

Example:

How much would you need to set aside each year for 25 years, at 10% interest, to have accumulated \$1,000,000 at the end of the 25 years?

Solution

$$A = \$1,000,000(A/F, 10\%, 25) = \$1,000,000(0.0102) = \$10,200$$

Uniform-Series, Capital-Recovery Factor
(Finding *A* when Given *P*)

Taking equation

$$P = A\left[\frac{(1+i)^N - 1}{i(1+i)^N}\right]$$

And solving for A, we find that

$$A = P\left[\frac{i(1+i)^N}{(1+i)^N - 1}\right]$$

The quantity $\left[\frac{i(1+i)^N}{(1+i)^N-1}\right]$ is called the *capital recovery factor*

$$\boldsymbol{A = P\left[\frac{i(1+i)^N}{(1+i)^N-1}\right] = P\left({}^{A}/_{P}\ , i\%, N\right)}$$

Example:

Solution

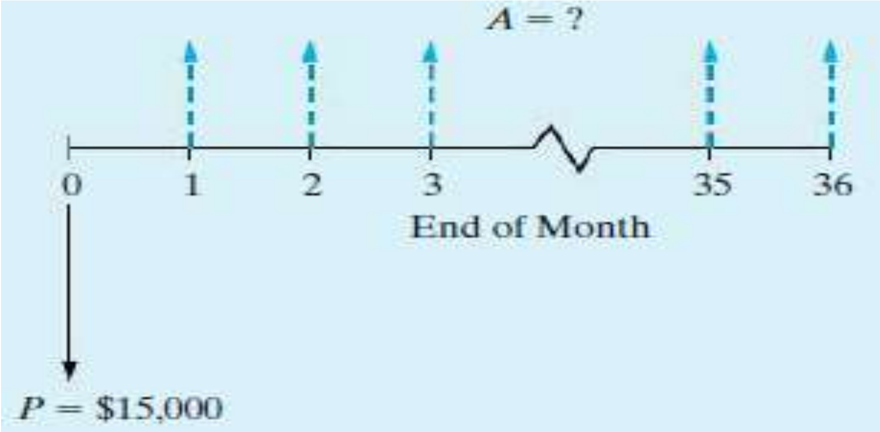
$$A = \$500,000(A/P, 10\%, 25) = \$500,000(0.1102) = \$55,100$$

Example 4-11: Computing Your Monthly Car Payment

You borrow $15,000 from your credit union to purchase a used car. The interest rate on your loan is 0.25% per month and you will make a total of 36 monthly payments. What is your monthly payment?

Solution

The cash-flow diagram shown below is drawn from the viewpoint of the bank. Notice that the present amount of $15,000 occurs one month (interest period) *before* the first cash flow of the uniform repayment series.



The amount of the car payment is easily calculated using Equation (4-15).

A = $15,000($A/P$, 1/4%, 36)

= $15,000(0.0291)

= $436.50 per month

Example:

An engineer who is about to retire has accumulated $50,000 in a savings account that that pays 6% per year, compounded annually. Suppose that the engineer wishes to withdraw a fixed sum of money at the end of each year for 10 years. What is the maximum amount that can be withdrawn?

Test Yourself

Acme Steamer purchased a new pump for $75,000. They borrowed the money for the pump from their bank at an interest rate of 0.5% per month and will make a total of 24 equal, monthly payments. How much will Acme's monthly payments be?

It can be challenging to solve for N or i.

- We may know P, A, and i and want to find N.

- We may know P, A, and N and want to find i.

Example: (Finding *N*)

Acme borrowed \$100,000 from a local bank, which charges them an interest rate of 7% per year. If Acme pays the bank \$8,000 per year, now many years will it take to pay off the loan?

Solution

$$\$100,000 = \$8,000(P/A, 7\%, N)$$

So,

$$(P/A, 7\%, N) = \frac{\$100,000}{\$8,000} = 12.5 = \frac{(1.07)^N - 1}{0.07(1.07)^N}$$

This can be solved by using the interest tables and interpolation, but we generally resort to a computer solution.

EXAMPLE 4-12

Your company has a \$100,000 loan for a new security system it just bought. The annual payment is \$8,880 and the interest rate is 8% per year for 30 years. Your company decides that it can afford to pay \$10,000 per year. After how many payments (years) will the loan be paid off?

Solution

The original loan payment was found using Equation (4-15).
A = \$100,000 (A/P, 8%, 30) = \$100,000 (0.0888) = \$8,800 per year
Now, instead of paying \$8,880 per year, your company is going to pay \$10,000 per year. Common sense tells us that less than 30 payments will be necessary to pay off the \$100,000 loan. Using Equation (4-11), we find
\$100,000 = \$10,000 (P/A, 8%,N)
(P/A, 8%,N) = 10.
We can now use the interest tables provided in Appendix C to find N. Looking down the Present Worth Factor column (P/A) of Table C-11, we see that
(P/A, 8%, 20) = 9.8181
and
(P/A, 8%, 21) = 10.0168.
So, if \$10,000 is paid per year, the loan will be paid off after 21 years instead of 30. The exact amount of the 21st payment will be slightly less than \$10,000 (but we'll save that solution for another example).

Example:(Finding *i*)

Jill invested $1,000 each year for five years in a local company and sold her interest after five years for $8,000. What annual rate of return did Jill earn?

Solution

$$\$8,000 = \$1,000(F/A, i\%, 5)$$

So,

$$(F/A, i\%, 5) = \frac{\$8,000}{\$1,000} = 8 = \frac{(1+i)^5 - 1}{i}$$

Again, this can be solved using the interest tables and interpolation, but we generally resort to a computer solution.

Example 4-13: Finding the Interest Rate to Meet an Investment Goal

After years of being a poor, debt-encumbered college student, you decide that you want to pay for your dream car in cash. Not having enough money now, you decide to specifically put money away each year in a "dream car" fund. The car you want to buy will cost $60,000 in eight years. You are going to put aside $6,000 each year (for eight years) to save for this. At what interest rate must you invest your money to achieve your goal of having enough to purchase the car after eight years?

Example:

A 40-years old person wants to accumulate \$500,000 by age of 65. how much will she need to save each month , starting one month from now, if the interest rate is 0.5% per month ?

There are specific spreadsheet functions to find *N* and *i*.

The Excel function used to solve for *N* is

NPER(*rate*, *pmt*, *pv*), which will compute the number of payments of magnitude *pmt* required to pay off a present amount (*pv)* at a fixed interest rate (*rate*).

One Excel function used to solve for *i* is

RATE(*nper*, *pmt*, *pv*, *fv*), which returns a fixed interest rate for an annuity of *pmt* that lasts for *nper* periods to either its present value (*pv*) or future value (*fv*).

Summary of Interest Formulas and Relationships for Discrete Compounding

Discrete compounding, means that the interest is compounded at the end of each finite length period, such as a month or a year.

Important

TABLE 4-3 Discrete Compounding-Interest Factors and Symbols[a]

To Find:	Given:	Factor by which to Multiply "Given"[a]	Factor Name	Factor Functional Symbol[b]
For single cash flows:				
F	P	$(1+i)^N$	Single payment compound amount	$(F/P, i\%, N)$
P	F	$\frac{1}{(1+i)^N}$	Single payment present worth	$(P/F, i\%, N)$
For uniform series (annuities):				
F	A	$\frac{(1+i)^N-1}{i}$	Uniform series compound amount	$(F/A, i\%, N)$
P	A	$\frac{(1+i)^N-1}{i(1+i)^N}$	Uniform series present worth	$(P/A, i\%, N)$
A	F	$\frac{i}{(1+i)^N-1}$	Sinking fund	$(A/F, i\%, N)$
A	P	$\frac{i(1+i)^N}{(1+i)^N-1}$	Capital recovery	$(A/P, i\%, N)$

$$(P/F,i\%,N) = \frac{1}{(F/P,i\%,N)};$$

$$(A/P,i\%,N) = \frac{1}{(P/A,i\%,N)};$$

$$(A/F,i\%,N) = \frac{1}{(F/A,i\%,N)};$$

$$(F/A,i\%,N) = (P/A,i\%,N)(F/P,i\%,N);$$

$$(P/A,i\%,N) = \sum_{k=1}^{N} (P/F,i\%,k);$$

$$(F/A,i\%,N) = \sum_{k=1}^{N} (F/P,i\%,N-k);$$

$$(A/F,i\%,N) = (A/P,i\%,N) - i.$$

Deferred Annuities (Uniform Series)

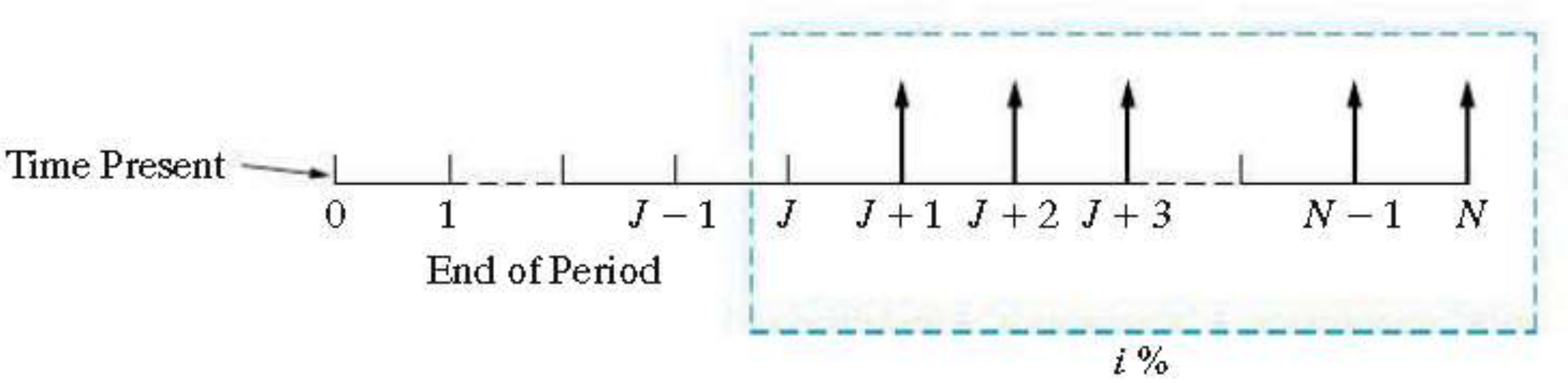
We need to be able to handle cash flows that do not occur until some time in the future.

- All annuities (uniform series) discussed to this point involve the first cash flow being made at the end of the first period, and they are called *ordinary annuities*.

- We need to be able to handle cash flows that do not occur until some time in the future.

- *Deferred annuities* are uniform series that do not begin until some time in the future.

- If the annuity is deferred J periods then the first payment (cash flow) begins at the end of period $J+1$.



General Cash-Flow Representation of a Deferred Annuity (Uniform Series)

Finding the value at time *0* of a deferred annuity is a two-step process.

1. Use (*P/A*, *i*%, *N*-*J*) find the value of the deferred annuity at the end of period *J* (where there are *N*-*J* cash flows in the annuity).
2. Use (*P/F*, *i*%, *J*) to find the value of the deferred annuity at time zero.

$$P_0 = A(P/A, i\%, N - J)(P/F, i\%, J)$$

EXAMPLE 4-14

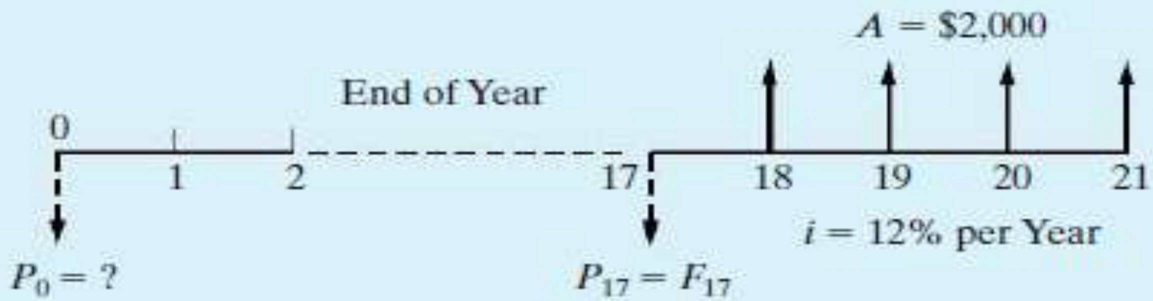
To illustrate the preceding discussion, suppose that a father, on the day his son is born, wishes to determine what lump amount would have to be paid into an account bearing interest of 12% per year to provide withdrawals of $2,000 on each of the son's 18th, 19th, 20th, and 21st birthdays.

Solution

The problem is represented in the following cash-flow diagram. One should first recognize that an ordinary annuity of four withdrawals of $2,000 each is involved and that the present equivalent of this annuity occurs at the 17th birthday when a $(P/A, i\%, N - J)$ factor is utilized. In this problem, $N = 21$ and $J = 17$. It is often helpful to use a *subscript* with P or F to denote the respective point in time.

Hence,

$$P_{17} = A(P/A, 12\%, 4) = \$2{,}000(3.0373) = \$6{,}074.60.$$



Note the dashed arrow in the cash-flow diagram denoting P_{17}. Now that P_{17} is known, the next step is to calculate P_0. With respect to P_0, P_{17} is a future equivalent, and hence it could also be denoted F_{17}. Money at a given point in time, such as the end of period 17, is the same regardless of whether it is called a present equivalent or a future equivalent. Hence,

$$P_0 = F_{17}(P/F, 12\%, 17) = \$6{,}074.60(0.1456) = \$884.46,$$

which is the amount that the father would have to deposit on the day his son is born.

Example: Present Equivalent of a Deferred Annuity

Suppose that a father, on the day his son is born, wishes to determine what lump amount would have to be paid into an account bearing interest of 12% per year to provide withdrawals of $2,000 on each of the son's 18th, 19th, 20th, and 21st birthdays.

Example 4-15: Deferred Future Value of an Annuity

When you take your first job, you decide to start saving right away for your retirement. You put $5,000 per year into the company's 401(k) plan, which averages 8% interest per year. Five years later, you move to another job and start a new 401(k) plan. You never get around to merging the funds in the two plans. If the first plan continued to earn interest at the rate of 8% per year for 35 years after you stopped making contributions, how much is the account worth?

Test yourself

Irene just purchased a new sports car and wants to also set aside cash for future maintenance expenses. The car has a bumper-to-bumper warranty for the first five years. Irene estimates that she will need approximately $2,000 per year in maintenance expenses for years 6-10, at which time she will sell the vehicle. How much money should Irene deposit into an account today, at 8% per year, so that she will have sufficient funds in that account to cover her projected maintenance expenses?

Equivalence Calculations Involving Multiple Interest Formulas

Example 4-16: Calculating Equivalent *P*, *F*, and *A* Values

Figure below depicts an example problem with a series of year-end cash flows extending over eight years. The amounts are \$100 for the first year, \$200 for the second year, \$500 for the third year, and \$400 for each year from the fourth through the eighth. These could represent something like the expected maintenance expenditures for a certain piece of equipment or payments into a fund. Note that the payments are shown at the end of each year. It is desired to find

(a) the present equivalent expenditure, *P*0;

(b) the future equivalent expenditure, *F*8;

(c) the annual equivalent expenditure, *A*

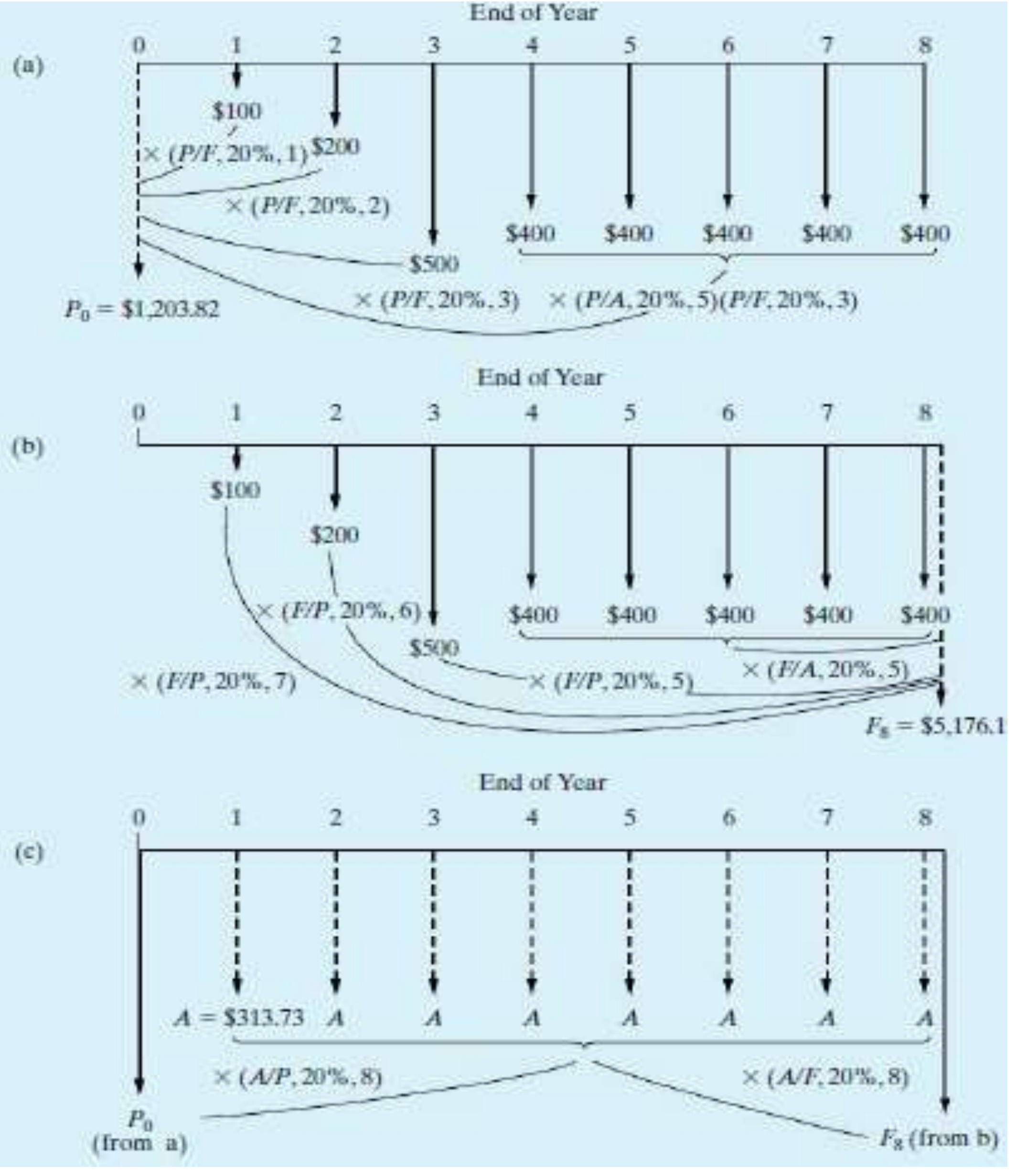
of these cash flows if the annual interest rate is 20%.

