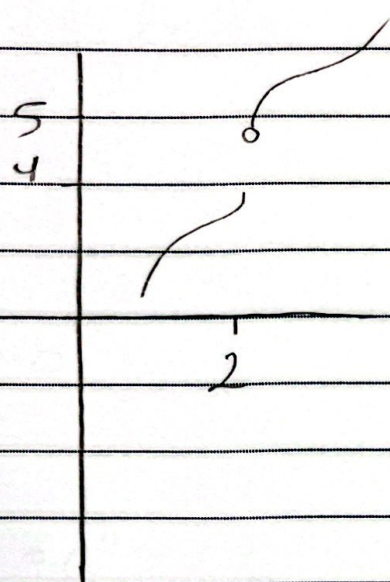


The limits of function

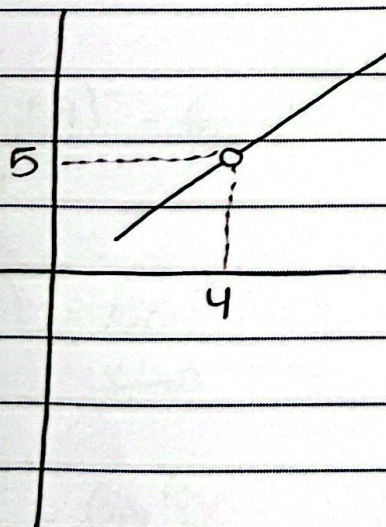
Chapter 2



$$\lim_{x \rightarrow 2} f(x) \begin{cases} \lim_{x \rightarrow 2^+} f(x) = 5 \\ \lim_{x \rightarrow 2^-} f(x) = 4 \end{cases}$$

$\lim_{x \rightarrow 2} f(x)$ does not exist

$$* \lim_{x \rightarrow a} f(x) = L \iff \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x)$$



$$\lim_{x \rightarrow 4} f(x) \begin{cases} \lim_{x \rightarrow 4^+} f(x) = 5 \\ \lim_{x \rightarrow 4^-} f(x) = 5 \end{cases}$$

$$\lim_{x \rightarrow 4} f(x) = 5$$

The limits of function

Chapter 2

* Calculating limits: $a, c \in \mathbb{R}$

$$1) \lim_{x \rightarrow a} c = c$$

$$\lim_{x \rightarrow 4} 10 = 10$$

$$2) \lim_{x \rightarrow a} x = a$$

$$\lim_{x \rightarrow 9} x = 9$$

$$3) \lim_{x \rightarrow a} x^n = a^n$$

$$\lim_{x \rightarrow 4} x^2 = 4^2 = 16, n = \text{عدد صحيح}$$

Then: $\lim_{x \rightarrow a} f(x) = L, \lim_{x \rightarrow a} g(x) = M, a, k \in \mathbb{R}$

$$1) \lim_{x \rightarrow a} (f \pm g) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = L \pm M \quad (\text{توزع})$$

$$2) \lim_{x \rightarrow a} (fg) = \lim_{x \rightarrow a} f \lim_{x \rightarrow a} g = LM$$

$$3) \lim_{x \rightarrow a} \frac{f}{g} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{L}{M}, M \neq 0$$

$$4) \lim_{x \rightarrow a} k f(x) = k \lim_{x \rightarrow a} f(x) = kL$$

$$5) \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} = \sqrt[n]{L} \quad \begin{matrix} \text{(neven)} \\ L > 0 \end{matrix}$$

The limits of Function

Chapter 2

Ex:

1) $\lim_{x \rightarrow 2} x^2 + 5x - 1 = 2^2 + 10 - 1 = 13$ * كثير الحدود دائما تعويض مباشر

* $\lim_{x \rightarrow a} P(x) = P(a)$ $P(x) = \text{poly}$
 $x \rightarrow a$

2) $f(x) = \begin{cases} \sqrt{x^2 - 1} & , x \geq 1 \text{ (يمين (+))} \\ \frac{5x^3 + 4}{x - 3} & , x < 1 \text{ (يسار (-))} \end{cases}$

a) $\lim_{x \rightarrow 3} f(x) = \lim_{x \rightarrow 3} \sqrt{x^2 - 1} = \sqrt{9 - 1} = 8$ تعويض مباشر

b) $\lim_{x \rightarrow 1} f(x)$
 $\lim_{x \rightarrow 1^+} \sqrt{x^2 - 1} = 0$
 $\lim_{x \rightarrow 1^-} \frac{5x^3 + 4}{x - 3} = \frac{9}{-2}$

$\lim_{x \rightarrow 1^+} f(x) \neq \lim_{x \rightarrow 1^-} f(x) \rightarrow \lim_{x \rightarrow 1} f(x) \text{ does not exist}$

Ex: $\lim_{x \rightarrow 2} f(x) = 3$, $\lim_{x \rightarrow 2} g(x) = 4$

Find:

The limits of function

Chapter 2

$$1) \lim_{x \rightarrow 2} f(x) + 5g(x) = \lim_{x \rightarrow 2} f(x) + 5 \lim_{x \rightarrow 2} g(x) = 3 + 5(4) = 23$$

$$2) \lim_{x \rightarrow 2} \frac{f(x) + g(x)}{f(x)} = \lim_{x \rightarrow 2} \frac{f(x)}{f(x)} + \lim_{x \rightarrow 2} \frac{g(x)}{f(x)} = 3 + \frac{4}{3} = \frac{13}{3}$$

Ex: Find k such that

limit $f(x)$: exist ∞
 $x \rightarrow 1$

$$f(x) = \begin{cases} x^2 + 1, & x > 1 \quad (+) \text{ اليمين} \\ 2kx + 5, & x \leq 1 \quad (-) \text{ اليسار} \end{cases}$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x)$$

$$\lim_{x \rightarrow 1^+} x^2 + 1 = \lim_{x \rightarrow 1^-} 2kx + 5$$

$$2 = 2k + 5$$

$$2k = -3$$

$$k = -\frac{3}{2}$$

The limits of function

chapter 2

* Calculating limits : $(\frac{0}{0})$

$$1) \lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 + x - 6} = \frac{2^2 - 4}{2^2 + 2 - 6} = \frac{0}{0} \quad \Delta$$

$$\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)(x+3)} = \lim_{x \rightarrow 2} \frac{(x+2)}{(x+3)} = \frac{4}{5} \quad \checkmark$$

$$2) \lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3} = \frac{0}{0} \quad \Delta$$

$$\lim_{x \rightarrow 3} \frac{\sqrt{x+1} - 2}{x - 3} \quad \text{اضرب بـ } \frac{\sqrt{x+1} + 2}{\sqrt{x+1} + 2}$$

$$\lim_{x \rightarrow 3} \frac{x+1 - 4}{(x-3)(\sqrt{x+1} + 2)} = \frac{x-3}{(x-3)(\sqrt{x+1} + 2)} = \frac{1}{\sqrt{x+1} + 2} = \frac{1}{4}$$

$$3) \lim_{x \rightarrow 2} \frac{\frac{3}{x} - \frac{3}{2}}{x - 2} = \frac{0}{0} \quad \Delta$$

$$\lim_{x \rightarrow 2} \frac{\frac{6-3x}{2x}}{x-2} = \lim_{x \rightarrow 2} \frac{\frac{3(2-x)}{2x}}{x-2} = \lim_{x \rightarrow 2} \frac{-3}{2x} = \frac{-3}{4}$$

$$4) \lim_{x \rightarrow 2} \frac{|x-2|}{x-2} = f(x) = \begin{cases} \frac{x-2}{x-2}, & x \geq 2 = 1 \\ \frac{x-2}{x-2}, & x < 2 = -1 \end{cases}$$

$$\lim_{x \rightarrow 2^+} |x-2| = 1, \quad \lim_{x \rightarrow 2^-} |x-2| = -1$$

$$\lim_{x \rightarrow 2} \frac{|x-2|}{x-2} \text{ does not exist}$$

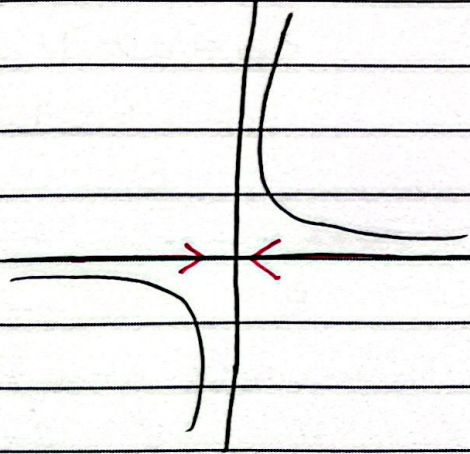
The limits of function

chapter 2

* Infinite limits : $\frac{c}{0}$, $c \in \mathbb{R}$, $c \neq 0$

$+\infty$ $-\infty$ $-\infty$, $+\infty$

1) $\lim_{x \rightarrow 0} \frac{1}{x} = \frac{1}{0} \begin{matrix} \nearrow +\infty \\ \searrow -\infty \end{matrix}$



2) $\lim_{x \rightarrow 7} \frac{1}{(x-7)^2} = \frac{1}{0} = +\infty$

$++++$
7

3) $\lim_{x \rightarrow 5} \frac{-1}{(x-5)^2} = \frac{-1}{0} = -\infty$

4) $\lim_{x \rightarrow 3} \frac{-1}{x-3} = \frac{-1}{0} \begin{matrix} \nearrow -\infty \\ \searrow +\infty \end{matrix}$

Ex: $f(x) = \frac{2-x}{(x-4)(x+2)}$

Find:

1) $\lim_{x \rightarrow 2} f(x) = 0 \checkmark$

$x \rightarrow 2$

2) $\lim_{x \rightarrow 4} f(x) = \frac{2-4}{0} = -\frac{2}{0}$

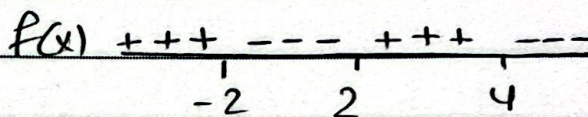
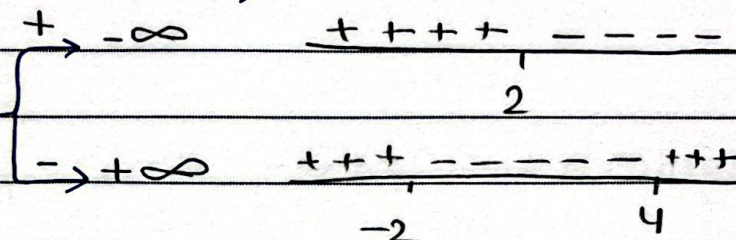
$x \rightarrow 4$

$x \rightarrow 4^+ \sim -\infty$

$x \rightarrow 4^- \sim +\infty$

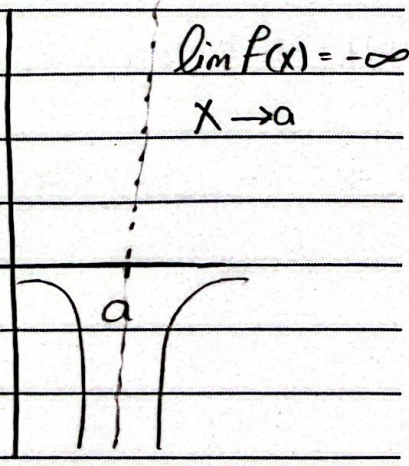
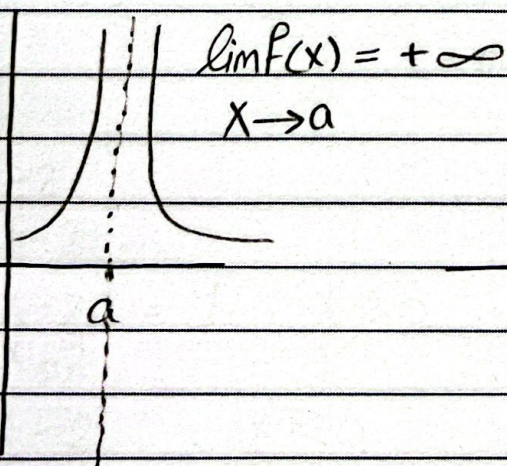
$\lim_{x \rightarrow -2} f(x) = \frac{4}{0}$

$x \rightarrow -2$



* Vertical asymptotes خطوط التقارب العمودية :

$x = a$ v.asy



Chapter 2

1) $f(x) = x^2 - 4$

$x^2 - 2x$

$$x^2 - 2x = 0$$

$$x(x-2) = 0$$

$$X=0, X=2$$

$$\lim_{x \rightarrow 0} \frac{x^2 - 4}{x^2 - 2x} = \frac{-4}{0} = \infty \rightarrow x=0 \text{ is vertical asymptotes}$$

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x^2 - 2x} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x(x-2)} = \lim_{x \rightarrow 2} \frac{x+2}{x} = \frac{4}{2} = 2 \quad X$$

* خطوات الحل :

(1) نخرج أصفار الحقام

(5) نفوذها في $\lim$ (إذا كان الجواب ∞) يكون U .

