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دفتر

تفاضل وتكامل 1 (كامل المادة)

إعداد : صفاء ابراهيم



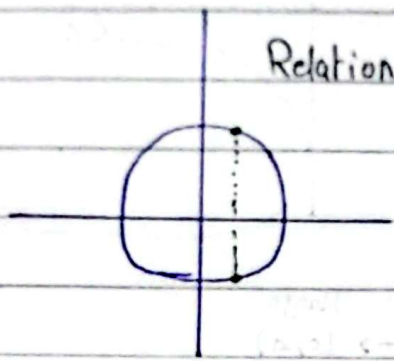
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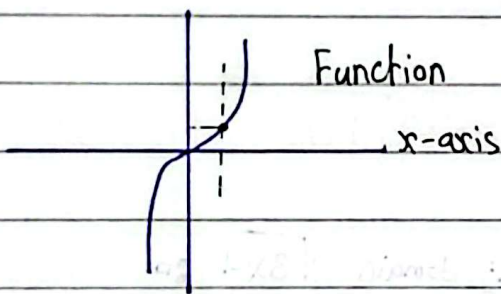
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Relation

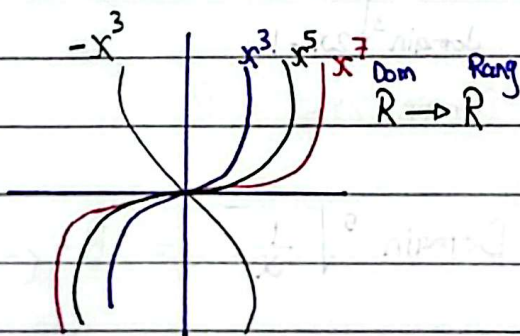
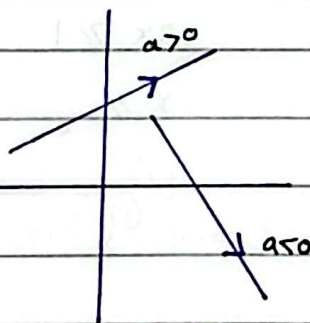


Function

x-axis

* $f(x) = ax + b$ ^{slope}

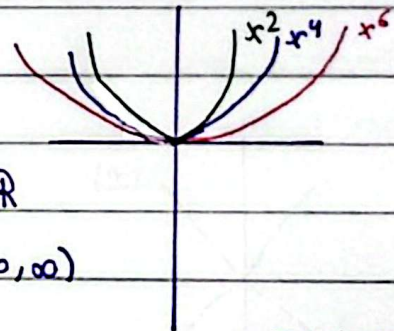
$R \rightarrow R$
Dom Range



Dom Range
 $R \rightarrow R$

Domain: R

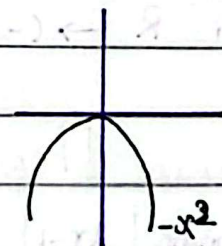
Range: $[0, \infty)$



x-axis ← Domain *
y-axis ← Range *

Domain: R

Range: $(-\infty, 0]$



* polynomials: Dom $\rightarrow \mathbb{R}$

$$f(x) = ax^2 + bx + c$$

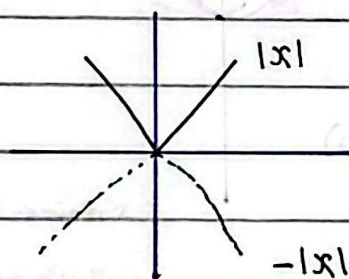
* Find domain and Range:
 $\mathbb{R}$

$$* f(x) = \frac{b}{2a}x - \frac{b^2}{4a}$$

Range $\frac{-b}{2a} = \frac{-2}{2(-1)} = 1$

Range $(-\infty, f(1)] \Rightarrow (-\infty, 1]$ Range

$|x|$

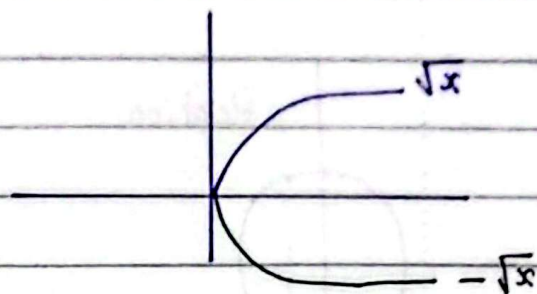


Domain Range
 $|x| : \mathbb{R} \rightarrow [0, \infty)$

$-|x| : \mathbb{R} \rightarrow (-\infty, 0]$

* Domain $|f(x)| = \text{dom } f(x)$

domain $|3x-1| = \mathbb{R}$



Domain Range
 $\sqrt{x} : [0, \infty) \rightarrow [0, \infty)$

$-\sqrt{x} : [0, \infty) \rightarrow (-\infty, 0]$

39)
 * Domain $\sqrt[n]{f(x)} = f(x) \geq 0$

Ex: domain $\sqrt{3x-1} \geq 0$

$$3x-1 \geq 0$$

$$3x \geq 1$$

$$x \geq \frac{1}{3} \quad [\frac{1}{3}, \infty)$$

52)
 * Domain $\sqrt[n]{f(x)} = \text{domain } f(x)$

Ex: domain $\sqrt[3]{2x-1}$

$$\text{Domain } 2x-1 = \mathbb{R}$$

Ex: Domain $\sqrt[5]{\frac{1}{x}} = \mathbb{R} - \{0\}$

$$* \text{Dom } \{f \pm g\}$$

$$\text{Dom } f \cap \text{Dom } g$$

$$* \text{Dom } \frac{f}{g} \Rightarrow \text{Dom } f \cap \text{Dom } g - \{g(x)=0\}$$

$$|x|=a \Rightarrow x=\pm a$$

$$|x|<a = -a < x < a$$

$$|x| \geq a = x \leq -a \text{ or } x \geq a$$

$$\sqrt{x^2} = |x|$$

Ex: Find Domain and Range:

$$1) \sqrt{4-x^2}$$

$$4-x^2 \geq 0$$

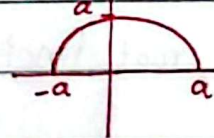
$$-x^2 \geq -4$$

$$\sqrt{x^2} \leq \sqrt{4}$$

$$|x| \leq 2$$

$$-2 \leq x \leq 2$$

$$\sqrt{a^2-x^2}$$



$$\text{ex: } \sqrt{25-x^2}$$

$$\text{Domain: } [-5, 5]$$

$$\text{Range: } [0, 5]$$

* chapter 1: Functions and models

Definition: $f: A \rightarrow B$, f is a function

from A to B if f is a relation that assigns every element $a \in A$ to a Unique element $b \in B$

denoted by $y = f(x)$

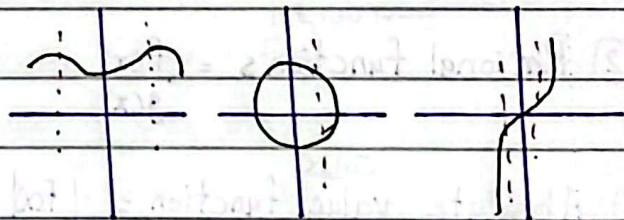
الاقتران هو علاقة بحيث يربط عنصر واحد فقط بالعنصر

* vertical line test:

if the vertical line intersects the graph at exactly one point then y is a function

* Example:

which of the following is function?



Function

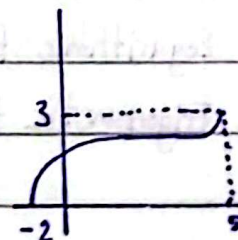
relation

function

* Example:

$$\text{Domain: } [-2, 5]$$

$$\text{Range: } [0, 3]$$



* Note that:

Domain : (Dom): the set of all possible input (x-axis)

Range: the set of all possible output (y-axis)

vertical line test
اختبار الخط العمودي
Polynomials كثيرات الحدود
integers كد صحيح

Rational functions $\frac{f(x)}{g(x)}$ اقرانان كسرية
Absolute value function القيمة المطلقة
root functions الجذور
Exponential function الاقتران الاسي
Logarithmic functions اللوغاريتم
Trigonometric functions الاقترانات المثلثية

• Types of functions :-

① ^{كثيرات الحدود} polynomials

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots$$

$n = 1, 2, 3 \dots$ +ve integers : كد صحيح موجب

$a_n \in \mathbb{R}$

$\mathbb{R} =$ كد حقيقي

② ^{اقرانان كسرية} Rational functions = $\frac{f(x)}{g(x)}$

③ ^{القيمة المطلقة} Absolute value function = $|f(x)|$

④ ^{الجذور} root functions, $f(x) = \sqrt[n]{g(x)}$

⑤ ^{الاسي} Exponential functions, $f(x) = a^x$

⑥ Logarithmic functions, $f(x) = \log x$

⑦ Trigonometric functions ^{الاقترانات المثلثية}

* Example : classify each functions :-

① $f(x) = 3x^2 - 7x^2 + x - 1$ polynomial function of degree 2

② $f(x) = x^2 - 2x + 5$ poly, Quadratic function

③ $f(x) = 2x + 1$ poly, linear.

④ $f(x) = x^3 - 2x^2 + 1$ poly, Cubic fun.

⑤ $f(x) = \frac{3x-1}{x^2+1}$ Rational function.

⑥ $f(x) = \sqrt{3x+1}$ root function.

⑦ $f = |x-1|$ Absolut value function.

* Even and odd functions :-

* Definition :

① A function f is said to be an even function, if: $f(-x) = f(x)$

Ex:) $f(x) = x^2$

$f(x) = x^4$

$f(x) = x^{\text{even}}$

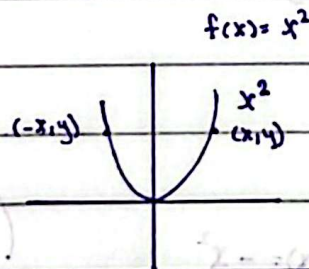
$f(x) = |x|$

$f(x) = \cos x \quad \cos(-x) = \cos x$

* note that :

even function

is symmetric about y-axis.



Quadratic function اقرون مربع

linear function اقرون خطي

Cubic function

اقرون تكعبي

Even function اقرون زوجي

odd function اقرون فردي

symmetric متماثل

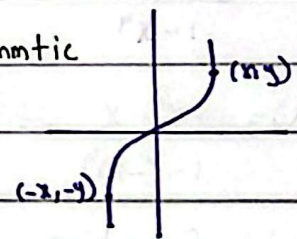
* not that :

$f(x) = x^3$

odd function symmetric

about $y=x$

"origin".



* Rule:

Even * Even = Even

Even * odd = odd

odd * Even = odd

odd * odd = Even

② A function f is said to be an

odd function if $f(-x) = -f(x)$.

Ex:) $f(x) = x$

$f(x) = x^3$

$f(x) = x^{\text{odd}} \rightarrow \text{odd}$

$f(x) = \sin x \quad \sin(-x) = -\sin(x)$

* Example: Determine whether each of the following function is even or odd or neither even nor odd

① $f(x) = \frac{3-x^4}{|x|}$

$f(-x) = \frac{3-(-x)^4}{|-x|} = \frac{3-x^4}{|x|} = f(x)$ even function

$$② f(x) = x^3 + x$$

$$f(-x) = -x^3 - x \quad \text{odd function}$$

$$③ f(x) = \frac{\sin x + x^3}{\cos x}$$

$$f(-x) = \frac{\sin(-x) + (-x)^3}{\cos(-x)} = \frac{-\sin x - x^3}{\cos x}$$

$$= - \left(\frac{\sin x + x^3}{\cos x} \right) \cos(-x) \quad \cos(x)$$

odd function

$$④ f(x) = \frac{x^5 + x}{1 - x^4}$$

$$f(-x) = \frac{-x^5 - x}{1 - (-x)^4} = \frac{-x^5 - x}{1 - x^4} = - \frac{x^5 + x}{1 - x^4}$$

odd function

$$⑤ f(x) = x + 5$$

$$f(-x) = -x + 5 \Rightarrow f(-x) \neq -f(x)$$

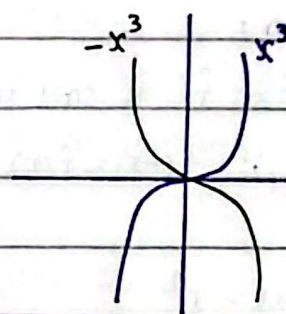
$$\neq -f(x)$$

neither even nor odd

② cubic function:

$$\text{Dom: } \mathbb{R}$$

$$\text{Rang: } \mathbb{R}$$

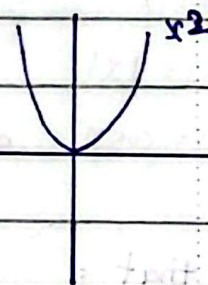


③ Quadratic function:

$$f(x) = x^2$$

$$\text{Dom: } \mathbb{R}$$

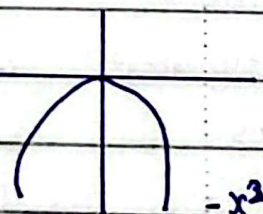
$$\text{Range: } [0, \infty)$$



$$f(x) = -x^2$$

$$\text{Dom: } \mathbb{R}$$

$$\text{Range: } (-\infty, 0]$$



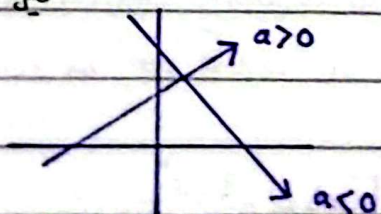
⊗ Domain and Range :-

① linear function / $y = ax + b$

a = slope $\downarrow$

$$\text{Dom: } \mathbb{R}$$

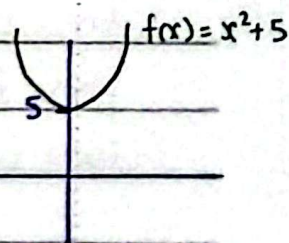
$$\text{Rang: } \mathbb{R}$$



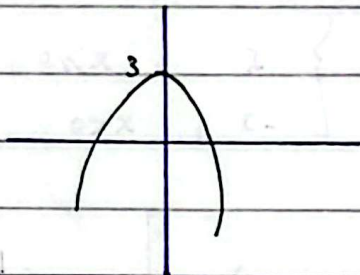
$$\text{Ex) } ① f(x) = x^2 + 5$$

$$\text{Dom: } \mathbb{R}$$

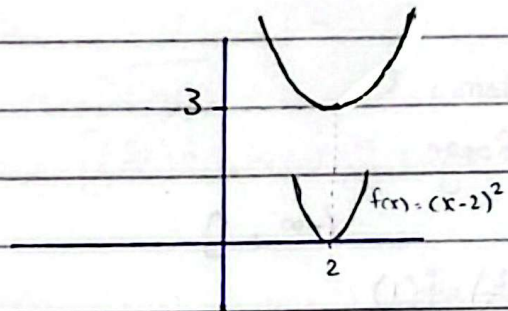
$$\text{Range: } [5, \infty)$$



Ex) ② $f(x) = -x^2 + 3$

Dom: $\mathbb{R}$ Range: $(-\infty, 3]$

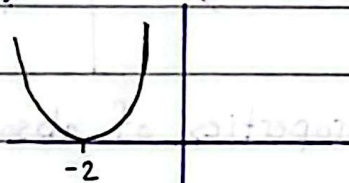
⑤ $f(x) = (x-2)^2 + 3$

Dom: $\mathbb{R}$ Range: $[3, \infty)$

③ $f(x) = (x+2)^2$

$x+2=0$

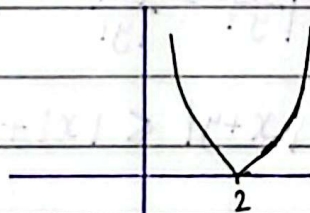
$x=-2$

Dom: $\mathbb{R}$ Range: $[0, \infty)$

④ $f(x) = (x-2)^2$

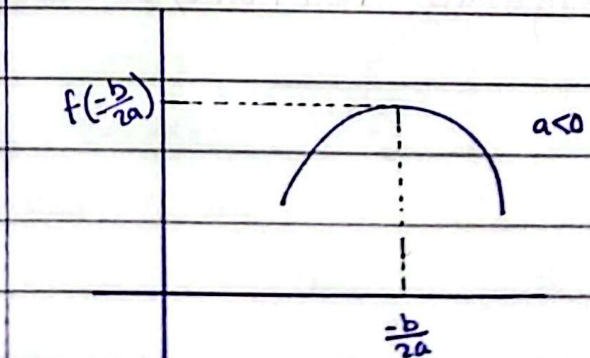
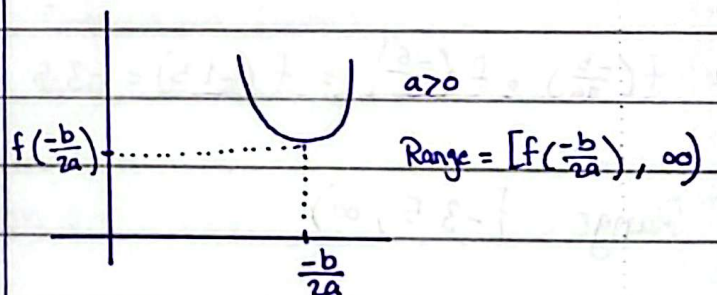
$x-2=0$

$x=2$

Dom: $\mathbb{R}$ Range: $[0, \infty)$

* note that: to find the range of quadratic function

$f(x) = ax^2 + bx + c \quad a \neq 0$



Range = $(-\infty, f(-\frac{b}{2a}))$

Ex) ① $f(x) = 2x - x^2$

Dom: $\mathbb{R}$

Range: $(-\infty, f(\frac{-b}{2a})]$
 $(-\infty, 1]$

$f(\frac{-2}{-2}) = f(1)$

$f(1) = 2 - 1 = 1$

② $f(x) = 2x^2 + 6x + 1$

Dom: $\mathbb{R}$

Range: $[f(\frac{-b}{2a}), \infty)$

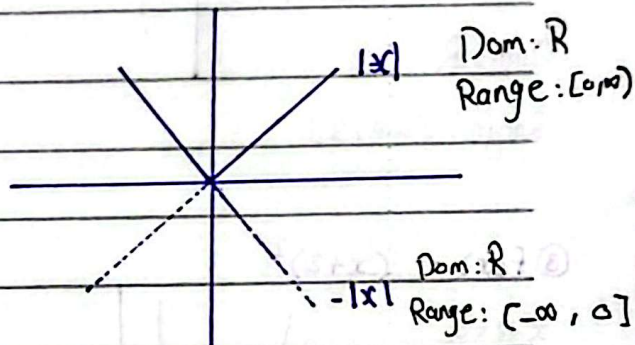
$f(\frac{-b}{2a}) = f(\frac{-6}{4}) = f(-1.5) = -3.5$

Range: $[-3.5, \infty)$

* Polynomial function $\Rightarrow$ Dom = $\mathbb{R}$ *

* Absolute value function

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



* properties of absolute value :-

① $\sqrt{x^2} = |x|$

② $|-x| = x$

③ $|xy| = |x||y|$

④ $|\frac{x}{y}| = \frac{|x|}{|y|}$

⑤ $|x+y| \leq |x| + |y|$

⑥ $|x| = a \Rightarrow x = \pm a$

⑦ $|x| \leq a \Rightarrow -a \leq x \leq a$
 $\therefore x \in [-a, a]$

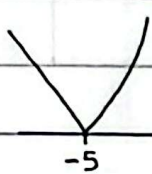
⑧ $|x| > a \Rightarrow x > a \text{ or } x < -a$
 $(a, \infty) \cup (-\infty, -a)$

* Example: find dom and Range :-

① $f(x) = |x+5|$

$x+5=0$

$x=-5$

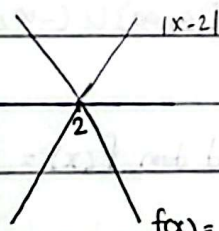


Dom: $\mathbb{R}$

Range: $[0, \infty)$

② $f(x) = -|x-2|$

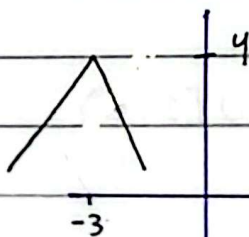
$x-2=0$
 $x=2$



Dom: $\mathbb{R}$

Range: $(-\infty, 0]$

③ $f(x) = 4 - |x+3|$



Dom: $\mathbb{R}$

Range: $(-\infty, 4]$

* Root function :

Rule:

① $f(x) = \sqrt[n]{g(x)}$ n : even number

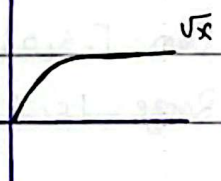
$\text{dom}(f) = \{x : g(x) \geq 0\}$

② $f(x) = \sqrt[n]{g(x)}$ n : odd number

$\text{dom}(f) = \text{dom}(g)$

* Example: Find dom and range

① $f(x) = \sqrt{x}$



Dom: $x \geq 0$ $[0, \infty)$

Range: $[0, \infty)$

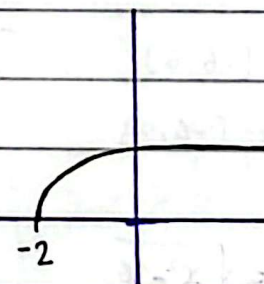
② $f(x) = \sqrt{x+2}$

Dom: $x+2 \geq 0$

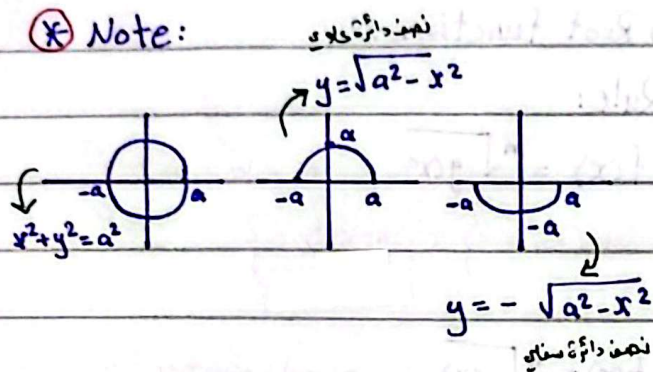
$x \geq -2$

$[-2, \infty)$

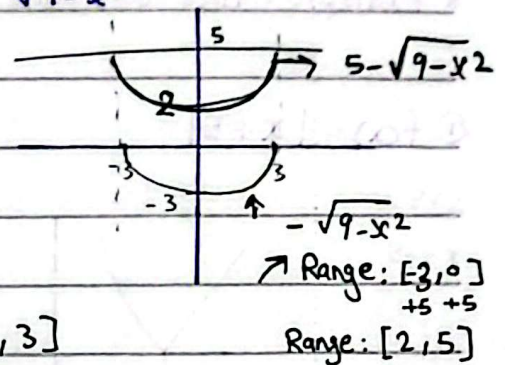
Range: $[0, \infty)$



* Note:



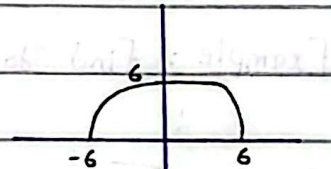
④ $y = 5 - \sqrt{9 - x^2}$



* Example: Find dom and range:

Range: $[2, 5]$

① $y = \sqrt{36 - x^2}$



Dom: $[-6, 6]$

Range: $[0, 6]$

$$36 - x^2 \geq 0$$

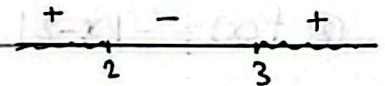
$$-x^2 \geq -36 \Rightarrow x^2 \leq 36$$

$$|x| \leq 6 \Rightarrow -6 \leq x \leq 6$$

⑤ find dom $f(x) = \sqrt{x^2 - 5x + 6}$

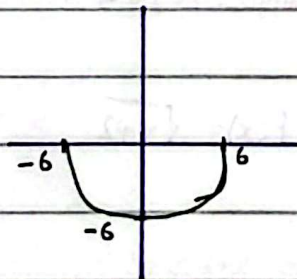
$$x^2 - 5x + 6 \geq 0$$

$$(x-3)(x-2) \geq 0$$



Dom: $[3, \infty) \cup (-\infty, 2]$

② $y = -\sqrt{36 - x^2}$



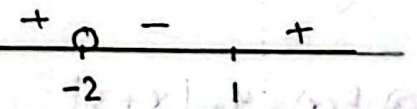
Dom: $[-6, 6]$

Range: $[-6, 0]$

⑥ find dom $f(x) = \sqrt{\frac{x-1}{x+2}}$

$$\frac{x-1}{x+2} \geq 0$$

critical points
 2 and 1



Dom: $(-\infty, -2) \cup [1, \infty)$

③ $y = \sqrt{x^2 - 36}$

$$x^2 - 36 \geq 0$$

Dom: $[6, \infty) \cup (-\infty, -6]$

$$\sqrt{x^2} \geq \sqrt{36}$$

range: $[0, \infty)$

$$|x| \geq 6$$

$$x \geq 6 \text{ or } x \leq -6$$

⑦ Find $\text{Dom } f(x) = \sqrt[3]{x-2}$

$$\text{Dom } f(x) = \text{Dom } (x-2) = \mathbb{R}$$

⑧ Find $\text{dom } f(x) = \sqrt{\frac{2}{x-1}}$

$$\text{Dom } f(x) = \text{Dom } \left(\frac{2}{x-1} \right)$$

$$= \mathbb{R} - \{1\}$$

* Find $\text{dom } f(x) = \sqrt[4]{|x+2|-5}$

$$|x+2|-5 \geq 0$$

$$|x+2| \geq 5$$

$$x+2 \geq 5 \quad \text{or} \quad x+2 \leq -5$$

$$x \geq 3 \quad \text{or} \quad x \leq -7$$

$$\text{Dom}: [3, \infty) \cup (-\infty, -7]$$

* New function from old go

→ Domain Rules:

① $\text{Dom } \{ (f \pm g)(x) \} = \text{Dom } f(x) \cap \text{Dom } g(x)$

② $\text{Dom } \{ (fg)(x) \} = \text{Dom } f(x) \cap \text{Dom } g(x)$

③ $\text{Dom } \left\{ \left(\frac{f}{g} \right)(x) \right\} = \{ \text{Dom } f(x) \cap \text{Dom } g(x) \} - \{ x: g(x)=0 \}$

* Example: Find domain:

① $f(x) = \sqrt{x-2}$

$$x-2 \geq 0 \Rightarrow x \geq 2$$

$$\text{Dom}: \mathbb{R} \cup [2, \infty)$$

② $f(x) = \sqrt{x+2}$

$$x+2 \geq 0$$

$$x \geq -2$$

$$[-2, \infty)$$

$$+ \frac{1}{\sqrt{3-x}}$$

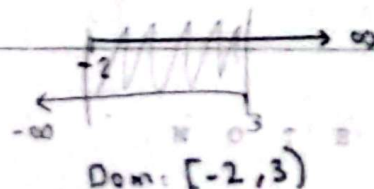
$$3-x > 0$$

$$x < 3$$

$$(-\infty, 3)$$

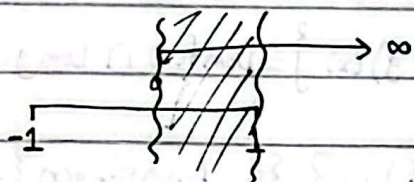
Dom:

$$[-2, \infty) \cup (-\infty, 3)$$



③ $f(x) = \sqrt{x}$ $x \geq 0$ $[0, \infty)$

$R \leftarrow 1 + \sqrt{1-x^2}$ $x \in [-1, 1]$



Dom: $[0, 1] - \{ \text{values where } 1 + \sqrt{1-x^2} = 0 \}$

ϕ $-1 \neq 0 \rightarrow \sqrt{1-x^2} = -1$

Dom: $[0, 1]$

④ $f(x) = \frac{(x-1)^2}{x-1} \rightarrow R$
 $x-1 \rightarrow -\{1\}$

Dom: $R - \{1\}$

⑤ $f(x) = \frac{x}{x-1} \rightarrow R$
 $|x-1| \rightarrow R - \{1\}$

$|x-1| = 0$

$|x-1| = 1$

$x-1 = 1$

or

$x-1 = -1$

$x = 2$

$x = 0$

Dom: $R - \{2, 0\}$

⑥ $f(x) = \sqrt{|x-1|-4} + \sqrt{2x-1}$ $x \geq \frac{1}{2}$

$|x-1|-4 \geq 0$

$|x-1| \geq 4$

$x-1 \geq 4$ or $x-1 \leq -4$

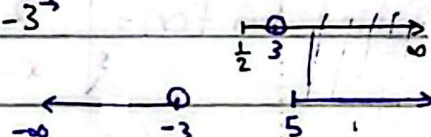
$x \geq 5$ or $x \leq -3$

$3-|x| \rightarrow R - \{3, -3\}$

$3-|x| = 0$

$|x| = 3$

$x = 3$ or $x = -3$



Dom: $[5, \infty)$

* Composition of functions:

Given two functions f and g , the composite (fog) defined by $fog(x) = f(g(x))$;

fog : f circle g

1	$\xrightarrow{g}$	9	$\xrightarrow{f}$	12
2	$\xrightarrow{g}$	8	$\xrightarrow{f}$	14
4	$\xrightarrow{g}$	2	$\xrightarrow{f}$	20
10		6		

$fog(1) = f(g(1)) = f(9) = 12$

$fog(2) = f(g(2)) = f(8) = 14$

$fog(4) = f(g(4)) = f(2) = 20$

$fog(10) = f(g(10)) = f(6) = \text{undefined}$

* Example: $f(x) = 10 - x^2$, $g(x) = \sqrt{x-1}$

find $f \circ g(x)$, $g \circ f(x) = ?$

$$\begin{aligned} \textcircled{1} \quad f \circ g(x) &= f(g(x)) = 10 - (\sqrt{x-1})^2 \\ &= 10 - x + 1 = 11 - x \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad g \circ f(x) &= g(f(x)) = g(10 - x^2) \\ &= \sqrt{(10 - x^2) - 1} = \sqrt{9 - x^2} \end{aligned}$$