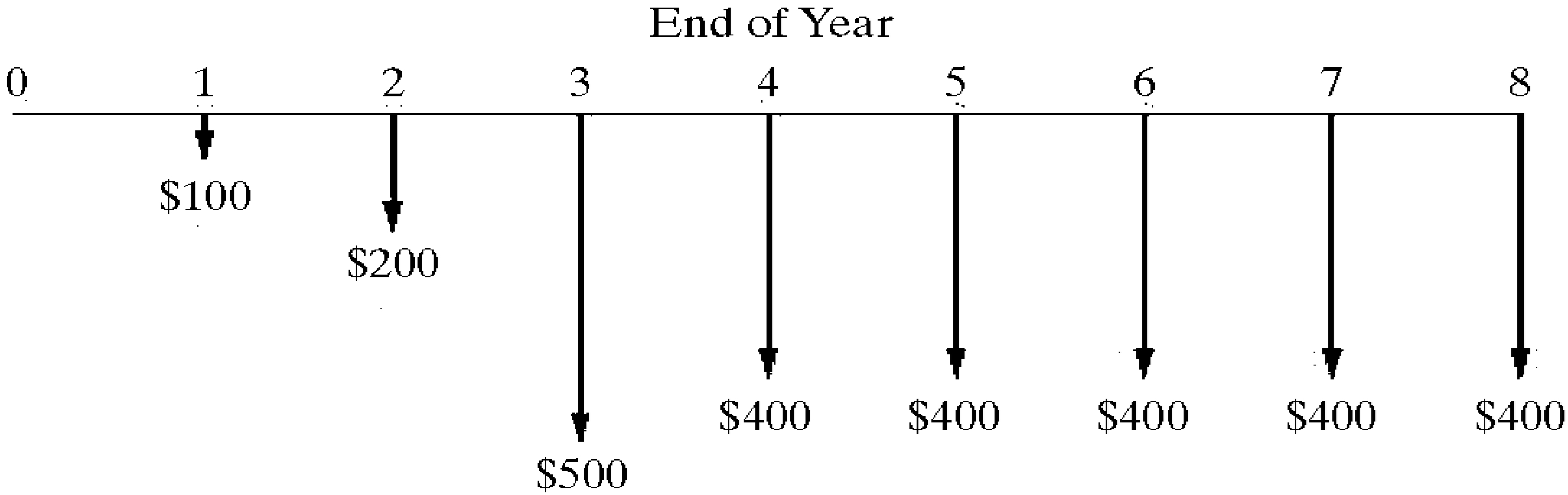
Solution

(a) To find the equivalent *P*0, we need to sum the equivalent values of all payments as of the beginning of the first year (time zero). The required movements of money through time are shown graphically in Figure 4-10(a).

$$
\begin{aligned}
P_0 = {} & F_1(P/F, 20\%, 1) && = \$100(0.8333) && = \$83.33 \\
& + F_2(P/F, 20\%, 2) && + \$200(0.6944) && + 138.8 \\
& + F_3(P/F, 20\%, 3) && + \$500(0.5787) && + 289.3 \\
& + A(P/A, 20\%, 5) \times (P/F, 20\%, 3) && + \$400(2.9900) \times (0.5787) && + 692.2
\end{aligned}
$$

(b) To find the equivalent $F8$, we can sum the equivalent values of all payments as of the end of the eighth year (time eight). Figure 4-10(b) indicates these





movements of money through time. However, since the equivalent $P0$ is already known to be $1,203.82, we can directly calculate

$F8 = P0(F/P, 20\%, 8) = \$1{,}203.82(4.2998) = \$5{,}176.19.$

(c) The equivalent A of the irregular cash flows can be calculated directly from either $P0$ or $F8$ as

$A = P0(A/P, 20\%, 8) = \$1{,}203.82(0.2606) = \$313.73$

or

$A = F8(A/F, 20\%, 8) = \$5{,}176.19(0.0606) = \$313.73.$

The computation of A from $P0$ and $F8$ is shown in Figure 4-10(c). Thus, we find that the irregular series of payments shown in Figure 4-10 is equivalent to $1,203.82 at time zero, $5,176.19 at time eight, or a uniform series of $313.73 at the end of each of the eight years.

Example 4-17: How Much is that Last Payment?

Your company has a $100,000 loan for a new security system it just bought. Your company decides that it can afford to pay $10,000 per year and the interest rate is 8% per year. After how many payments (years) will the loan be paid off?

We determined that the loan could be paid in full after 21 years if the annual payment was $10,000.

As with most real-life loans, the final payment will be something different (usually less) than the annuity amount. This is due to the effect of rounding in the interest calculations—you can't pay in fractions of a cent! For this example, determine the amount of the 21st (and final) payment on the $100,000 loan when 20 payments of $10,000 have already been made. The interest rate remains at 8% per year.




Example 4-18: Crisis in the Gulf

How vividly do you remember the biggest man-made environmental catastrophe in American history—millions of gallons of oil flowing unchecked into the Gulf of Mexico from an undersea well? In response to this tragedy, British Petroleum (BP) will make payments into a fund to pay for some of the damages to the Gulf Coast resulting from their massive oil spill in April and following months of 2010. BP will pay \$3 billion at the end of the third quarter of 2010 and another \$2 billion in the fourth quarter of 2010. BP will then make payments of \$1.25 billion each quarter thereafter until a total of \$20 billion has been paid into the fund. If the opportunity cost of capital (interest rate) is 3% per quarter, what is the equivalent value of this payment stream at the beginning of the third quarter of 2010?

Example 4-19: Determining an Unknown Annuity Amount

Two receipts of $1,000 each are desired at the EOYs 10 and 11. To make these receipts possible, four EOY annuity amounts will be deposited in a bank at EOYs 2, 3, 4, and 5. The bank's interest rate (*i*) is 12% per year.

(a) Draw a cash-flow diagram for this situation.

(b) Determine the value of *A* that establishes equivalence in your cash-flow diagram.

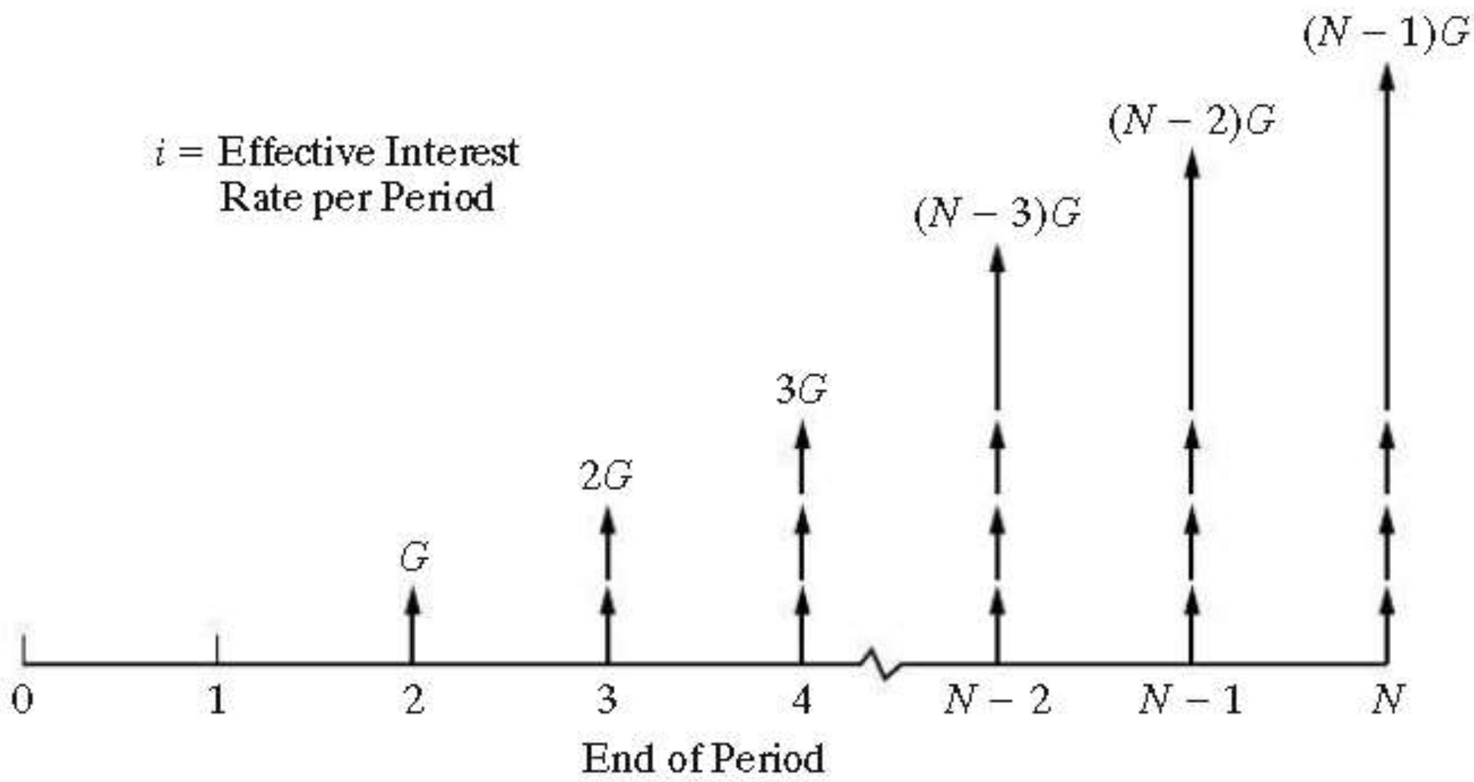
(c) Determine the lump-sum value at the end of year 11 of the completed cash-flow diagram based on your answers to Parts (a) and (b).

Uniform (Arithmetic) Gradient of Cash Flows

Sometimes cash flows change by a constant amount each period.

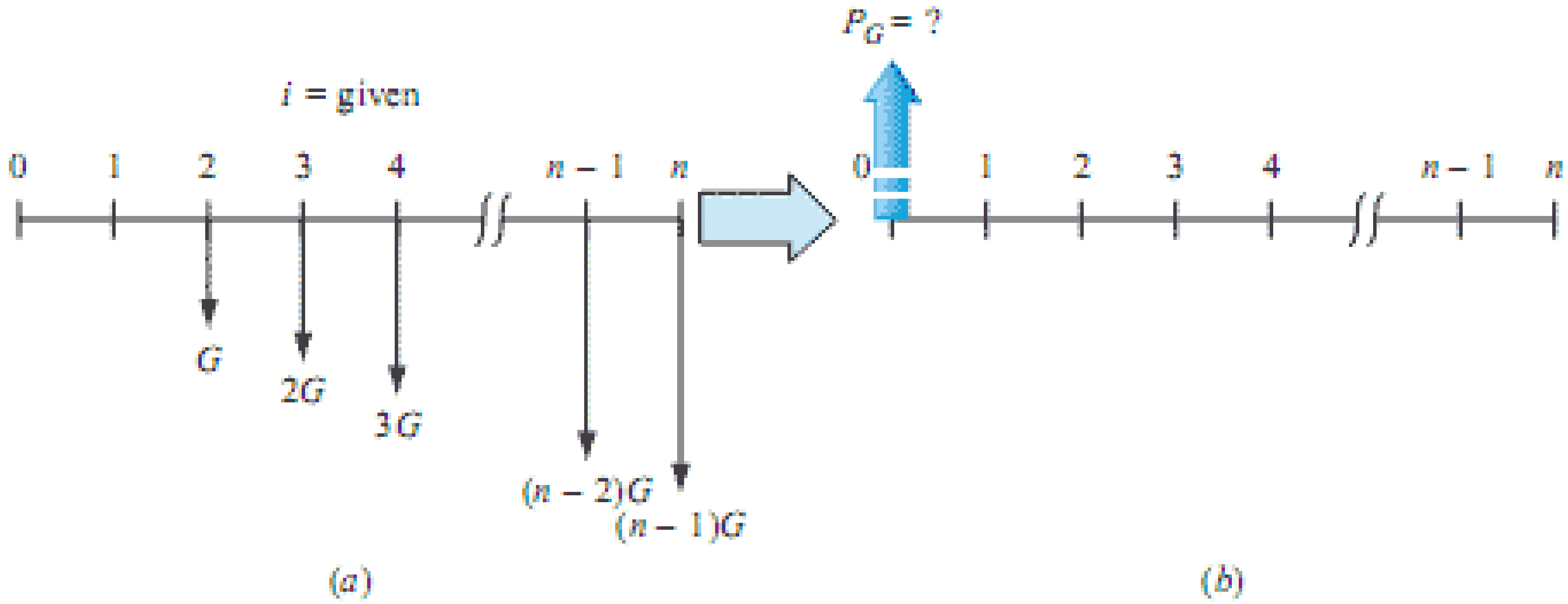
- A *gradient series* is a series of annual payments in which each payment is greater that the previous one by a constant amount, *G*.
- We can model these situations as a *uniform gradient* of cash flows. The table below shows such a gradient.

End of Period	Cash Flows
1	0
2	G
3	$2G$
:	:
N	$(N-1)G$



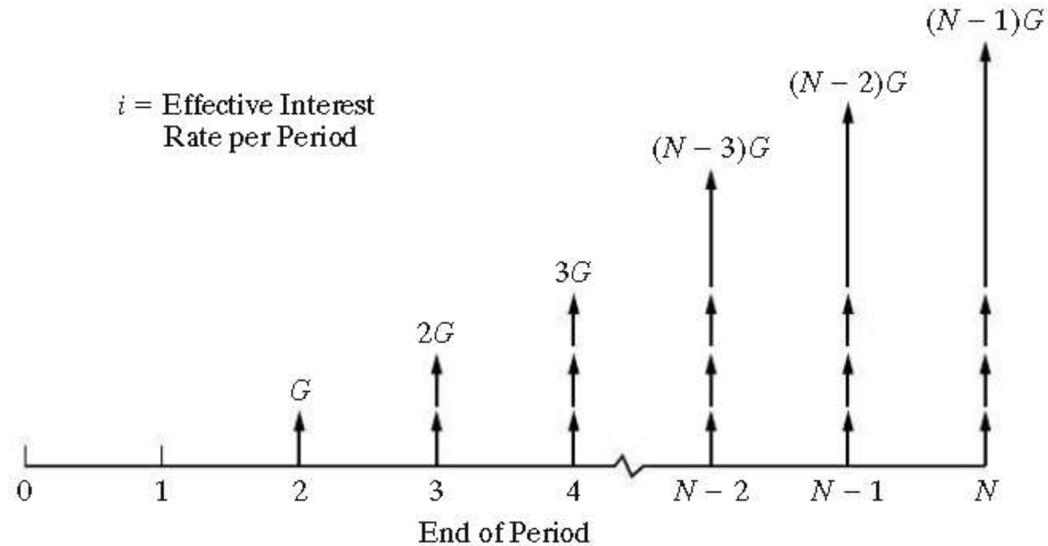
Cash-Flow Diagram for a Uniform Gradient Increasing by G Dollars per Period

Finding *P* when Given *G*



- It is easy to find the present value of a uniform gradient series.
- Similar to the other types of cash flows, there is a formula (albeit quite complicated) we can use to find the present value, and a set of factors developed for interest tables.

Be sure to notice that the direct use of gradient conversion factors applies when there is no cash flow at the end of period one



$$P = \frac{G(1)}{(1+i)^2} + \frac{G(2)}{(1+i)^3} + \frac{G(3)}{(1+i)^4} + \ \dots + \frac{G(N-2)}{(1+i)^{N-1}} + \frac{G(N-1)}{(1+i)^N}$$

If we add in the dummy term $G(0)/(1 + i)^1$ to represent the "missing" cash flow at time one, we can rewrite the above equation as:

$$P = G\sum_{n=1}^{N}\frac{(n-1)}{(1+i)^n}$$

$$\rightarrow \boldsymbol{P = G\left\{\frac{1}{i}\left[\frac{(1+i)^N-1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\}}$$

The quantity

$$\left\{\frac{1}{i}\left[\frac{(1+i)^N-1}{i(1+i)^N}-\frac{N}{(1+i)^N}\right]\right\}$$

is called the *gradient to present equivalent conversion factor.*

It can also be expressed as $(1/i)[(P/A,\ i\%,N) - N(P/F,\ i\%,N)]$.

$$\boldsymbol{P = G\left\{\frac{1}{i}\left[\frac{(1+i)^N-1}{i(1+i)^N}-\frac{N}{(1+i)^N}\right]\right\} = G\,(P/G, i\%, N)}$$

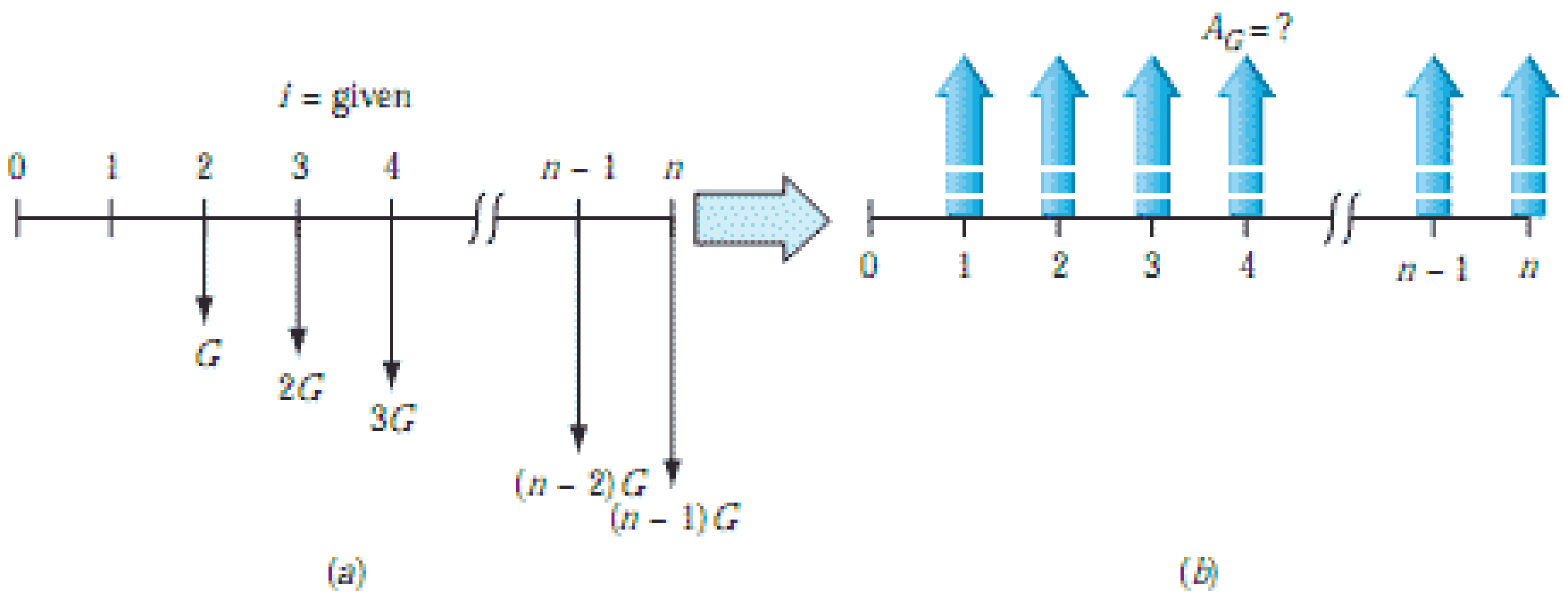
Example:

An engineer is planning for a 15-year retirement. In order to supplement his pension and offset the anticipated effects of inflation, he intends to withdraw $5000 at the end of the first year, and to increase the withdrawal by $1,000 at the end of each successive year. How much money must the engineer have in his savings account at the start of his retirement, if money earns 6% per year, compounded annually?

Example:

How much money must initially be deposited in a savings account paying 5% per year, compounded annually, to provide for ten annual withdrawals that start at $6,000 and decrease by $500 each year?

Finding *A* when Given *G*



$$A = P(A/P, i\%, N)$$

$$= G\left\{\frac{1}{i}\left[\frac{(1+i)^N - 1}{i(1+i)^N} - \frac{N}{(1+i)^N}\right]\right\}(A/P, i\%, N)$$

$$= \boldsymbol{G\left[\frac{1}{i} - \frac{N}{(1+i)^{N}-1}\right]}$$

The quantity $\left[\frac{1}{i} - \frac{N}{(1+i)^N - 1}\right]$ is called the *gradient to uniform series conversion factor*.

$$A = G(A/G, i\%, N)$$

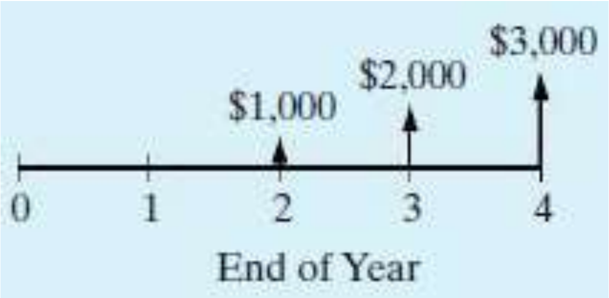
Example 4-20: Using the Gradient Conversion Factors to Find *P* and *A*

suppose that certain EOY cash flows are expected to be \$1,000 for the *second* year, \$2,000 for the third year, and \$3,000 for the fourth year and that, if interest is 15% per year, it is desired to find

(a) present equivalent value at the beginning of the first year,

(b) uniform annual equivalent value at the end of each of the four years.

Solution



Observe that this schedule of cash flows fits the model of the arithmetic gradient formulas with G = \$1,000 and N = 4. Note that there is no cash flow at the end of the first period.