(3) إذا كان الجواب عددية كونها $not U$.

2) $f(x) = x - 2$

$|x| - 2$

$$|x| \Rightarrow x = 0 \quad - \quad +$$

$$P(x) = \begin{cases} x-2 & x \geq 0 \\ x-2 & x < 0 \end{cases} = 1, x \geq 0$$

$$\frac{x-2}{-x-2}, x < 0$$

$$\underline{-x - 2 = 0 \rightarrow x = -2 \text{ V. asy}}$$

$$|x| - 2 = 0 \rightarrow x = \pm 2$$

$$\lim F(x), \lim f(x)$$

$$x \rightarrow 2 \qquad x \rightarrow -2$$

Vertical asymptotes

Chapter 2

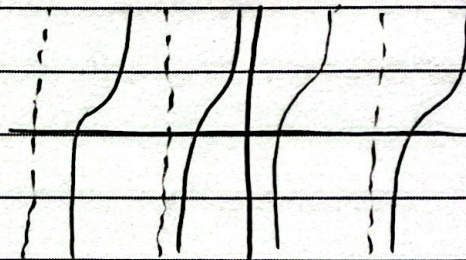
1) $f(x) = x^2 + 5x + 7$ (not v. asy)

2) $f(x) = \frac{1}{x-7}$, $x=7$ (v. asy)

3) $f(x) = \frac{1}{(x-1)(x-2)}$, $x=1, x=2$ (v. asy)

4) $f(x) = \frac{5}{(x-3)(x)(x+4)}$, $x=3, x=0, x=-4$ (v. asy)

5) $f(x) = \tan x$
 $x = \frac{\pi}{2} + n\pi$ (v. asy)



6) $f(x) = \log_b x$, $b > 1$

$x=0$ (v. asy)

$f(x) = \ln x$

$x=0$ (v. asy)

$\lim_{x \rightarrow 0^+} \ln x = -\infty$

$x \rightarrow 0^+$

$\lim_{x \rightarrow 0^+} \log_b x = -\infty$



Continuity

Chapter 2

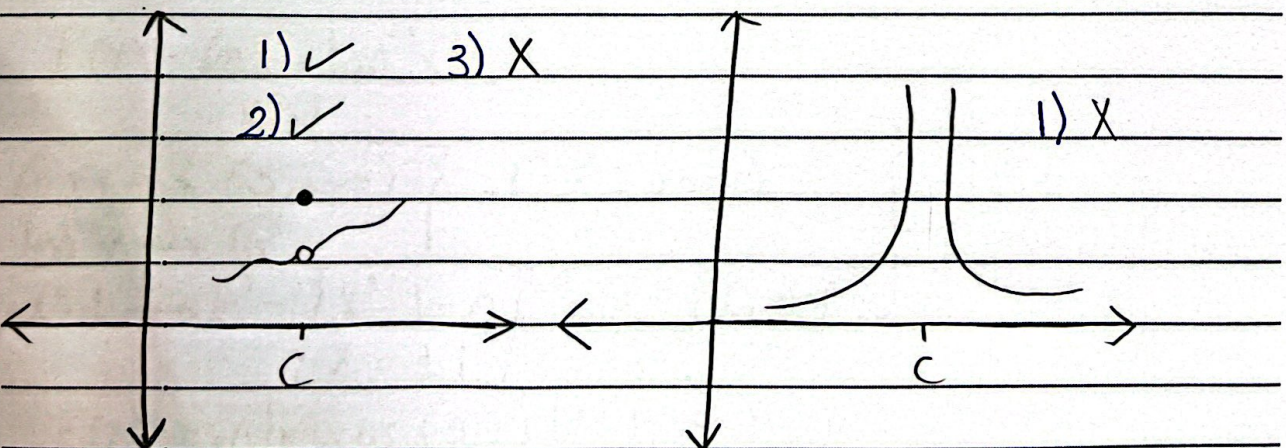
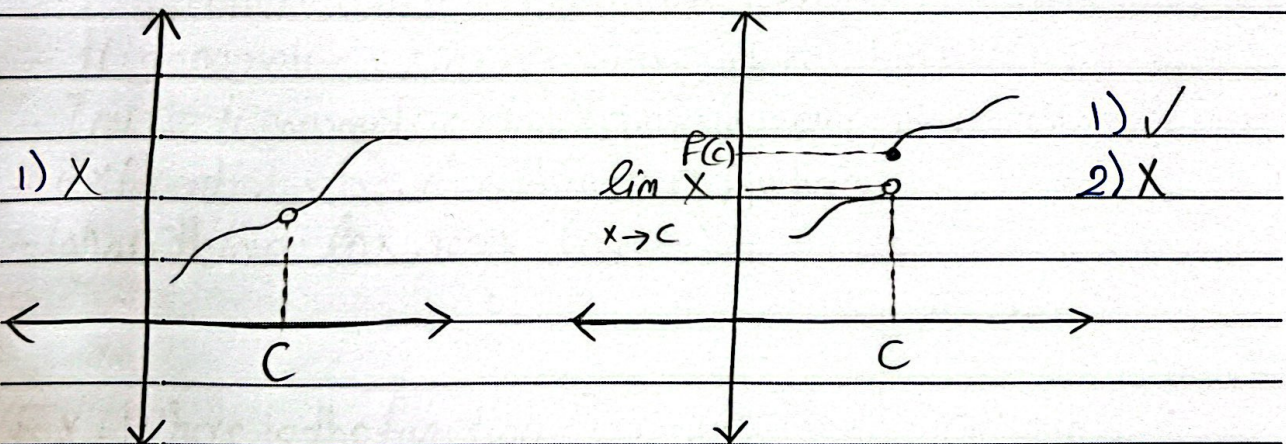
* Definition : A Function f is said to be continuous at $x=c$ provided the following conditions are satisfied :

- 1) $f(c)$ is defined ($c \in D_f$)
- 2) $\lim_{x \rightarrow c} f(x)$ exists ($\lim_{x \rightarrow c^+} f = \lim_{x \rightarrow c^-} f$)
- 3) $\lim_{x \rightarrow c} f(x) = f(c)$

* إذا الشرح الثالث تحقق معناه الأول والثاني تحققاً

إذا الشرح الأول ما تحقق ما ينكسر إذا الشرح الثاني ما تحقق ما ينكسر

* ملاحظة : f is not continuous at $x=c$, f discontinuous at $x=c$



Continuity

chapter 2

* نظريات الاصل :

1) If the functions f & g are continuous at $x=c$

a) $f \pm g$ is continuous at $x=c$

b) fg is continuous at $x=c$

c) $\frac{f}{g}$ is continuous at $x=c$, $g(c) \neq 0$

2) The following types of functions are continuous at every number in their domains

- Polynomials: $x^2 + 5x + 7$, x^3 , ... (everywhere)

- Rational: $\frac{1}{x} \rightarrow \mathbb{R} - \{0\}$

- root functions: $\sqrt{x}$ continuous $[0, \infty)$, $\sqrt[3]{x}$ continuous everywhere

- trigonometric: $\sin x$, $\cos x$ everywhere, $\tan x$ $\mathbb{R} - \{\frac{\pi}{2} + n\pi\}$

- Inverse trigonometric: $\tan^{-1} x$ continuous everywhere, $\sin^{-1} x$, $\cos^{-1} x$ $[-1, 1]$

- exponential: b^x , e^x continuous everywhere

- logarithmic: $\ln x$, $\log_b x$ $(0, \infty)$

Ex: Where is the function $f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

$$f(x) = \frac{\ln x + \tan^{-1} x}{x^2 - 1}$$

$$\left. \begin{array}{l} \ln x \rightarrow (0, \infty) \\ \tan^{-1} x \rightarrow \mathbb{R} \\ x^2 - 1 \rightarrow \mathbb{R} - \{x^2 - 1 = 0\} = \mathbb{R} - \{1, -1\} \end{array} \right\} \rightarrow \cap (0, \infty) \rightarrow \cap = (0, 1) \cup (1, \infty)$$

$$f(x) = \text{continuous } (0, 1) \cup (1, \infty)$$

Ex: $f(x) = \sin^{-1}(1+2x)$

Def? $-1 \leq 1+2x \leq 1$

$-2 \leq 2x \leq 0$

$-1 \leq x \leq 0$

Def = $[-1, 0] \rightarrow f(x)$ continuous $[-1, 0]$

Ex: $f(x) = \frac{1}{1+e^{\frac{1}{x}}}$

$-1 = e^{\frac{1}{x}}$ X

$\frac{1}{x} \rightarrow x=0$ Def = $\mathbb{R} - \{0\}$

$f(x)$ continuous $\mathbb{R} - \{0\}$

* Continuity On an interval : اتصال على الفترة المغلقة
 f continuous $[a, b]$ if

1) f continuous (a, b)

2) $\lim_{x \rightarrow a^+} f(x) = f(a)$ (f continuous from right at $x=a$)

3) $\lim_{x \rightarrow b^-} f(x) = f(b)$ (f continuous from left at $x=b$)

Ex: $f(x) = \begin{cases} x^2 - 1, & 0 \leq x < 1 \\ 5, & x = 1 \end{cases}$ Is $f(x)$ continuous on the closed interval $[0, 1]$?

✓ Is $f(x)$ continuous on $(0, 1)$? yes : $x^2 - 1$ Poly everywhere

✓ Is $f(x)$ continuous at $x=0$ (right) ?

$$\lim_{x \rightarrow 0^+} f(x) \stackrel{?}{=} f(0)$$

$$x \rightarrow 0^+$$

$$\lim_{x \rightarrow 0^+} x^2 - 1 \stackrel{?}{=} 0 - 1$$

$$x \rightarrow 0^+$$

$$-1 = -1$$

✗ Is $f(x)$ continuous at $x=1$? (left) ?

$$\lim_{x \rightarrow 1^-} f(x) \stackrel{?}{=} f(1)$$

$$x \rightarrow 1^-$$

$$\lim_{x \rightarrow 1^-} x^2 - 1 = 0$$

$$x \rightarrow 1^-$$

$$0 \neq 5$$

No, $f(x)$ continuous $[0, 1)$

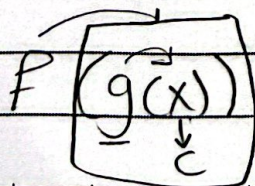
* Continuity of Composition :

Theorem 1 : f continuous everywhere, g continuous everywhere Then $f \circ g, g \circ f$ continuous everywhere

1) $f(x) = \sin x$, $g(x) = |x| \rightarrow \sin |x|$, $|\sin x|$ continuous everywhere

2) $\tan^{-1}(e^x) \rightarrow \tan^{-1} x$ continuous everywhere
continuous everywhere e^x continuous everywhere

Theorem 2 : If g continuous at $x=c$ & $f(x)$ continuous at $g(c)$ then $f \circ g$ continuous at $x=c$



$f \circ g$ continuous $[c]$

Continuity of composition

Chapter 2

Ex: $f(x) = \frac{1}{x - \frac{1}{2}}$, $g(x) = \frac{1}{2x}$

f is continuous $\mathbb{R} - \{\frac{1}{2}\}$

g is continuous $\mathbb{R} - \{0\}$

1) Is $f \circ g$ continuous at $x=0$? dis continuous at $x=0$

Is g continuous at $x=0$? No

2) Is $f \circ g$ continuous at $x=1$? dis continuous at $x=1$

Is g continuous at $x=1$? Yes

Is f continuous at $g(1) = \frac{1}{2}$? No

3) Is $f \circ g$ continuous at $x = \frac{1}{2}$?