فقط استبدال $g(x) = f(x) \leftarrow g \circ f(x)$ $\leftarrow$ $g(x) = f(x)$
 $f(x) = f(f)$ $\leftarrow$ $f \circ f(x)$ $\leftarrow$ $f(x) = f(f)$
 $\sqrt[3]{x}$, x^3 $\leftarrow$ x^3 , $\sqrt[3]{x}$

* Example: $f(x) = x^2 - 1$, $g(x) = \sqrt{3-x}$
Find

$$\begin{aligned} \textcircled{1} \quad f \circ g(-1) &= f(g(-1)) = f(2) \\ &= 3 \end{aligned}$$

$$\textcircled{2} \quad g \circ f(1) = g(f(1)) = g(0) = \sqrt{3}$$

very important Example: let $f \circ g(x) = 6x^2 - 10x + 5$, $f(x) = 2x + 1$

find $g(0) = ?$

$$f(g(x)) = 6x^2 - 10x + 5$$

$$2g(x) + 1 = 6x^2 - 10x + 5$$

$$2g(0) + 1 = 5 \Rightarrow 2g(0) = 4$$

$$g(0) = 2$$

very important

* Example: if $f(3x+5) = 6x+11$
then $f(x) = ??$

$$3x+5 = y \Rightarrow 3x = y-5$$

$$x = \frac{y-5}{3}$$

$$f(y) = 6\left(\frac{y-5}{3}\right) + 11$$

$$f(y) = 2y - 10 + 11$$

$$f(y) = 2y + 1 \quad \leftarrow \text{فقط استبدال}$$

$$f(x) = 2x + 1$$

* Example: if $f \circ g(x) = x^2 + 6x + 6$ and
 $g(x) = x+1$ find $f(x) = ??$

$$f(g(x)) = x^2 + 6x + 6$$

$$f(x+1) = x^2 + 6x + 6$$

$$y = x+1$$

$$x = y-1$$

$$f(y) = (y-1)^2 + 6(y-1) + 6$$

$$f(y) = y^2 - 2y + 1 + 6y - 6 + 6$$

$$f(y) = y^2 + 4y + 1$$

$$f(x) = x^2 + 4x + 1$$

* Example: if $f \circ g(x) = 3x^2 + 3x + 2$,
 $f(x) = 3x + 5$ find $g(x)$??

$$f(g(x)) = 3x^2 + 3x + 2$$

$$3(g(x)) + 5 = 3x^2 + 3x + 2$$

$$3g(x) = 3x^2 + 3x - 3$$

$$g(x) = x^2 + x - 1$$

* Dom $f \circ g(x)$

* Dom $(f \circ g(x)) = \{x : x \in \text{Dom } g(x) \text{ and } g(x) \in \text{Dom } f(x)\}$ *

* Example: if $\text{Dom } f(x) = [1, 4]$ then
 $\text{Dom } \{f(3x+4)\}$ is

$$\text{Dom } \{3x+4\} = \mathbb{R}$$

$$(3x+4) \in [1, 4]$$

$$\begin{matrix} 1 & \leq & 3x+4 & \leq & 4 \\ -4 & & -4 & & -4 \end{matrix}$$

$$\begin{matrix} -3 & \leq & 3x & \leq & 0 \\ \frac{-3}{3} & & \frac{3x}{3} & & \frac{0}{3} \end{matrix}$$

$$-1 \leq x \leq 0$$

$$\text{Dom: } [-1, 0]$$

* Example: if $f(x) = \sqrt{1-x}$, $g(x) = 2x$
 find dom $f \circ g(x)$.

$$\textcircled{1} f \circ g(x) = f(2x) = \sqrt{1-2x}$$

$$\text{Dom}(f \circ g(x)) = 1-2x \geq 0$$

$$-2x \geq -1$$

$$x \leq \frac{1}{2} \quad \left[-\infty, \frac{1}{2}\right]$$

$$\textcircled{2} \text{Dom } g(x) = 2x : \mathbb{R}$$

$$\mathbb{R} \cap \left(-\infty, \frac{1}{2}\right]$$

$$\text{Dom: } \left(-\infty, \frac{1}{2}\right]$$

* Example: if $f(x) = x^2 - 1$, $g(x) = \sqrt{3-x}$
 find dom $f \circ g(x)$ and dom $g \circ f(x)$

$$\textcircled{1} \text{Dom } f \circ g(x) : \rightarrow f(\sqrt{3-x}) = 3-x-1 = 2-x$$

$$\text{Dom: } \text{Dom}(2-x) \cap \text{Dom } \sqrt{3-x}$$

$$: \mathbb{R} \cap 3-x \geq 0 \Rightarrow x \leq 3 \quad (-\infty, 3]$$

$$\text{Dom: } (-\infty, 3]$$

$$\textcircled{2} \text{Dom } g \circ f(x) = g(x^2-1) = \sqrt{3-x^2+1} = \sqrt{4-x^2}$$

$$\text{Dom } \sqrt{4-x^2} \cap \text{Dom}(x^2-1)$$

$$[-2, 2] \cap \mathbb{R}$$

$$\text{Dom: } [-2, 2]$$

* Example: find dom $f(x) = \sqrt{4-\sqrt{x}}$

$$f \circ g(x) = \sqrt{4-\sqrt{x}} \Rightarrow \text{Dom: } 4-\sqrt{x} \geq 0$$

$$-\sqrt{x} \geq -4$$

$$(\sqrt{x})^2 \leq (4)^2$$

$$x \leq 16 \quad (-\infty, 16]$$

$$\text{Dom: } \text{Dom}(4-\sqrt{x}) \cap \text{Dom}\sqrt{x}$$

$$(-\infty, 16] \cap x \geq 0 \quad [0, \infty)$$

$$\text{Dom: } [0, 16]$$

* Example: $f(x) = \frac{1+x}{1-x}$, $g(x) = \frac{x}{1-x}$

find dom $f \circ g(x)$

$$\textcircled{1} f \circ g(x) = f\left(\frac{x}{1-x}\right) = \frac{\frac{1-x}{1-x} \cdot \frac{x}{1-x}}{\frac{1-x}{1-x} - \frac{x}{1-x}} =$$

$$= \frac{\frac{1}{1-x}}{\frac{1-2x}{1-x}} = \frac{1}{1-2x}$$

$$\text{Dom} \left(\frac{1}{1-2x} \right) \begin{matrix} \leftarrow R \\ R - \left\{ \frac{1}{2} \right\} \end{matrix} \quad \begin{matrix} 1-2x=0 \\ 2x=1 \\ x=\frac{1}{2} \end{matrix}$$

$$\text{Dom} \left(\frac{x}{1-x} \right) \begin{matrix} \leftarrow R \\ R - \{1\} \end{matrix}$$

$$\begin{aligned} \text{Dom } f \circ g(x) &= \text{Dom} \left(\frac{1}{1-2x} \right) \cap \left(\frac{x}{1-x} \right) \\ &= R - \left\{ \frac{1}{2} \right\} \cap R - \{1\} \\ &= R - \left\{ \frac{1}{2}, 1 \right\} \end{aligned}$$

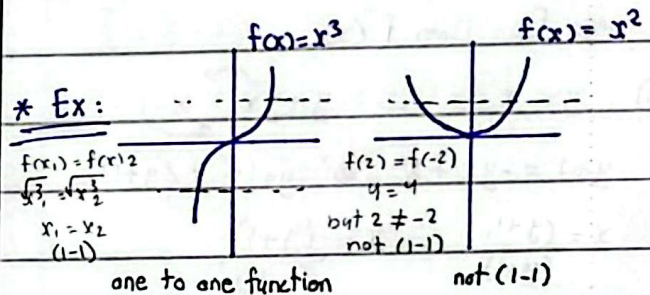
* one to one function:

A function f is called a one to one function (1-1) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

اختبار الخط الافقي

→ by Horizontal line test, A function is

1-1 iff no horizontal line intersects its graph more than once.



* Thm: if $f(x)$ 1-1 function $\Leftrightarrow f(x)$ has inverse.

* inverse function $\equiv f^{-1}$.

$$* f^{-1}(x) \neq \frac{1}{f(x)}$$

Ex: if $f(x) = 3x - 5$ find $f^{-1}(x)$

$$y = 3x - 5$$

$$3x = y + 5$$

$$x = \frac{y+5}{3}$$

$$f^{-1}(x) = \frac{x+5}{3}$$

wednesday

15/10/2025

Subject

Date

No.

* Note that:

$$\textcircled{1} f: A \rightarrow B \quad \begin{matrix} \text{Dom} & \text{Range} \\ A & B \end{matrix}$$

$$f^{-1}: B \rightarrow A \quad \begin{matrix} \text{Dom} & \text{Range} \\ B & A \end{matrix} \quad \text{Dom } f^{-1}(x) = \text{Rang } f$$

* Ex: find the rang of $f(x) = \frac{x-1}{1+x}$

* range $f(x) = \text{Dom } f^{-1}(x)$

$$\textcircled{1} y = \frac{x-1}{1+x} \Rightarrow y + yx = x - 1$$

$$y + 1 = -yx + x \Rightarrow y + 1 = x(1 - y)$$

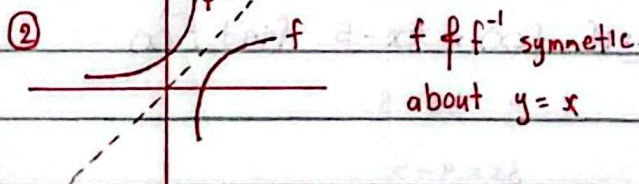
$$x = \frac{y+1}{1-y}$$

$$f^{-1}(x) = \frac{1+x}{-x+1} \rightarrow \mathbb{R}$$

$$-x+1 \leftarrow \mathbb{R} - \{1\}$$

$$\text{Dom } f^{-1}(x): \mathbb{R} - \{1\}$$

$$\text{Rang } f(x) = \text{Dom } f^{-1}(x) = \mathbb{R} - \{1\}$$



$$\textcircled{3} f \circ f^{-1}(x) = x \quad f(f^{-1}(x)) = x$$

$$f^{-1} \circ f(x) = x \quad f^{-1}(f(x)) = x$$

* Ex: let $f(x) = x^3 + 4x + 1$, if $f^{-1}(c^3) = c$
the $c = ?$

$$f(f^{-1}(c^3)) = f(c)$$

$$c^3 = f(c)$$

$$c^3 = c^3 + 4c + 1$$

$$4c + 1 = 0 \Rightarrow \boxed{c = -\frac{1}{4}}$$

Ex: let $f(x) = \frac{2x^3}{x^4 + 1}$, find x such that $1 = f(x)$

$$1 = f(x) \Rightarrow f(1) = f(f^{-1}(x))$$

$$f(1) = x$$

$$\boxed{1 = x}$$

Ex: $f(x) = x^3 + 4x + 1$ find $f^{-1}(6)$

$$f(?) = 6$$

$$f(1) = 6$$

$$f^{-1}(6) = 1$$

OR

$f^{-1}(6) = y$

$6 = f(y)$

$6 = y^3 + 4y + 1$

$y^3 + 4y - 5 = 0$

$y = 1$

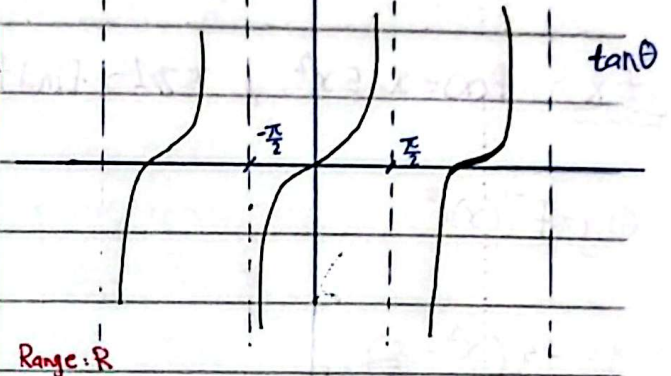
$f^{-1}(6) = 1$

* هذه الطريقة مدهشة
أحد زياراتي

* قد أستخدمه فقط
بالتجريب

* Trigonometric functions *

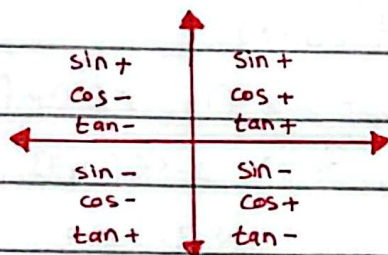
0	$\frac{\pi}{4}$ 45	$\frac{\pi}{3}$ 60°	$\frac{\pi}{6}$ 30°	الزاوية
0	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	cos θ
1	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	sin θ
2π 360°	$\frac{3\pi}{2}$ 270°	π 180°	$\frac{\pi}{2}$ 90°	الزاوية
1	0	-1	0	cos θ
0	-1	0	1	sin θ