(a)The present equivalent can be calculated as
$P0 = G(P/G, 15\%, 4) = \$1{,}000(3.79) = \$3{,}790.$

(b) The annual equivalent can be calculated from Equation (4-26) as
$A = G(A/G, 15\%, 4) = \$1{,}000(1.3263) = \$1{,}326.30.$

Of course, once $P0$ is known, the value of A can be calculated as
$A = P0(A/P, 15\%, 4) = \$3{,}790(0.3503) = \$1{,}326.30.$

Example 4-21: Present Equivalent of an Increasing Arithmetic Gradient Series

suppose that we have cash flows as follows:

End of Year	Cash Flows($)
1	5,000
2	6,000
3	7,000
4	8,000

Also, assume that we wish to calculate their present equivalent at i = 15% per year, using gradient conversion factors.

Solution

The schedule of cash flows is depicted in the left-hand diagram of Figure 4-14. The right two diagrams of Figure 4-14 show how the original schedule can be broken into two separate sets of cash flows, an annuity series of $5,000 plus an arithmetic gradient of $1,000 that fits the general gradient model for which factors are tabled. The summed present equivalents of these two separate sets of cash flows equal the present equivalent of the original problem. Thus, using the symbols shown in Figure 4-14, we have

$P_{0T} = P_{0A} + P_{0G}$

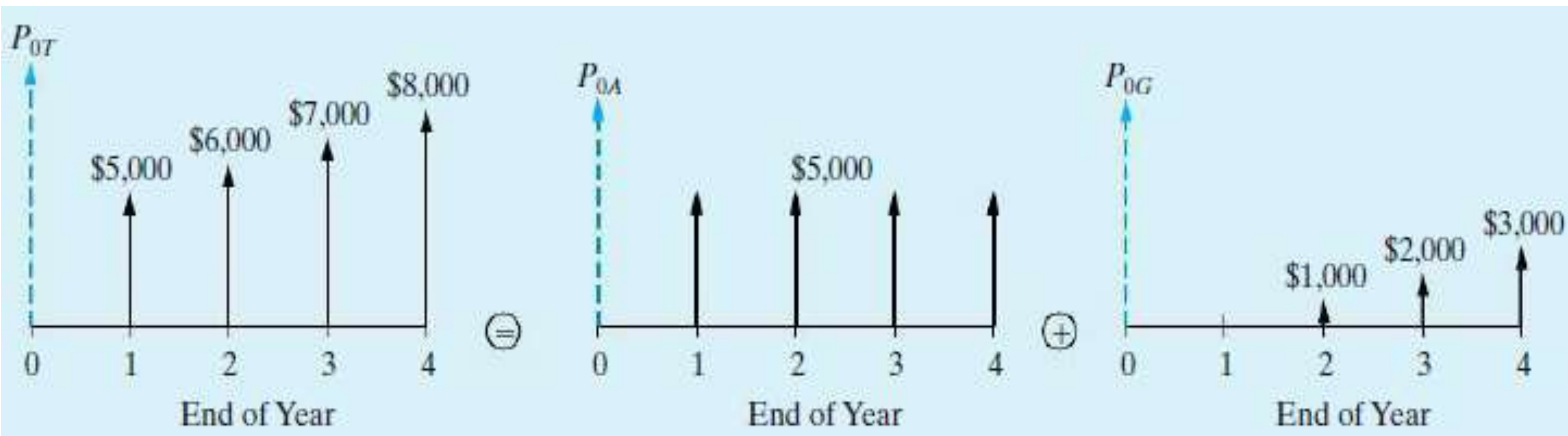
= A(P/A, 15%, 4) + G(P/G, 15%, 4)

= $5,000(2.8550) + $1,000(3.79) = $14,275 + 3,790 = $18,065.

The annual equivalent of the original cash flows could be calculated with the aid of Equation (4-26) as follows:

$A_T = A + A_G$ = \$5,000 + \$1,000(A/G, 15%, 4) = \$6,326.30.

A_T is equivalent to P_{0T} because \$6,326.30(P/A, 15%, 4) = \$18,061, which is the same value obtained previously (subject to round-off error).



Example 4-22: Present Equivalent of a Decreasing Arithmetic Gradient Series

suppose that we have cash flows as follows:

End of Year	Cash Flows($)
1	8,000
2	7,000
3	6,000
4	5,000

Calculate the present equivalent at i = 15% per year, using arithmetic gradient interest factors.

Solution

The right two diagrams of Figure 4-15 show how the uniform gradient can be broken into two separate cash-flow diagrams. In this example, we are *subtracting* an arithmetic gradient of $1,000 from an annuity series of $8,000.

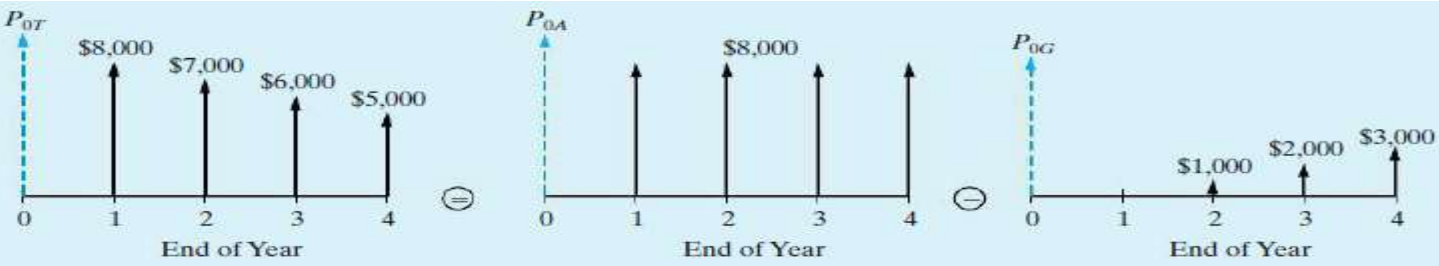


Figure 4-15 Breakdown of Cash Flows for Example 4-22

So,

$$
\begin{aligned}
P_{0T} &= P_{0A} - P_{0G} \\
&= A(P/A, 15\%, 4) - G(P/G, 15\%, 4) \\
&= \$8{,}000(2.8550) - \$1{,}000(3.79) \\
&= \$22{,}840 - \$3{,}790 = \$19{,}050.
\end{aligned}
$$

Again, the annual equivalent of the original decreasing series of cash flows can be calculated by the same rationale:

$$
\begin{aligned}
A_T &= A - A_G \\
&= \$8{,}000 - \$1{,}000(A/G, 15\%, 4) \\
&= \$6{,}673.70.
\end{aligned}
$$

Examples

Example 1: A person borrowed one hundred thousand dollars, which will be repaid at ten equal annual installments, at an annual interest rate of 15%. Calculate the value of each installment if the first one is to be repaid after four years

Example 2: A person bought a machine which cost him thirty thousand JD.

the following table summarized his income from the machine

End of Year	Cash Flows($)
1	5,000
2	7,000
3	8,000
4	7,000

If he wants to sell the machine at the end of the fourth year, what is the least amount he can accept to sell the machine?

Example 3- Debt Rescheduling : A person borrowed JD50000 , which will be repaid at 20 successive equal annual installments, at an annual interest rate of 14%. If he paid the first 12 installments on time, and he didn't paid the next three installments (13, 14, 15) on time, so as a penalty he have to pay the remaining amount in the period (16-20). Calculate the value of the new installment if he repaid the remaining amount in the dates they agreed upon.

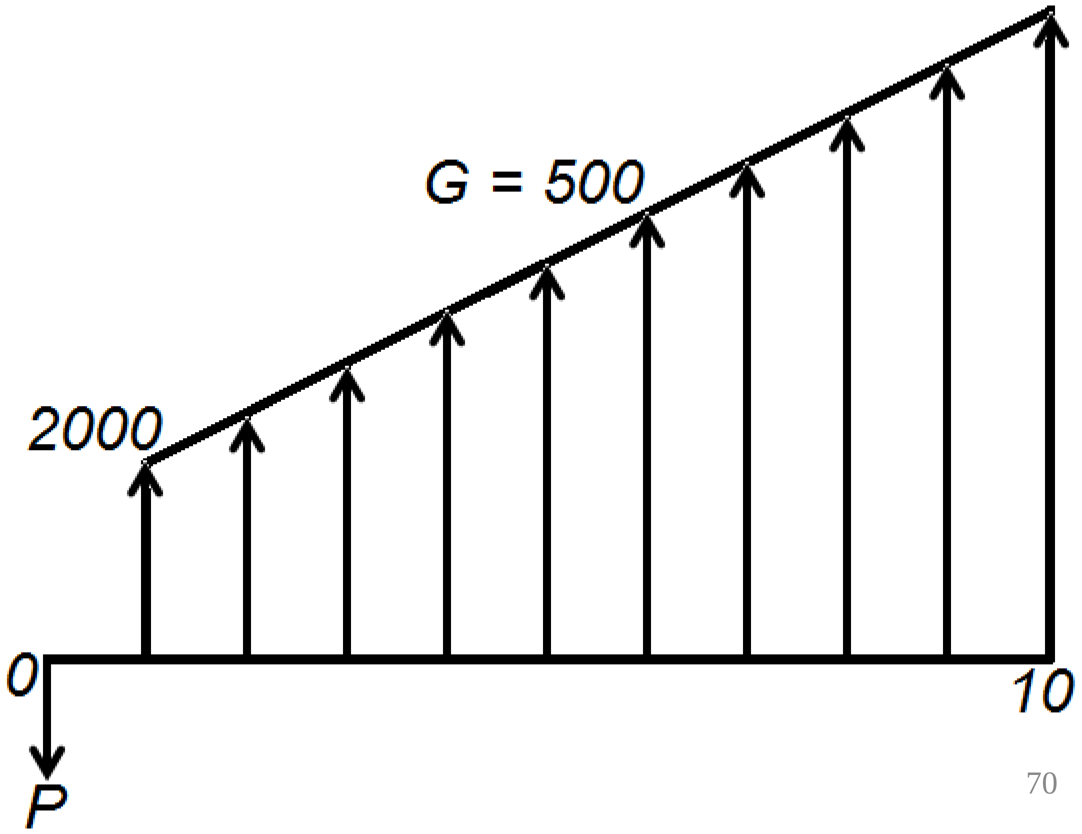
Example 4: A person bought a car for JD20,000 he payed JD5,000 immediately while the remaining amount will be repaid at 4 equal annual payments, at an annual interest rate of 12%, calculate:

1. The value of each installment.

2. If he paid the first two installments but he could not pay the rest of the amount on time agreed upon, so his penalty was to repay the remaining amount in four equal installments, starting in the fifth years after the purchase of the car, find the value of the new installment

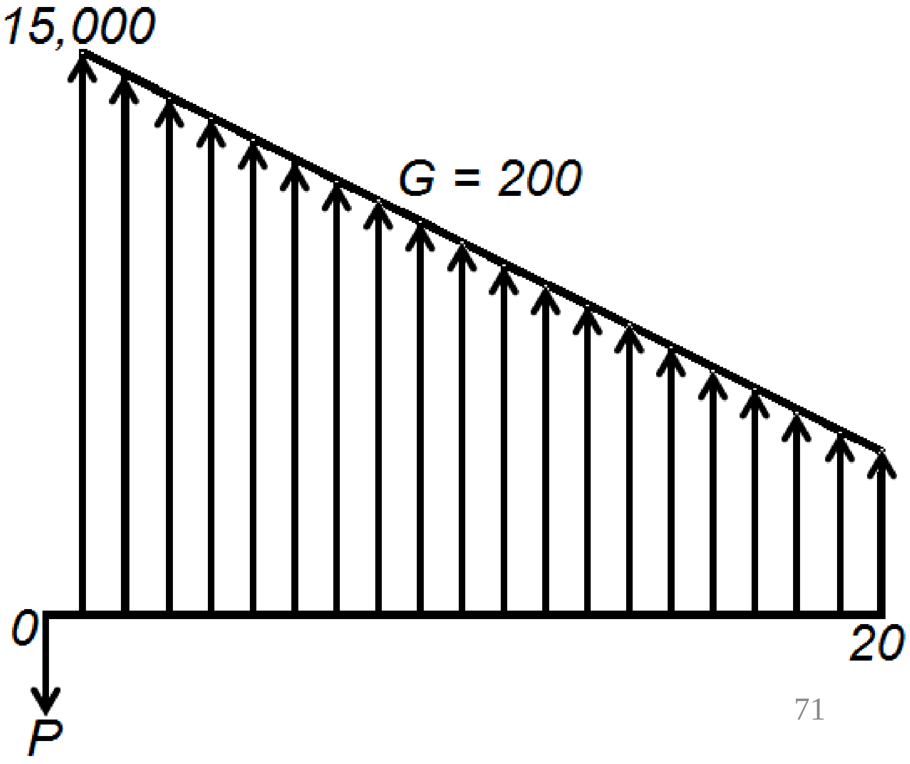
Example 5: For the shown cash flow

1. What is the value of the last payment
2. Find the present value of the shown cash flow diagram if i=8%

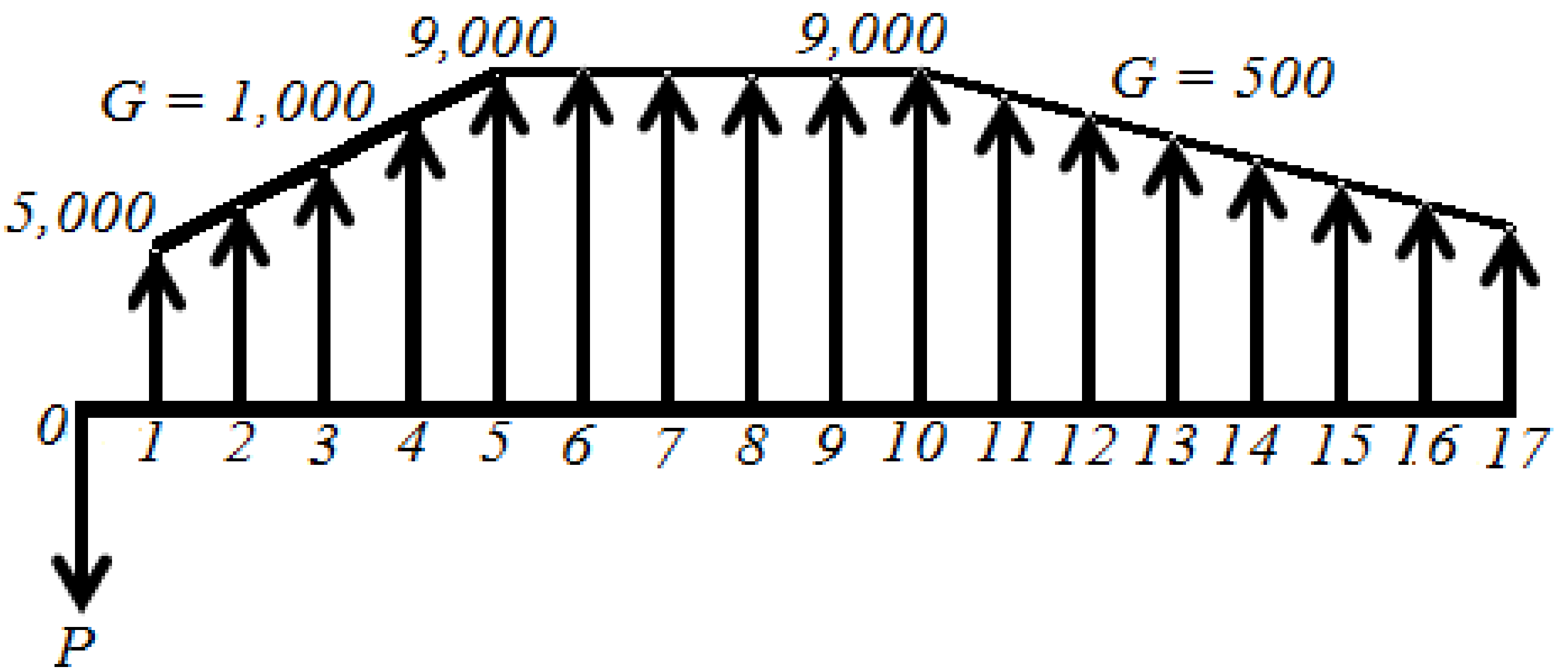


Example 6: For the shown cash flow

1. Find the present value of the shown cash flow diagram if i=9%
2. What is the future value of this cash flow?



Example 7:Find the present value of the following cash flow diagram if *i*= 6%.



Example 8: A person deposited JD 500 annually for five consecutive years, if his balance at the end of the fifth year becomes JD 4,000. Find the interest rate for this balance

Example 9: Mr. Franklin wants to save for a new sports car that he expects will cost $38,000 four and one-half years from now. How much money will he have to save each year and deposit in a savings account that pays 6¼% per year, compounded annually, to buy the car in four and one-half years?

Example 10: A person invested JD 50,000, after 4 years he invested an additional JD10,000, after how many years the total amount will become JD 200,000. Knowing that the interest rate per year is 10%.

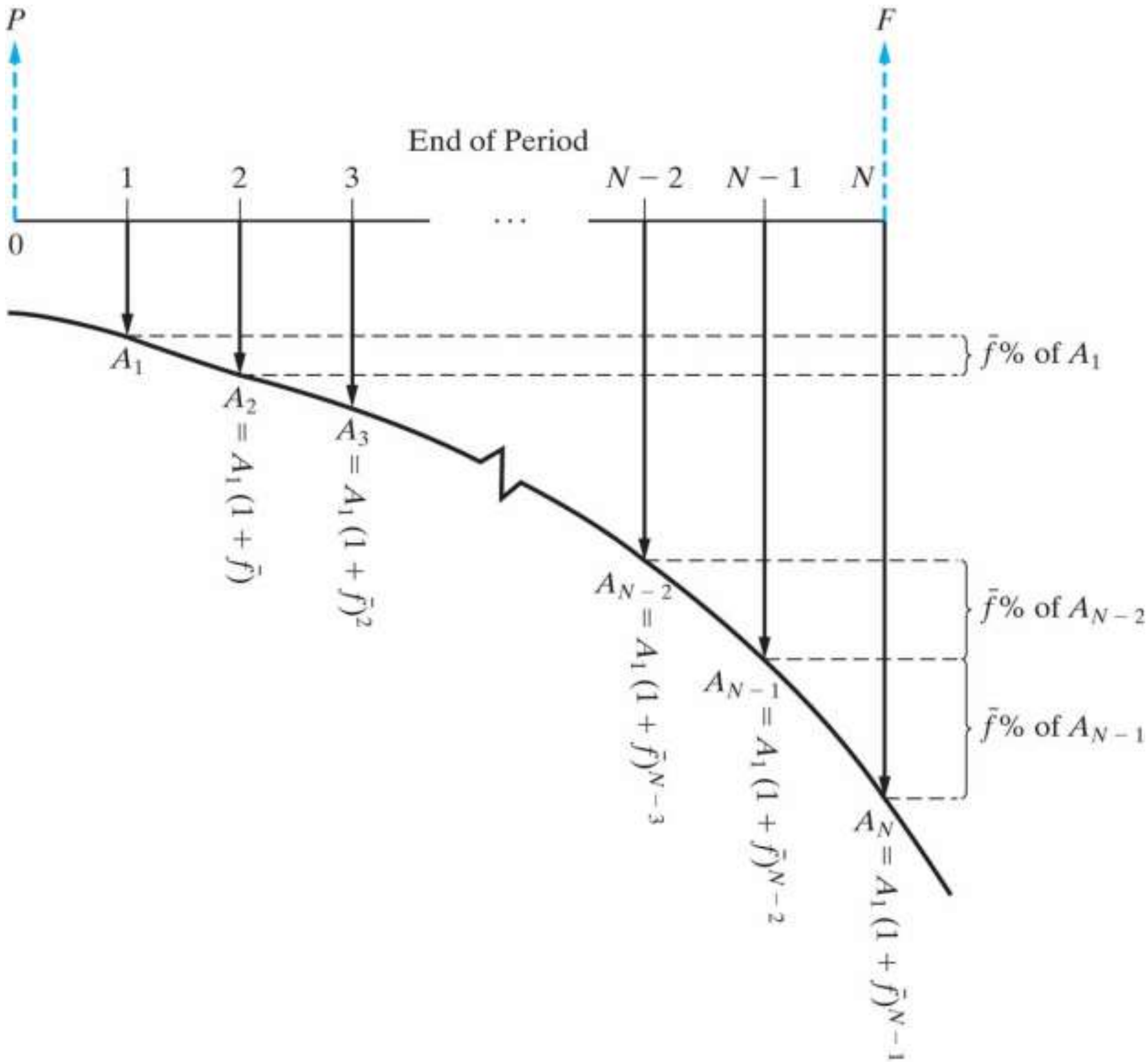
Example 11: A person invested JD 50,000, after 4 years he invested an additional JD10,000, if after 10 years the total amount of these investments become JD 200,000. Find the annual interest rate for this investment.