Is g continuous at $x = \frac{1}{2}$? Yes

Is $f(x)$ continuous at $g(\frac{1}{2}) = 1$? Yes

$f \circ g$ continuous at $x = \frac{1}{2}$

Theorem 3: If $\lim_{x \rightarrow c} g(x) = L$ and $f(x)$ is continuous at L , Then

$$\lim_{x \rightarrow c} (f(g(x))) = f(\lim_{x \rightarrow c} g(x)) = f(L)$$

↓
continuous at L

1) $\lim_{x \rightarrow 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}\right)$

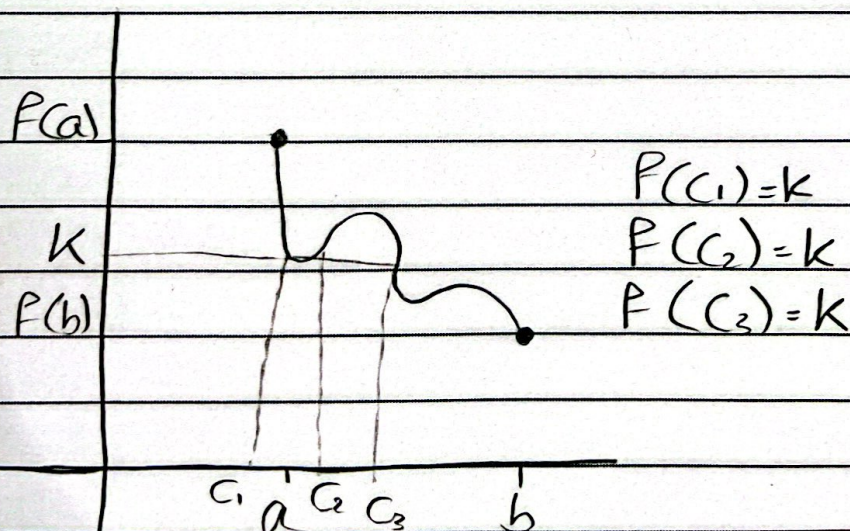
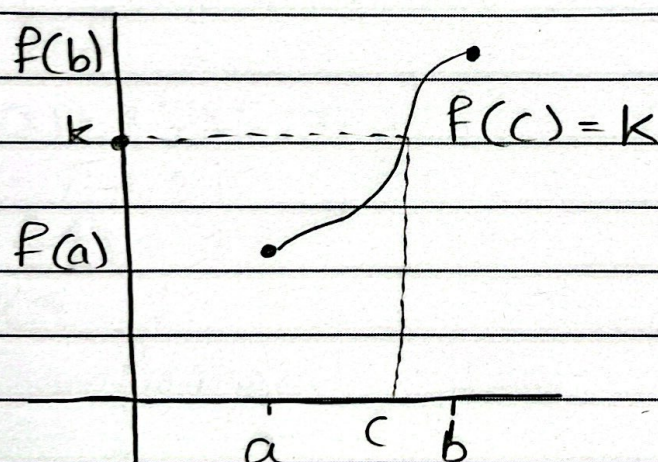
$$= \sin^{-1}\left(\lim_{x \rightarrow 1} \frac{1}{1+\sqrt{x}}\right) = \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$2) \lim_{x \rightarrow 1} \cos(x^2 - 1) = \cos(\lim_{x \rightarrow 1} x^2 - 1) = \cos(\lim_{x \rightarrow 1} (x-1)(x+1))$$

$$= \cos(\lim_{x \rightarrow 1} x + 1) = \cos 2$$

* The Intermediate Value Theorem: نظرية القيمة المتوسطة
المتصلة

IF f is continuous on a closed Interval $[a, b]$ and k any number between $f(a)$ and $f(b)$. Then there is at least one number c in the Interval $[a, b]$ such that $f(c) = k$.



The intermediate Value theorem

chapter 2

Ex: I.F $P(x) = x^5 + 7x^4 + 3$

Show that there is a number c such that $P(c) = 100$ in the interval $[0, 2]$

$$P(0) = 3$$

$$P(2) = 147$$

$$3 < 100 < 147$$

by IVT, c such that $P(c) = 100$

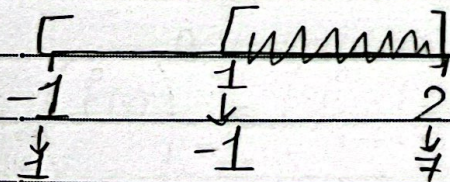
Ex: Show that $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[-1, 2]$

$$P(x) = x^3 + x^2 - 2x - 1$$

$$[-1, 2]$$

$$P(-1) = 1$$

$$P(2) = 7$$



$$P(1) = -1$$

$$-1 < 0 < 7$$

by IVT at least c

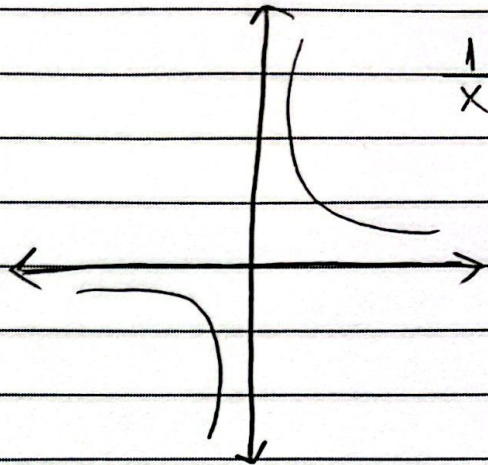
$$P(c) = 0$$

Limits at infinity $\begin{matrix} x \rightarrow +\infty \\ x \rightarrow -\infty \end{matrix}$

Chapter 2

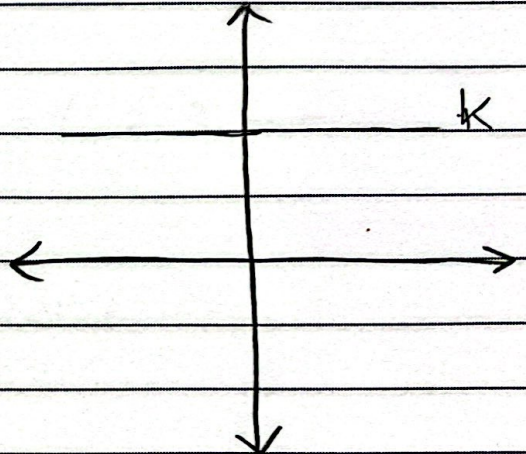
$$1) \lim_{x \rightarrow +\infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$



$$2) \lim_{\substack{x \rightarrow +\infty \\ x \rightarrow -\infty}} k = k = \text{القيمة الثابتة}$$

$$\text{Ex: } \lim_{x \rightarrow +\infty} \pi = \pi, \lim_{x \rightarrow -\infty} 7 = 7$$



$$3) \lim_{\substack{x \rightarrow +\infty \\ x \rightarrow -\infty}} (f(x))^n = \left(\lim_{\substack{x \rightarrow +\infty \\ x \rightarrow -\infty}} f(x) \right)^n, \text{ n Positive integer } 1, 2, 3, 4$$

$$\text{Ex: } \lim_{x \rightarrow +\infty} f(x) = 3, \text{ Find } \lim_{x \rightarrow +\infty} (f(x))^4 ?$$

$$\left(\lim_{x \rightarrow +\infty} f(x) \right)^4 = 3^4 = 81$$

$$4) \lim_{\substack{x \rightarrow +\infty \\ x \rightarrow -\infty}} k f(x) = k \lim_{\substack{x \rightarrow +\infty \\ x \rightarrow -\infty}} f(x)$$

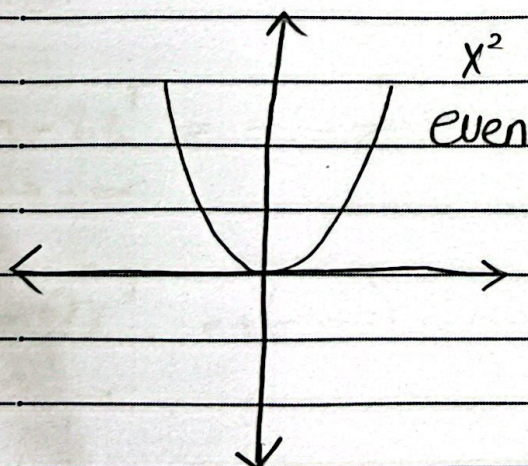
$$\text{Ex: } \lim_{x \rightarrow -\infty} f(x) = 2, \quad \lim_{x \rightarrow -\infty} 5f(x) ?$$

$$5 \lim_{x \rightarrow -\infty} f(x) = 5 \times 2 = 10$$

$$* \text{ Limit of } x^n \quad x \rightarrow +\infty, x \rightarrow -\infty$$

$$1) \lim_{x \rightarrow +\infty} x^n = +\infty, \quad n = 1, 2, 3, \dots$$

$$2) \lim_{x \rightarrow -\infty} x^n = \begin{cases} +\infty, & n \text{ even } 2, 4, 6, \dots \\ -\infty, & n \text{ odd } 1, 3, 5, \dots \end{cases}$$

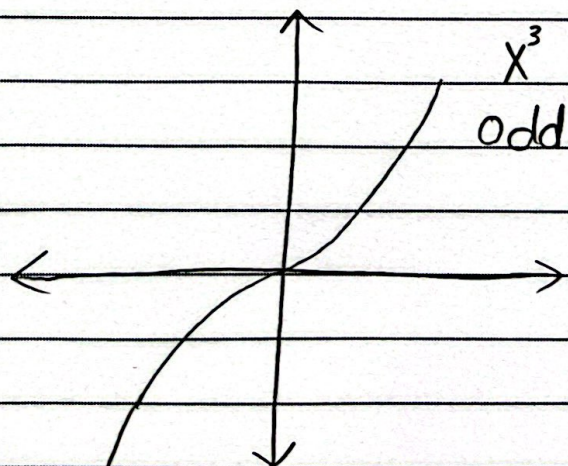


$$\lim_{x \rightarrow +\infty} x^2 = +\infty$$

$$x \rightarrow +\infty$$

$$\lim_{x \rightarrow -\infty} x^2 = +\infty$$

$$x \rightarrow -\infty$$



$$\lim_{x \rightarrow +\infty} x^3 = +\infty$$

$$x \rightarrow +\infty$$

$$\lim_{x \rightarrow -\infty} x^3 = -\infty$$

$$x \rightarrow -\infty$$

Limits of Polynomials

Chapter 2

$$\ast \lim_{\substack{x \rightarrow +\infty \\ x \rightarrow -\infty}} (C_0 + C_1x + C_2x^2 + \dots + C_nx^n) = \lim_{\substack{x \rightarrow +\infty \\ x \rightarrow -\infty}} C_nx^n$$

$$\text{Ex: } \lim_{x \rightarrow +\infty} 7x^5 - 4x^2 + 10x^3 + 8x - 19 = \lim_{x \rightarrow +\infty} 7x^5 = +\infty$$

بعلامات أكبر قوة

$$\text{Ex: } \lim_{x \rightarrow -\infty} 1 - x^2 - 3x^3 + 4x^4 - 5x^5 + 10x^6 = \lim_{x \rightarrow -\infty} 10x^6 = +\infty$$

$$\text{Ex: } \lim_{x \rightarrow -\infty} (-3x^3 + 2x^2 + 8)^4 = \lim_{x \rightarrow -\infty} (3x^3)^4 = \lim_{x \rightarrow -\infty} 81x^{12} = +\infty$$

* Limits of Rational Functions $x \rightarrow +\infty$, $x \rightarrow -\infty$:

الطريقة السريعة :

بناءً على أكبر من البسط والأكبر من المقام

$$1) \lim_{x \rightarrow -\infty} \frac{3x+5}{6x-8} = \lim_{x \rightarrow -\infty} \frac{3x}{6x} = \lim_{x \rightarrow -\infty} \frac{1}{2} = \frac{1}{2}$$

$$2) \lim_{x \rightarrow +\infty} \frac{5x^3 - 2x^2 + 1}{1 - 3x} = \lim_{x \rightarrow +\infty} \frac{5x^3}{-3x} = \lim_{x \rightarrow +\infty} -\frac{5}{3}x^2 = -\infty$$

$$3) \lim_{x \rightarrow -\infty} \frac{4x^2 - x}{2x^3 - 5} = \lim_{x \rightarrow -\infty} \frac{4x^2}{2x^3} = \lim_{x \rightarrow -\infty} \frac{2}{x} = 0$$

$$\text{Ex: } \lim_{x \rightarrow -\infty} \frac{3x+5}{6x-8} = \frac{-\infty}{-\infty} \quad \nabla \quad 0$$

$$\lim_{x \rightarrow -\infty} x \left(3 + \frac{5}{x} \right) = \lim_{x \rightarrow -\infty} 3 + \frac{5}{x} = 3 = \frac{3}{1} = \frac{3}{2}$$

$$1) \lim_{x \rightarrow +\infty} \frac{2x+1}{6x-1} = \lim_{x \rightarrow +\infty} \frac{2x+1}{6x-1} = \lim_{x \rightarrow +\infty} \frac{2x}{6x} = \frac{1}{3}$$

$$2) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+2}}{3x-6} = \frac{+\infty}{-\infty} \quad \nabla \quad 0$$

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{\sqrt{x^2(1+\frac{2}{x^2})}}{x(3-\frac{6}{x})} &= \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{1+\frac{2}{x^2}}}{x(3-\frac{6}{x})} \\ &= \lim_{x \rightarrow -\infty} \frac{-x \sqrt{1+\frac{2}{x^2}}}{x(3-\frac{6}{x})} = \frac{-1}{3} \end{aligned}$$