Range: $\mathbb{R}$

$$\text{Dom: } \mathbb{R} - \left\{ \frac{\pi}{2} + n\pi \right\}$$

$$\{n = 0, \pm 1, \pm 2, \dots\}$$



composition function

Ex: Find dom {cos(2x-1)}

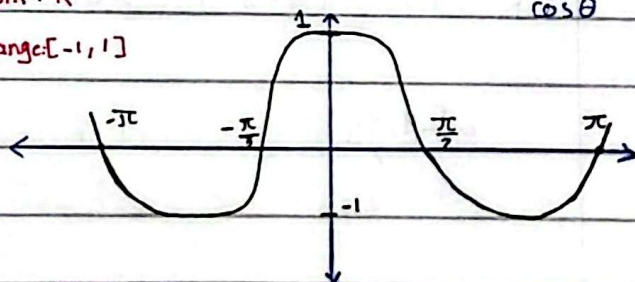
$$f(x) = \cos(x), g(x) = 2x-1$$

$$f \circ g(x) = \cos(2x-1)$$

$$\mathbb{R} \cap \mathbb{R} = \mathbb{R}$$

Dom: $\mathbb{R}$

Range: $[-1, 1]$



Ex: Find dom {sin(√x-5)}

$$\text{Dom } \{\sqrt{x-5}\}$$

$$\cap \text{Dom}(\sin)$$

$$x-5 \geq 0$$

$$\mathbb{R}$$

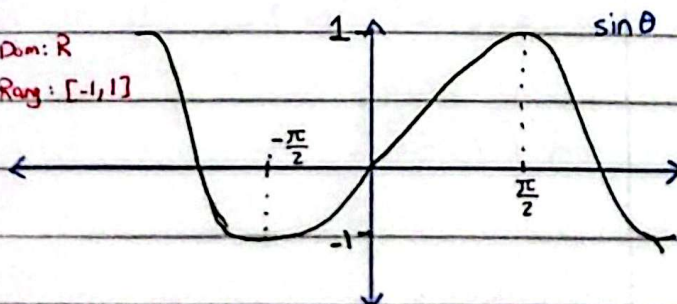
$$x \geq 5$$

$$[5, \infty)$$

$$\text{Dom } \sin(\sqrt{x-5}) = [5, \infty)$$

Dom: $\mathbb{R}$

Range: $[-1, 1]$



Ex: Find the range of

$$a) f(x) = \frac{3}{5 + \cos x}$$

$$\frac{-1}{+5} \leq \cos x \leq \frac{1}{+5}$$

$$4 \leq \cos x + 5 \leq 6$$

نقلب المتباينة

$$\frac{1}{4} \geq \frac{1}{\cos x + 5} \geq \frac{1}{6}$$

$$\frac{3}{4} \geq \frac{3}{\cos x + 5} \geq \frac{3}{6}$$

$$\text{Range: } \left[\frac{3}{6}, \frac{3}{4} \right] = \left[\frac{1}{2}, \frac{3}{4} \right]$$

$$b) f(x) = 2 \sin^2 x + 3$$

$$(-1 \leq \sin x \leq 1)^2$$

$$2 \times [0 \leq \sin^2 x \leq 1]$$

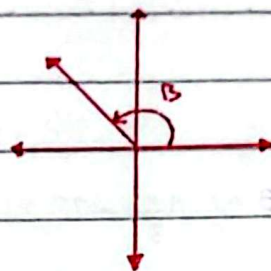
$$3 + (0 \leq 2 \sin^2 x \leq 2)$$

$$3 \leq 2 \sin^2 x + 3 \leq 5$$

$$\text{Range: } [3, 5]$$

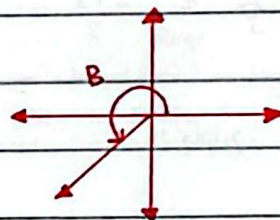
$$0 \leq \sin^2 x \leq 1$$

$$0 \leq \left| \frac{\sin x}{\cos x} \right| \leq 1$$



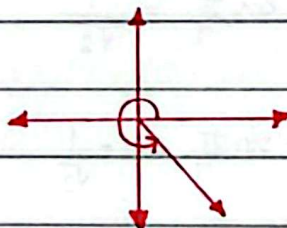
الزاوية
Q₂ → Q₁

$$\theta = \pi - B$$



Q₃ → Q₁

$$\theta = B - \pi$$



Q₄ → Q₃

$$\theta = 2\pi - B$$

Ex: Find $\sin B$ and $\cos B$ for:

$$(1) B = \frac{5\pi}{6}$$

$$\theta = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$$

$$\cos \frac{5\pi}{6} = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

$$\sin \frac{5\pi}{6} = \sin \frac{\pi}{6} = \frac{1}{2}$$

سؤال ٢١١

$$(2) B = \frac{4\pi}{3}$$

$$\theta = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

$$\cos \frac{4\pi}{3} = -\cos \frac{\pi}{3} = -\frac{1}{2}$$

$$\sin \frac{4\pi}{3} = -\sin \frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$(3) B = \frac{7\pi}{4} \quad \text{الربع الرابع}$$

$$\theta = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$$

$$\cos \frac{7\pi}{4} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin \frac{7\pi}{4} = -\sin \frac{\pi}{4} = -\frac{1}{\sqrt{2}}$$

سؤال ٢٢٢

(*) Identities 80

$$* \sin^2 x + \cos^2 x = 1$$

$$* \sin 2x = 2 \sin x \cos x$$

$$* \cos 2x = \cos^2 x - \sin^2 x \quad \begin{matrix} \cos 2x = 1 - 2\sin^2 x \\ \cos 2x = 2\cos^2 x - 1 \end{matrix}$$

$$* \sin^2 x = \frac{1}{2} (1 - \cos 2x)$$

$$* \cos^2 x = \frac{1}{2} (1 + \cos 2x)$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$

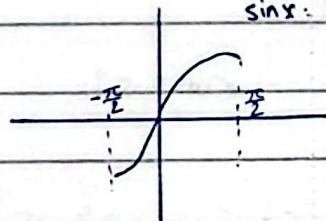
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$* 1 + \tan^2 x = \sec^2 x$$

$$* 1 + \cot^2 x = \csc^2 x$$

* Inverse trigonometric functions.

$$\sin x: \begin{matrix} \text{Dom} \\ [-\frac{\pi}{2}, \frac{\pi}{2}] \end{matrix} \rightarrow \begin{matrix} \text{Range} \\ [-1, 1] \end{matrix}$$

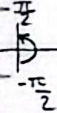


$$(1) \sin^{-1}: \begin{matrix} \text{Dom} \\ [-1, 1] \end{matrix} \rightarrow \begin{matrix} \text{Range} \\ [-\frac{\pi}{2}, \frac{\pi}{2}] \end{matrix}$$

$$\therefore \text{Dom } \sin^{-1} x = [-1, 1]$$

$$\text{Range of } \sin^{-1} x = [-\frac{\pi}{2}, \frac{\pi}{2}]$$

الربع الثاني والربع الثالث

Ex: find domain of $f(x) = \sin^{-1}(2x+1)$ composition function

$$2x+1 \in \text{Dom } \sin^{-1}$$

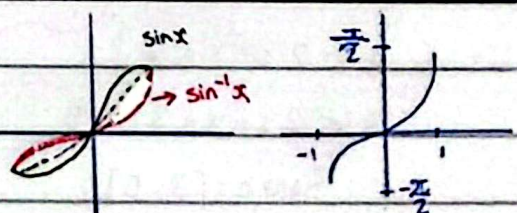
$$-1 \leq 2x+1 \leq 1$$

$$-2 \leq 2x \leq 0$$

$$-1 \leq x \leq 0$$

$$\therefore \text{Dom } [-1, 0] \cap \mathbb{R}$$

(2)



$$(3) f(f^{-1}(x)) = x, \text{ if } x \in \text{Dom } f^{-1}(x)$$

$$f^{-1}(f(x)) = x, \text{ if } x \in \text{Dom } f(x)$$

مثلاً $\sin(\sin^{-1}(x)) = x$ if $x \in [-1, 1]$

بمعنى آخر $\sin^{-1}(\sin(x)) = x$ if $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

④ $\sin^{-1}(x)$ is odd function since
 $\sin^{-1}(-x) = -\sin^{-1}x$

Ex: find the exact value of so

① $\sin^{-1}(\frac{1}{2}) = \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$
 $= \frac{\pi}{6}$

② $\sin^{-1}(-1) = -\frac{\pi}{2}$
 or $-\sin(1) = -\frac{\pi}{2}$

③ $\sin^{-1}(-\frac{\sqrt{3}}{2}) = -\sin^{-1}(\frac{\sqrt{3}}{2})$
 $= -\frac{\pi}{3}$

④ $\sin^{-1}(\sin(\frac{\pi}{6})) = \frac{\pi}{6}$
 $\in [-\frac{\pi}{2}, \frac{\pi}{2}]$

⑤ $\sin(\sin^{-1}(\frac{1}{2})) = \frac{1}{2}$
 $\in [-1, 1]$

⑥ $\sin^{-1}(\sin(\frac{2\pi}{3}))$ ② $\theta = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$
 $\sin \frac{2\pi}{3} = \sin \frac{\pi}{3}$
 $\sin^{-1}(\sin(\frac{\pi}{3})) = \frac{\pi}{3}$

⑦ $\sin^{-1}(\sin(\frac{5\pi}{4}))$

$\theta = \frac{5\pi}{4} - \pi = \frac{\pi}{4}$

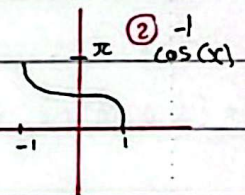
$\sin \frac{5\pi}{4} = -\sin \frac{\pi}{4}$

$\sin^{-1}(-\sin(\frac{\pi}{4})) = -\frac{\pi}{4}$

* $\cos^{-1}(x)$:

① $\cos x: [0, \pi] \rightarrow [-1, 1]$

$\cos^{-1}: [-1, 1] \rightarrow [0, \pi]$



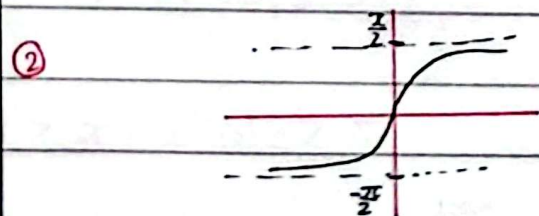
③ $\cos(\cos^{-1}(x)) = x$ if $x \in [-1, 1]$
 $\cos^{-1}(\cos x) = x$ if $x \in [0, \pi]$

④ $\cos^{-1}(x)$ is neither even nor odd
 $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

* $\tan^{-1}x$

① $\tan x: (-\frac{\pi}{2}, \frac{\pi}{2}) \rightarrow \mathbb{R}$

$\tan^{-1}x: \mathbb{R} \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})$



$$(3) \tan(\tan^{-1}x) = x, \text{ if } x \in \mathbb{R}$$

$$\tan^{-1}(\tan x) = x, \text{ if } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$(4) \tan^{-1}x \text{ is odd function}$$

$$\tan^{-1}(-x) = -\tan^{-1}x$$

Ex: find dom $\cos^{-1}(3x-1)$

$$-1 \leq 3x-1 \leq 1$$

$$0 \leq 3x \leq 2$$

$$0 \leq x \leq \frac{2}{3}$$

$$\therefore \text{Dom } [0, \frac{2}{3}]$$

Ex: Find the range of the function ~~Ex~~

$$f(x) = \pi + 2|\tan^{-1}x|$$

$$-\frac{\pi}{2} \leq \tan^{-1}x \leq \frac{\pi}{2}$$

$$0 \leq |\tan^{-1}x| \leq \frac{\pi}{2}$$

$$\left(0 \leq |\tan^{-1}x| \leq \frac{\pi}{2}\right) \times 2$$

$$0 \leq 2|\tan^{-1}x| \leq \pi$$

$$\pi \leq 2\tan^{-1}x + \pi \leq 2\pi$$

$$\text{Range: } (\pi, 2\pi)$$

↓
القيمة القصوى
لـ $\tan^{-1}$

Ex: find the exact value of

أو قيم الجيب العكسي
أو الجيب العكسي

$$(1) \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \in [0, \pi]$$

$$(2) \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$(3) \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$$(4) \sec^{-1}(-2) = \text{Note } \sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$$

$$= \cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

$$(5) \tan^{-1}(-1) = -\tan^{-1}(1) = -\frac{\pi}{4}$$

$$(6) \cos^{-1}\left(\cos\left(\frac{12\pi}{7}\right)\right)$$

$$\theta = 2\pi - \frac{12\pi}{7} = +\frac{2\pi}{7}$$

$$\cos^{-1}\left(\cos\left(\frac{12\pi}{7}\right)\right) = \frac{2\pi}{7}$$

tan و sin و cos

$$\textcircled{7} \tan^{-1}(\tan(\frac{\pi}{8})) = \frac{\pi}{8}$$

حل تنقيح إيجاد
 $\in (-\frac{\pi}{2}, \frac{\pi}{2})$
 مفتوح

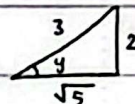
$$\textcircled{2} \cot(\sin^{-1}(\frac{2}{3}))$$

$$\textcircled{1} \sin(\sin^{-1}(\frac{2}{3})) = \sin y$$

$$\textcircled{8} \cos^{-1}(\cos(\frac{7\pi}{6})) = \cos^{-1}(\cos(-\frac{\pi}{6}))$$

الربع الثالث
 $\in [0, \pi]$
 $\theta = \frac{7\pi}{6} - \pi = -\frac{\pi}{6}$
 $= \pi - \cos^{-1}(\cos(\frac{\pi}{6}))$
 $= \frac{\pi - \pi}{6}$
 $= \frac{5\pi}{6}$

$$\frac{\text{المقابل}}{\text{الوتر}} = \frac{2}{3} = \sin y$$



$$\cot y = \frac{\text{المجاور}}{\text{المقابل}}$$

$$\tan y = \frac{\text{المقابل}}{\text{المجاور}}$$

$$\cot y = \frac{\sqrt{5}}{2}$$

* Identities for inverse trigonometric functions

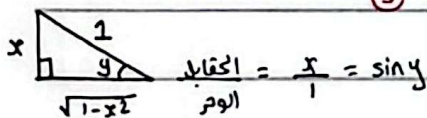
Ex: Find the exact value for 80

طريقة الحل [نظريات تنقيح لا Dam]

$$\textcircled{1} \cos(\sin^{-1} x) \quad \textcircled{1} \sin^{-1} x = y$$

$$\textcircled{2} \sin(\sin^{-1}(x)) = \sin y$$

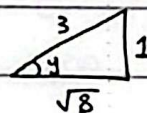
$$\textcircled{3} \sin y = x$$



$$\cos(y) = \frac{\text{المجاور}}{\text{الوتر}} = \frac{1}{\sqrt{1-x^2}}$$

$$\sin(\sin^{-1}(\frac{1}{3})) = \sin y$$

$$\frac{\text{المقابل}}{\text{الوتر}} = \frac{1}{3} = \sin y$$



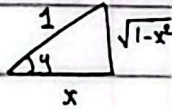
$$= -\tan(y) = -\frac{\text{المقابل}}{\text{المجاور}}$$

$$= -\frac{1}{\sqrt{8}}$$

④ $\sin(2\cos^{-1}x)$

$\cos(\cos^{-1}x) = y$

$\frac{\text{مقابل}}{\text{مجاور}} = \frac{x}{1} = \cos y$



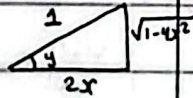
$$\begin{aligned} \Rightarrow \sin(2y) &= 2\sin y \cos y \\ &= 2\left(\frac{\sqrt{1-x^2}}{1}\right)\left(\frac{x}{1}\right) \end{aligned}$$

$$\boxed{\sin(2y) = 2x\sqrt{1-x^2}}$$

⑤ $\sin(\pi + \cos^{-1}(2x))$

$\cos y = \cos^{-1}(2x)$

$\cos y = \frac{2x}{1} = \frac{\text{مجاور}}{\text{مقابل}}$



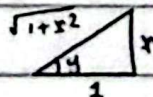
$$\begin{aligned} \Rightarrow \sin(\pi + y) &= \sin \pi \cos y + \sin y \cos \pi \\ &= -\sin y \\ &= -\sqrt{1-4x^2} \end{aligned}$$

⑥ $\sec(\tan^{-1}x)$

$\tan(\tan^{-1}x) = y$

$\frac{\text{مقابل}}{\text{مجاور}} = \frac{x}{1} = \tan y$

$\cos = \frac{\text{مجاور}}{\text{مقابل}}$

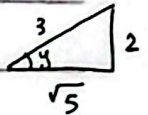


$$\begin{aligned} \Rightarrow \sec(y) &= \frac{\text{مقابل}}{\text{مجاور}} \\ &= \sqrt{1+x^2} \end{aligned}$$