Interest Rates that Vary with Time

Sometimes cash flows change by a constant rate, $\bar{f}$, each period--this is a *geometric gradient series*.

This table presents a geometric gradient series. It begins at the end of year 1 and has a rate of growth, $\bar{f}$, of 20%.

End of Year	Cash Flows ($)
1	1,000
2	1,200
3	1,440
4	1,728



P
F
End of Period
0
1
2
3
…
$N-2$
$N-1$
N
A_1
$A_2 = A_1(1+\bar{f})$
$A_3 = A_1(1+\bar{f})^2$
$A_{N-2} = A_1(1+\bar{f})^{N-3}$
$A_{N-1} = A_1(1+\bar{f})^{N-2}$
$A_N = A_1(1+\bar{f})^{N-1}$
$\bar{f}$% of A_1
$\bar{f}$% of A_{N-2}
$\bar{f}$% of A_{N-1}

We can find the present value of a geometric series by using the appropriate formula below.

If $\bar{f} \neq i$

$$\frac{A_1[1-(P/F,i\%,N)(F/P,\bar{f}\%,N]}{1-\bar{f}}$$

If $\bar{f} = i$

$$A_1 N(P/F,i\%,1)$$

Where A_1 is the initial cash flow in the series.

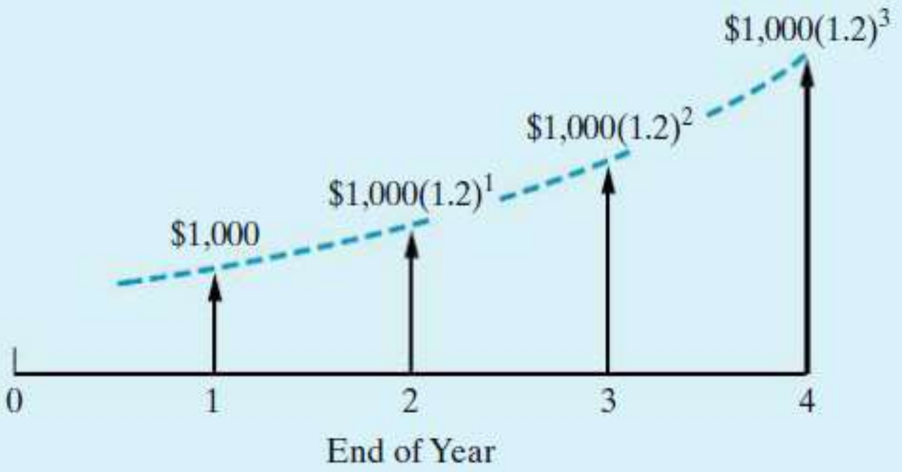
$$P = \begin{cases} \dfrac{A_1[1-(1+i)^{-N}(1+\bar{f})^N]}{i-\bar{f}} & \bar{f} \neq i \\ A_1 N(1+i)^{-1} & \bar{f} = i, \end{cases}$$

or

$$P = \begin{cases} \dfrac{A_1[1-(P/F, i\%, N)(F/P, \bar{f}\%, N)]^*}{i-\bar{f}} & \bar{f} \neq i \\ A_1 N(P/F, i\%, 1) & \bar{f} = i. \end{cases}$$

EXAMPLE 4-23

Consider the following EOY geometric sequence of cash flows and determine the *P*, *A*, and *F* equivalent values. The rate of increase is 20% per year after the first year, and the interest rate is 25% per year.



$$P = \frac{\$1{,}000\,[1-(P/F,25\%,4)(F/P,20\%,4)]}{0.25-0.20}$$

$$\begin{aligned} P &= \frac{\$1{,}000}{0.05}\left[1-(0.4096)(2.0736)\right] \\ &= \$20{,}000(0.15065) \\ &= \$3{,}013; \end{aligned}$$

$$A = \$3{,}013(A/P,25\%,4) = \$1{,}275.70;$$

$$F = \$3{,}013(F/P,25\%,4) = \$7{,}355.94.$$

EXAMPLE 4-24

Suppose that the geometric gradient in Example 4-23 begins with $1,000 at EOY one and *decreases* by 20% per year after the first year. Determine P, A, and F under this condition.

Solution The value of f is –20% in this case. The desired quantities are as follows:

$$P = \frac{\$1{,}000[1 - (P/F, 25\%, 4)(F/P, -20\%, 4)]}{0.25 - (-0.20)}$$

$$= \frac{\$1{,}000}{0.45}\left[1 - (0.4096)(1 - 0.20)^4\right]$$

$$= \$2{,}222.22(0.83222)$$

$$= \$1{,}849.38;$$

$$A = \$1{,}849.38(A/P, 25\%, 4) = \$783.03;$$

$$F = \$1{,}849.38(F/P, 25\%, 4) = \$4{,}515.08.$$

H.W

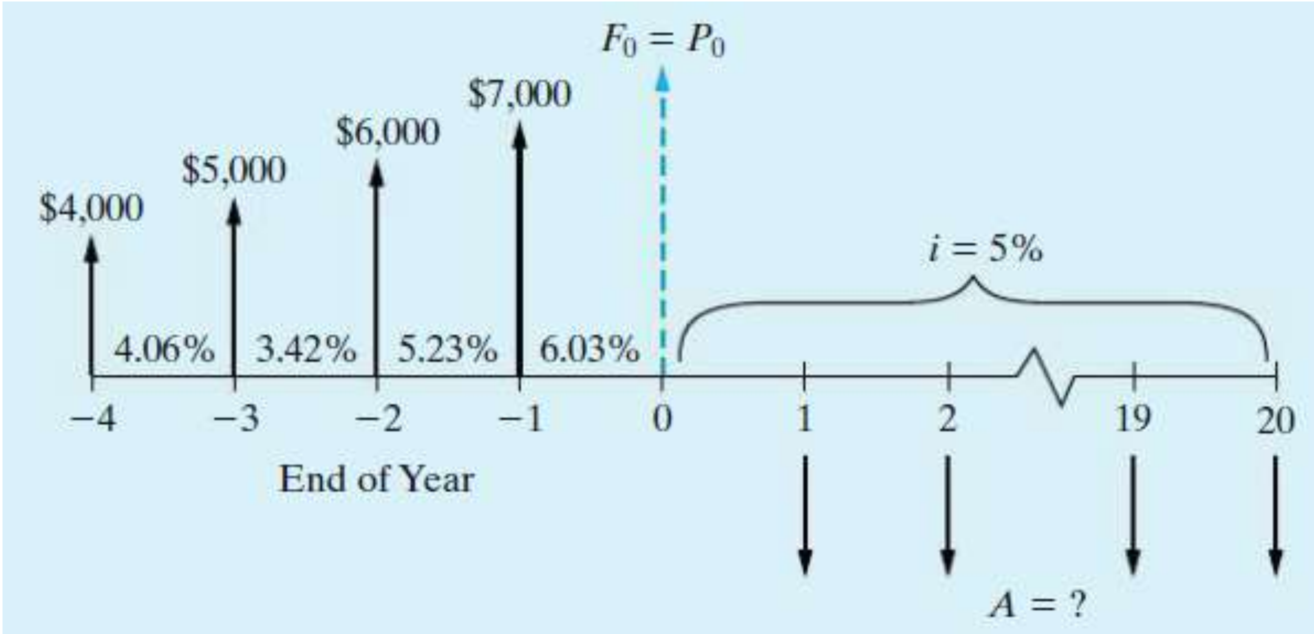
Check EXAMPLE 4-26

When the interest rate on a loan can vary with time, it is necessary to take this into account when determining the future equivalent value of the loan.

We often must resort to moving cash flows one period at a time, reflecting the interest rate for that single period.

Example 4-27: Compounding with Changing Interest Rates

Ashea Smith is a 22-year-old senior who used the Stafford loan program to borrow $4,000 four years ago when the interest rate was 4.06% per year. $5,000 was borrowed three years ago at 3.42%. Two years ago, she borrowed $6,000 at 5.23%, and last year $7,000 was borrowed at 6.03% per year. Now she would like to consolidate her debt into a single 20-year loan with a 5% fixed annual interest rate. If Ashea makes annual payments (starting in one year) to repay her total debt, what is the amount of each payment?



$$F_{-3} = \$4{,}000(F/P, 4.06\%, 1) + \$5{,}000 = \$4{,}000(1.0406)$$
$$+ \$5{,}000 = \$9{,}162.40$$
$$F_{-2} = \$9{,}162.40(F/P, 3.42\%, 1) + \$6{,}000 = \$15{,}475.75$$
$$F_{-1} = \$15{,}475.75(F/P, 5.23\%, 1) + \$7{,}000 = \$23{,}285.13$$
$$F_0 = \$23{,}285.13(F/P, 6.03\%, 1) = \$24{,}689.22$$

$$A = \$24{,}689.22(A/P, 5\%, 20) = \$24{,}689.22(0.0802) = \$1{,}980.08 \text{ per year}$$

Moral

Borrow as little as possible and repay as quickly as possible to reduce interest expense

- The present equivalent of a cash flow occurring at the end of period N can be computed with the equation below, where i_k is the interest rate for the k^{th} period. (the symbol Π means “the product of”):

$$P = \frac{F_N}{\prod_{k=1}^{N}(1 + i_k)}$$

Example:

If F_4 = \$2,500 and i_1=8%, i_2=10%, and i_3=11%, then

P = \$2,500[(P/F, 8%, 1)(P/F, 10%, 1)(P/F, 11%, 1)]

= \$2,500[(0.9259)(0.9091)(0.9009)] = \$1,896.

Nominal and Effective Interest Rates

- Many financial transactions requires that interest be compounded more often than once a year (e.g., quarterly, monthly, weekly, daily, etc.). In such situations, there are two expressions for the interest rate. The *nominal* interest rate, *r*, is expressed on an annual basis; this is the rate that is normally quoted when describing an interest-bearing transaction. The *effective* interest rate, *i*, is the rate that corresponds to the actual interest period.
- It has become customary to quote interest rates on an annual basis, followed by the compounding period if different from one year in length.

The effective interest rate is obtained by dividing the nominal interest rate, r, by the number of interest periods per year, M:

$$i = \frac{r}{M}$$

Example : a bank claims to pay interest to its depositors at the rate of 6% per year, compounded quarterly. What are the nominal and effective interest rates?

- For example, if the interest rate is 6% per interest period and the interest period is six months, it is customary to speak of this rate as **“12% compounded semiannually.”**
- The annual rate is known as a *nominal* rate.
- A nominal interest rate is represented by *r*. But the actual (or effective) annual rate on the principal is not 12%, but something greater, because compounding occurs twice during the year.
- The more frequent the compounding the greater the *effective* interest

For instance, consider a principal amount of \$1,000 to be invested for three years at a nominal rate of 12% compounded semiannually. The interest earned during the first six months would be \$1,000 × (0.12/2) = \$60.

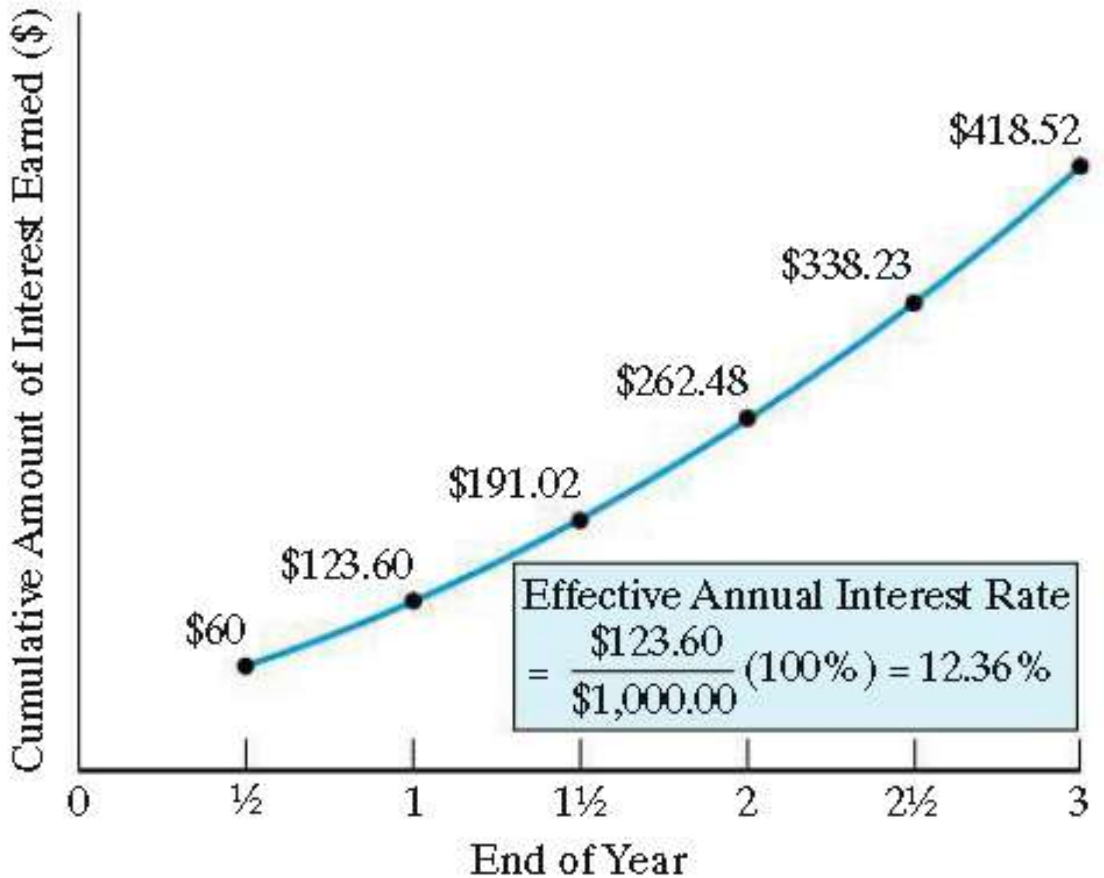
Total principal and interest at the beginning of the second six-month period is $P + Pi$ = \$1,000 + \$60 = \$1,060.

The interest earned during the second six months would be \$1,060 × (0.12/2) = \$63.60.

Then total interest earned during the year is

\$60.00 + \$63.60 = \$123.60.

Finally, the *effective* annual interest rate for the entire year is(\$123.60/ \$1,000) × 100 = 12.36%.



$1,000 Compounded at a Semiannual Frequency
(r = 12%, M = 2)

$1,000 Compounded at a Monthly Frequency
(r = 12%, M = 12)

TABLE 4-4 Effective Interest Rates for Various Nominal Rates and Compounding Frequencies

Compounding Frequency	Number of Compounding Periods per Year, *M*	Effective Rate (%) for Nominal Rate of					
		6%	8%	10%	12%	15%	24%
Annually	1	6.00	8.00	10.00	12.00	15.00	24.00
Semiannually	2	6.09	8.16	10.25	12.36	15.56	25.44
Quarterly	4	6.14	8.24	10.38	12.55	15.87	26.25
Bimonthly	6	6.15	8.27	10.43	12.62	15.97	26.53
Monthly	12	6.17	8.30	10.47	12.68	16.08	26.82
Daily	365	6.18	8.33	10.52	12.75	16.18	27.11

The actual or exact rate of interest earned on the principal for one year is known as the *effective rate*. It should be noted that effective interest rates are always expressed on an annual basis, unless specifically stated otherwise.

When Interest Periods Coincide with Payment Periods

When the interest periods and the payment periods coincide, it is possible to make direct use both of the compound interest formulas developed previously and the compound interest tables presented in Appendix C, provided the interest rate, *i*, is taken to be the *effective* interest rate *for that interest period*. Moreover, the number of years, *n*, must be replaced by the total number of interest periods, *Mn*.

Example: An engineer plans to borrow $3,000 from his company credit union, to be repaid in 24 equal monthly installments. The credit union charges interest at the rate of 1% per month on the unpaid balance. How much money must the engineer repay each month?

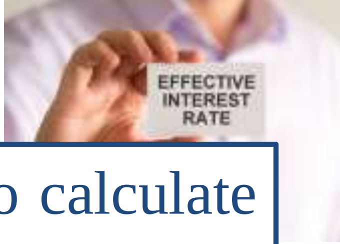
Example: An engineer wishes to purchase an \$80,000 home by making a down payment of \$20,000 and borrowing the remaining \$60,000 which he will repay on a monthly basis over the next 30 years. If the bank charges interest at the rate of 9½ % per year, compounded monthly, how much money must the engineer repay each month?

When Interest Periods are Smaller than Payment Periods

If the interest periods are smaller than the payment periods, then the interest may be compounded several times between payments.

One way to handle problems of this type is to determine the effective interest rate for the given interest period, and then treat each payment separately.

Example: An engineer deposits $1,000 in a savings account at the end of each year. If the bank pays interest at the rate of 6% per year, compounded quarterly, how much money will have accumulated in the account after 5 years?



Another procedure, which is usually more convenient, is to calculate an effective interest rate for the given payment period, and then to proceed as though the interest periods and the payment periods coincided.

This effective interest rate can be determined as:

$$i = \left(1 + \frac{r}{\alpha}\right)^{\alpha} - 1$$

Where α represents the number of interest periods per payment period and r is the nominal interest rate for that payment period. If the payment period is one year, then α = M, and we obtain the following expression for the *effective annual interest rate*:

$$i = \left(1 + \frac{r}{M}\right)^{M} - 1$$

Rework previous example by using an effective annual interest rate

EXAMPLE 4-28

A credit card company charges an interest rate of 1.375% per month on the unpaid balance of all accounts. The annual interest rate, they claim, is 12(1.375%) = 16.5%. What is the effective rate of interest per year being charged by the company?

Solution

Equation (4-32) is used to compute the effective rate of interest in this example:

$$i = \left(1 + \frac{0.165}{12}\right)^{12} - 1$$

$$= 0.1781, \text{ or } 17.81\%/\text{year}.$$

Note that $r = 12(1.375\%) = 16.5\%$, which is the APR. In general, it is true that $r = M(r/M)$, where r/M is the interest rate per period.

When Interest Periods are Larger than Payment Periods

If the interest periods are larger than the payment periods, some of the payments may not have been deposited for an entire interest period. Such payments do not earn any interest during that interest period. In other words, interest is earned only by those payments that have been deposited or invested for the entire interest period.

Situations of this type can be treated in the following manner:

1. Consider all *deposits* that were made during the interest period to have been made at the *end* of the interest period (and therefore to have earned no interest during that interest period).
2. Consider all *withdrawals* that were made during the interest period to have been made at the *beginning* of the interest period (again earning no interest).
3. Then proceed as though the interest periods and payment periods coincided.

Example: A person has $4,000 in a savings account at the beginning of a calendar year; the bank pays interest at 6% per year, compounded quarterly. Table below shows the transactions carried out during the calendar year; the second column gives the effective dates according to rules 1 and 2 above. Find the balance in the account at the end of the calendar year.

Date	Effective Date	Deposit	Withdrawal
Jan. 10	Jan. 1		$175
Feb. 20	Mar. 31	$1,200	
Apr. 12	Apr. 1		1,500
May 5	June 30	65	
May 13	June 30	115	
May 24	Apr. 1		50
June 21	Apr. 1		250
Aug.10	Sept. 30	1,600	
Sept. 12	July 1		800
Nov. 27	Oct. 1		350
Dec. 17	Dec. 31	2,300	
Dec. 29	Oct. 1		750

The APR is a nominal interest rate and *does not* account for compounding that may occur, or be appropriate, during a year.

→ APR: The nominal annual interest rate is also called the Annual Percentage Rate, or (APR)

→ APR = (r/M) x M

(r/M) is the interest rate per period

$I_{per.} = r/M$

Example: Finding effective annual interest rates.

1. For an 18% nominal rate, compounded quarterly, the effective annual interest is.

$$i = (1 + \frac{0.18}{4})^4 - 1 = 19.25\%$$

2. For a 7% nominal rate, compounded monthly, the effective annual interest is

$$i = (1 + \frac{0.07}{12})^{12} - 1 = 7.23\%$$

Example 4-29: Future Equivalent when Interest Is Compounded Quarterly

Suppose that a $100 lump-sum amount is invested for 10 years at a nominal interest rate of 6% compounded quarterly. How much is it worth at the end of the 10th year?