Zero

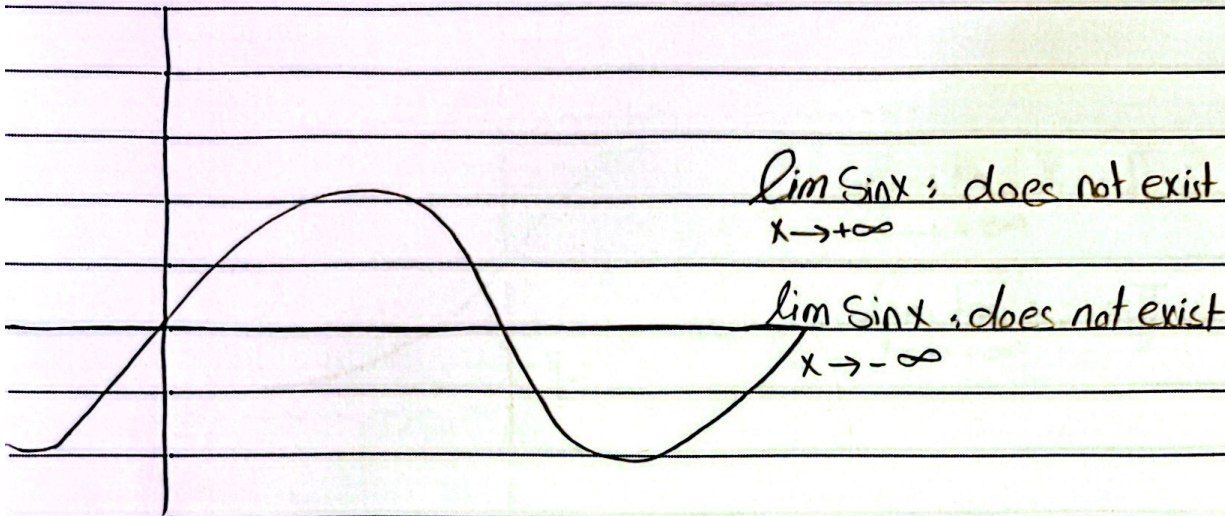
$$3) \lim_{x \rightarrow +\infty} \sqrt{x^6+5x^3} - x^3 = \infty - \infty \quad \nabla \quad 0$$

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{x^6+5x^3} - x^3}{\sqrt{x^6+5x^3} + x^3}$$

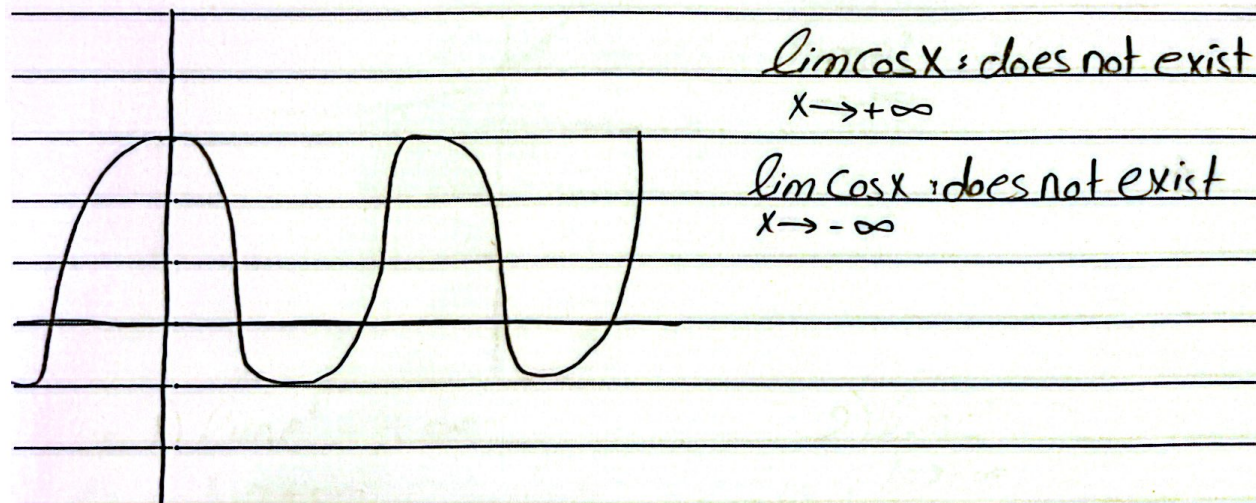
$$= \lim_{x \rightarrow +\infty} \frac{\sqrt{x^6+5x^3} - x^3}{\sqrt{x^6+5x^3} + x^3} = \lim_{x \rightarrow +\infty} \frac{5x^3}{\sqrt{x^6(1+\frac{5}{x^3})} + x^3}$$

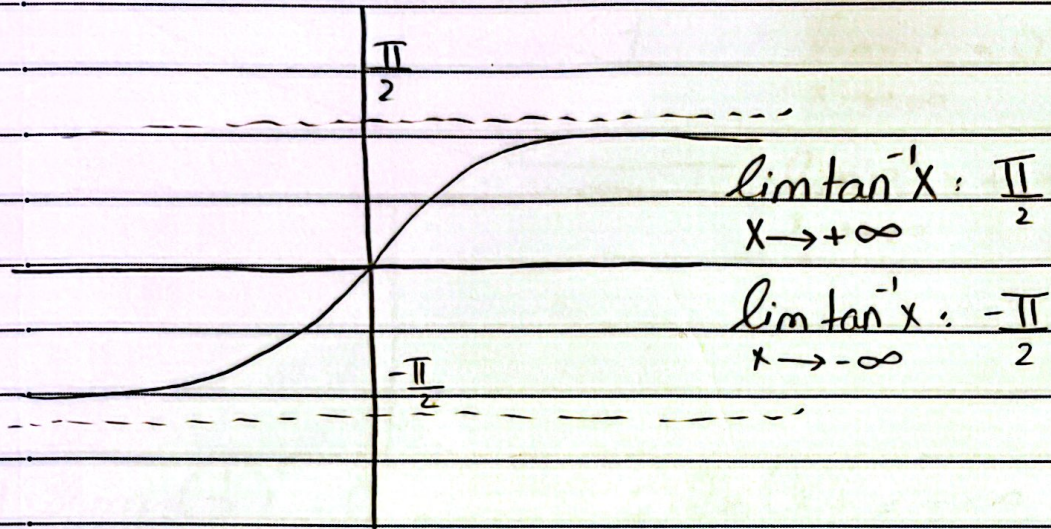
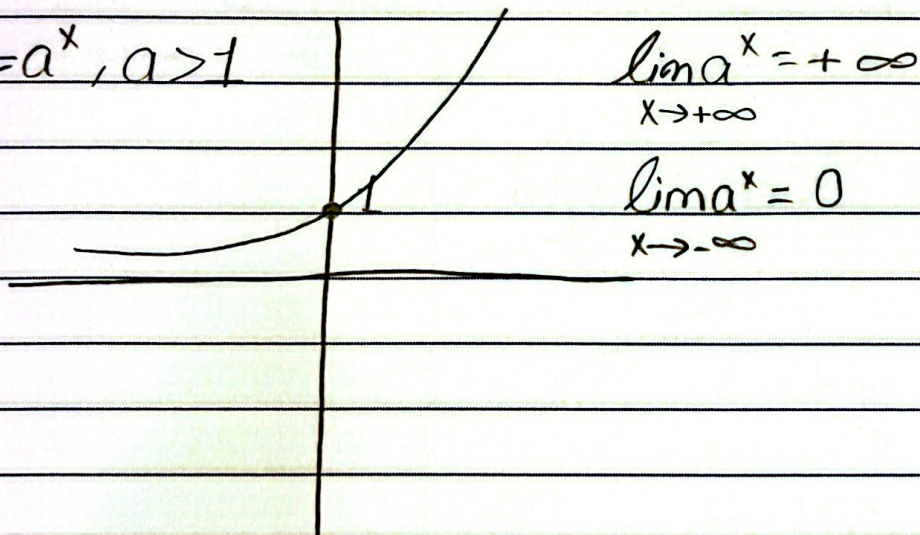
$$= \lim_{x \rightarrow +\infty} \frac{5x^3}{x^3(\sqrt{1+\frac{5}{x^3}} + 1)} = \frac{5}{\sqrt{1+0} + 1} = \frac{5}{2}$$

1) $\sin x$:



2) $\cos x$:



3) $\tan^{-1} x$ 4) $f(x) = a^x, a > 1$ 

Ex: 1) $\lim_{x \rightarrow +\infty} e^x = +\infty$

2) $\lim_{x \rightarrow -\infty} e^x = 0$

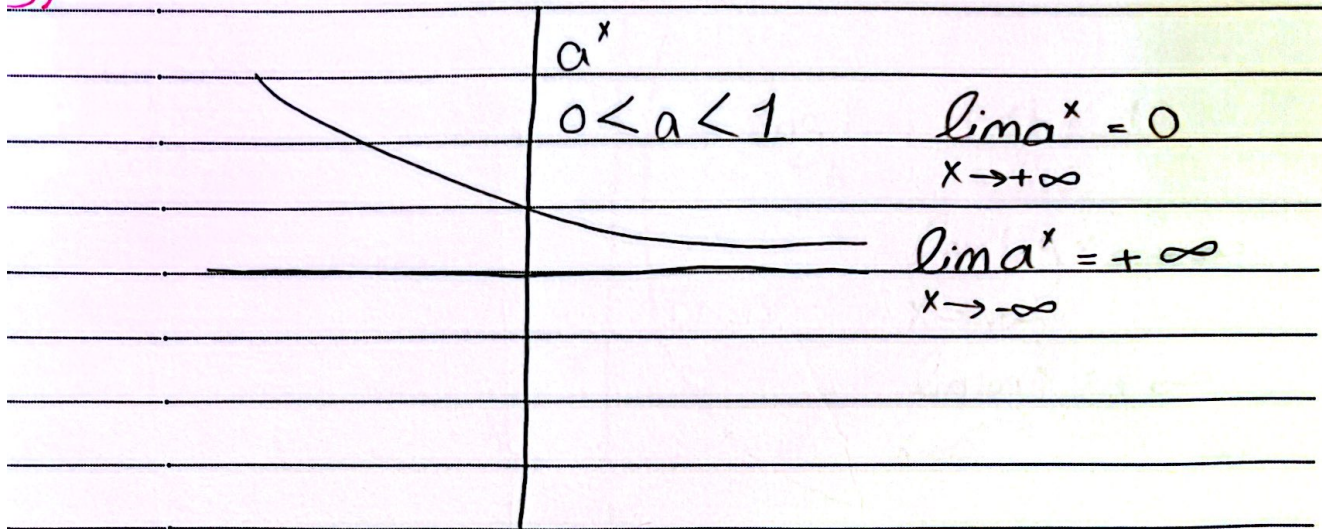
3) $\lim_{x \rightarrow +\infty} 2^x = +\infty$

4) $\lim_{x \rightarrow -\infty} 5^x = 0$

Limits of Polynomials

chapter 2

5)



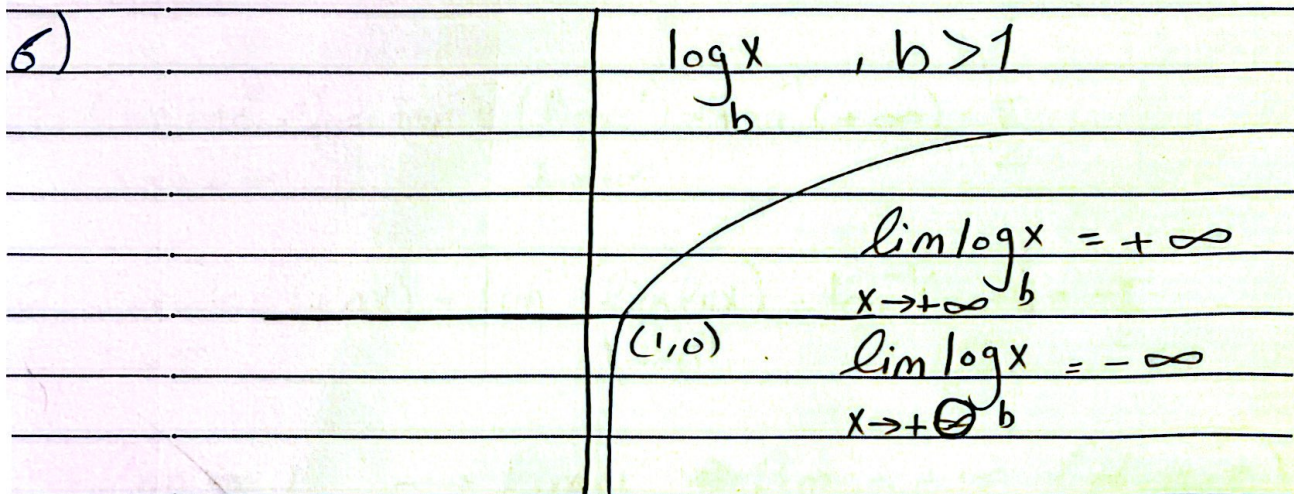
Ex: 1) $\lim_{x \rightarrow +\infty} \left(\frac{1}{2}\right)^x = 0$

2) $\lim_{x \rightarrow -\infty} \left(\frac{1}{3}\right)^x = +\infty$

3) $\lim_{x \rightarrow +\infty} e^{-x} = 0$

4) $\lim_{x \rightarrow -\infty} e^{-x} = +\infty$

6)