⑦ $\sin(2\sin^{-1}(\frac{2}{3}))$

$\sin(\sin^{-1}(\frac{2}{3})) = y$

$\frac{\text{مقابل}}{\text{مجاور}} = \frac{2}{3} = \sin y$



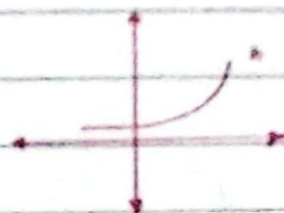
$$\begin{aligned} \Rightarrow \sin(2y) &= 2\sin y \cos y \\ &= 2\left(\frac{2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) \end{aligned}$$

$$\sin(2y) = \frac{4\sqrt{5}}{9}$$

* Exponential function

Defn: $f(x) = b^x$, $b > 0$, $b \neq 1$ is called exponential function

Range: $(0, \infty)$

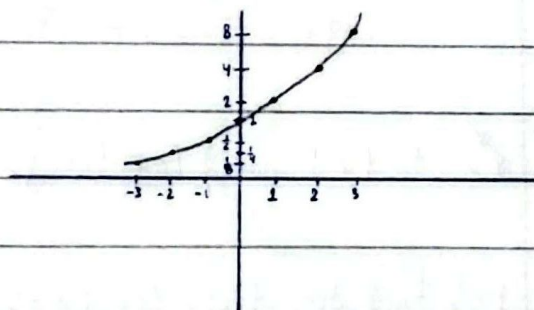


$$f(x) = b^x, b > 1$$

Dom: $\mathbb{R}$, Range: $(0, \infty)$
لا يتجاوز

Ex) sketch $f(x) = 2^x$

x	-3	-2	-1	0	1	2	3
f(x)	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

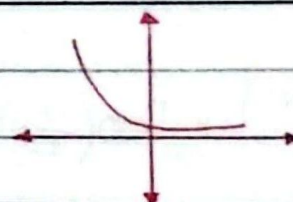


$$f(x) = b^x$$

$0 < b < 1$

Dom: $\mathbb{R}$

Range: $(0, \infty)$

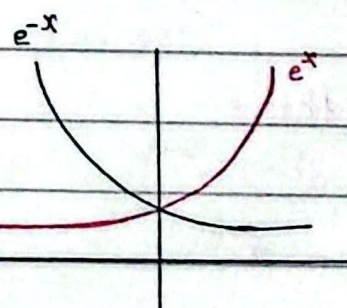
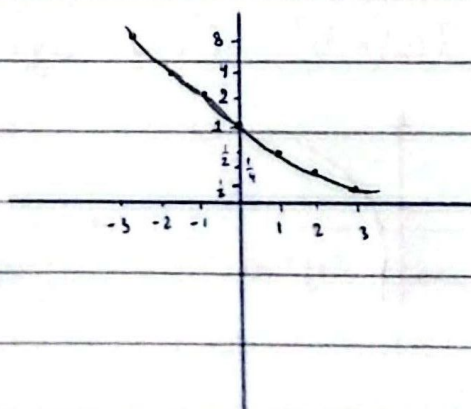


Defn: The natural exponential function ^{الطبيعي}

$$f(x) = e^x \text{ where } e = 2.718 \dots$$

Ex: sketch $f(x) = \left(\frac{1}{2}\right)^x$

x	-3	-2	-1	0	1	2	3
f(x)	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$



نلاحظ:

* note that: $\text{Dom} \{b^{f(x)}\} = \text{Dom} \{f(x)\}$

$$\text{Dom} \{e^{f(x)}\} = \text{Dom} \{f(x)\}$$

Ex: find doms

① $f(x) = 3^{x+1}$

$\text{Dom}(3^{x+1}) = \text{Dom}(x+1) = \mathbb{R}$

② $f(x) = 2^{\frac{1}{x-1}}$

$\text{Dom}(2^{\frac{1}{x-1}}) = \text{Dom}\left(\frac{1}{x-1}\right) = \mathbb{R} - \{1\}$

③ $f(x) = e^{\sqrt{x-1}}$

$\text{Dom}(e^{\sqrt{x-1}}) = \text{Dom} \sqrt{x-1} = x-1 \geq 0$

$x \geq 1$

$\text{Dom} = [1, \infty)$

* note that :

1. $a^x \cdot a^y = a^{x+y}$

2. $\frac{a^x}{a^y} = a^{x-y}$

3. $(a^x)^y = a^{x \cdot y}$

4. $a^{-x} = \frac{1}{a^x}$

5. $(a \cdot b)^x = a^x \cdot b^x$

6. $a^{x/y} = \sqrt[y]{a^x}$

7. $a^0 = 1$

* note that:

$f(x) = b^x \Rightarrow f^{-1}(x) = \log_b x$

$\therefore$ Logarithmic function: $f(x) = \log_b x$

$b > 0, b \neq 1$

Now ① $b^x : \mathbb{R} \rightarrow (0, \infty)$ f

$\log_b x : (0, \infty) \rightarrow \mathbb{R}$ f^{-1}

** to find dom $\{\log_b f(x)\}$: set $f(x) > 0$

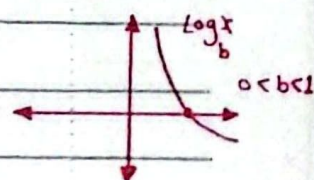
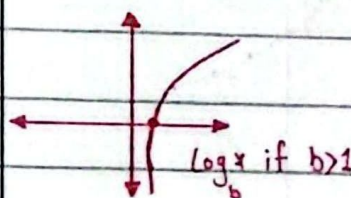
Ex) Find dom $f(x) = \log_3(x+5)$

$x+5 > 0$

$x > -5$

$\text{Dom} = (-5, \infty)$

② $f(x) = b^x, f^{-1}(x) = \log_b x$



$$\textcircled{3} f \circ f^{-1}(x) = x \quad f = b^x, f^{-1} = \log_b$$

$$b^{\log_b x} = x$$

$$f^{-1} \circ f(x) = x$$

$$\log_b b^x = x$$

$$\textcircled{6} \log_b b = 1$$

$$\textcircled{7} \ln e = \log_e e = 1$$

$$\textcircled{8} \log_b 1 = 0$$

$$\textcircled{9} \ln 1 = 0$$

* note that :

$$\textcircled{1} \log_{10} x = \log x$$

$$\textcircled{2} \log_e x = \ln x \quad \text{"natural log"}$$

* note that:

$$f(f^{-1}(x)) = x$$

$$\text{now } f(x) = e^x, f^{-1}(x) = \ln x$$

$$\frac{\ln x}{e^x} = x \quad / \quad \ln e^x = x$$

* Algebraic properties of Logarithms [خصائص اللوغاريتم]

if $b > 0$ & $b \neq 1$, $x, y > 0$ and $r \in \mathbb{R}$. Then

$$\textcircled{1} \log_b(xy) = \log_b x + \log_b y$$

$$\log(x+y) \neq \frac{\log(x)}{\log(y)}$$

$$\textcircled{2} \log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$$

$$\textcircled{3} \log_b x^r = r \log_b x$$

$$\textcircled{4} \log_b \frac{1}{y} = \log_b y^{-1} = -\log_b y$$

$$\textcircled{5} \log_y x = \frac{\log_b x}{\log_b y}, \quad b \neq 1$$

$$= \frac{\ln x}{\ln y} = \frac{\log x}{\log y}$$

find $x = ??$

$$\text{Ex)) } \ln x = 5$$

$$e^{\ln x} = e^5$$

$$x = e^5$$

$$\text{Ex 8 } \ln e^x = 5$$

$$\ln e^x = \ln 5$$

$$x = \ln 5$$

Ex 8 if $f(x) = 2^{3x-1}$ find

① $f(0) = 2^{-1} = \frac{1}{2}$

② $f(2) = 2^5 = 32$

③ $f(\frac{1}{3}) = 2^0 = 1$

Ex 8 if $g(x) = e^{2x}$ find

① $g(\frac{1}{2}) = e^1 = e = 2.718...$

② $g(0) = e^0 = 1$

③ $g(-1) = e^{-2} = \frac{1}{e^2}$

Ex 8 Find the value of :

① $\log_2 16 = \log_2 2^4 = 4 \log_2 2 = 4$

OR : $\log_2 16 = x \Rightarrow 2^x = 16 \Rightarrow 2^x = 2^4$
 $x=4$

② $\log_4 4 = 1$

③ $\log_{\frac{1}{1000}} 1 = \log_{10} 10^{-3} = -3 \log_{10} 10 = -3$

④ $\log_5 1 = 0 \Rightarrow \log_5 1 = x \Rightarrow 5^x = 1 \Rightarrow \boxed{x=0}$

⑤ $\log_9 3 = x \Rightarrow 9^x = 3 \Rightarrow x = \frac{1}{2}$

$(3^2)^x = 3 \Rightarrow 3^{2x} = 3$
 $3^{2x} = 3^1 \Rightarrow 2x = 1$
 $x = \frac{1}{2}$

⑥ $\ln e^3 = 3$

⑦ $e^{\ln 5} = 5$

⑧ $\ln \sqrt[3]{e} = \frac{1}{3} \ln e = \frac{1}{3}$

⑨ $\ln \frac{1}{e} = \ln e^{-1} = -\ln e = -1$

⑩ $e^{-2 \ln 3} = e^{\ln 3^{-2}} = \frac{1}{9}$

⑪ $\log_2 6 - \log_2 15 + \log_2 20$

$\log_2 \left(\frac{6}{15} \right) + \log_2 20$

$\log_2 \left(\frac{6}{15} \times 20 \right) = \log_2 8 = \log_2 2^3$
 $= 3 \log_2 2 = 3$

Ex 2 find the domain of

① $f(x) = e^{\sqrt{x^2-9}}$ = Dom $\sqrt{x^2-9}$

$x^2-9 \geq 0 \Rightarrow \sqrt{x^2} \geq \sqrt{9} \Rightarrow |x| \geq 3$

$x \geq 3$ or $x \leq -3$

Dom = $[3, \infty) \cup (-\infty, -3]$

② $f(x) = 3^{\frac{7-x}{2-x}}$ Dom $\left(\frac{7-x}{2-x}\right) \in \mathbb{R}$

$\therefore \mathbb{R} - \{2\}$

③ $f(x) = \log(x^2-25)$

$x^2-25 > 0$

$\sqrt{x^2} > \sqrt{25}$

$|x| > 5 \quad x > 5 \text{ or } x < -5$

$\therefore (5, \infty) \cup (-\infty, -5)$

④ $f(x) = \ln(7-x)$

$7-x > 0$

$x < 7 \quad \therefore (-\infty, 7)$

⑤ $f(x) = \frac{1}{\ln(4-x)}$ $\leftarrow \mathbb{R}$

$4-x > 0$

$x < 4 \quad (-\infty, 4)$

$\ln(f(x)) = 0$
 $f(x) = 1$

$\mathbb{R} \cap (-\infty, 4) = \{ \ln(4-x) = 0 \}$
 $4-x = 1$

$x = 3$

$\therefore \mathbb{R} \cap (-\infty, 4) = \{3\} \quad \therefore (-\infty, 4) - \{3\}$

⑥ $f(x) = \ln(x^2+4)$

$x^2+4 > 0 \quad \therefore \text{Dom } \mathbb{R}$

⑦ $f(x) = \ln|x|$

$|x| > 0 \quad |x| = 0 \rightarrow x = 0$

$\therefore \mathbb{R} - \{0\}$

⑧ $f(x) = \ln(x^2-4)$

$x^2-4 > 0$

$\sqrt{x^2} > \sqrt{4}$

$|x| > 2$

$x > 2 \text{ or } x < -2$

$\therefore (2, \infty) \cup (-\infty, -2)$

⑨ $f(x) = \ln(\ln x)$

$x > 0$

$\therefore (0, \infty) \cap (1, \infty)$

$\ln x > 0 \Rightarrow e^{\ln x} > 1$

$x > 1 \quad (1, \infty)$

Dom $(1, \infty)$

⑩ $f(x) = \frac{1}{e^x - e^{2x}} \leftarrow \mathbb{R}$

$\leftarrow \mathbb{R} - \{0\}$

$e^x - e^{2x} = 0$

$e^x(1 - e^x) = 0$

$e^x \neq 0 \quad | \quad 1 - e^x = 0 \Rightarrow \ln e^x = \ln 1$

$\ln 1 = x$

$x = 0$

$\therefore \mathbb{R} - \{0\}$

⑪ $f(x) = \log_3 x = \frac{\log x}{\log 3} \leftarrow \mathbb{R}$

$x > 0$

$\log x \leftarrow x > 0$

$(0, \infty) - \log x = 0$
 $x = 1$

$\therefore (0, \infty) - \{1\}$

Ex: Find the range of

① $f(x) = e^{2x+10}$

① $\rightarrow$ $\ln$ $\rightarrow$ $f^{-1}(x)$

$\ln y = \ln e^{2x+10} \Rightarrow \ln y = 2x+10$

$2x = \ln y - 10$

$x = \frac{\ln y - 10}{2}$

$f^{-1}(x) = \frac{\ln x - 10}{2}$

$\ln x > 0$
 $x > 0$

Dom $f^{-1} = \text{Range } f(x) = (0, \infty)$

② $f(x) = 2^x + 5$

$y = 2^x + 5$

$\log_2(y-5) = \log_2(2^x)$

$\log_2(y-5) = \log_2 2^x$

$\log_2(y-5) = x$

$f^{-1}(x) = \log_2(x-5)$

Dom $f^{-1}(x) = \text{Range } f(x) = x-5 > 0$

$\therefore (5, \infty)$

$x > 5$

Ex: Solve for x

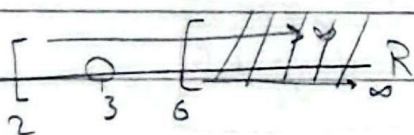
$$\textcircled{1} (x^2-9)(e^{-x}-2) \log(x-6) = 0$$