Solution

There are four compounding periods per year, or a total of $4 \times 10 = 40$ interest periods. The interest rate per interest period is $6\%/4 = 1.5\%$. When the values

are used in Equation (4-3), one finds that

$$F = P(F/P, 1.5\%, 40) = \$100.00(1.015)^{40} = \$100.00(1.814) = \$181.40.$$

Alternatively, the effective interest rate from Equation (4-32) is 6.14%. Therefore,

$$F = P(F/P, 6.14\%, 10) = \$100.00(1.0614)^{10} = \$181.40.$$

Example 4-30: Computing a Monthly Auto Payment

Stan Moneymaker has a bank loan for \$10,000 to pay for his new truck. This loan is to be repaid in equal *end-of-month* installments for five years with a nominal interest rate of 12% compounded monthly. What is the amount of each payment?

Solution

The number of installment payments is $5 \times 12 = 60$, and the interest rate per month is $12\%/12 = 1\%$. When these values are used in Equation (4-15), one finds that

$$A = P(A/P, 1\%, 60) = \$10{,}000(0.0222) = \$222.$$

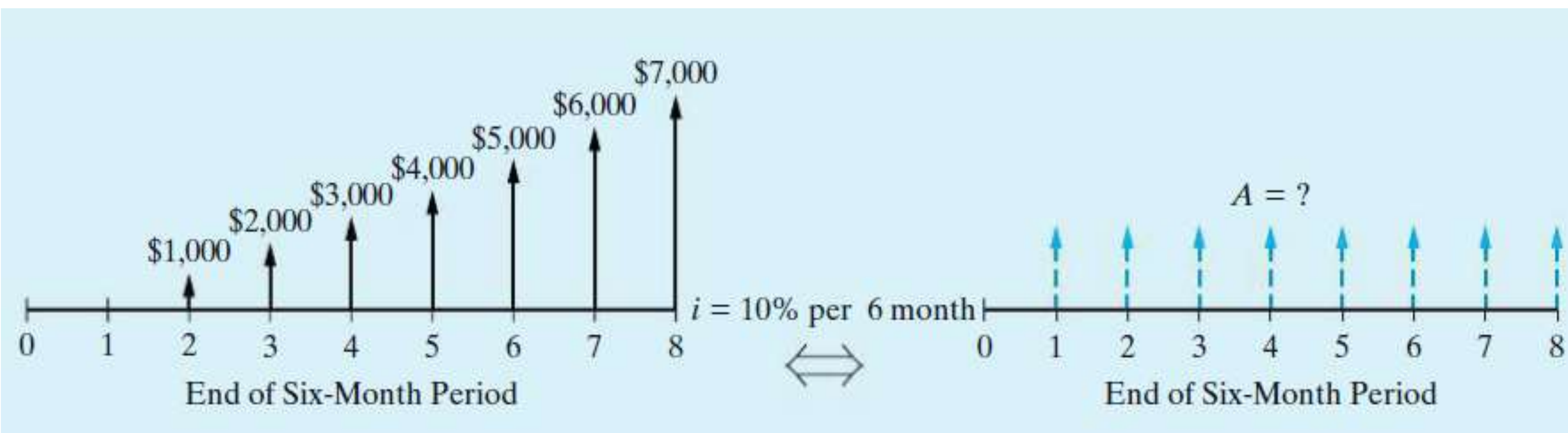
Notice that there is a cash flow at the end of each month (interest period), including month 60, in this example.

Example 4-31: Uniform Gradient Series and Semiannual Compounding

Certain operating savings are expected to be 0 at the end of the first six months, to be \$1,000 at the end of the second six months, and to increase by \$1,000 at the end of each six-month period thereafter, for a total of four years. It is desired to find the equivalent uniform amount, *A*, at the end of each of the eight six-month periods if the nominal interest rate is 20% compounded semiannually.

Solution

A cash-flow diagram is given below, and the solution is $A = G(A/G, 10\%, 8) = \$1,000(3.0045) = \$3,004.50$.



Example 4-32: Finding the Interest Rate on a Loan

A loan of $15,000 requires monthly payments of $477 over a 36-month period of time. These payments include both principal and interest.

(a) What is the nominal interest rate (APR) for this loan?

(b) What is the effective interest rate per year?

(c) Determine the amount of unpaid loan principal after 20 months.

Solution

(a) We can set up an equivalence relationship to solve for the unknown interest rate since we know that $P = \$15{,}000$, $A = \$477$, and $N = 36$ months.

$$\$477 = \$15{,}000(A/P, i_{mo}, 36)$$

$$(A/P,\ i_{mo}, 36) = 0.0318$$

We can now look through Appendix C to find values of i that have an $(A/P,\ i,\ 36)$ value close to 0.0318. From Table C-3 ($i = 3/4\%$), we find $(A/P, 3/4\%, 36) = 0.0318$. Therefore,

$$i_{mo} = 0.75\% \text{ per month}$$

and

$$r = 12 \times 0.75\% = 9\% \text{ per year, compounded monthly.}$$

(b) Using Equation (4-32),

$$i_{eff} = \left(1 + \frac{0.09}{12}\right)^{12} - 1 = 0.0938 \text{ or } 9.38\% \text{ per year.}$$

(c) We can find the amount of the unpaid loan principal after 20 months by finding the equivalent value of the remaining 16 monthly payments as of month 20.

$$P_{20} = \$477(P/A, 3/4\%, 16) = \$477(15.0243) = \$7{,}166.59$$

After 20 payments have been made, almost half of the original principal amount remains. Notice that we used the monthly interest rate of $3/4\%$ in our calculation since the cash flows are occurring monthly.

Interest can be compounded continuously.

- Interest is typically compounded at the end of discrete periods.
- In most company's cash is always flowing, and should be immediately put to use.
- We can allow compounding to occur continuously throughout the period.
- The effect of this compared to discrete compounding is small in most cases.

We can use the effective interest formula to derive the interest factors.

$$i = (1 + \frac{r}{M})^M - 1$$

As the number of compounding periods gets larger (*M* gets larger), we find that

$$i = e^r - 1$$

Continuous compounding interest factors.

$$(P/F, \underline{r}\%, N) = e^{-rN}$$

$$(F/A, \underline{r}\%, N) = \frac{e^{rN} - 1}{e^r - 1}$$

$$(P/A, \underline{r}\%, N) = \frac{e^{rN} - 1}{e^{rN}(e^r - 1)}$$

The other factors can be found from these.

TABLE 4-5 Continuous Compounding and Discrete Cash Flows: Interest Factors and Symbols[a]

To Find:	Given:	Factor by which to Multiply "Given"	Factor Name	Factor Functional Symbol
For single cash flows:				
F	P	e^{rN}	Continuous compounding compound amount (single cash flow)	$(F/P, \underline{r}\%, N)$
P	F	e^{-rN}	Continuous compounding present equivalent (single cash flow)	$(P/F, \underline{r}\%, N)$
For uniform series (annuities):				
F	A	$\frac{e^{rN}-1}{e^r-1}$	Continuous compounding compound amount (uniform series)	$(F/A, \underline{r}\%, N)$
P	A	$\frac{e^{rN}-1}{e^{rN}(e^r-1)}$	Continuous compounding present equivalent (uniform series)	$(P/A, \underline{r}\%, N)$
A	F	$\frac{e^r-1}{e^{rN}-1}$	Continuous compounding sinking fund	$(A/F, \underline{r}\%, N)$
A	P	$\frac{e^{rN}(e^r-1)}{e^{rN}-1}$	Continuous compounding capital recovery	$(A/P, \underline{r}\%, N)$

EXAMPLE 4-33

You have \$10,000 to invest for two years. Your bank offers 5% interest, compounded continuously for funds in a money market account. Assuming no additional deposits or withdrawals, how much money will be in that account at the end of two years?

Solution

F = \$10,000 (F/P, r = 5%, 2) = \$10,000 $e^{(0.05)(2)}$ = \$10,000 (1.1052) = \$11,052

Comment

If the interest rate was 5% compounded annually, the account would have been worth F = \$10,000 (F/P, 5%, 2) = \$10,000 (1.1025) = \$11,025.

EXAMPLE 4-34

Suppose that one has a present loan of $1,000 and desires to determine what equivalent uniform EOY payments, *A*, could be obtained from it for 10 years if the nominal interest rate is 20% compounded continuously ($M = \infty$).

Solution

Here we utilize the formulation $A = P(A/P, r\%, N)$.

Since the (A/P) factor is not tabled for continuous compounding, we substitute its inverse (P/A), which is tabled in Appendix D. Thus,

$A = P \times 1\ (P/A, 20\%, 10) = \$1{,}000 \times 1\ 3.9054 = \$256$.

Note that the answer to the same problem, with discrete annual compounding

($M = 1$), is $A = P(A/P, 20\%, 10) = \$1{,}000(0.2385) = \$239$.

EXAMPLE 4-35

An individual needs $12,000 immediately as a down payment on a new home. Suppose that he can borrow this money from his insurance company. He must repay the loan in equal payments every six months over the next eight years. The nominal interest rate being charged is 7% compounded continuously. What is the amount of each payment?

Solution

The nominal interest rate per six months is 3.5%. Thus, A each six months is $\$12{,}000(A/P, \underline{r} = 3.5\%, 16)$. By substituting terms in Equation (4-38) and then using its inverse, we determine the value of A per six months to be $997:

$$A = \$12{,}000\left[\frac{1}{(P/A, \underline{r} = 3.5\%, 16)}\right] = \frac{\$12{,}000}{12.038} = \$997.$$

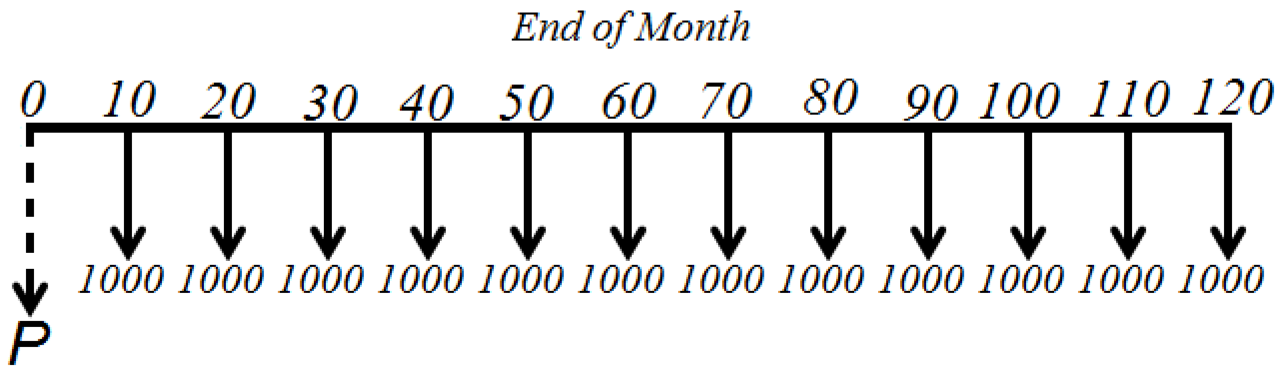
Examples

Example 1: A person borrowed \$40,000 at an interest rate of 12% compounded monthly, the debt will be repaid at 20 equal monthly installments.

1. Calculate the value of each installment if the first one is to be repaid after one month.
2. Calculate the value of each installment if the first one is to be repaid after one year

Example 2: A person invested $10,000 at an interest rate of 15% compounded monthly, for 150 months. Calculate the lump-sum value at the end of the period

Example 3: find the present worth (P) of the shown cash flow if the interest is 12% compounded monthly.



Example 4: A person borrowed $100,000 , which will be repaid at 20 successive equal annual installments, at an annual interest rate of 15%. if the first one is to be repaid after one year. How much does he repaid from the capital after 15 years? (how much remain from the capital)

Example 5: A man plans to buy a $150,000 house, he wants to make a down payment of $30,000 and to take out a 30-years mortgage for the remaining $120,000 at 10% per year, compounded monthly. How much he repay each month?

Example 6: How much money must be deposited in a savings account each month to accumulates \$10,000 at the end of 5 years, if the bank pays interest at the rate of 6% per year, compounded (a) monthly? (b) semiannually? (c) quarterly? (d) daily?

Evaluating a Single Project

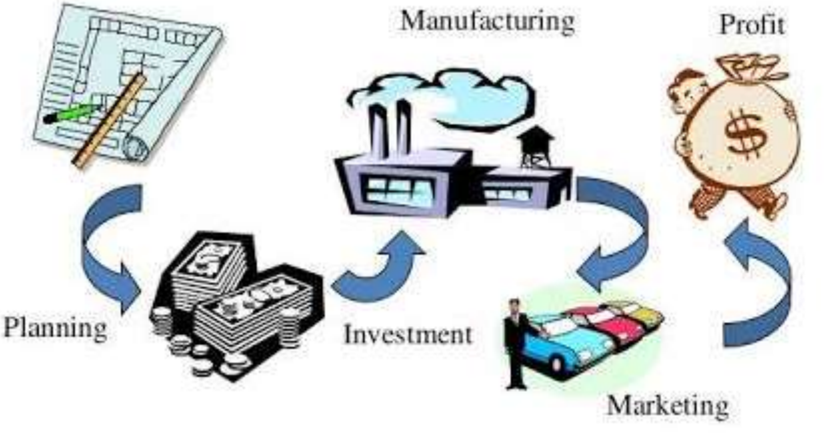
Introduction

A basic question that will be answered in this chapter is whether a proposed capital investment and its associated expenditures can be recovered by revenue (or savings) over time *in addition to* a return on the capital that is sufficiently attractive in view of the risks involved and the potential alternative uses.

The objective of this chapter is to discuss and critique contemporary methods for determining project profitability.

Proposed capital projects can be evaluated in several ways:

- Present worth (PW)
- Future worth (FW)
- Annual worth (AW)
- Internal rate of return (IRR)
- External rate of return (ERR)
- Payback period (generally not appropriate as a primary decision rule)



Determining the Minimum Attractive Rate of Return (MARR)

- To be attractive, a capital project must provide a return that exceeds a minimum level established by the organization. This minimum level is reflected in a firm's Minimum Attractive Rate of Return (MARR).

- Many elements contribute to determining the MARR.

1. The amount of money available for investment, and the source and cost of these funds.

2. The number of good projects available for investment and their purpose .

3. The amount of perceived risk associated with investment opportunities available to the firm.

4. The type of organization involved.

The Present Worth Method

- It is the most-used method

- The present worth (PW) is found by discounting all cash inflows and outflows to the present time at an interest rate that is generally the MARR.

- A positive PW for an investment project means that the project is acceptable (it satisfies the MARR).

PW Decision Rule:
If PW (i = MARR) ≥ 0, the project is economically justified.

$$PW(i\%) = F_0(1+i)^0 + F_1(1+i)^{-1} + F_2(1+i)^{-2} + \ldots + F_k(1+i)^{-k} + \ldots + F_N(1+i)^{-N}$$

$$= \sum_{k=0}^{N} F_k(1+i)^{-k}$$

Here;

i = effective interest rate, or MARR, per compounding period;

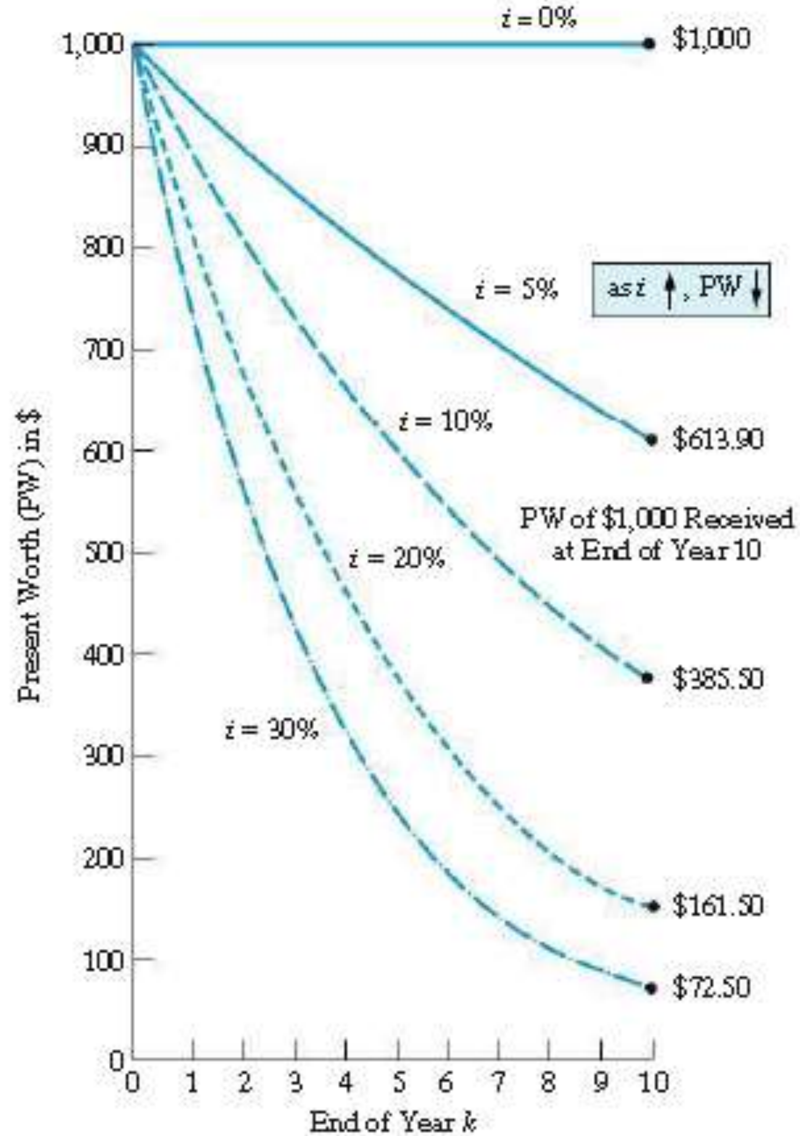
k = index for each compounding period ($0 \le k \le N$);

F_k = future cash flow at the end of period k;

N = number of compounding periods in the planning horizon (i.e., study period).

Assumption: *constant interest rate* throughout the life of a particular project

It is important to observe that the higher the interest rate and the farther into
the future a cash flow occurs, the lower its PW is.



PW of $1,000 Received at the End of Year *k* at an Interest Rate of *i*% per Year

Present Worth Example

Example: Consider a project that has an initial investment of \$50,000 and that returns \$18,000 per year for the next four years. If the MARR is 12%, is this a good investment?

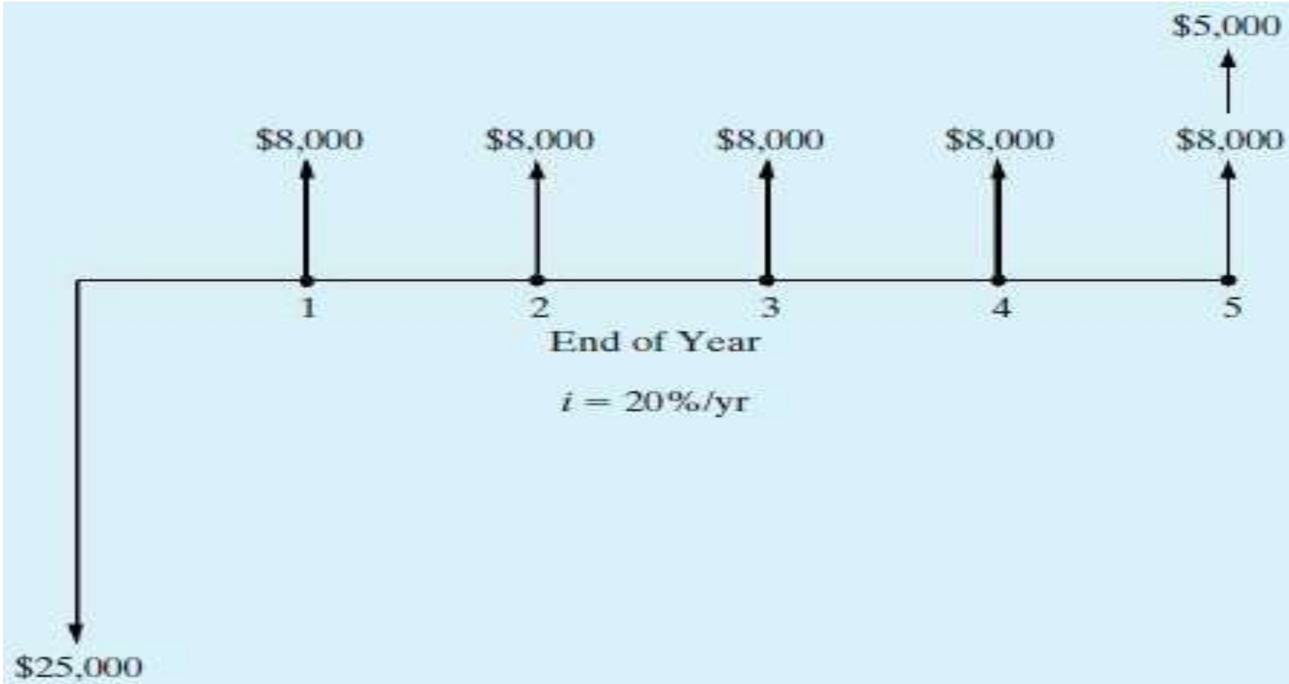
$$PW = -50{,}000 + 18{,}000\ (P/A,\ 12\%,\ 4)$$

$$PW = -50{,}000 + 18{,}000\ (3.0373)$$

$PW = \$4{,}671.40$ → This is a good investment!