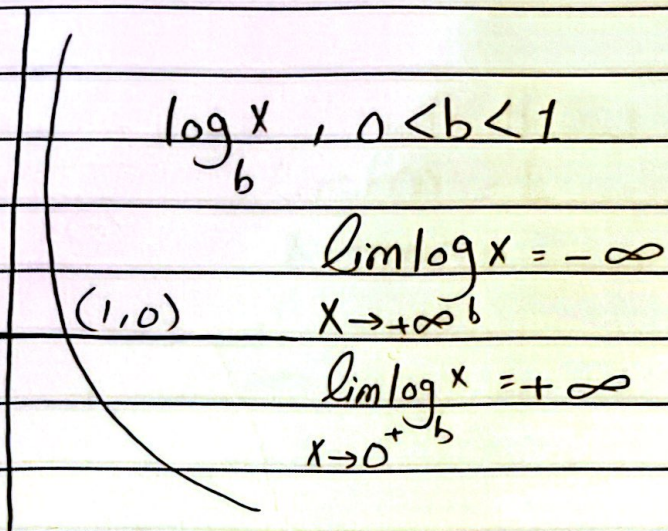


Ex: 1) $\lim_{x \rightarrow +\infty} \log x = +\infty$

2) $\lim_{x \rightarrow +\infty} \ln x = +\infty$

3) $\lim_{x \rightarrow 0^+} \ln x = -\infty$

7)



Ex: 1) $\lim_{x \rightarrow +\infty} \log_{0.5} x = -\infty$

2) $\lim_{x \rightarrow 0^+} \log_{\frac{1}{4}} x = +\infty$

Examples:

1) $\lim_{x \rightarrow +\infty} \tan^{-1} e^x = \tan^{-1} x (\lim_{x \rightarrow +\infty} e^x) = \tan^{-1} (+\infty) = \frac{\pi}{2}$

2) $\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x) = \tan^{-1}(\lim_{x \rightarrow 0^+} \ln x) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

3) $\lim_{x \rightarrow 0} \frac{1}{1+e^{\frac{1}{x}}} \rightarrow \lim_{x \rightarrow 0} \frac{1}{x} \Rightarrow \lim_{x \rightarrow 0^+} = +\infty$ (d.n.e)
 $\lim_{x \rightarrow 0^-} = -\infty$

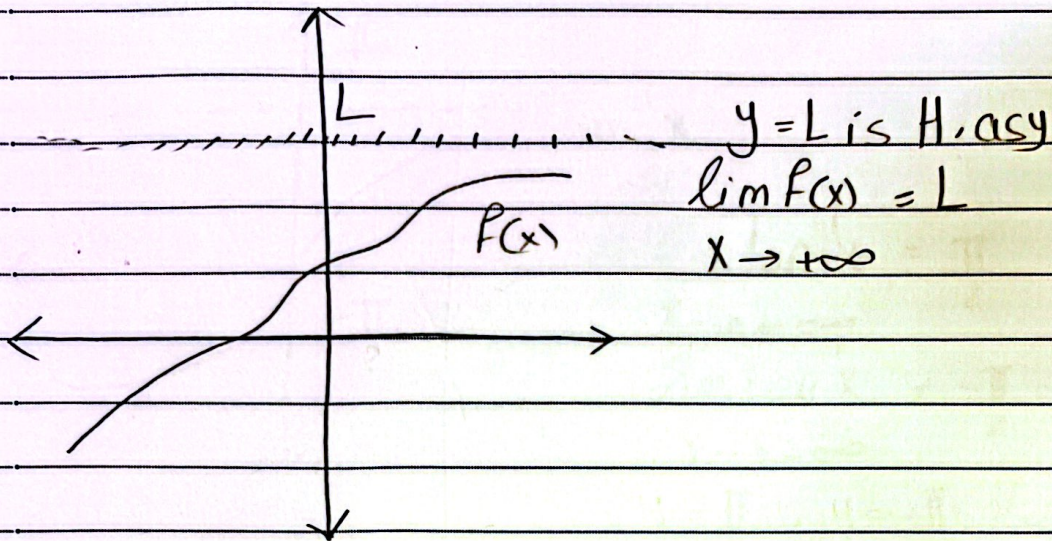
$\lim_{x \rightarrow 0^+} \frac{1}{1+e^{\frac{1}{x}}} = \lim_{x \rightarrow 0^+} \frac{1}{1+e^{+\infty}} = \frac{1}{1+\infty} = 0$

$\lim_{x \rightarrow 0^-} \frac{1}{1+e^{\frac{1}{x}}} = \lim_{x \rightarrow 0^-} \frac{1}{1+e^{-\infty}} = \frac{1}{1+0} = 1$

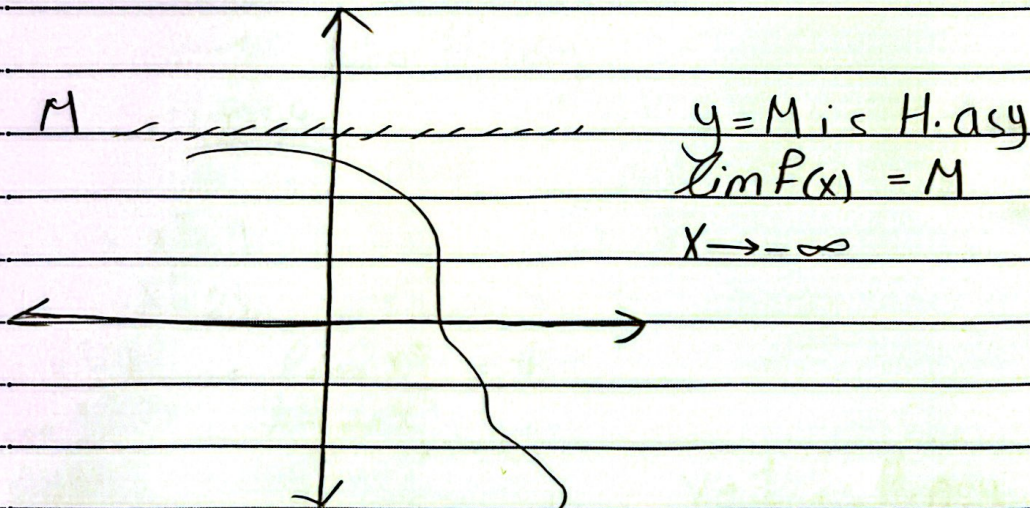
Horizontal asymptotes

chapter 2

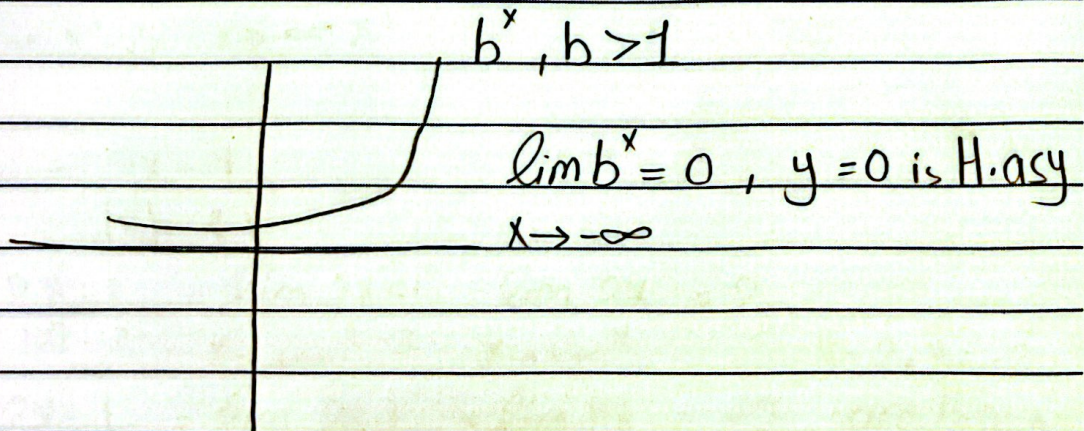
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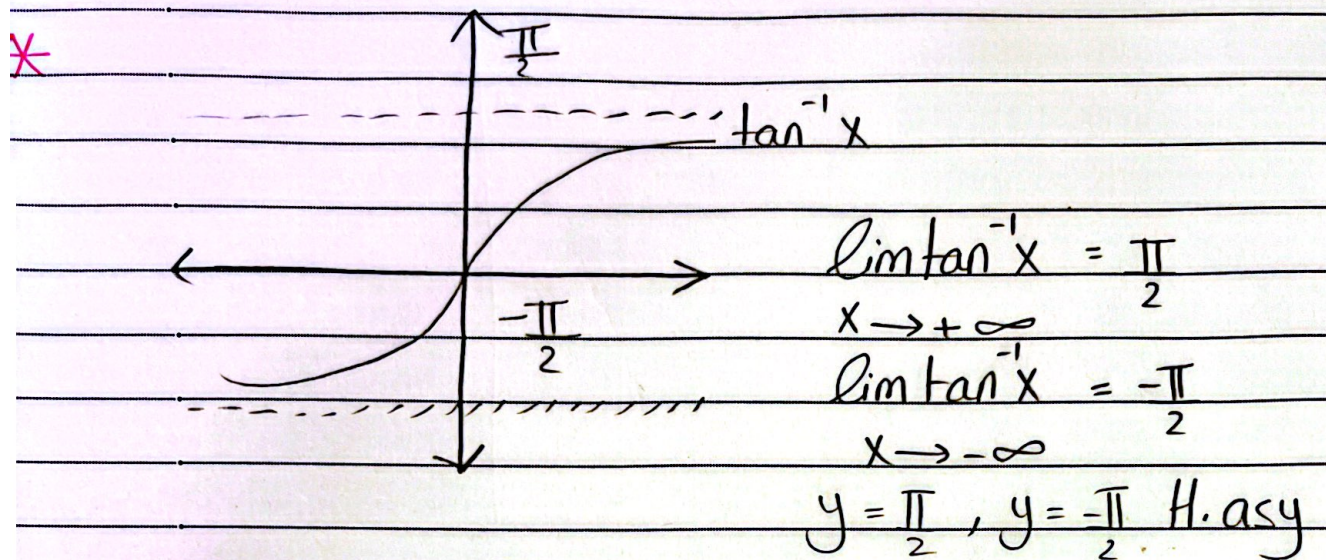


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Horizontal asymptotes

Chapter 2



* $P(x) = x^2 + x + 1$ No H.asy

* Find the H.asy

1) $P(x) = \frac{x^2 - 4}{x^2 - 2x}$

$$\lim_{x \rightarrow +\infty} \frac{x^2 - 4}{x^2 - 2x} = \lim_{x \rightarrow +\infty} \frac{x^2}{x^2} = 1$$

$y = 1$ is H.asy

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 4}{x^2 - 2x} = \lim_{x \rightarrow -\infty} \frac{x^2}{x^2} = 1$$

2) $P(x) = \frac{2x - 1}{|x| - 3}$

$$\lim_{x \rightarrow +\infty} \frac{2x - 1}{|x| - 3} = \lim_{x \rightarrow +\infty} \frac{2x - 1}{x - 3} = \lim_{x \rightarrow +\infty} \frac{2x}{x} = 2$$

$y = 2, y = -2$

$$\lim_{x \rightarrow -\infty} \frac{2x - 1}{|x| - 3} = \lim_{x \rightarrow -\infty} \frac{2x - 1}{-x - 3} = \lim_{x \rightarrow -\infty} \frac{2x}{-x} = -2$$

are H.asy

Limit of the trigonometric

Chapter 2

$$1) \lim_{x \rightarrow \pi} \sin x = \sin \pi = 0 \quad (\text{تقریباً صاف})$$

$$2) \lim_{x \rightarrow \frac{\pi}{4}} \cos x = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$3) \lim_{x \rightarrow \pi} \tan x = \tan \pi = 0$$

$$4) \lim_{x \rightarrow \frac{\pi}{2}} \tan x = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{\cos x} = \frac{1}{0} \quad \begin{array}{c} + + + \\ \frac{\pi}{2} \\ - - - \end{array}$$

$$\lim_{x \rightarrow \frac{\pi}{2}^+} \tan x = -\infty$$

$$\lim_{x \rightarrow \frac{\pi}{2}^-} \tan x = +\infty$$

does not exist

$$5) \lim_{x \rightarrow 2\pi^-} x \csc x = \lim_{x \rightarrow 2\pi^-} \frac{x}{\sin x} = \frac{2\pi}{0} = \frac{2\pi}{0} = \frac{2\pi}{0} = -\infty$$

Thm: $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$1) \lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