$\downarrow$ $\downarrow$ $\downarrow$
 $\mathbb{R}$ $\mathbb{R}$ $(x-2)$

$\textcircled{1}$ اول خطوة نطلع Dom

$\textcircled{2}$ نخرج x ونستثني القيم الخارج المجال
 $\log(x-6) \leftarrow x-6 > 0 \quad x > 6 \quad \therefore (6, \infty)$
 $\log(x-2) \leftarrow x-2 > 0 \quad x > 2 \quad \therefore (2, \infty)$
 $x-2=1$
 $\boxed{x=3}$

$$(6, \infty) \cap (2, \infty) = \{3\}$$



$$\text{Dom} \therefore (6, \infty)$$

$$\rightarrow x^2-9=0 \Rightarrow x^2=9 \Rightarrow \boxed{x=\pm 3} \quad \text{X} \quad \begin{matrix} \text{لا} \\ \text{يكون} \\ \text{في} \\ \text{المجال} \end{matrix}$$

$$\rightarrow e^{-x}-2=0$$

$$e^{-x}=2 \Rightarrow \ln e^{-x} = \ln 2 \Rightarrow -x = \ln 2$$

$$\boxed{x = -\ln 2} \quad \text{X} \quad \notin (6, \infty)$$

$$\rightarrow \frac{\log(x-6)}{\log(x-2)} = 0 \Rightarrow$$

$\frac{a}{b} = 0 \text{ if } a=0$

$$\log(x-6)=0$$

$$x-6=1 \Rightarrow \boxed{x=7} \checkmark \in (6, \infty)$$

$$\boxed{x=7}$$

$$\textcircled{2} \frac{1}{3-e^{2x}} = 4$$

$$1 = 12 - 4e^{2x} \quad \therefore \text{Dom } \mathbb{R}$$

$$4e^{2x} = 11$$

$$\ln e^{2x} = \ln \frac{11}{4} \Rightarrow 2x = \ln \frac{11}{4}$$

$$\frac{2x}{2} = \frac{\ln 11 - \ln 4}{2}$$

$$x = \frac{1}{2} \ln(11/4)$$

$$x = \ln \sqrt{\ln(11/4)}$$

$$\textcircled{3} 3e^{-2x} = 5 \quad \therefore \text{Dom} = \mathbb{R}$$

$$\ln e^{-2x} = \ln \frac{5}{3}$$

$$-2x = \ln(5/3)$$

$$x = \frac{\ln(5/3)}{-2}$$

$$x = -\frac{1}{2} \ln(5/3)$$

$$= \frac{1}{2} \ln(5/3)^{-1} = \frac{1}{2} \ln\left(\frac{3}{5}\right) \quad \text{or}$$

$$= \ln \sqrt{3/5}$$

Monday 27/10/2023

④ $x^2 \ln x - 16 \ln x = 0$

Dom $\therefore (0, \infty)$

$\ln x (x^2 - 16) = 0$

$\ln x = 0$

$x = 1$

$x \in (0, \infty)$

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

$x = -4 \notin (0, \infty) \quad x = 4 \in (0, \infty)$

الجواب $x = 1 \quad x = 4 \quad x \in \{1, 4\}$

⑤ $\log x^{\frac{3}{2}} - \log \sqrt{x} = 5$

Dom $\therefore (0, \infty)$

$\log_{10} \frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}} = 5$

$\log_{10} x^{\frac{3}{2} - \frac{1}{2}} = 5$

$\log_{10} x = 5 \Rightarrow x = 10^5 \in (0, \infty)$

⑥ $e^{-2x} - \frac{1}{9} = 0 \quad \therefore \text{Dom } \mathbb{R}$

$e^{-2x} = \frac{1}{9}$

$\ln e^{-2x} = \ln \left(\frac{1}{9} \right)$

$-2x = \ln \left(\frac{1}{9} \right)$

$x = -\frac{1}{2} \ln \left(\frac{1}{9} \right)$

$x = \ln 3$

⑦ $2^{x-5} = 3 \quad \therefore \text{Dom } \mathbb{R}$

$\log_2 2^{x-5} = \log_2 3$

$(x-5) \log 2 = \log 3$

$(x-5) = \frac{\log 3}{\log 2}$

$x = \frac{\log 3}{\log 2} + 5 \in \mathbb{R}$

$x = \frac{\log 3}{2} + 5$

8) $e^{2x} - e^x = 6 \quad \therefore \text{Dom } \mathbb{R}$

$e^{2x} - e^x - 6 = 0$

$(e^x + 2)(e^x - 3) = 0$

$e^x - 3 = 0$

$e^x = 3$

$x = \ln 3$

$x = \ln 3$

$e^x + 2 = 0$

$e^x = -2$

NOTEBOOK

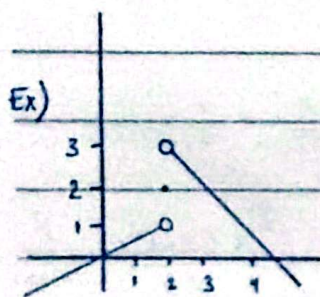
* limits :

* Defn: $\lim_{x \rightarrow a} f(x) = l$ - if $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = l$ then $\lim_{x \rightarrow a} f(x) = l$ "exist"- if $\lim_{x \rightarrow a^+} f(x) \neq \lim_{x \rightarrow a^-} f(x)$ then $\lim_{x \rightarrow a} f(x)$ does not "exist" [d.n.e]

$$\textcircled{3} \lim_{x \rightarrow a} \left(\frac{f}{g} \right)(x) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{M}{L}, L \neq 0$$

$$\textcircled{4} \lim_{x \rightarrow a} (f(x))^n = \left(\lim_{x \rightarrow a} f(x) \right)^n = (L)^n$$

$$\textcircled{5} \lim_{x \rightarrow a} c = c \quad "c \text{ is constant}"$$

is $\lim_{x \rightarrow 2} f(x)$ exist?

$$\lim_{x \rightarrow 2^-} f(x) = 1$$

$$\lim_{x \rightarrow 2^+} f(x) = 3$$

$$\lim_{x \rightarrow 2^-} f(x) \neq \lim_{x \rightarrow 2^+} f(x)$$

 $\therefore \lim_{x \rightarrow 2} f(x)$ does not exist.
Ex) if $\lim_{x \rightarrow -1} f(x) = 3$, $\lim_{x \rightarrow -1} g(x) = 2$ find

$$\textcircled{1} \lim_{x \rightarrow -1} (f(x) + g(x)) = \lim_{x \rightarrow -1} f(x) + \lim_{x \rightarrow -1} g(x) = 3 + 2 = 5$$

$$\textcircled{2} \lim_{x \rightarrow -1} \frac{f(x)}{g(x)} = \frac{3}{2} = 1.5$$

Ex) find $\lim_{x \rightarrow 1} (2x^5 - 4x + 5) = 2(1)^5 - 4(1) + 5 = 3$

* Thm: if $\lim_{x \rightarrow a} f(x) = M$, $\lim_{x \rightarrow a} g(x) = L$ then

* note that :

$$\textcircled{1} \lim_{x \rightarrow a} (f+g)(x) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x) = M + L$$

$$\textcircled{2} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{M}{L}$$

$$\textcircled{3} \lim_{x \rightarrow a} (f \cdot g)(x) = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = M \cdot L$$

$$\text{Ex) } \lim_{x \rightarrow 4} \frac{3x+5}{\sqrt{x}+3} = \frac{17}{5}$$

$$\text{Ex) } \lim_{x \rightarrow 2} \frac{\frac{5}{3} - \frac{5}{x+1}}{x-2} = \frac{\frac{4}{3} - \frac{5}{3}}{0} = \frac{0}{0} = 0$$

$$\boxed{2} \lim_{g(x)} \frac{f(x)}{g(x)} = \frac{0}{\text{number}} = 0$$

$$= \lim_{x \rightarrow 2} \frac{\frac{5x+5}{3x+3} - \frac{15}{3x+3}}{x-2} = \lim_{x \rightarrow 2} \frac{\frac{5x-10}{3x+3}}{x-2}$$

$$\text{Ex) } \lim_{x \rightarrow \frac{1}{2}} \frac{2x-1}{2x} = \frac{1-1}{1} = \frac{0}{1} = 0$$

$$= \lim_{x \rightarrow 2} \frac{5x-10}{(3x+3)(x-2)} = \lim_{x \rightarrow 2} \frac{5(x-2)}{(3x+3)(x-2)}$$

$$\boxed{3} \lim_{g(x)} \frac{f(x)}{g(x)} = \frac{0}{0} \left\} \begin{array}{l} \text{indeterminate form} \\ \text{L'Hôpital's rule} \end{array} \right.$$

$$= \lim_{x \rightarrow 2} \frac{5}{(3x+3)} = \frac{5}{9}$$

$$\text{Ex) } \lim_{x \rightarrow 3} \frac{x^2-9}{x-3} = \frac{3^2-9}{3-3} = \frac{0}{0} = 0$$

$$\text{Ex) } \lim_{x \rightarrow 0} \frac{(x-2)^2-4}{x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x \rightarrow 3} (x+3) = 3+3 = 6$$

$$\lim_{x \rightarrow 0} \frac{x^2-4x+4}{x} = \frac{0}{0} = \lim_{x \rightarrow 0} \frac{(x-4)}{1} = \lim_{x \rightarrow 0} (x-4)$$

$$= -4$$

$$\text{Ex) } \lim_{x \rightarrow 4} \frac{2x+8}{x^2+x-12} = \frac{-8+8}{16-4-12} = \frac{0}{0} = 0$$

$$\text{Ex) } \lim_{x \rightarrow 4} \frac{|x-4|}{x-4} = \frac{0}{0} = \lim_{x \rightarrow 4} \frac{(x-4)}{(x-4)} = \lim_{x \rightarrow 4} 1 = 1$$

$$\lim_{x \rightarrow 4} \frac{2(x+4)}{(x-3)(x+4)} = \lim_{x \rightarrow 4} \frac{2}{(x-3)} = \frac{2}{1} = 2$$

$$\lim_{x \rightarrow 4} \begin{cases} \frac{x-4}{x-4} = 1, & x > 4 \\ \frac{-(x-4)}{x-4} = -1, & x < 4 \end{cases}$$

$$\text{Ex) } \lim_{x \rightarrow 1} \frac{x^{\frac{3}{2}} - x}{x^{\frac{1}{2}} - 1} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 4^+} f(x) = 1$$

$$\lim_{x \rightarrow 1} \frac{x(x^{\frac{1}{2}} - 1)}{(x^{\frac{1}{2}} - 1)} = 1$$

$$\lim_{x \rightarrow 4^-} f(x) = -1$$

$$\therefore \lim_{x \rightarrow 4^-} f(x) \neq \lim_{x \rightarrow 4^+} f(x)$$

$$\therefore \lim_{x \rightarrow 4} f(x) = \text{d.n.e}$$

$$\text{Ex) } \lim_{x \rightarrow 3^0} \frac{|x-3|}{2x-6} = \frac{0}{0}$$

$$= \lim_{x \rightarrow 3^+} \frac{x-3}{2x-6} = \frac{x-3}{2(x-3)} = \frac{1}{2}$$

$$\text{Ex) } \lim_{x \rightarrow 1} \frac{\sqrt[4]{x+15} - 2}{x-1} = \frac{0}{0}$$

$$\sqrt[4]{x+15} = y \Rightarrow x+15 = y^4 \quad \text{اكتشف 2 اذ 3 يفصل فنضرب}$$

$$\therefore \boxed{x = y^4 - 15}$$

$$\text{If } x \rightarrow 1 \Rightarrow y^4 \sqrt[4]{1+15} = \sqrt[4]{16} = 2$$

$$\therefore \lim_{y \rightarrow 2} \frac{y-2}{y^4-15-1} = \lim_{y \rightarrow 2} \frac{y-2}{y^4-16} = \lim_{y \rightarrow 2} \frac{(y-2)}{(y^2+4)(y^2-4)}$$

$$\lim_{y \rightarrow 2} \frac{(y-2)}{(y^2+4)(y-2)(y+2)} = \frac{1}{32}$$

*note that:

$$\textcircled{1} a^2 - b^2 = (a-b)(a+b)$$

$$\textcircled{2} a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

$$\textcircled{3} a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$\text{Ex) } \lim_{x \rightarrow 1} \frac{2 - \sqrt{x+3}}{x^2 + 2x - 3} = \frac{0}{0} = 0$$

$$\lim_{x \rightarrow 1} \frac{2 - \sqrt{x+3}}{x^2 + 2x - 3} \cdot \frac{(2 + \sqrt{x+3})}{(2 + \sqrt{x+3})}$$

$$\lim_{x \rightarrow 1} \frac{4 - (x+3)}{x^2 + 2x - 3} \quad (4) \leftarrow \begin{matrix} \text{نضرب} \\ \text{بـ} \\ \text{1} \end{matrix}$$

$$= \frac{1}{4} \lim_{x \rightarrow 1} \frac{4 - x - 3}{(x+3)(x-1)}$$

$$= \frac{1}{4} \lim_{x \rightarrow 1} \frac{(1-x)^{-1}}{(x+3)(x-1)} = \frac{1}{4} \lim_{x \rightarrow 1} \frac{-1}{x+3}$$

$$= \frac{1}{4} \cdot \frac{-1}{4} = \frac{-1}{16}$$

$$\text{Ex) } \lim_{x \rightarrow 3} \frac{3 - \sqrt[3]{2x+21}}{x^2 - 9} = \frac{0}{0}$$

$$\lim_{x \rightarrow 3} \frac{3 - \sqrt[3]{2x+21}}{x^2 - 9} \cdot \left(9 + 3\sqrt[3]{2x+21} + \sqrt[3]{2x+21} \right)$$

$$\lim_{x \rightarrow 3} \frac{9 - (2x+21)}{x^2 - 9} \quad (6)$$

$$\frac{1}{6} \lim_{x \rightarrow 3} \frac{2(x-3)}{(x+3)(x-3)}$$

الصفحة
البر

$$\text{Ex) } \lim_{x \rightarrow 3} \frac{3 - \sqrt[3]{2x+21}}{x^2-9} = \frac{0}{0}$$

$$\lim_{x \rightarrow 3} \frac{3 - \sqrt[3]{2x+21}}{x^2-9} \cdot \frac{(9 + 3\sqrt[3]{2x+21} + (\sqrt[3]{2x+21})^2)}{(9 + 3\sqrt[3]{2x+21} + (\sqrt[3]{2x+21})^2)}$$

$$= \lim_{x \rightarrow 3} \frac{(27 - (2x+21))}{(x^2-9)(27)}$$

$$= \frac{1}{27} \lim_{x \rightarrow 3} \frac{(6-2x)}{(x-3)(x+3)} = \frac{1}{27} \lim_{x \rightarrow 3} \frac{-2}{(x+3)}$$

$$= \frac{1}{27} \cdot \frac{-2}{6} = -\frac{1}{81}$$

$$\text{Ex) } \lim_{x \rightarrow 6} \sqrt{x-2} = \sqrt{6-2} = \sqrt{4} = 2 \checkmark$$

$$\text{Ex) } \lim_{x \rightarrow 1} \sqrt{x-3} = \sqrt{1-3} = \sqrt{-2} = \text{d.n.e}$$

$$\text{Ex) } \lim_{x \rightarrow 2} \sqrt{2-x} \neq 0$$

$$\lim_{x \rightarrow 2^-} \sqrt{2-x} = 0$$

$$\lim_{x \rightarrow 2^+} \sqrt{2-x} = \text{d.n.e}$$

$$\therefore \lim_{x \rightarrow 2} \neq \lim_{x \rightarrow 2^-} = \lim_{x \rightarrow 2^+} \text{d.n.e}$$

$$\text{Ex) } f(x) = \begin{cases} \sqrt{x-4} & , x > 4 \\ 8-2x & , x \leq 4 \end{cases}$$

$$\textcircled{1} \text{ find } \lim_{x \rightarrow 4} f(x) = \lim_{x \rightarrow 4} 8-2x = 8$$

$$\textcircled{2} \text{ find } \lim_{x \rightarrow 4} f(x)$$

$$\lim_{x \rightarrow 4^+} \sqrt{x-4} = 0$$

$$\lim_{x \rightarrow 4^-} 8-2x = 0$$

$$\therefore \lim_{x \rightarrow 4} f(x) = 0 \Rightarrow \lim_{x \rightarrow 4^+} f(x) = \lim_{x \rightarrow 4^-} f(x)$$