Example 5-1: Evaluation of New Equipment Purchase Using PW

A piece of new equipment has been proposed by engineers to increase the productivity of a certain manual welding operation. The investment cost is $25,000, and the equipment will have a market value of $5,000 at the end of a study period of five years. Increased productivity attributable to the equipment will amount to $8,000 per year after extra operating costs have been subtracted from the revenue generated by the additional production. If the firm's MARR is 20% per year, is this proposal a sound one? Use the PW method.



Solution

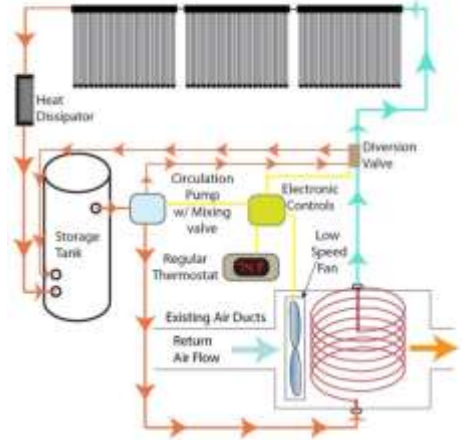
PW = PW of cash inflows − PW of cash outflows,

Or

PW(20%) = \$8,000(*P*/*A*, 20%, 5) + \$5,000(*P*/*F*, 20%, 5) − \$25,000 = \$934.29

Because PW(20%) ≥ 0, this equipment is economically justified.

Example 5-2 : Present Worth of a Space-Heating System



A retrofitted space-heating system is being considered for a small office building. The system can be purchased and installed for \$110,000, and it will save an estimated 300,000 kilowatt-hours (kWh) of electric power each year over a six-year period. A kilowatt-hour of electricity costs \$0.10, and the company uses a MARR of 15% per year in its economic evaluations of refurbished systems. The market value of the system will be \$8,000 at the end of six years, and additional annual operating and maintenance expenses are negligible. Use the PW method to determine whether this system should be installed.

Solution

To find the PW of the proposed heating system, we need to find the present equivalent of all associated cash flows. The estimated annual savings in electrical power is worth 300,000 kWh×\$0.10/kWh = \$30,000 per year. At a MARR of 15%, we get

$$PW(15\%) = -\$110{,}000 + \$30{,}000\,(P/A, 15\%, 6) + \$8{,}000\,(P/F, 15\%, 6)$$
$$= -\$110{,}000 + \$30{,}000(3.7845) + \$8{,}000(0.4323) = \$6{,}993.40.$$

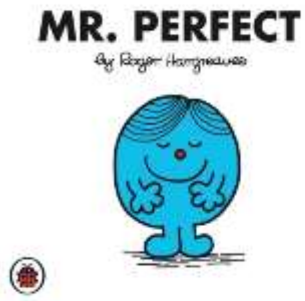
Since $PW(15\%) \geq 0$, we conclude that the retrofitted space-heating system should be installed

Assumptions of the PW Method

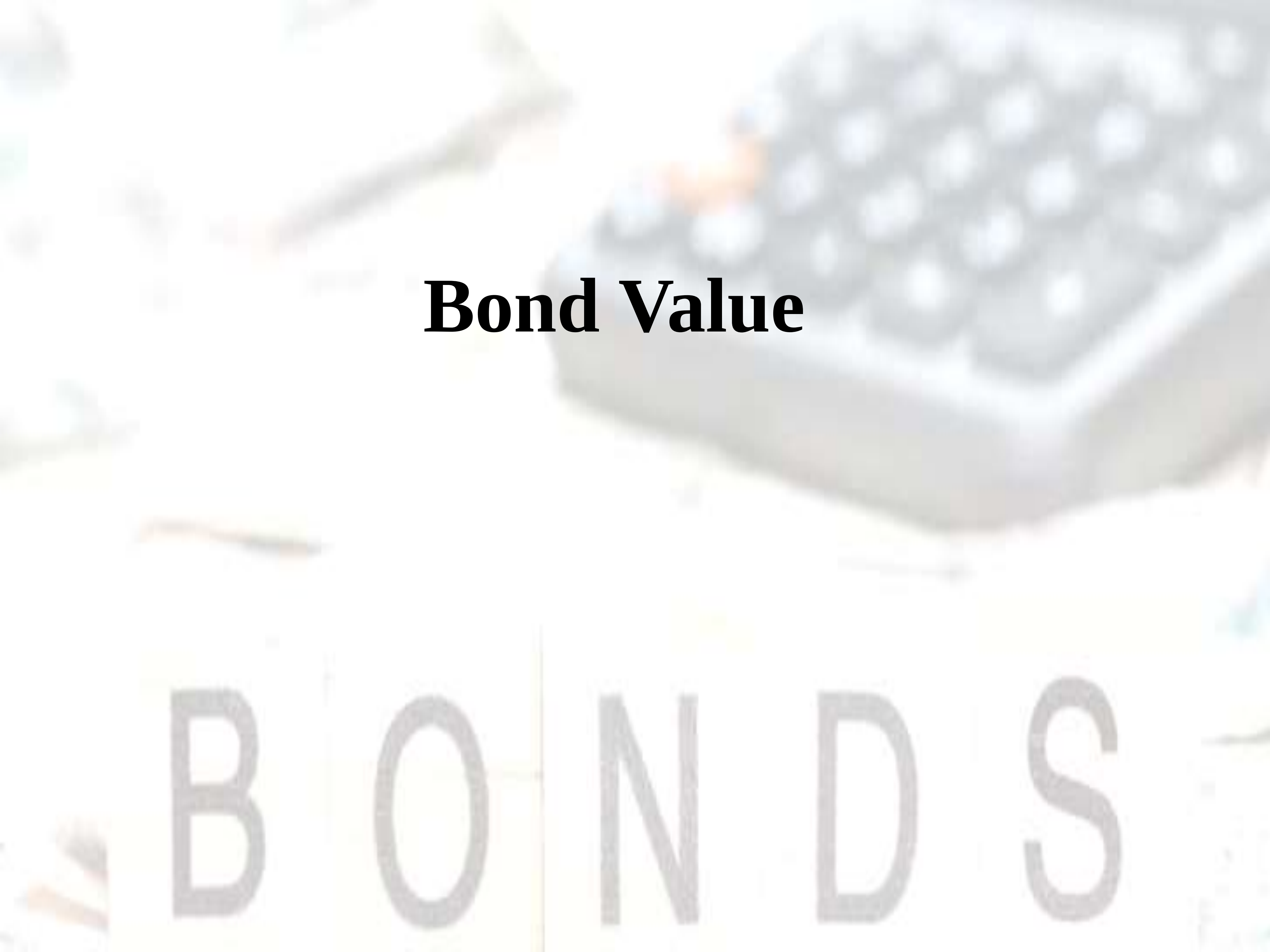
There are several noteworthy assumptions that we make when using PW to model the wealth-creating promise of a capital investment opportunity.

First, it is assumed that we know the future with certainty (we don't live in a certain world!).

Second, it is assumed we can borrow and lend money at the same interest rate (i.e., capital markets are perfect).



Bond Value



A bond is an IOU (I owe you) where you agree to lend the bond issuer money for a specified length of time (say, 10 years). In return, you receive periodic interest payments (e.g., quarterly) from the issuer plus a promise to return the face value of the bond when it matures.

A bond provides an excellent example of commercial value as being the PW of the future net cash flows that are expected to be received through ownership of an interest-bearing certificate.

The value of a bond, at any time, is the PW of future cash receipts.

For a bond, let

Z = face, or par, value;

C = redemption or disposal price (usually equal to Z);

r = bond rate (nominal interest) per interest period;

N = number of periods before redemption;

i = bond *yield* rate per period;

V_N = value (price) of the bond N interest periods prior to redemption—this is a PW measure of merit.

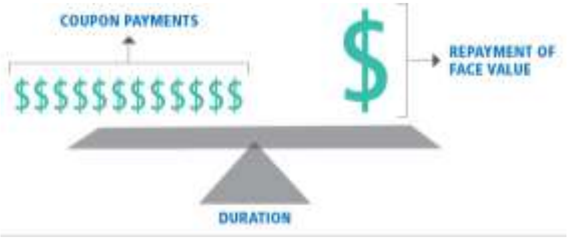
The owner of a bond is paid two types of payments by the borrower.

The first consists of the series of periodic interest payments he or she will receive until the bond is retired. There will be N such payments, each amounting to rZ. These constitute an annuity of N payments.

In addition, when the bond is retired or sold, the bondholder will receive a single payment equal in amount to C.

The PW of the bond is the sum of PWs of these two types of payments at the bond's yield rate (i%):

$$V_N = C(P/F, i\%,N) + rZ(P/A, i\%,N).$$



Bond example

What is the value of a 6%, 10-year bond with a par (and redemption) value of $20,000 that pays dividends semi-annually, if the purchaser wishes to earn an 8% return?

$$V_N = \$20{,}000\ (P/F, 4\%, 20) + (0.03)\$20{,}000\ (P/A, 4\%, 20)$$

$$V_N = \$20{,}000\ (0.4564) + (0.03)\$20{,}000\ (13.5903)$$

$$V_N = \$17{,}282.18$$

Example 5-3: Stan Moneymaker Wants to Buy a Bond

Stan Moneymaker has the opportunity to purchase a certain U.S. Treasury bond that matures in eight years and has a face value of \$10,000. This means that Stan will receive \$10,000 cash when the bond's maturity date is reached. The bond stipulates a fixed nominal interest rate of 8% per year, but interest payments are made to the bondholder every three months; therefore, each payment amounts to 2% of the face value. Stan would like to earn 10% nominal interest (compounded quarterly) per year on his investment, because interest rates in the economy have risen since the bond was issued. How much should Stan be willing to pay for the bond?

Solution

To establish the value of this bond, in view of the stated conditions, the PW of future cash flows during the next eight years (the study period) must be evaluated. Interest payments are quarterly. Because Stan Moneymaker desires to obtain 10% *nominal interest per year* on the investment, the PW is computed at i = 10%/4 = 2.5% per quarter for the remaining 8(4) = 32 quarters of the bond's life:

VN = \$10,000(P/F, 2.5%, 32) + \$10,000(0.02)(P/A, 2.5%, 32) = \$4,537.71 + \$4,369.84 = \$8,907.55.

Thus, Stan should pay no more than \$8,907.55 when 10% nominal interest per year is desired.

Example5-4 : Current Price and Annual Yield of Bond Calculations

A bond with a face value of $5,000 pays interest of 8% per year. This bond will be redeemed at par value at the end of its 20-year life, and the first interest payment is due one year from now.

(a) How much should be paid now for this bond in order to receive a yield of 10% per year on the investment?

(b) If this bond is purchased now for $4,600, what annual yield would the buyer receive?

Solution

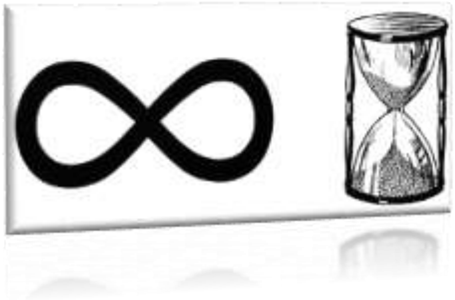
(a) By using Equation (5-2), the value of VN can be determined:

VN = \$5,000(P/F, 10%, 20) + \$5,000(0.08)(P/A, 10%, 20)
= \$743.00 + \$3,405.44 = \$4,148.44.

(b) Here, we are given VN = \$4,600, and we must find the value of i% in Equation (5-2):

\$4,600 = \$5,000(P/F, i'%, 20) + \$5,000(0.08)(P/A, i'%, 20).

To solve for i'%, we can resort to an iterative trial-and-error procedure (e.g., try 8.5%, 9.0%), to determine that i'% = 8.9% per year.



The Capitalized-Worth Method

- Capitalized worth is a special variation of present worth.
- Capitalized worth is the present worth of all revenues or expenses over an infinite length of time.
- If only expenses are considered this is sometimes referred to as *capitalized cost.*
- The capitalized worth method is especially useful in problems involving endowments and public projects with indefinite lives.

The CW of a perpetual series of end-of-period uniform payments A, with interest at i% per period, is $A(P/A, i\%, \infty)$. From the interest formulas, it can be seen that $(P/A, i\%, N) \rightarrow 1/i$ as N becomes very large. Thus, CW = A/i for such a series, as can also be seen from the relation

$$CW(i\%) = PW_{N\rightarrow\infty} = A(P/A, i\%, \infty)$$

$$= A\left[\lim_{N\rightarrow\infty} \frac{(1+i)^N - 1}{i(1+i)^N}\right] = A\left(\frac{1}{i}\right)$$

Hence, the CW of a project with interest at i% per year is the annual equivalent of the project over its useful life divided by i (as a decimal).

The AW of a series of payments of amount \$X at the end of each kth period with interest at i% per period is \$X(A/F, i%, k). The CW of such a series can thus be calculated as \$X(A/F, i%, k)/i.

Example 5-5 : Determining the Capitalized Worth of a Bridge

A new bridge across the Cumberland River is being planned near a busy highway intersection in the commercial part of a mid-western town. The construction (first) cost of the bridge is $1,900,000 and annual upkeep is estimated to be $25,000. In addition to annual upkeep, major maintenance work is anticipated every eight years at a cost of $350,000 per occurrence. The town government's MARR is 8% per year.

(a) For this problem, what analysis period (*N*) is, practically speaking, defined as forever?

(b) If the bridge has an expected life of 50 years, what is the capitalized worth (CW) of the bridge over a 100-year study period?

Solution

(a) A practical approximation of "forever" (infinity) is dependent on the interest rate. By examining the $(A/P, i\%, N)$ factor as N increases in the Appendix C tables, we observe that this factor approaches a value of i as N becomes large.

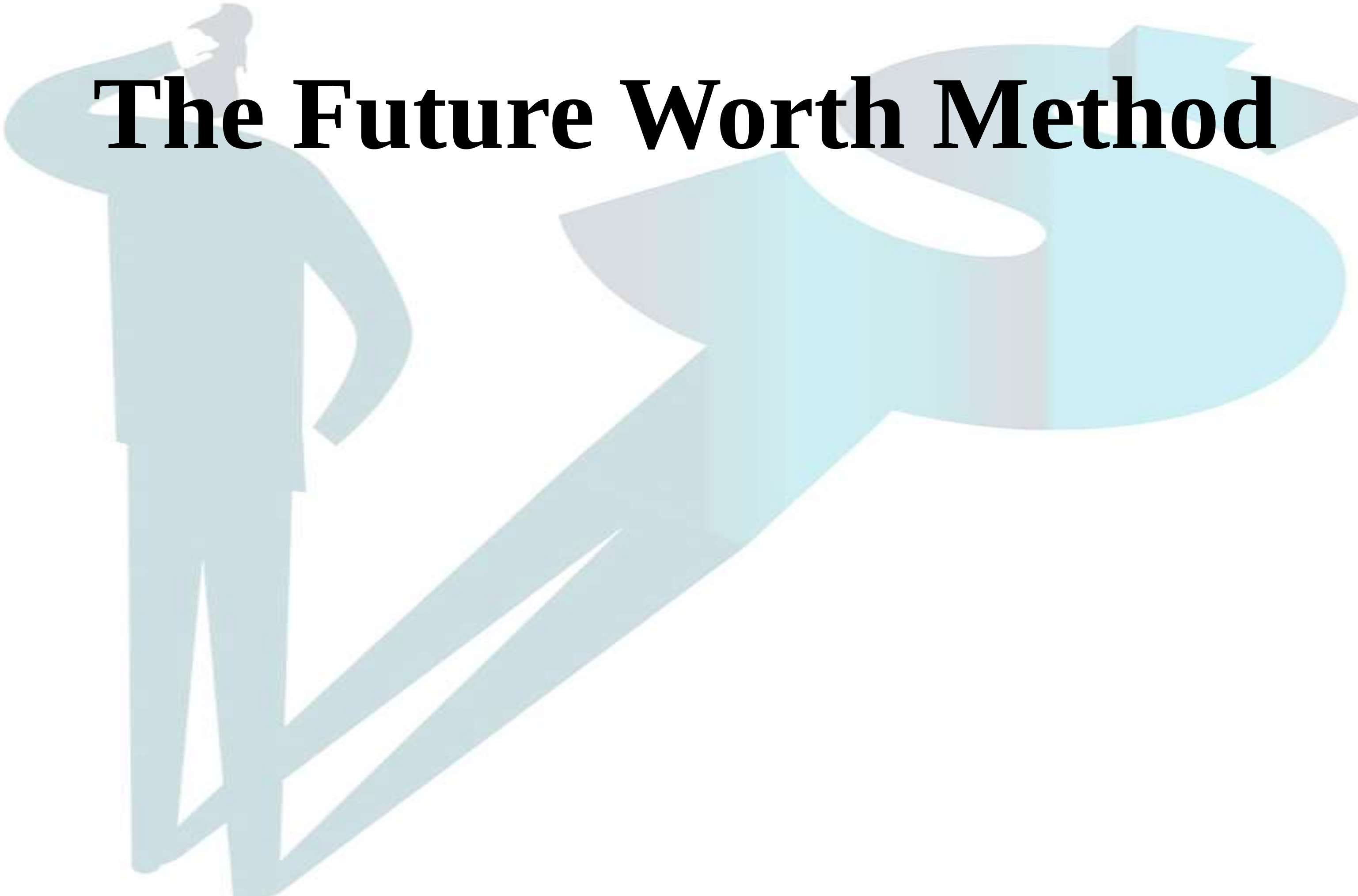
For $i = 8\%$ (Table C-11), the $(A/P, 8\%, 100)$ factor is 0.08. So, $N = 100$ years is, for practical purposes, "forever" in this example.

(b) The CW is determined as follows:

CW(8%) = −\$1,900,000 − \$1,900,000 $(P/F, 8\%, 50)$ −[\$350,000 $(A/F, 8\%, 8)$]/0.08 − \$25,000/0.08.

The CW turns out to be −\$2,664,220 over a 100-year study period, assuming the bridge is replaced at the end of year 50 for \$1,900,000.

The Future Worth Method



- Future Worth (FW) method is an alternative to the PW method.
- Looking at FW is appropriate since the primary objective is to maximize the futعre wealth of owners of the firm.
- FW is based on the equivalent worth of all cash inflows and outflows at the end of the study period at an interest rate that is generally the MARR.
- Decisions made using FW and PW will be the same.

FW Decision Rule:
If FW (i = MARR) ≥ 0, the project is economically justified.

Future worth example.

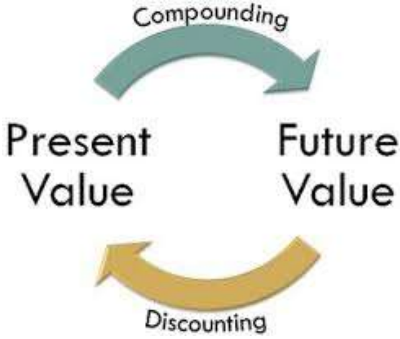
A \$45,000 investment in a new conveyor system is projected to improve throughput and increasing revenue by \$14,000 per year for five years. The conveyor will have an estimated market value of \$4,000 at the end of five years. Using FW and a MARR of 12%, is this a good investment?

$$FW = -\$45{,}000(F/P, 12\%, 5)+\$14{,}000(F/A, 12\%, 5)+\$4{,}000$$

$$FW = -\$45{,}000(1.7623)+\$14{,}000(6.3528)+\$4{,}000$$

$FW = \$13{,}635.70$ → This is a good investment!

Example5-6: The Relationship between FW and PW



A piece of new equipment has been proposed by engineers to increase the productivity of a certain manual welding operation. The investment cost is \$25,000, and the equipment will have a market value of \$5,000 at the end of a study period of five years. Increased productivity attributable to the equipment will amount to \$8,000 per year after extra operating costs have been subtracted from the revenue generated by the additional production. If the firm's MARR is 20% per year.