Limit of the trigonometric

chapter 2

$$2) \lim_{x \rightarrow 0} \frac{ax}{\sin bx} = \frac{a}{b}$$

$$3) \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} = \frac{a}{b}$$

$$4) \lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \frac{a}{b}$$

$$5) \lim_{x \rightarrow 0} \frac{ax}{\tan bx} = \frac{a}{b}$$

$$6) \lim_{x \rightarrow 0} \frac{\tan ax}{\tan bx} = \frac{a}{b}$$

$$7) \lim_{x \rightarrow 0} \frac{\tan ax}{\sin bx} = \frac{a}{b}$$

$$8) \lim_{x \rightarrow 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$$

Ex:

$$1) \lim_{x \rightarrow 0} \frac{\sin 7x}{5x} = \frac{7}{5}$$

Limit of the trigonometric

Chapter 2

$$2) \lim_{x \rightarrow 0} \frac{\sin x}{\sin \pi x} = \frac{1}{\pi}$$

$$3) \lim_{x \rightarrow 0} \frac{\tan 8x}{\sin 10x} = \frac{8}{10}$$

$$4) \lim_{x \rightarrow 0} \frac{3x + \tan x}{\sin x + x} = \lim_{x \rightarrow 0} \frac{x(3 + \frac{\tan x}{x})}{x(\sin x + 1)} = \frac{3+1}{1+1} = 2$$

$$5) \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{\sin^2 x}{x(1 + \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \frac{\sin x}{1 + \cos x} = \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \lim_{x \rightarrow 0} \frac{\sin x}{1 + \cos x}$$

$$= 1 \cdot 0 = 0$$

$$* f(x) = \begin{cases} 1 + \sin x & , x < 0 \\ \cos x & , 0 < x \leq \pi \\ \sin x & , x > \pi \end{cases}$$

Find :

$$1) \lim_{x \rightarrow 0} F(x) = \left[\begin{array}{l} \lim_{x \rightarrow 0^+} \cos x = 1 \\ \lim_{x \rightarrow 0^-} 1 + \sin x = 1 \end{array} \right] \rightarrow \lim_{x \rightarrow 0} F(x) = 1$$

$$2) \lim_{x \rightarrow \pi} F(x) = \left[\begin{array}{l} \lim_{x \rightarrow \pi^+} \sin x = \sin \pi = 0 \\ \lim_{x \rightarrow \pi^-} \cos x = \cos \pi = -1 \end{array} \right] \rightarrow \lim_{x \rightarrow \pi} F(x) = \text{d.n.e}$$

Ex: Suppose that $\lim_{x \rightarrow 0} F(x) = 4$

Find $\lim_{x \rightarrow 0} \tan 8x$

$$y = 4x \rightarrow x = \frac{y}{4}$$

$$F \rightarrow 0$$

$$y \rightarrow 0$$

$$\lim_{y \rightarrow 0} \tan 8 \left(\frac{y}{4} \right) = \lim_{y \rightarrow 0} 4 \tan 2y = 4 \lim_{y \rightarrow 0} \tan 2y \times \lim_{y \rightarrow 0} \frac{1}{F(y)}$$

$$= 4 \lim_{y \rightarrow 0} \tan 2y \times \lim_{y \rightarrow 0} \frac{1}{F(y)}$$

$$= (4)(2) \left(\frac{1}{4} \right) = 2$$

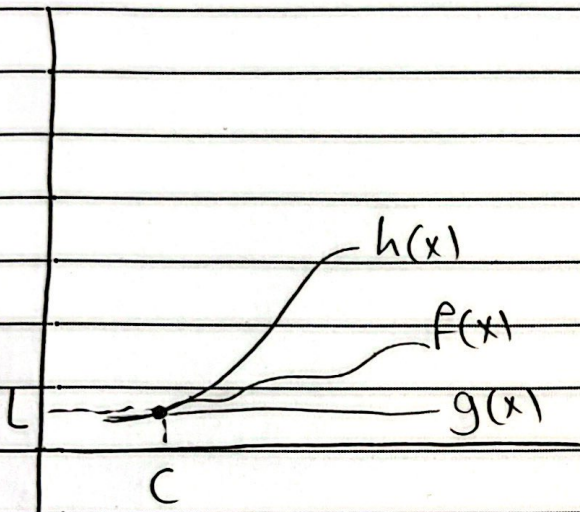
The squeeze Theorem

chapter 2

* The squeeze Theorem نظرية الضغط: let f, g, h be functions such that $g(x) \leq f(x) \leq h(x)$

$$\text{if } \lim_{x \rightarrow c} h(x) = \lim_{x \rightarrow c} g(x) = L$$

$$\text{Then } \lim_{x \rightarrow c} f(x) = L$$



Ex: $\lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0$ * d.n.e غير موجود

$$g(x) \leq x^2 \sin \frac{1}{x} \leq h(x)$$

$$-1 \leq \sin \frac{1}{x} \leq 1 \quad x \neq 0 \quad x^2 > 0$$

$$-x^2 \leq x^2 \sin \frac{1}{x} \leq x^2$$

$$\lim_{x \rightarrow 0} \downarrow \quad \quad \quad \downarrow \quad \quad \quad \lim_{x \rightarrow 0} \rightarrow$$

$$0 \quad \quad \quad \lim_{x \rightarrow 0} x^2 \sin \frac{1}{x} = 0 \quad \quad \quad 0$$

by squeeze Thm

The derivative of Function

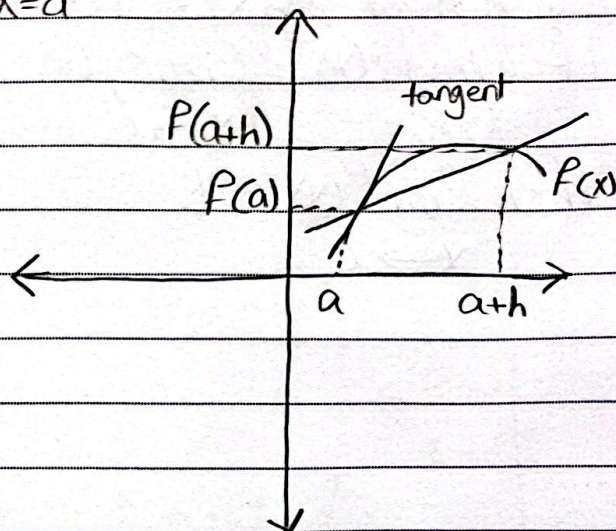
Chapter 3

$$F(x) \rightarrow F'(a) = \frac{dy}{dx} \Big|_{x=a} = \frac{d}{dx} \Big|_{x=a}$$

$$F'(a) = \lim_{h \rightarrow 0} \frac{F(a+h) - F(a)}{h}$$

$$\text{Slope of tangent} = F'(a)$$

$$F'(a) = \lim_{x \rightarrow a} \frac{F(x) - F(a)}{x - a}$$



Ex: Find the derivative of a the function $F(x) = x^2$?

$$F'(x) = \lim_{h \rightarrow 0} \frac{F(x+h) - F(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \rightarrow 0} \frac{2xh + h^2}{h} \quad \frac{0}{0} ?$$

$$\lim_{h \rightarrow 0} h(2x + h) = \lim_{h \rightarrow 0} 2x + h$$

$$F'(x) = 2x$$

$$F'(5) = 2(5) = 10$$

Differentiation Rules

chapter 3

* Differentiation Rules :

$$1) \frac{d}{dx}(c) = 0, c \in \mathbb{R}$$

$$\text{Ex: } f(x) = 10 \rightarrow f'(x) = 0$$

$$2) \frac{d}{dx}(x^n) = nx^{n-1}$$

$$\text{Ex: } f(x) = x^5 \rightarrow f'(x) = 5x^4$$

$$y = x^{-7} \rightarrow y' = -7x^{-8}$$

$$3) \frac{d}{dx}(cf) = cf'$$

$$\text{Ex: } y = 7x^2 \rightarrow y' = 7(2)x = 14x$$

$$4) \frac{d}{dx}(f \pm g) = f' \pm g'$$

$$\text{Ex: } f(x) = \sqrt{x} + \sqrt[3]{x^2} + \frac{12}{x} + 7$$

$$f(x) = x^{\frac{1}{2}} + x^{\frac{2}{3}} + 12x^{-1} + 7$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{2}{3}x^{-\frac{1}{3}} - 12x^{-2}$$

Differentiation Rules

Chapter 3

$$5) \frac{d}{dx} (fg) = fg' + g f'$$

$$\text{Ex: } y = (8x+3)(x^2-3x)$$

$$\frac{dy}{dx} = (8x+3)(2x-3) + (x^2-3x)(8)$$

$$6) \frac{d}{dx} \left(\frac{f}{g} \right) = \frac{g f' - f g'}{g^2}$$

$$\text{Ex 1: } f(x) = \frac{x^2}{x^3+5x}$$

$$f'(x) = \frac{(2x)(x^3+5x) - (x^2)(3x^2+5)}{(x^3+5x)^2}$$

$$\text{Ex 2: } f(x) = \frac{x^4-1}{8x^3+2x^2}$$

$$f'(x) = \frac{(4x^3)(8x^3+2x^2) - (x^4-1)(24x^2+4x)}{(8x^3+2x^2)^2}$$

$$7) \frac{d}{dx} \left(\frac{c}{g} \right) = \frac{-c g'}{g^2}$$

$$\text{Ex: } f(x) = \frac{7}{x^3+1}$$

$$f'(x) = \frac{-7(3x^2)}{(x^3+1)^2} = \frac{-21x^2}{(x^3+1)^2}$$

Examples:

1) $P(x) = \frac{7x-1}{x^2+5}$, Find $P'(2)$

$$P'(x) = \frac{(7)(x^2+5) - (7x-1)(2x)}{(x^2+5)^2}$$

$$P'(2) = \frac{(7)(4+5) - (14-1)(4)}{(4+5)^2} = \frac{63 - 52}{81} = \frac{11}{81}$$

2) $P(x) = (10x-1)(8x^2+5)$, Find $P'(1)$

$$P'(x) = (10)(8x^2+5) + (10x-1)(16x)$$

$$\begin{aligned} P'(1) &= (10)(13) + (9)(16) \\ &= 130 + 144 \\ &= 274 \end{aligned}$$

* Chain Rule القاعدة السلسلة :

$$\frac{d}{dx} P(g(x)) = P'(g(x))g'(x)$$

Ex: $P(x) = x^2 + x$, $g(x) = \frac{4}{x^2+1}$, Find $(P \circ g)'(1)$?

$$\begin{aligned} \frac{d}{dx} (P(g(x))) &= P'(g(x))g'(x) \\ &= P'(g(1))g'(1) \\ &= 5 \cdot -2 = -10 \end{aligned}$$

$$\begin{aligned} g(1) &= 2 \\ g'(1) &= -2 \\ P'(2) &= 5 \end{aligned}$$

$$* \frac{d}{dx} (g(x))^n = n (g(x))^{n-1} g'(x)$$

Ex: Find y'

$$1) y = (x^2 + 5)^{10}$$

$$y' = 10 (x^2 + 5)^9 (2x)$$

$$2) y = (x^7 - 2)^{-5}$$

$$y' = -5 (x^7 - 2)^{-6} (7x^6)$$

$$3) y = \sqrt[3]{x^2 + 1} \rightarrow y = (x^2 + 1)^{\frac{1}{3}}$$

$$y' = \frac{1}{3} (x^2 + 1)^{-\frac{2}{3}} (2x)$$

$$4) y = \sqrt{x^3 + x^2} \rightarrow y = (x^3 + x^2)^{\frac{1}{2}}$$

$$y' = \frac{1}{2} (x^3 + x^2)^{-\frac{1}{2}} (3x^2 + 2x)$$

$$= \frac{3x^2 + 2x}{2\sqrt{x^3 + x^2}}$$

* فيه قانون أسهل لاشتقاق الجذر:

$$y = \sqrt{f(x)} \Rightarrow y' = \frac{f'(x)}{2\sqrt{f(x)}}$$

Chain Rule

chapter 3

* IF $P(2x+5) = x^3 + 2x^2 + 4$, Find $P'(3)$

$$P'(2x+5)(2) = 3x^2 + 4x$$

$$2x+5 = 3$$

$$P'(3)(2) = -1$$

$$2x = -2$$

$$P'(3) = \frac{-1}{2}$$

$$x = -1$$

* $\frac{d}{dx} (P(x^3)) = 12x^{17}$, Find $P'(x)$

$$P'(x^3)(3x^2) = 12x^{17}$$

$$P'(x^3) = \frac{12x^{17}}{3x^2}$$

$$P'(x^3) = 4x^{15}$$

$$x^3 = y$$

$$P'(y) = 4(y^{\frac{1}{3}})^{15}$$

$$x = y^{\frac{1}{3}}$$

$$P'(y) = 4y^5$$

$$P'(x) = 4x^5$$

$$* \frac{d}{dx} (\sin(P(x))) = \cos P(x) P'(x)$$

$$\text{Ex: } y = \sin(x^3) = 3x^2 \cos x^3$$

$$* \frac{d}{dx} (\cos(P(x))) = -\sin P(x) P'(x)$$

$$\text{Ex: } y = \cos(5x^2 + 10) \rightarrow y' = -\sin(5x^2 + 10) (10x)$$

$$* \frac{d}{dx} (\tan(P(x))) = \sec^2(P(x)) P'(x)$$

$$\text{Ex: } y = \sqrt{\tan x} \rightarrow y = (\tan x)^{\frac{1}{2}} \rightarrow y' = \frac{\sec^2 x}{2\sqrt{\tan x}}$$

$$* \frac{d}{dx} (\cot(P(x))) = -\csc^2(P(x)) P'(x)$$

$$\text{Ex: } y = \cot^3(x^2) = (\cot(x^2))^3 = 3\cot^2(x^2) (-\csc^2(x^2)) (2x)$$

$$* \frac{d}{dx} (\sec P(x)) = \sec P(x) \tan P(x) P'(x)$$

$$\text{Ex: } y = \sec\left(\frac{2}{x+1}\right) \rightarrow y' = \sec \frac{2}{x+1} \tan \frac{2}{x+1} \left(\frac{-2}{(x+1)^2}\right)$$

$$* \frac{d}{dx} (\csc P(x)) = -\csc P(x) \cot P(x) P'(x)$$

$$\text{Ex: } y = \csc x^3 \Rightarrow y' = -\csc x^3 \cot x^3 (3x^2)$$

Derivative of exponential function

Chapter 3

$$* \frac{d}{dx} (b^{f(x)}) = b^{f(x)} f'(x) \ln b$$

Ex: 1) $y = 5^{x^2} \rightarrow y' = 5^{x^2} (2x) \ln 5$

2) $y = 2^{\cos x} \rightarrow y' = 2^{\cos x} (-\sin x) \ln 2$

3) $y = 3^x \rightarrow y' = 3^x \ln 3$

$$* \frac{d}{dx} (e^{f(x)}) = e^{f(x)} f'(x)$$

Ex: 1) $y = e^{x \tan x} \rightarrow y' = e^{x \tan x} (x \sec^2 x + \tan x)$

2) $y = e^{\sqrt{1-5x^3}} \rightarrow y' = e^{\sqrt{1-5x^3}} \left(\frac{-15x^2}{2\sqrt{1-5x^3}} \right)$