$$\text{Ex) } f(x) = \begin{cases} kx^2 & , x \leq 2 \\ 2x+k & , x > 2 \end{cases}$$

$$\text{find } k \text{ such that } \lim_{x \rightarrow 2} f(x) \text{ exist}$$

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$$

$$\lim_{x \rightarrow 2^-} 2x+k = \lim_{x \rightarrow 2^+} kx^2$$

$$4+k = 4k$$

$$4 = 4k - k \Rightarrow 4 = 3k$$

$$k = \frac{4}{3}$$

الربع
في المنطقة
d.n.e :
منه : لا يوجد

$$\boxed{4} \lim_{x \rightarrow a} \frac{f(x)}{g(x)}, f(a) \neq 0, g(a) = 0$$

$$\text{Ex) } \lim_{x \rightarrow 0} \frac{x^4 + 5}{x^2} = \frac{5}{0}$$

دائماً لا يكون تقويمي عدد
 ∞
 $-\infty$
 d.n.e
 لا جواب

$$\lim_{x \rightarrow 0^+} \frac{x^4 + 5}{x^2} = +\infty$$

$$\lim_{x \rightarrow 0^-} \frac{x^4 + 5}{x^2} = +\infty$$

$$\text{Ex) } \lim_{x \rightarrow 2^+} \frac{x+1}{x^2-4} = \frac{3}{0}$$

① تأخذ اصغر البسط

واصغر المقام

وندرس الإشارة

نوجد قيم x من 2 في الاقتران
 حتى نطلع الإشارة

-2	-1	2
من المقام	من المقام	من المقام

$$= +\infty$$

$$\therefore \lim_{x \rightarrow 0} \frac{x^4 + 5}{x^2} = +\infty$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$$

$$\text{Ex) } \lim_{x \rightarrow 1} \frac{x}{x^2-1} = \frac{1}{0}$$

لا يوجد اصغر البسط

$$\text{Ex) } \lim_{x \rightarrow 0^-} \frac{3}{x^2-x} = \frac{3}{0}$$

$$x^2-x=0 \quad \text{اصغر المقام}$$

$$x(x-1)=0$$

$$\boxed{x=0} \quad \boxed{x=1}$$

$$= +\infty$$

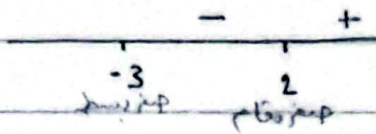
$$\lim_{x \rightarrow 1^+} \frac{x}{x^2-1} = +\infty$$

$$\lim_{x \rightarrow 1^-} \frac{x}{x^2-1} = -\infty$$

$$\lim_{x \rightarrow 1^+} \frac{x}{x^2-1} \neq \lim_{x \rightarrow 1^-} \frac{x}{x^2-1}$$

$$\therefore \lim_{x \rightarrow 1} \frac{x}{x^2-1} = \text{d.n.e}$$

$$\text{Ex) } \lim_{x \rightarrow 2} \frac{x+3}{x-2} = \frac{5}{0}$$



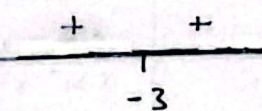
$$\lim_{x \rightarrow 2^+} \frac{x+3}{x-2} = +\infty$$

$$\lim_{x \rightarrow 2^-} \frac{x+3}{x-2} = -\infty$$

$$\lim_{x \rightarrow 2^+} \frac{x+3}{x-2} \neq \lim_{x \rightarrow 2^-} \frac{x+3}{x-2}$$

$$\lim_{x \rightarrow 2} \frac{x+3}{x-2} = \text{d.n.e.}$$

$$\text{Ex) } \lim_{x \rightarrow -3} \frac{1}{(x+3)^2} = \frac{1}{0}$$

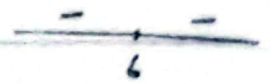


$$\lim_{x \rightarrow -3^+} \frac{1}{(x+3)^2} = +\infty$$

$$\lim_{x \rightarrow -3^-} \frac{1}{(x+3)^2} = +\infty$$

$$\therefore \lim_{x \rightarrow -3} \frac{1}{(x+3)^2} = +\infty$$

$$\text{Ex) } \lim_{x \rightarrow 6} \frac{-1}{(x-6)^2} = \frac{-1}{0}$$

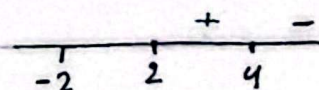


$$\lim_{x \rightarrow 6^+} \frac{-1}{(x-6)^2}$$

$$\lim_{x \rightarrow 6^-} \frac{-1}{(x-6)^2}$$

$$\therefore \lim_{x \rightarrow 6} \frac{-1}{(x-6)^2} = -\infty$$

$$\text{Ex) } \lim_{x \rightarrow 4} \frac{2-x}{x^2-2x-8} = \frac{-2}{0}$$



$$\lim_{x \rightarrow 4^+} \frac{2-x}{x^2-2x-8} = -\infty$$

$$\lim_{x \rightarrow 4^-} \frac{2-x}{x^2-2x-8} = +\infty$$

$$\therefore \lim_{x \rightarrow 4} \frac{2-x}{x^2-2x-8} = \text{d.n.e.}$$

* limits of infinity *

* $\lim_{x \rightarrow \infty} f(x)$ means that the behaviour

fast when x becomes very large ^{number} +ve

big

1

$$\frac{\infty}{\infty} = \infty$$

$$\frac{\infty}{\infty} = -\infty$$

$$\frac{1}{x} \quad 0 = \lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} = 0$$

2

$$\lim_{x \rightarrow \infty} e^x = \infty$$

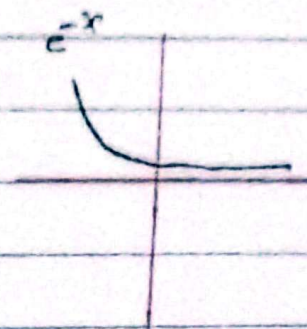
$$\lim_{x \rightarrow -\infty} e^x = 0$$

* if $a > 1$, $\lim_{x \rightarrow \infty} a^x = \infty$, $\lim_{x \rightarrow -\infty} a^x = 0$

$$\text{Ex) } \lim_{x \rightarrow \infty} 2^x = \infty$$

$$\lim_{x \rightarrow -\infty} 2^x = 0$$

$$\frac{1}{2^x}$$



$$\lim_{x \rightarrow \infty} e^{-x} = 0$$

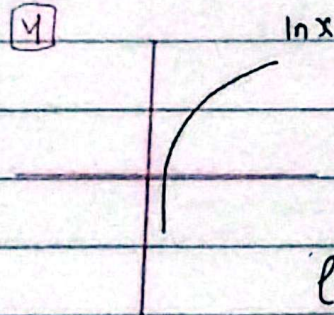
$$\lim_{x \rightarrow -\infty} e^{-x} = \infty$$

* if $0 < a < 1$, $\lim_{x \rightarrow \infty} a^x = 0$, $\lim_{x \rightarrow -\infty} a^x = \infty$

$$\lim_{x \rightarrow -\infty} a^x = \infty$$

$$\text{Ex) } \lim_{x \rightarrow \infty} 2^{-x} = 0, \quad \lim_{x \rightarrow -\infty} 2^{-x} = \infty$$

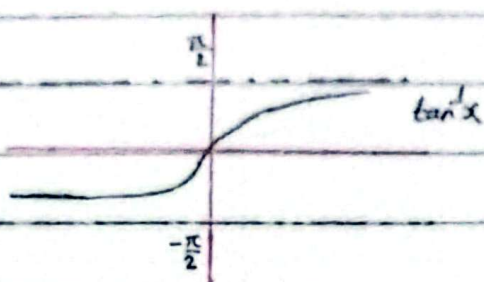
4



$$\lim_{x \rightarrow \infty} \ln x = \infty$$

$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

5



$$\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$$

$$\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$$\text{Ex) } \lim_{x \rightarrow \infty} x^3 = \infty$$

$$\text{Ex) } \lim_{x \rightarrow \infty} -x^2 = -\infty$$

$$\text{Ex) } \lim_{x \rightarrow -\infty} 3x^5 = -\infty$$

* note that : $\lim_{x \rightarrow \infty} (a_n + a_1 x + a_2 x^2 + \dots + a_n x^n)$
 $= \lim_{x \rightarrow \infty} a_n x^n$
 $x \rightarrow \infty$
 $x \rightarrow -\infty$

هو أدنى من $\lim_{x \rightarrow \infty}$ و هو أدنى من $\lim_{x \rightarrow -\infty}$

polynomial مع $\lim_{x \rightarrow \infty}$

$$\text{Ex) } \lim_{x \rightarrow \infty} (3x - 4x^3 + 5) = \lim_{x \rightarrow \infty} -4x^3 = -\infty$$

$$\text{Ex) } \lim_{x \rightarrow \infty} (3x^4 + 3x^2 - 10) = \lim_{x \rightarrow \infty} 3x^4 = \infty$$

$$\text{Ex) } \lim_{x \rightarrow -\infty} (7x - 3x^6 + 1) = \lim_{x \rightarrow -\infty} -3x^6 = -\infty$$

$$\text{Ex) } \lim_{x \rightarrow \infty} \frac{3x^2 - 1}{x + 7} = \lim_{x \rightarrow \infty} \frac{3x^2}{x} =$$

$$\lim_{x \rightarrow \infty} 3x = \infty$$

$$\text{Ex) } \lim_{x \rightarrow -\infty} \frac{7 - 2x + x^3}{3x^3 - 5}$$

$$= \lim_{x \rightarrow -\infty} \frac{x^3}{3x^3} = \lim_{x \rightarrow -\infty} \frac{1}{3} = \frac{1}{3}$$

$$\text{Ex) } \lim_{x \rightarrow \infty} \frac{7 - x^2}{5 + 3x^5} = \lim_{x \rightarrow \infty} \frac{-x^2}{3x^5} =$$

$$= \lim_{x \rightarrow \infty} \frac{-1}{3x^3} = 0 \quad \text{و } 0 = \frac{\text{عدد}}{\infty \pm}$$

1. درجة البسط أقل من درجة المقام دائماً جواً، صفر.
 2. درجة البسط تساوي درجة المقام وعامل أدنى البسط على المقام هو أدنى البسط.
 3. درجة البسط أعلى من درجة المقام جواً، ∞ أو $-\infty$.

$$\lim_{x \rightarrow \infty} \frac{\text{Poly}}{\text{Poly}}$$

* note that:

$$* \sqrt{x^2} = |x| \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$* \sqrt{x^4} = |x^2| = x^2$$

$$* \sqrt{x^6} = (\sqrt{x^2})^3 = (|x|)^3 = \begin{cases} x^3, & x \geq 0 \\ -x^3, & x < 0 \end{cases}$$

Ex) $\lim_{x \rightarrow \infty} \sqrt{x^6 - 5} - x^3 = \infty - \infty = ?$

$$\lim_{x \rightarrow \infty} \sqrt{x^6 - 5} - x^3 \left(\frac{\sqrt{x^6 - 5} + x^3}{\sqrt{x^6 - 5} + x^3} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x^6 - 5 - x^6}{\sqrt{x^6 - 5} + x^3} = \lim_{x \rightarrow \infty} \frac{-5}{\sqrt{x^6 - 5} + x^3}$$

$$= \lim_{x \rightarrow \infty} \frac{-5}{\sqrt{x^6} + x^3} = \frac{-5}{\infty} = 0$$

Ex) $\lim_{x \rightarrow \infty} \frac{\sqrt{5x^2 - 2}}{x + 3}$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{5x^2}}{x} = \lim_{x \rightarrow \infty} \frac{\sqrt{5} |x|}{x}$$

$$= \lim_{x \rightarrow \infty} \frac{\sqrt{5} x}{x} = \sqrt{5}$$

Ex) $\lim_{x \rightarrow \infty} \frac{e^{-x} - 3e^{2x}}{e^{2x} + 5e^x}$

$$= \lim_{x \rightarrow \infty} \frac{e^{-x} - 3e^{2x}}{e^{2x} + 5e^x} = \lim_{x \rightarrow \infty} \frac{-3e^{1x}}{5e^x}$$

$$= \lim_{x \rightarrow \infty} \frac{-3}{5} e^{-x} = 0$$

lim * إذا كان من النوع $\frac{\infty}{\infty}$ أو $\frac{0}{0}$

Ex) $\lim_{x \rightarrow \infty} \frac{5x^2 - 2}{x + 3}$

$$= \lim_{x \rightarrow \infty} \frac{5x^2 - 2}{x + 3} = \lim_{x \rightarrow \infty} \frac{5x^2}{x}$$

$$= \lim_{x \rightarrow \infty} 5x = \infty$$

Ex) $\lim_{x \rightarrow \infty} \frac{5e^{-2x} - 3e^x}{e^{2x} - 4e^{-2x}}$

$$= \frac{5e^{-\infty} - 3e^{\infty}}{e^{\infty} - 4e^{-\infty}} = \lim_{x \rightarrow \infty} \frac{-3e^x}{e^{2x}}$$

$$= \frac{-3}{e^x} = 0$$

القيمة الصغرى

$$\begin{aligned} \text{Ex) } \lim_{x \rightarrow -\infty} \frac{5e^{2x} - 3e^x}{e^{2x} - 4e^{-2x}} \\ = \frac{5e^{\infty} - 3e^{\infty}}{e^{\infty} - 4e^{\infty}} = \lim_{x \rightarrow -\infty} \frac{5e^{-2x}}{-4e^{-2x}} \\ = \frac{-5}{4} \end{aligned}$$

$$\text{Ex) } \lim_{x \rightarrow \infty} \ln x = \infty$$

$$\text{1) } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\text{2) } \lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$$

$$\text{3) } \lim_{x \rightarrow 0} \frac{bx}{\sin ax} = \frac{b}{a}$$

$$\text{4) } \lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \frac{a}{b}$$

$$\text{5) } \lim_{x \rightarrow 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$$

$$\text{Ex) } \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 4x} = \frac{3}{4}$$

ما يقرب من ∞

$$\text{Ex) } \lim_{x \rightarrow \infty} \frac{\ln 2x}{\ln 3x} = \frac{\infty}{\infty} = \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow \infty} \frac{\ln 2 + \ln x}{\ln 3 + \ln x}$$

$$= \lim_{x \rightarrow \infty} \frac{\ln x \left(\frac{\ln 2}{\ln x} + 1 \right)}{\ln x \left(\frac{\ln 3}{\ln x} + 1 \right)}$$

$$= \lim_{x \rightarrow \infty} \frac{0 + 1}{0 + 1} = 1$$

* Horizontal asymptote: the line $y = L$ is

called horizontal asymptote to $f(x)$ if $\lim_{x \rightarrow \infty} f(x) = L$

or $\lim_{x \rightarrow -\infty} f(x) = L$

$$(2) f(x) = \frac{x-3}{3x-5}$$

$$(1) \lim_{x \rightarrow \infty} \frac{x-3}{3x-5} = \frac{x}{3x}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{3} = \frac{1}{3}$$

$$\lim_{x \rightarrow -\infty} = \frac{1}{3}$$

$\therefore y = \frac{1}{3}$ is horizontal asymptote

as $x \rightarrow \infty$
 $x \rightarrow -\infty$

Ex) find a horizontal asymptote for:

$$(1) f(x) = \frac{\sqrt{x^4-1}}{x+5}$$

$$(1) \lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{\sqrt{x^4-1}}{x+5} = \lim_{x \rightarrow \infty} \frac{\sqrt{x^4}}{x} = \lim_{x \rightarrow \infty} \frac{x^2}{x}$$

$$= \lim_{x \rightarrow \infty} x = \infty$$

asymptote

is not a horizontal asymptote

$$(2) \lim_{x \rightarrow -\infty} \frac{\sqrt{x^4+1}}{x+5} = \lim_{x \rightarrow -\infty} \frac{\sqrt{x^4}}{x} = \lim_{x \rightarrow -\infty} \frac{x^2}{x}$$

$$= \lim_{x \rightarrow -\infty} x = -\infty$$

$\therefore$ there is no horizontal asymptote.

$$(3) f(x) = \frac{\sqrt{4x^2+1}}{2x-3}$$

$$(1) \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2+1}}{2x-3} = \lim_{x \rightarrow \infty} \frac{\sqrt{4x^2}}{2x}$$

$$= \lim_{x \rightarrow \infty} \frac{2|x|}{2x} = \lim_{x \rightarrow \infty} \frac{2x}{2x} = 1$$

$$(2) \lim_{x \rightarrow -\infty} \frac{\sqrt{4x^2+1}}{2x-3} = \lim_{x \rightarrow -\infty} \frac{2|x|}{2x}$$

$$\lim_{x \rightarrow -\infty} \frac{2(-x)}{2x} = -1$$

$\therefore$ there is two H.A

$y = 1$ as $x \rightarrow \infty$ and

$y = -1$ as $x \rightarrow -\infty$

NOTEBOOK

④ $f(x) = e^x$

① $\lim_{x \rightarrow \infty} e^x = \infty$

② $\lim_{x \rightarrow -\infty} e^x = 0$

$\therefore y = 0$ is H.A as $x \rightarrow -\infty$

* vertical asymptote.