- Evaluate the FW of the potential improvement project.
- Show the relationship between FW and PW for this example.

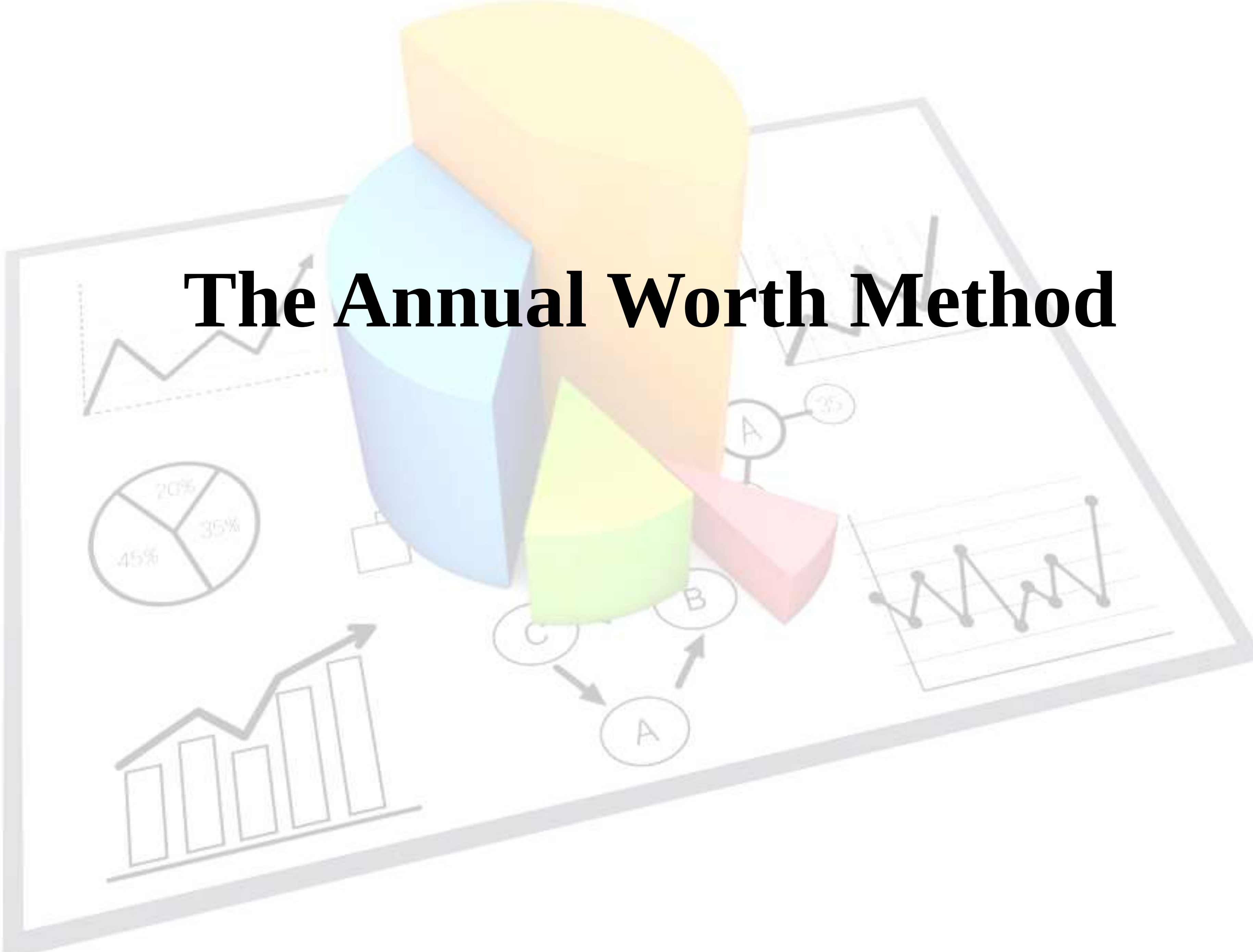
Solution

FW(20%) = −\$25,000(F/P, 20%, 5) + \$8,000(F/A, 20%, 5) + \$5,000 = \$2,324.80.

Again, the project is shown to be a good investment (FW ≥ 0). The PW is a multiple of the equivalent FW value:

PW(20%) = \$2,324.80(P/F, 20%, 5) = \$934.29.

The Annual Worth Method



- Annual Worth (AW) is another way to assess projects.
- Annual worth is an equal periodic series of dollar amounts that is *equivalent* to the cash inflows and outflows, at an interest rate that is generally the MARR.
- The AW of a project is annual equivalent revenue or savings $\underline{R}$ minus annual equivalent expenses $\underline{E}$, less its annual capital recovery (*CR*) amount.

$$AW(i\%) = \underline{R} - \underline{E} - CR(i\%)$$

- the AW of a project is equivalent to its PW and FW. That is, AW = PW(*A*/*P*, *i*%,*N*), and AW = FW(*A*/*F*, *i*%,*N*).

AW Decision Rule:
If AW (i = MARR) ≥ 0, the project is economically justified.

Capital Recovery (CR)

- Capital recovery reflects the capital cost of the asset.

- *CR* is the annual equivalent cost of the capital invested.

- The *CR* covers the following items.
  - Loss in value of the asset.
  - Interest on invested capital (at the MARR).

- The *CR* distributes the initial investment cost for the project (*I*) and the salvage (market) value at the end of the study period (*S*) across the life of the asset (N).

$$CR(i\%) = I(A/P, i\%, N) - S(A/F, i\%, N)$$

A project requires an initial investment of $45,000, has a salvage value of $12,000 after six years, incurs annual expenses of $6,000, and provides an annual revenue of $18,000. Using a MARR of 10%, determine the AW of this project.

$$AW(10\%) = \underline{R} - \underline{E} - CR(10\%)$$

$$CR(10\%) = 45,000(A/P, 10\%, 6) - 12,000(A/F, 10\%, 6)$$

$$CR(10\%) = 8,777$$

$$AW(10\%) = 18,000 - 6,000 - 8,777 = \$3,223$$

Since the *AW* is positive, it's a good investment.

Example5-8: Using AW to Evaluate the Purchase of New Equipment

A piece of new equipment has been proposed by engineers to increase the productivity of a certain manual welding operation. The investment cost is $25,000, and the equipment will have a market value of $5,000 at the end of a study period of five years. Increased productivity attributable to the equipment will amount to $8,000 per year after extra operating costs have been subtracted from the revenue generated by the additional production. If the firm's MARR is 20% per year.

Use the AW method to determine whether the equipment should be recommended.

Solution

The AW Method applied to Example 5-1 yields the following:

$$\text{AW}(20\%) = \underbrace{\$8{,}000}_{R-E} - \underbrace{[\$25{,}000(A/P, 20\%, 5) - \$5{,}000(A/F, 20\%, 5)]}_{\text{CR amount [Equation (5-5)]}}$$

$$= \$8{,}000 - \$8{,}359.50 + \$671.90$$

$$= \$312.40.$$

Because its AW(20%) is positive, the equipment more than pays for itself over a period of five years, while earning a 20% return per year on the unrecovered investment. In fact, the annual equivalent “surplus” is \$312.40, which means that the equipment provided more than a 20% return on beginning-of-year unrecovered investment. This piece of equipment should be recommended as an attractive investment opportunity. Also, we can confirm that the AW(20%) is equivalent to PW(20%)=\$934.29 in Example 5-1 and FW(20%)=\$2,324.80 in Example 5-6. That is,

AW(20%) = \$934.29(A/P, 20%, 5) = \$312.40, and also
AW(20%) = \$2,324.80(A/F, 20%, 5) = \$312.40.

When revenues are *absent* in Equation

$$AW(i\%) = \underline{R} - \underline{E} - CR(i\%)$$

We designate this metric as EUAC(i%) and call it “equivalent uniform annual cost.” A low- valued EUAC(i%) is preferred to a high-valued EUAC(i%).

Example 5-9: Equivalent Uniform Annual Cost of a Corporate Jet

A corporate jet costs \$1,350,000 and will incur \$200,000 per year in fixed costs (maintenance, licenses, insurance, and hangar rental) and \$277 per hour in variable costs (fuel, pilot expense, etc.). The jet will be operated for 1,200 hours per year for five years and then sold for \$650,000. The MARR is 15% per year.

(a) Determine the capital recovery cost of the jet.

(b) What is the EUAC of the jet?

- **Solution**

(a) CR = \$1,350,000 (A/P, 15%, 5) − \$650,000 (A/F, 15%, 5) = \$306,310.

(b) The total annual expense for the jet is the sum of the fixed costs and the variable costs.

E = \$200,000 + (1,200 hours)(\$277/hour) = \$532,400

EUAC(15%) = \$532,400 + \$306,310 = \$838,710

Example 5-10: Determination of Annual Savings by Using the AW Method

A retrofitted space-heating system is being considered for a small office building. The system can be purchased and installed for $110,000. The market value of the system will be $8,000 at the end of six years, and additional annual operating and maintenance expenses are negligible. What is the minimum annual electrical power savings (in kWh) required to make this project economically acceptable? The MARR = 15% per year and electricity costs $0.10 per kWh.

Solution

To make this project acceptable, the annual power savings must be at least as great as the annual CR amount. Using Equation (5-5),

$$\text{CR} = \$110{,}000(A/P, 15\%, 6) - \$8{,}000\,(A/F, 15\%, 6) = \$28{,}148.40.$$

This value (\$28,148.40) is the minimum annual dollar savings needed to justify the space-heating system. This equates to

$$\frac{\$28{,}148.40}{\$0.10/\text{kWh}} = 281{,}480 \text{ kWh per year.}$$

If the space-heating system can save 281,480 kWh per year, it is economically justified (exactly 15% is earned on the beginning-of-year unrecovered investment). Any savings greater than 281,480 kWh per year (such as the original estimate of 300,000 kWh per year) will serve to make this project even more attractive.

Example: Avoid Getting Fleeced on an Auto Lease

Automobile leases are built around three factors: negotiated sales price, residual value, and interest rate. The residual value is what the dealership expects the car's value will be when the vehicle is returned at the end of the lease period. The monthly cost of the lease is the capital recovery amount determined by using these three factors.

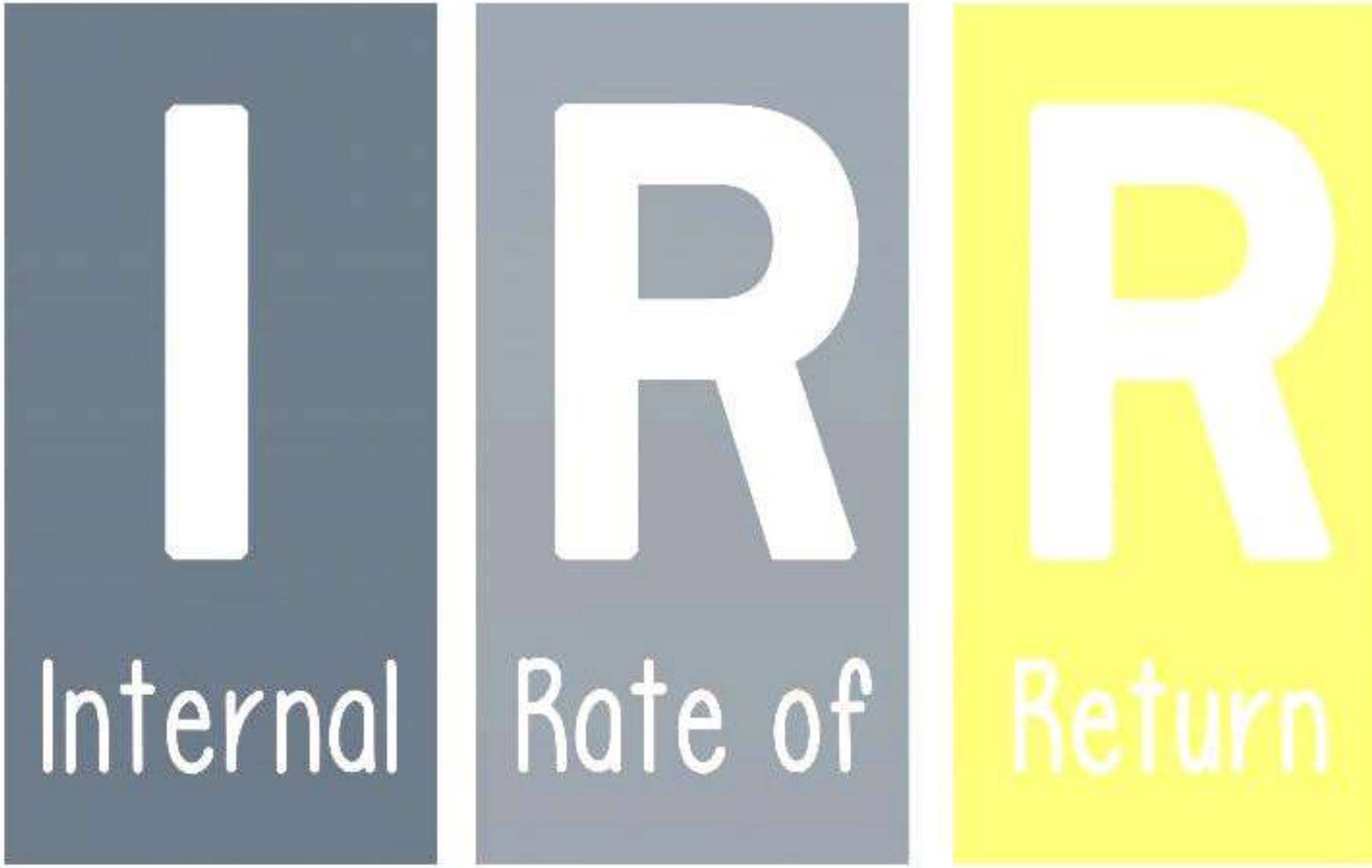
(a) Determine the monthly lease payment for a car that has an agreed-upon sales price of $34,995, an APR of 9% compounded monthly, and an estimated residual value of $20,000 at the end of a 36-month lease.* An up-front payment of $3,000 is due when the lease agreement (contract) is signed.

(b) If the estimated residual value is raised to $25,000 by the dealership to get your business, how much will the monthly payment be?

Solution

(a) The effective sales price is \$31,995 (\$34,995 less the \$3,000 due at signing). The monthly interest rate is 9%/12 = 0.75% per month. So, the capital recovery amount is:
CR = \$31,995(A/P, 0.75%, 36) − \$20,000(A/F, 0.75%, 36) = \$1,017.44 − \$486 = \$531.44 per month.

(b) The capital recovery amount is now \$1,017.44−\$25,000 (A/F, 0.75%, 36) = \$409.94 per month. But the customer might experience an actual residual value of less than \$25,000 and have to pay the difference in cash when the car is returned after 36 months. This is the "trap" that many experience when they lease a car, so be careful not to drive the car excessively or to damage it in any way.



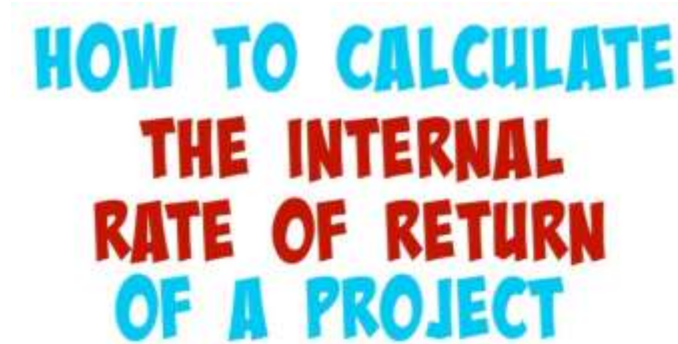
The Internal Rate of Return Method (IRR)

- The internal rate of return (IRR) method is the most widely used rate of return method for performing engineering economic analysis.

- It is also called the *investor's method*, the *discounted cash flow* method, and the *profitability index*.

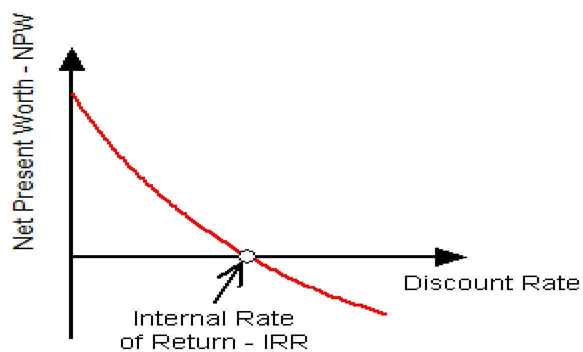
- If the IRR for a project is greater than the MARR, then the project is *acceptable*.

How the IRR works



This method solves for the interest rate that equates the equivalent worth of an alternative’s cash inflows (receipts or savings, R) to the equivalent worth of cash outflows (expenditures, including investment costs, E). The resultant interest rate is termed the *Internal Rate of Return (IRR)*.

The *IRR* is sometimes referred to as the *breakeven interest rate*.



- IRR is positive for a single alternative only if:
  - both receipts and expenses are present in the cash flow diagram
  - the sum of inflows exceeds the sum of outflows

- Using a PW formulation, we see that the IRR is the $i'\%*$ at which

$$\sum_{k=0}^{N} R_k(P/F, i'\%, k) = \sum_{k=0}^{N} E_k(P/F, i'\%, k)$$

where ; R_k = net revenues or savings for the kth year;

E_k = net expenditures, including any investment costs for the kth year;

N = project life (or study period).

Note: FW or AW can be used instead of PW

IRR Decision Rule:

If IRR ≥ MARR, the project is economically justified.

- Solving for the IRR is a bit more complicated than PW, FW, or AW

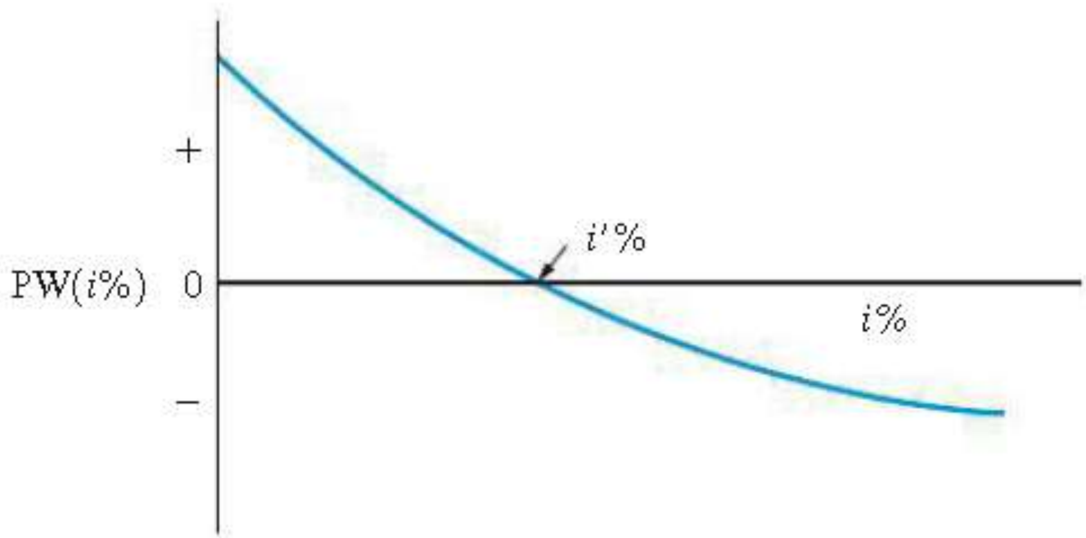
- The method of solving for the *i'%* that equates revenues and expenses normally involves trial-and-error calculations or solving numerically using mathematical software.

- A popular variation of previous Equation for computing the IRR for an alternative is to determine the *i′* at which its *net* PW is zero. In equation form, the IRR is the value of *i′* at which

$$PW = \sum_{k=0}^{N} R_k(P/F, i'\%, k) - \sum_{k=0}^{N} E_k(P/F, i'\%, k) = 0$$

For an alternative with a single investment cost at the present time followed by a series of positive cash inflows over *N*, a graph of PW versus the interest rate typically has the general convex form shown in Figure below. The point at which PW = 0 defines i'%, which is the project's IRR.

The value of i'% can also be determined as the interest rate at which FW = 0 or AW= 0.