3) $y = e^x \rightarrow y' = e^x$

4) $y = e^{-x} + \pi^{-x} \rightarrow y' = e^{-x}(-1) + (\pi^{-x})(-1) \ln \pi$

Ex: If $f(x) = e^x g(x)$, $g(0) = 2$, $g'(0) = 5$, Find $f'(0)$?

$$f'(x) = e^x g(x) + e^x g'(x)$$

$$f'(0) = 2 + 5$$

$$f'(0) = 7$$

Derivatives of logarithmic

chapter 3

$$* \frac{d}{dx} \left(\log_b F(x) \right) = \frac{F'(x)}{F(x) \ln b}, \quad F(x) > 0$$

Ex: Find y' :

$$1) y = \log_2(x^3 + 5) \rightarrow y' = \frac{3x^2}{(x^3 + 5) \ln 2}$$

$$2) y = \log \sin x \rightarrow y' = \frac{\cos x}{\sin x \ln 10} = \frac{\cot x}{\ln 10}$$

$$* \frac{d}{dx} \ln(F(x)) = \frac{F'(x)}{F(x)}, \quad F(x) > 0$$

Ex: Find y' :

$$1) y = \ln x \rightarrow y' = \frac{1}{x}$$

$$2) y = \ln(x^2 + 1) \rightarrow y' = \frac{2x}{x^2 + 1}$$

$$3) y = \ln(\csc e^x) \rightarrow y' = \frac{-e^x \sin e^x}{\csc e^x} = -e^x \tan e^x$$

$$4) y = \ln(\ln x), \quad \left. \frac{dy}{dx} \right|_{x=e}$$

$$y' = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \ln x} \Big|_{x=e} = \frac{1}{e}$$

Derivatives of logarithmic

chapter 3

$$5) y = \ln |\sin x + x|, \quad \frac{d \ln |x|}{dx} = \frac{1}{x}$$

$$y' = \frac{\cos x + 1}{\sin x + x}$$

Ex: Find y' .

$$1) y = \log_7 x$$

$$y' = \frac{1}{x \ln 7}$$

$$2) y = \log_7 7$$

$$y' = \frac{\ln 7}{\ln x}$$

$$y' = \frac{(-\ln 7) \left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$= -\frac{\ln 7}{x (\ln x)^2}$$

EX: Find y' .

$$1) y = \ln \left(\frac{x^2 \sin x}{\sqrt{1+x}} \right)$$

$$y = 2 \ln x + \ln \sin x - \frac{1}{2} \ln(1+x)$$

$$y' = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}$$

$$2) y = \log_4 \left(\frac{\cos x}{(x-1)^3(x^2+1)^4} \right)$$

$$y = \log_4 \cos x - 3 \log_4 (x-1) + 4 \log_4 (x^2+1)$$

$$y' = \frac{-\sin x}{\cos x \ln 4} - \frac{3}{(x-1) \ln 4} + \frac{8x}{(x^2+1) \ln 4}$$

$$y' = \frac{1}{\ln 4} \left(-\tan x - \frac{3}{x-1} + \frac{8x}{x^2+1} \right)$$

* Derivatives of Inverse Trigonometric functions:

$$\frac{d}{dx} \left(\sin^{-1}(f(x)) \right) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$$

$$\frac{d}{dx} \left(\cos^{-1}(f(x)) \right) = \frac{-f'(x)}{\sqrt{1-(f(x))^2}}$$

Find y' :

Find y' :

$$y = \sin^{-1} x \rightarrow y' = \frac{1}{\sqrt{1-x^2}}$$

$$y = \cos^{-1} x^2 \rightarrow y' = \frac{-2x}{\sqrt{1-x^4}}$$

$$y = \sin^{-1} \ln x \rightarrow y' = \frac{1/x}{\sqrt{1-(\ln x)^2}}$$

$$\frac{d}{dx} \left(\tan^{-1}(f(x)) \right) = \frac{f'(x)}{1+(f(x))^2}$$

$$\frac{d}{dx} \left(\cot^{-1}(f(x)) \right) = \frac{-f'(x)}{1+(f(x))^2}$$

$$y = \tan^{-1} x \rightarrow y' = \frac{1}{1+x^2}$$

$$y = \cot^{-1}(x^2+1) = \frac{-2x}{1+(x^2+1)^2}$$

$$y = \tan^{-1} e^x \rightarrow y' = \frac{e^x}{1+(e^x)^2}$$

Derivatives of inverse trigonometric

chapter 3

$$\frac{d}{dx} \left(\sec^{-1}(f(x)) \right) = \frac{f'(x)}{|f(x)|\sqrt{f(x)^2-1}}$$

$$\frac{d}{dx} \left(\csc^{-1}(f(x)) \right) = \frac{-f'(x)}{|f(x)|\sqrt{f(x)^2-1}}$$

Find y' :

Find y' :

$$y = \sec^{-1} x \rightarrow y' = \frac{1}{|x|\sqrt{x^2-1}}$$

$$y = \csc^{-1}(\ln x) \rightarrow y' = \frac{-\frac{1}{x}}{|\ln x|\sqrt{(\ln x)^2-1}}$$

$$y = \sec^{-1} x^3 \rightarrow y' = \frac{3x^2}{|x^3|\sqrt{(x^3)^2-1}}$$

* Higher Derivatives Well known:

$$f, f', f'', f''', f^{(4)}, f^{(5)}, \dots, f^{(n)}$$

$$y, \frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \dots, \frac{d^n}{dx^n}$$

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \dots, \frac{d^ny}{dx^n}$$

Ex: $f(x) = x^5, f^{(4)}, f^{(5)}, f^{(6)}$

$$f'(x) = 5x^4$$

$$f''(x) = 20x^3$$

$$f'''(x) = 60x^2$$

$$f^{(4)}(x) = 120x$$

$$f^{(5)}(x) = 120$$

$$f^{(6)}(x) = 0$$

$$f^{(7)}(x) = 0$$

Higher derivatives

Chapter 3

$$P(x) = a_0 + a_1x + \dots + a_nx^n$$

$$P^{(n)}(x) = n! , P^{(m)}(x) = 0 , m > n$$

$$\text{Ex: } P(x) = x^7 , \text{ Find } P^{(7)} , P^{(8)}$$

$$P^{(7)}(x) = 7! , P^{(8)}(x) = 0$$

$$\text{Ex: } y = 3^x , \text{ Find } \left. \frac{d^{10}y}{dx^{10}} \right|_{x=0} = ?$$

$$y' = 3^x \ln 3$$

$$y'' = 3^x \ln 3 \ln 3 = 3^x (\ln 3)^2$$

$$y''' = 3^x (\ln 3)^3$$

$$\frac{d^{10}y}{dx^{10}} = 3^x (\ln 3)^{10}$$

$$\left. \frac{d^{10}y}{dx^{10}} \right|_{x=0} = (\ln 3)^{10}$$

$$\text{Ex: } P(4) = 3 , P'(4) = 2 , P''(4) = 5 , \text{ Find } \left(\frac{P'}{P} \right)'(4) = ?$$

$$\left(\frac{P'}{P} \right)'(x) = \frac{P(x)P''(x) - P'(x)^2}{(P(x))^2}$$

$$= \frac{P(4)P''(4) - (P'(4))^2}{(P(4))^2}$$

$$= \frac{15 - 4}{9} = \frac{11}{9}$$

Higher derivatives

Chapter 3

Ex: $y = \cos^3 x + 5 \tan x$, Find y'' ?

$$y' = -3 \cos^2 x \sin x + 5 \sec^2 x$$

$$y'' = +6 \cos x \sin x + -3 \cos^3 x + 10 \sec^2 x \tan x$$

* Implicit differentiation في المشتقات الضمنية:

1) $x^3 + y^3 = 1$

$$3x^2 + 3y^2 y' = 0$$

$$3y^2 y' = -3x^2$$

$$y' = \frac{-3x^2}{3y^2} \rightarrow y' = -\frac{x^2}{y^2}$$

2) $e^y \sin x = x + x^3 y^2$

$$y' e^y \sin x + e^y \cos x = 1 + 3x^2 y^2 + 2xy y' x^3$$

$$y' e^y \sin x - 2xy y' x^3 = 1 + 3x^2 y^2 - e^y \cos x$$

$$y' (e^y \sin x - 2xy x^3) = 1 + 3x^2 y^2 - e^y \cos x$$

$$y' = \frac{1 + 3x^2 y^2 - e^y \cos x}{e^y \sin x - 2xy x^3}$$

Implicit differentiation

chapter 3

$$\text{Ex: } F(1) = 3$$

$$F'(1) = 2$$

$$\text{and } F(y^2x) = 2yF(x) - 1, \text{ Find } \left. \frac{dy}{dx} \right|_{(x,y)=(1,1)}$$

$$F(y^2x) = 2yF(x) - 1$$

$$F'(y^2x)(y^2 + 2yy'x) = 2yF'(x) + F(x)2y'$$

$$F'(1)(1 + 2y') = 2F'(1) + 2F(1)y'$$

$$2(1 + 2y') = 4 + 6y'$$

$$1 + 2y' = 2 + 3y'$$

$$\left. \frac{dy}{dx} \right|_{(x,y)=(1,1)} = -1$$

$$1) y = x^2 \rightarrow y' = 2x$$

$$2) y = 2^x \rightarrow y' = 2^x \ln 2$$

$$3) y = x^x$$

$$\ln y = \ln x^x$$

$$\ln y = x \ln x$$

$$\frac{y'}{y} = x \cdot \frac{1}{x} + \ln x$$

$$\frac{y'}{y} = 1 + \ln x$$

$$y' = x^x(1 + \ln x)$$

Implicit differentiation

chapter 3

Ex:

$$1) y = (x^2 + 1)^{\sin x}$$

$$\ln y = \sin x \ln(x^2 + 1)$$

$$\frac{y'}{y} = \sin x \left(\frac{2x}{x^2 + 1} \right) + (\ln(x^2 + 1)) \cos x$$

$$y' = y \left(\frac{2x \sin x + \cos x \ln(x^2 + 1)}{x^2 + 1} \right)$$

$$y' = (x^2 + 1)^{\sin x} \left(\frac{2x \sin x + \ln(x^2 + 1) \cos x}{x^2 + 1} \right)$$

$$2) x^y = y^x, \text{ Find } y'$$

$$\ln x^y = \ln y^x$$

$$y \ln x = x \ln y$$

$$y \frac{1}{x} + \ln x y' = x \frac{y'}{y} + \ln y$$

$$\ln x y' - \left(\frac{x}{y} \right) y' = \ln y - \frac{y}{x}$$

$$y' \left(\ln x - \frac{x}{y} \right) = \ln y - \frac{y}{x}$$

$$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$$

Implicit differentiation

chapter 3

3) $y = (x^3 - 2x)^{\ln x}$, find y'

$$\ln y = \ln x \ln(x^3 - 2x)$$

$$\frac{y'}{y} = (\ln x) \frac{(3x^2 - 2)}{x^3 - 2x} + \ln(x^3 - 2x) * \frac{1}{x}$$

$$y' = y \left((\ln x) \frac{3x^2 - 2}{x^3 - 2x} + \frac{\ln(x^3 - 2x)}{x} \right)$$

$$y' = (x^3 - 2x)^{\ln x} \left((\ln x) \frac{3x^2 - 2}{x^3 - 2x} + \frac{\ln(x^3 - 2x)}{x} \right)$$

* The relationship between differentiability and continuity:

العلاقة بين الاتصال وقابلية الاشتقاق:

Theorem: If a function $f(x)$ is differentiable at c then f continuous at c

diff $\rightarrow$ cts

not cts $\rightarrow$ not diff

cts $\rightarrow$??