Plot of PW versus Interest Rate

Example 5-12: Economic Desirability of a Project Using the IRR Method

AMT, Inc., is considering the purchase of a digital camera for the maintenance of design specifications by feeding digital pictures directly into an engineering workstation where computer-aided design files can be superimposed over the digital pictures. Differences between the two images can be noted, and corrections, as appropriate, can then be made by design engineers. The capital investment requirement is $345,000 and the estimated market value of the system after a six-year study period is $115,000. Annual revenues attributable to the new camera system will be $120,000, whereas additional annual expenses will be $22,000. You have been asked by management to determine the IRR of this project and to make a recommendation. The corporation’s MARR is 20% per year.

Solution by Linear Interpolation

In this example, we can easily see that the sum of positive cash flows ($835,000) exceeds the sum of negative cashflows($455,000). Thus, it is likely that a positive valued IRR can be determined. By writing an equation for the PW of the project's total net cash flow and setting it equal to zero, we can compute the IRR:

$PW = 0 = -\$345{,}000 + (\$120{,}000 - \$22{,}000)(P/A, i'\%, 6) + \$115{,}000(P/F, i'\%, 6)$

$i'\% = ?$

To use linear interpolation, we first need to try a few values for i'. A good starting point is to use the MARR.

At $i' = 20\%$: $PW = -\$345{,}000 + \$98{,}000(3.3255) + \$115{,}000(0.3349) = +\$19{,}413$

Since the PW is positive at 20%, we know that $i' > 20\%$.

At $i' = 25\%$: $PW = -\$345{,}000 + \$98{,}000(2.9514) + \$115{,}000(0.2621)$

$= -\$25{,}621$

Now that we have both a positive and a negative PW, the answer is bracketed ($20\% \leq i'\% \leq 25\%$). The dashed curve in Figure 5-5 is what we are linearly approximating. The answer, $i'\%$, can be determined by using the similar triangles represented by dashed lines in Figure 5-5.

$$\frac{\text{line BA}}{\text{line BC}} = \frac{\text{line dA}}{\text{line de}}.$$

Here, BA is the line segment $B - A = 25\% - 20\%$. Thus,

$$\frac{25\% - 20\%}{\$19{,}413 - (-\$25{,}621)} = \frac{i'\% - 20\%}{\$19{,}413 - \$0}$$

$$i' \approx 22.16\%.$$

Because the IRR of the project (22.16%) is greater than the MARR, the project is acceptable.

Example 5-13 *See the Book* : Evaluation of New Equipment Purchase, Using the Internal Rate of Return Method

A piece of new equipment has been proposed by engineers to increase the productivity of a certain manual welding operation. The investment cost is $25,000, and the equipment will have a market (salvage) value of $5,000 at the end of its expected life of five years. Increased productivity attributable to the equipment will amount to $8,000 per year after extra operating costs have been subtracted from the value of the additional production. Determine the IRR of the proposed equipment. Is the investment a good one? Recall that the MARR is 20% per year.

Installment Financing

Example 5-14 (HW) : Do You Know What Your Effective Interest Rate Is?

In 1915, Albert Epstein allegedly borrowed \$7,000 from a large New York bank on the condition that he would repay 7% of the loan every three months, until a total of 50 payments had been made. At the time of the 50th payment, the \$7,000 loan would be completely repaid. Albert computed his annual interest rate to be [0.07(\$7,000) × 4]/\$7,000 = 0.28 (28%).

(a) What true *effective* annual interest rate did Albert pay?

(b) What, if anything, was wrong with his calculation?

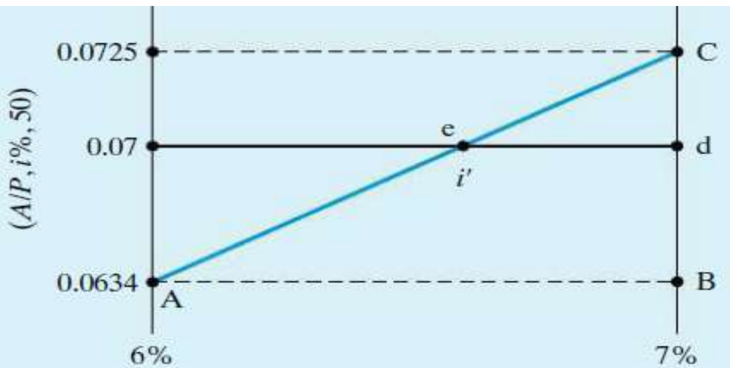
Solution

(a) The true interest rate per quarter is found by equating the equivalent value of the amount borrowed to the equivalent value of the amounts repaid.

Equating the AW amounts per quarter, we find \$7,000(A/P, i'%/quarter, 50 quarters) = 0.07(\$7,000) per quarter, (A/P, i'%, 50) = 0.07.

Linearly interpolating to find i'%/quarter by using similar triangles is the next step:

$$(A/P, 6\%, 50) = 0.0634,$$
$$(A/P, 7\%, 50) = 0.0725.$$



$$\frac{\text{line de}}{\text{line BA}} = \frac{\text{line Cd}}{\text{line CB}}$$

$$\frac{7\% - i'\%}{7\% - 6\%} = \frac{0.0725 - 0.07}{0.0725 - 0.0634},$$

$$i'\% = 7\% - 1\%\left(\frac{0.0025}{0.0091}\right),$$

or $i'\% \simeq 6.73\%$ per quarter.

Now we can compute the effective i'% per year that Albert was paying:

$i'\% = [(1.0673)^4 - 1]100\% \simeq 30\%$ per year.

(b) Even though Albert's answer of 28% is close to the true value of 30%, his calculation is insensitive to how long his payments were made. For instance, he would get 28% for an answer when 20, 50, or 70 quarterly payments of \$490 were made! For 20 quarterly payments, the true effective interest rate is 14.5% per year, and for 70 quarterly payments, it is 31% per year. As more payments are made, the true annual effective interest rate being charged by the bank will increase, but Albert's method would not reveal by how much.

Example 5-15 (H.W): Be Careful with "Fly-by-Night" Financing!

The Fly-by-Night finance company advertises a "bargain 6% plan" for financing the purchase of automobiles. To the amount of the loan being financed, 6% is added for each year money is owed. This total is then divided by the number of months over which the payments are to be made, and the result is the amount of the monthly payments. For example, a woman purchases a $10,000 automobile under this plan and makes an initial cash payment of $2,500. She wishes to pay the $7,500 balance in 24 monthly payments:

Purchase price	=	$10,000
−Initial payment	=	2,500
= Balance due, (P_0)	=	7,500
+6% finance charge = 0.06 × 2 years × $7,500	=	900
= Total to be paid	=	8,400
∴ Monthly payments (A) = $8,400/24	=	$350

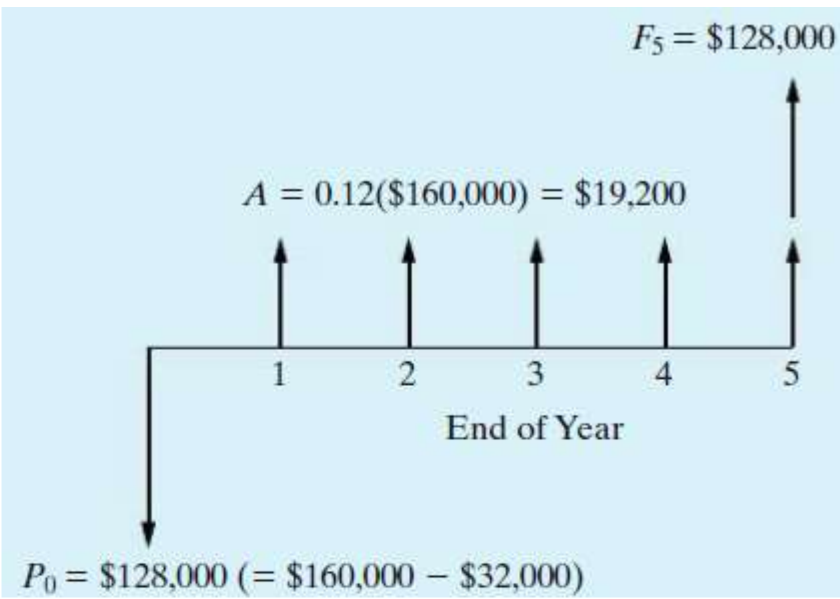
What effective annual rate of interest does she actually pay?

Example 5-16: Effective Interest Rate for Purchase of New Aircraft Equipment

A small airline executive charter company needs to borrow \$160,000 to purchase a prototype synthetic vision system for one of its business jets. The SVS is intended to improve the pilots' situational awareness when visibility is impaired. The local (and only) banker makes this statement: "We can loan you \$160,000 at a very favorable rate of 12% per year for a five-year loan. However, to secure this loan, you must agree to establish a checking account (with no interest) in which the *minimum* average balance is \$32,000. In addition, your interest payments are due at the end of each year, and the principal will be repaid in a lump-sum amount at the end of year five." What is the true effective annual interest rate being charged?

Solution

The cash-flow diagram from the banker's viewpoint appears below. When solving for an unknown interest rate, it is good practice to draw a *cashflow diagram* prior to writing an equivalence relationship. Notice that $P_0 = \$160{,}000 - \$32{,}000 = \$128{,}000$. Because the bank is requiring the company to open an account worth \$32,000, the bank is only \$128,000 out of pocket. This same principal applies to *F*5 in that the company only needs to repay \$128,000 since the \$32,000 on deposit can be used to repay the original principal.



The interest rate (IRR) that establishes equivalence between positive and negative cash flows can now easily be computed:*

$P0 = F5(P/F, i'\%, 5) + A(P/A, i'\%, 5)$,
$\$128{,}000 = \$128{,}000(P/F, i'\%, 5) + \$19{,}200(P/A, i'\%, 5)$.

If we try $i' = 15\%$, we discover that \$128,000 = \$128,000. Therefore, the true
effective interest rate is 15% per year.

Challenges in Applying the IRR Method.

- It is computationally difficult without proper tools.
- In rare instances multiple rates of return can be found. (See Appendix 5-A.)
- The IRR method must be carefully applied and interpreted when comparing two more mutually exclusive alternatives (e.g., do not directly compare internal rates of return).

Reinvesting Revenue—the External Rate of Return (ERR)

- The IRR assumes revenues generated are reinvested at the IRR—which may not be an accurate situation.

- The ERR takes into account the interest rate, ε, external to a project at which net cash flows generated (or required) by a project over its life can be reinvested (or borrowed). This is usually the MARR.

- If the ERR happens to equal the project's IRR, then using the ERR and IRR produce identical results.

The ERR procedure

- Discount all the net cash *outflows* to time *0* at ε% per compounding period.

- Compound all the net cash *inflows* to period *N* at at ε%.

- Solve for the ERR, the interest rate that establishes equivalence between the two quantities

ERR is the $i'\%$ at which

$$\sum_{k=0}^{N} E_k(P/F, \varepsilon\%, k)(F/P, i'\%, N) = \sum_{k=0}^{N} R_k(F/P, \varepsilon\%, N-k)$$

Where;

R_k =excess of receipts over expenses in period k,

E_k =excess of expenses over receipts in period k,

N =project life or number of periods, and

ε =external reinvestment rate per period.

ERR Decision Rule:

If ERR ≥ MARR, the project is economically justified.

The ERR method has two basic advantages over the IRR method:

1. It can usually be solved for directly, without needing to resort to trial and error.

2. It is not subject to the possibility of multiple rates of return.

Applying the ERR method

For the cash flows given below, find the ERR when the external reinvestment rate is ε = 12% (equal to the MARR).

Year	0	1	2	3	4
Cash Flow	-$15,000	-$7,000	$10,000	$10,000	$10,000

Expenses $15,000 + 7,000(P/F, 12\%, 1) = 21,250$

Revenue $10,000(F/A, 12\%, 3) = 33,744$

Solving, we find

$$21,250(F/P, i'\%, 4) = 33,744$$

$$i' = 16.67\% > 12\%$$

Example 5-17: Calculation of the ERR

Referring to Example (**Evaluation of New Equipment Purchase, Using the Internal Rate of Return Method)**, suppose that ε =MARR = 20% per year. What is the project's ERR, and is the project acceptable?

Solution

By utilizing Equation (5-8), we have the following relationship to solve for *i′*:

$$\$25{,}000(F/P, i'\%, 5) = \$8{,}000(F/A, 20\%, 5) + \$5{,}000,$$

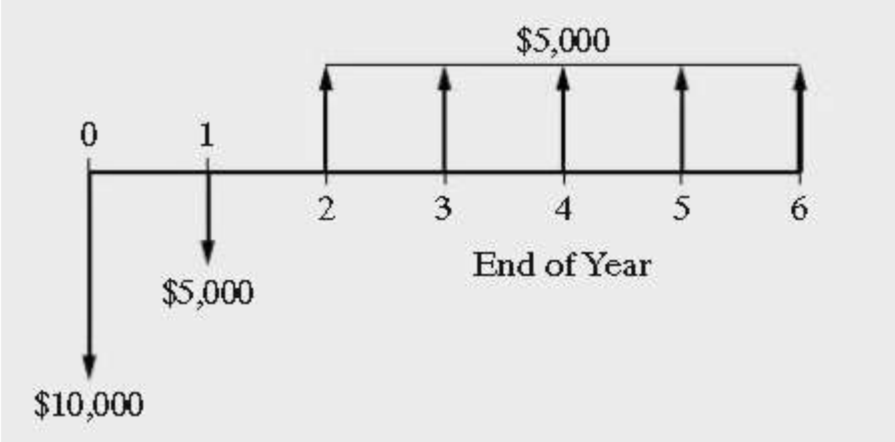
$$(F/P, i'\%, 5) = \frac{\$64{,}532.80}{\$25{,}000} = 2.5813 = (1 + i')^5,$$

$$i' = 20.88\%.$$

Because *i′* > MARR, the project is justified, but just barely.

Example 5-18: Determining the Acceptability of a Project, Using ERR

When ε=15% and MARR = 20% per year, determine whether the project (whose net cash-flow diagram appears next) is acceptable. Notice in this example that the use of an ε% different from the MARR is illustrated. This might occur if, for some reason, part or all of the funds related to a project are "handled" outside the firm's normal capital structure.



Solution

$E0$ = \$10,000 (k = 0),
$E1$ = \$5,000 (k = 1),
Rk = \$5,000 for k = 2, 3, . . . , 6,

[\$10,000 + \$5,000(P/F, 15%, 1)](F/P, i'%, 6) = \$5,000(F/A, 15%, 5); i'% = 15.3%.

The i'% is less than the MARR = 20%; therefore, this project would be unacceptable according to the ERR method.

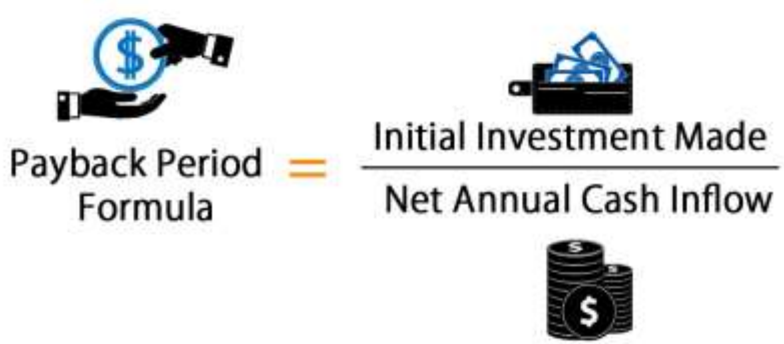
The Payback (Payout) Period Method

The payback period method is simple, but possibly misleading.

- The simple payback period is the number of years required for cash *inflows* to just equal cash *outflows*.

- It is a measure of *liquidity* rather than a measure of profitability.

- A low- valued pay back period is considered desirable



Payback is simple to calculate.

- The payback period is the *smallest* value of θ ($\theta \leq N$) for which the relationship below is satisfied.

$$\sum_{k=1}^{\theta}(R_k - E_k) - I \geq 0$$

- For *discounted* payback future cash flows are discounted back to the present, so the relationship to satisfy becomes

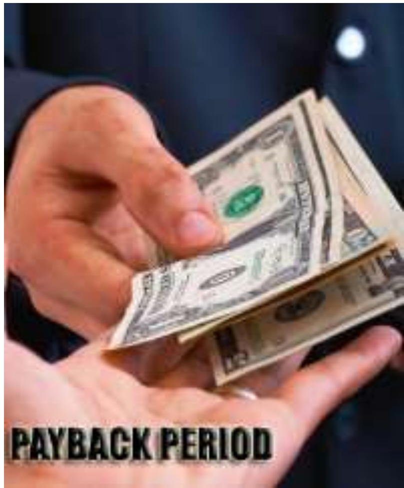
$$\sum_{k=1}^{\theta'}(R_k - E_k)(P/F, i\%, k) - I \geq 0$$

Problems with the payback period method:

- It doesn’t reflect any cash flows occurring after θ, or θ'.
- It doesn’t indicate anything about project desirability except the speed with which the initial investment is recovered.
- Recommendation: use the payback period only as supplemental information in conjunction with one or more of the other methods in this chapter.

Example: Finding the simple and discounted payback period for a set of cash flows.

End of Year	0	1	2	3	4	5
Net Cash Flow	$-42,000	$12,000	$11,000	$10,000	$10,000	$9,000



The cumulative cash flows in the table were calculated using the formulas for simple and discounted payback.

From the calculations $\theta = 4$ years and $\theta' = 5$ years.

End of Year	Net Cash Flow	Cumulative PW at *0%*	Cumulative PW at *6%*
0	-$42,000	-$42,000	-$42,000
1	$12,000	-$30,000	-$30,679
2	$11,000	-$19,000	-$20,889
3	$10,000	-$9,000	-$12,493
4	$10,000	$1,000	-$4,572
5	$9,000		$2,153

Col.3:

EOF1= -42,000-12,000=-30,000

EOF2 : 11,000-30,000=-19,000

EOF3 : 10,000-19,000=-9,000

EOF4 : 10,000-9,000=1,000>0

θ = 4

Col.4:

12,000 (p/F,6%,1) =

12,000* 0.9434= 113,208

11,000 (p/F,6%,2) = 11,000*0.8900= 9,790

10,000*(P/F,6%,3) =

10,000*0.8396= 8,396

10,000*(P/F,6%,4) =

10,000*0.7921 =7,921

9,000*(P/F,6%,5) =

9,000*0.7473 = 6,725.7

End of Year	Net Cash Flow	Cumulative PW at *0%*	Present worth at i = 6%	Cumulative PW at *6%*
0	-$42,000	-$42,000	-$42,000	-$42,000
1	$12,000	-$30,000	11,320.8	-$30,679
2	$11,000	-$19,000	9,790	-$20,889
3	$10,000	-$9,000	8,396	-$12,493
4	$10,000	$1,000	7,921	-$4,572
5	$9,000		6,725.7	$2,153

$\theta = 4$, because cumulative balance turns positive at EOY 4

Col.5:

EOF1:

-42,000 + 11,320.8 = -30,679.2

EOF2 :

9,790 -30,679.2 = -20,889.2

EOF3 :

8,396 -20,889.2 = -12,493.2

EOF4 :

7,921-12,493.2 = -4,572.2 <0

EOF5 :

6,725.7 -4,572.2 = 2,153.5 >0

End of Year	Net Cash Flow	Cumulative PW at *0%*	Present worth at i = 6%	Cumulative PW at *6%*
0	-$42,000	-$42,000	-$42,000	-$42,000
1	$12,000	-$30,000	11,320.8	-$30,679
2	$11,000	-$19,000	9,790	-$20,889
3	$10,000	-$9,000	8,396	-$12,493
4	$10,000	$1,000	7,921	-$4,572
5	$9,000		6,725.7	$2,153

$\theta' = 5$, because cumulative discounted balance turns positive at EOY 5

Example 5-19: Determining the Simple Payback Period

A public school is being renovated for $13.5 million. The building has geothermal heating and cooling, high-efficiency windows, and a solar array that permits the school to sell electricity back to the local electric utility. The annual value of these benefits is estimated to be $2.7 million. In addition, the residual value of the school at the end of its 40-year life is negligible. What is the simple payback period and internal rate of return for the renovated school?

Solution

The simple payback period is

$$\frac{\$13.5 \text{ million}}{\$2.7 \text{ million/year}} = 5 \text{ years.}$$

This is fairly good for a publically sponsored project. The IRR can be computed using the equation

$$0 = -\$13.5 \text{ million} + \$2.7 \text{ million } (P/A, i'\%, 40),$$

yielding $i'\%$ (the IRR) $= 20\%$ per year. The IRR indicates the project is profitable for a MARR of 20% per year or less.