Ex: Is $f(x)$ differentiable at $x=1$, $f(x) = \begin{cases} x^2 + 1, & x \leq 1 \\ 2x, & x > 1 \end{cases}$

$$\lim_{x \rightarrow 1^+} 2x = 2$$

$$\lim_{x \rightarrow 1^-} x^2 + 1 = 2$$

$$f(1) = 2$$

$\rightarrow f$ cts at $x=1$

$$f(x) = \begin{cases} 2x, & x < 1 \\ x, & x > 1 \end{cases}$$

$f(x)$ diff at $x=1$

$$f'(1) = f'(1) \\ 2 = 2 \checkmark$$

$$* f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x, & x > 1 \end{cases} \quad \text{Is } f(x) \text{ diff at } x=1?$$

$$\lim_{x \rightarrow 1^+} 2x = 2$$

$$x \rightarrow 1^+ \neq$$

$f(x)$ discts at $x=1$

$$\lim_{x \rightarrow 1^-} = 3$$

$f(x)$ not diff at $x=1$

$$x \rightarrow 1^-$$

$$* \text{Is } f(x) = |x-5| \text{ diff at } x=5?$$

$$f(x) = \begin{cases} x-5, & x \geq 5 \\ 5-x, & x < 5 \end{cases}$$

$$|x-5| = x-5 = 0$$

$$x=5$$

$$\begin{array}{ccc} - & + & + \\ 5-x & 5 & x-5 \end{array}$$

$$\lim_{x \rightarrow 5^+} x-5 = 0$$

$$x \rightarrow 5^+$$

$$\lim_{x \rightarrow 5^-} 5-x = 0$$

$$x \rightarrow 5^-$$

$$f(5) = 0$$

$f(x)$ cts at $x=5$

$$f'(x) = \begin{cases} 1, & x > 5 \rightarrow f'(5) = 1 \\ -1, & x < 5 \rightarrow f'(5) = -1 \end{cases} \neq \text{not diff at } x=5$$

$f(x) = |x-5|$ not diff at $x=5$
diff $\mathbb{R} - \{5\}$

Equation of tangent and normal line

Chapter 3

$$* f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ kx - k, & x > 1 \end{cases}$$

Find the Value of k

1) IF $f(x)$ continuous? $k \in \mathbb{R}$

2) IF $f(x)$ differentiable? $k = 2$

3) IF $f(x)$ continuous but not diff?

$$\mathbb{R} - \{2\}$$

$$1) \lim_{x \rightarrow 1^+} kx - k = \lim_{x \rightarrow 1^-} x^2 - 1$$

$$x \rightarrow 1^+$$

$$x \rightarrow 1^-$$

$$0 = 0 \quad ?? \quad k \in \mathbb{R} = (-\infty, \infty)$$

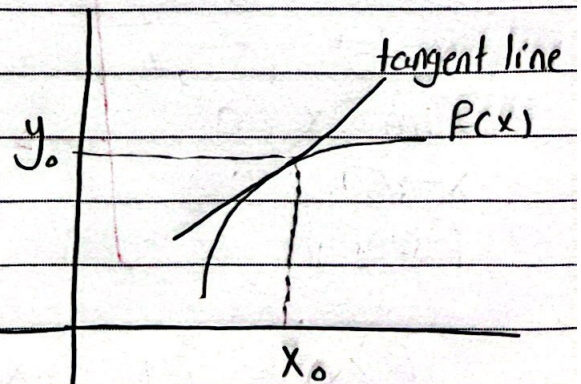
$$2) f'(x) = \begin{cases} 2x, & x \leq 1 \\ k, & x > 1 \end{cases}$$

$$f'(1) = f'(1) \\ + \quad k = 2 \quad -$$

* Equation of tangent line : معادلة المماس

$$y - y_0 = m(x - x_0)$$

$$m = \text{slope} = f'(x_0)$$



Equation of normal line
(perpendicular to the tangent line)

معادلة المماس العمودي على المماس

$$y - y_0 = -\frac{1}{m}(x - x_0)$$

Equation of tangent and normal line

Chapter 3

1) Find the slope of $f(x) = 2 + 3\tan^{-1} 2x$ at $x=1$

$$f'(x) = 3 \left(\frac{2}{1+4x^2} \right) = \frac{6}{1+4x^2}$$

$$f'(1) = \frac{6}{5} \text{ (slope of } f(x) \text{ at } x=1)$$

2) Find the slope of normal line of $f(x) = x^3$ at $x=-1$

$$f'(x) = 3x^2$$

$$f'(-1) = 3 \text{ (slope of tangent line)}$$

$$\text{Slope of normal line} = -\frac{1}{3}$$

3) Find the equation of tangent line and normal line

$$f(x) = \frac{2}{1+e^{-x}} \text{ at } x=0$$

$$* (x_0, y_0) ?$$
$$(0, 1)$$

$$* m = \text{slope} = f'(x) = \frac{-2(-e^{-x})}{(1+e^{-x})^2} = \frac{2e^{-x}}{(1+e^{-x})^2}$$

$$f'(0) = \frac{2e^0}{(1+e^0)^2} = \frac{1}{2}$$

$$* \text{equation of tangent line} \rightarrow y - 1 = \frac{1}{2}(x - 0) \rightarrow y = \frac{1}{2}x + 1$$

$$* \text{equation of normal line} \rightarrow y - 1 = -\frac{2}{1}(x - 0) \rightarrow y = -2x + 1$$

4) For what value of x , the graph of $f(x) = 2x^3 - 6x$ have a horizontal tangent? $\rightarrow$ slope $= m = 0$
 $f'(x) = 0$

$$f'(x) = 6x^2 - 6 = 0$$

$$x^2 = 1$$

$$x = \pm 1 \in \mathbb{R}$$

at $x = \pm 1$ horizontal tangent

* Hyperbolic Functions :

1) Hyperbolic Sine : $\sinh x = \frac{e^x - e^{-x}}{2}$

2) Hyperbolic Cosine : $\cosh x = \frac{e^x + e^{-x}}{2}$

3) Hyperbolic tangent = $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4) Hyperbolic cotangent = $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5) Hyperbolic secant = $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

6) Hyperbolic cosecant = $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

Hyperbolic Functions

Chapter 3

Ex:

$$1) \sinh 0 = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0$$

$$2) \cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{10}{6}$$

$$3) \operatorname{sech}(\ln 3) = \frac{1}{\cosh(\ln 3)} = \frac{1}{\frac{10}{6}} = \frac{6}{10}$$

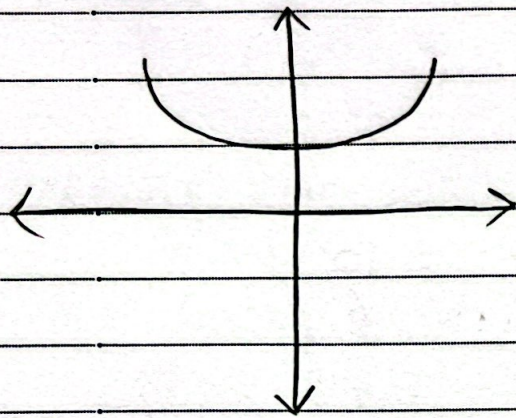
$$4) \sinh(\ln 5) = \frac{e^{\ln 5} - e^{-\ln 5}}{2} = \frac{5 - \frac{1}{5}}{2} = \frac{24}{10}$$

$$5) \cosh(\ln 5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2} = \frac{5 + \frac{1}{5}}{2} = \frac{26}{10}$$

$$6) \tanh(\ln 5) = \frac{\sinh(\ln 5)}{\cosh(\ln 5)} = \frac{\frac{24}{10}}{\frac{26}{10}} = \frac{24}{26}$$

$$7) \operatorname{csch}(\ln 5) = \frac{1}{\sinh(\ln 5)} = \frac{1}{\frac{24}{10}} = \frac{10}{24}$$

* Graphs of hyperbolic Functions:



$$y = \cosh x$$

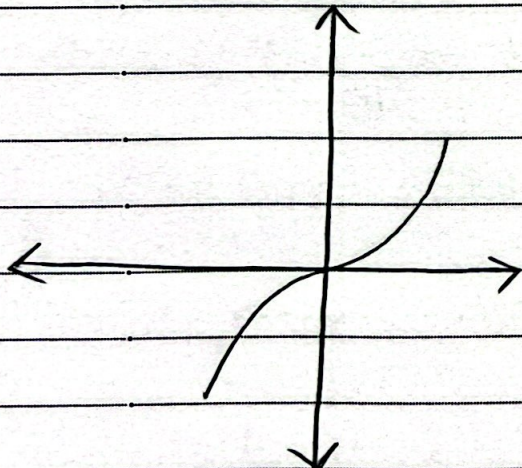
Domain: $\mathbb{R}$, Range: $[1, \infty)$

Even ($\cosh(-x) = \cosh(x)$)

$$\lim_{x \rightarrow +\infty} \cosh x = +\infty$$

$$\lim_{x \rightarrow -\infty} \cosh x = +\infty$$

No Horizontal and Vertical asy



$$y = \sinh x$$

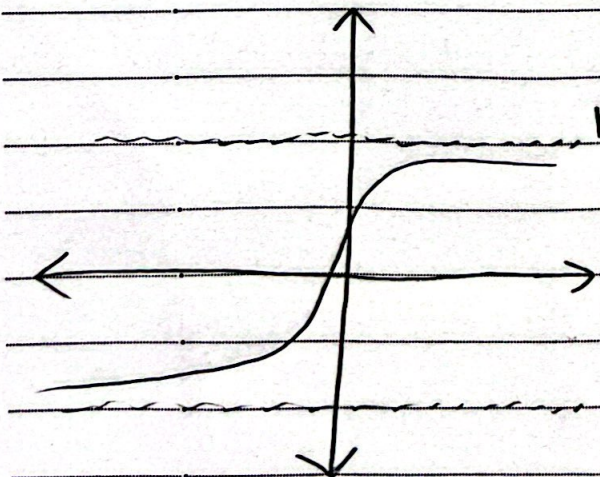
Domain: $\mathbb{R}$, Range: $\mathbb{R}$

Odd ($\sinh(-x) = -\sinh(x)$)

$$\lim_{x \rightarrow +\infty} \sinh x = +\infty$$

$$\lim_{x \rightarrow -\infty} \sinh x = -\infty$$

No Horizontal and Vertical asy



$$y = \tanh x$$

Domain: $\mathbb{R}$, Range: $(-1, 1)$

Odd ($\tanh(-x) = -\tanh(x)$)

$$\lim_{x \rightarrow +\infty} \tanh x = 1$$

$$\lim_{x \rightarrow -\infty} \tanh x = -1$$

Horizontal asy ✓

No Vertical asy

Ex: If $\sinh x = \frac{1}{2}$, find $\cosh x$?

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh^2 x - \frac{1}{4} = 1$$

$$\cosh^2 x = \frac{5}{4} \longrightarrow \cosh x = \pm \sqrt{\frac{5}{4}} = \pm \frac{\sqrt{5}}{2}$$

+ since $\cosh x \geq 1$

* Derivative of hyperbolic Functions :

$$\frac{d}{dx} (\sinh x) = \frac{d}{dx} \left(\frac{e^x - e^{-x}}{2} \right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

$$1) \frac{d}{dx} (\sinh x) = \cosh x$$

$$2) \frac{d}{dx} (\cosh x) = \sinh x$$

$$3) \frac{d}{dx} (\tanh x) = \operatorname{sech}^2 x$$

$$4) \frac{d}{dx} (\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$5) \frac{d}{dx} (\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$6) \frac{d}{dx} (\coth x) = -\operatorname{csch}^2 x$$

Hyperbolic identities

Chapter 3

* Find y' :

1) $y = x \sinh x$

$$y' = x \cosh x + \sinh x$$

2) $y = \cosh(\ln x)$

$$y' = \frac{1}{x} \sinh(\ln x)$$

$$= \frac{1}{x} * \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{1}{x} * \frac{x - \frac{1}{x}}{2} = \frac{x^2 - 1}{\frac{x}{2x}} = \frac{x^2 - 1}{2x^2}$$

3) $y = \ln(\tanh x)$

$$y' = \frac{\operatorname{sech}^2 x}{\tanh x} = \frac{1}{\cosh^2 x} * \frac{\cosh x}{\sinh x} = \frac{1}{\cosh x \sinh x} = \frac{1}{\frac{1}{2} \sinh 2x} = 2 \operatorname{csch} 2x$$

4) $y = 3^{\coth x}$

$$y' = (3^{\coth x}) (-\operatorname{csch}^2 x) \ln 3$$