* the line $x = a$ is called vertical asymptote if

$$\lim_{x \rightarrow a} f(x) = \begin{cases} \infty \\ -\infty \\ \text{d.n.e} \end{cases}$$

$x \rightarrow a^+$
 $x \rightarrow a^-$

⑤ $f(x) = 2 + \tan^{-1}(2x)$

① $\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} 2 + \tan^{-1}(2x)$

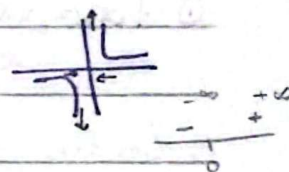
$= 2 + \frac{\pi}{2}$

② $\lim_{x \rightarrow -\infty} 2 + \tan^{-1}(2x) = 2 - \frac{\pi}{2}$

$\therefore y = 2 + \frac{\pi}{2}$

$y = 2 - \frac{\pi}{2}$ are H.A

ex) $f(x) = \frac{1}{x}$
 $x = 0$



$x = 0$ is v.A because.

$\lim_{x \rightarrow 0^+} \frac{1}{x} = +\infty$

$\lim_{x \rightarrow 0^-} \frac{1}{x} = -\infty$

$\lim_{x \rightarrow 0} = \text{d.n.e}$

⑥ $f(x) = 4\tan^{-1}x - 1$

① $\lim_{x \rightarrow \infty} 4\tan^{-1}x - 1 = 4\left(\frac{\pi}{2}\right) - 1 = 2\pi - 1$

② $\lim_{x \rightarrow -\infty} 4\tan^{-1}x - 1 = 4\left(-\frac{\pi}{2}\right) - 1 = -2\pi - 1$

$y = 2\pi - 1$ and $y = -2\pi - 1$ are H.A

Ex) $f(x) = \frac{1}{x^2}$

$x = 0$ is v.A

$\lim_{x \rightarrow 0^+} = \infty$

$x \rightarrow 0^- = \infty$

$x \rightarrow 0 = \infty$



Ex) find the vertical asymptote of :-

$$① f(x) = \frac{4}{x^2 - 16}$$

$$x^2 - 16 = 0 \Rightarrow x = 4, -4 \text{ are v. A}$$

$$x = 4, x = -4$$

$$\lim_{x \rightarrow 4} \frac{4}{x^2 - 16} = \frac{+}{-4} \frac{-}{4} \frac{+}{+}$$

$$x \rightarrow 4^+ = -\infty$$

$$x \rightarrow 4^+ = \infty$$

$$x \rightarrow 4^- = \infty$$

$$x \rightarrow 4^- = -\infty$$

$$x \rightarrow -4 = \text{d.n.e}$$

$$x \rightarrow -4 = \text{d.n.e}$$

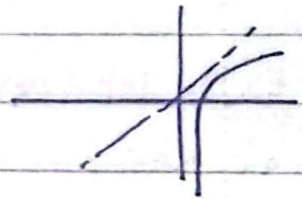
$$③ f(x) = \frac{x-1}{|x|-1}$$

$$= \begin{cases} \frac{x-1}{x-1}, & x \geq 1 \\ \frac{x-1}{-x-1}, & x < 1 \end{cases}$$

$$-x-1=0$$

$$x = -1 \text{ v. A}$$

$$④ f(x) = \ln x$$



$$\lim_{x \rightarrow 0^+} \ln x = -\infty$$

$$\therefore x=0 \text{ v. A}$$

$$[\ln \text{ negative } \ln \text{ is not defined}]$$

$$② f(x) = \frac{x-3}{x^2-9} = \frac{x-3}{(x-3)(x+3)} = \frac{1}{x+3}$$

$$\lim_{x \rightarrow -3} \frac{1}{x+3}$$

$$\therefore x = -3 \text{ is v. A}$$

$$\lim_{x \rightarrow -3^+} = \infty$$

$$\lim_{x \rightarrow -3^-} = -\infty$$

$$x \rightarrow -3 = \text{d.n.e}$$

$$x \rightarrow -3 = \text{d.n.e}$$

$$\lim_{x \rightarrow 1} (x-1)$$

$$x-1=0$$

$$\lim_{x \rightarrow 1^+} \ln(x-1) = -\infty$$

هذا هو الجواب الصحيح

* Continuity

الاستمرارية

→ Definition: $f(x)$ is Continuous (cont) at $x=a$ if:

- ① $f(a)$ defined
- ② $\lim_{x \rightarrow a} f(x)$ exist
- ③ $\lim_{x \rightarrow a} f(x) = f(a)$

Ex) let $f(x) = 3 + e^x + x^2$ is f cont at $x=0$?

- ① $f(0) = 3 + e^0 + 0 = 4$
- ② $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} 3 + e^x + x^2$
 $= 3 + 1 + 0 = 4$
- ③ $f(0) = \lim_{x \rightarrow 0} f(x)$

$\therefore f(x)$ cont at $x=0$.

$$\text{Ex) let } f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \\ 4, & x = 3 \end{cases}$$

is f cont at $x=3$ or not?

- ① $f(3) = 4$
- ② $\lim_{x \rightarrow 3} f(x) = \frac{x^2-9}{x-3} = \frac{0}{0}$ نصل دالة ليست مكونة من صفرين

$$= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)}{x-3} = \lim_{x \rightarrow 3} (x+3) = 6$$

$$f(3) \neq \lim_{x \rightarrow 3} f(x)$$

$\therefore f(x)$ not cont at $x=3$
 discont

* Remark: if $f(x)$ not cont at $x=a$ it is called discontinuous at $x=a$ (discont)

Ex) is $f(x) = \frac{|x|}{x}$ cont?

① بعد القربى
الانفصال

$$f(x) = \begin{cases} \frac{x}{x}, & x \geq 0 \\ -\frac{x}{x}, & x < 0 \end{cases} = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0^+} f(x) = 1, \quad \lim_{x \rightarrow 0^-} f(x) = -1$$

$$\therefore \lim_{x \rightarrow 0} f(x) \text{ d.n.e}$$

→ $f(x)$ dis cont at $x=0$

$$\textcircled{2} \lim_{x \rightarrow 2^+} f(x) = f(2)$$

$$4a+1 = -11$$

$$4a = -12$$

$$a =$$

$$\boxed{a = -3}$$

$$\textcircled{3} \lim_{x \rightarrow 2^-} f(x) = f(2)$$

$$8+b = -11$$

$$b = -11-8 = -19$$

$$\boxed{b = -19}$$

Ex) find the values of a & b such that:

$$e. f(x) = \begin{cases} ax^2+1, & x > 2 \\ -11, & x = 2 \\ x^3+b, & x < 2 \end{cases} \text{ is cont?}$$

since $f(x)$ cont $\Rightarrow f(x)$ cont at $x=2$

$$\textcircled{2} f(2) = \lim_{x \rightarrow 2^+} f(x)$$

$$\textcircled{1} \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x)$$

$$\boxed{4a+1 = 8+b}$$

* Defn : $f(x)$ is cont on the open interval (a, b) if $f(x)$ is cont for all $C \in (a, b)$

هذا يعني ان $f(x)$ متصلة في كل نقطة داخل الفترة المفتوحة (a, b)

(a, b) or $(-\infty, b)$ or (a, ∞) or $(-\infty, \infty)$

* Defn : $f(x)$ is (cont every) where iff $f(x)$ is cont on $(-\infty, \infty)$

Then: polynomials are continuous every where.

Monday 17/11/2025

* Thm: polynomials are continuous everywhere.

Ex) if $f(x) = \frac{x+3}{x^2+ax+1}$ find

the values of a that makes $f(x)$ cont everywhere.

Ex) let $f(x) = |x|$. show that $f(x)$ is cont everywhere

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

$$\textcircled{1} \lim_{x \rightarrow 0} f(x) \stackrel{?}{=} f(0)$$

$$0 = 0 \quad \therefore \text{Cont at } x=0$$

and x poly $x \geq 0$
 $-x$ poly $x < 0$

$\therefore f(x) = |x|$ cont everywhere

$$x^2+ax+1 \neq 0$$

لا يحل

الجزء > صفر

$$b^2-4ac < 0$$

$$f(x) = \frac{ax^2+bx+c}{x^2+ax+1}$$

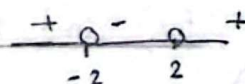
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$$a^2 - (4 \times 1 \times 1) < 0$$

$$a^2 - 4 < 0$$

$$(a-2)(a+2) < 0$$

$$\therefore a \in (-2, 2)$$



* Remark: if $f(x) = \frac{p(x)}{q(x)}$, where $p(x), q(x)$ are polynomials then $f(x)$ is cont everywhere except when $q(x)=0$

Ex) $f(x) = \frac{x^3+3}{x^3-x}$ cont every

$$\text{where } x^3-x=0$$

$$x(x^2-1)=0$$

$$\boxed{x=0} \quad \boxed{x=1} \quad \boxed{x=-1}$$

$\therefore$ cont every where $x \in \{0, \pm 1\}$

* Defn:

$$\textcircled{3} \lim_{x \rightarrow 3^-} f(x) = f(3)$$

$$0 = 0$$

 $\therefore f(x)$ cont $x=3$

from left

① $f(x)$ is cont at $x=c$ from the right iff

$$\lim_{x \rightarrow c^+} f(x) = f(c)$$

$$\therefore f(x) \text{ cont } [-3, 3]$$

② $f(x)$ is cont at $x=c$ from the left iff

$$\lim_{x \rightarrow c^-} f(x) = f(c)$$

* Thrm:

let $f(x), g(x)$ be cont at $x=c$, then:

③ $f(x)$ is cont on $[a, b]$ iff

① $f(x)$ is cont on (a, b)

② $f(x)$ is cont at a from the right

③ $f(x)$ is cont at b from the left

① $f+g$ cont at $x=c$

② $f-g$ cont at $x=c$

③ $f \cdot g$ cont at $x=c$

④ $\frac{f}{g}$ cont at $x=c$ if $g(c) \neq 0$

e.g) let $f(x) = \sqrt{9-x^2}$. show that $f(x)$ is cont on $[-3, 3]$

* Thrm:

① if $g(x)$ is cont at $x=c$, and $f(x)$ is cont at $g(x)$, then $f \circ g$ is cont at $x=c$

① let $c \in (-3, 3)$ then

$$\lim_{x \rightarrow c} f(x) = \lim_{x \rightarrow c} \sqrt{9-x^2} = \sqrt{9-c^2}$$

② if $g(x)$ is cont every where and $f(x)$ is cont every where then $f \circ g$ is cont every where.

$$f(c) = \sqrt{9-c^2}$$

$$\therefore f(c) = \lim_{x \rightarrow c} f(x) \therefore f(x) \text{ cont on } (-3, 3)$$

* Thrm:

if $f(x)$ is one to one and cont on its domain, the $f^{-1}(x)$ is cont on the range of f

$$\textcircled{2} \lim_{x \rightarrow -3^+} f(x) = f(-3)$$

$$0 = 0$$

$\therefore f(x)$ cont at $x=-3$ from right

* Remark :

the following types of functions are cont

at every number in their domains:-

1. polynomials.

2. rational functions.

3. root functions.

4. trigonometric functions.

5. inverse trigonometric functions.

6. exponential functions.

7. logarithmic functions.

e.g) discuss the continuity of:

$$① f(x) = x^2 - 3x + 5$$

cont on $\mathbb{R}$

$$② f(x) = \frac{x+1}{x^2-4} \quad \text{cont on } \mathbb{R} - \{\pm 2\}$$

$$x^2 - 4 = 0$$

$$\frac{x^2 - 4}{x = \pm 2}$$

$$③ f(x) = \frac{2x}{x^2+1} \quad \text{cont on } \mathbb{R} - \{\pm 2\}$$

$$x^2 + 1 \neq 0$$

cont on $\mathbb{R}$

$$④ f(x) = \frac{2x+1}{|x|+2} \quad \text{cont on } \mathbb{R}$$

cont on $\mathbb{R}$

cont on $\mathbb{R}$

$$④ f(x) = 2x+1 \rightarrow \mathbb{R}$$

$$|x|+2 \neq 0 \quad |x|+2 \rightarrow \mathbb{R}$$

$$|x| \neq -2$$

cont on $\mathbb{R}$

$$⑤ f(x) = \frac{2x+1}{12+x} \quad \mathbb{R} - \{x+2=0\}$$

$$\text{cont on } \mathbb{R} - \{-2\}$$

$$⑥ f(x) = \frac{2x+1}{|x|-2} \quad \text{cont on } \mathbb{R} - \{2/-2\}$$

$$|x| = 2$$

$$\downarrow$$

$$x = 2 \text{ or } x = -2$$

$$⑦ f(x) = \frac{\ln(2x-4)}{\sqrt{9-x^2}} \rightarrow 2x-4 > 0$$

$$x > 2$$

$$(2, \infty)$$

$$\sqrt{9-x^2} = 0$$

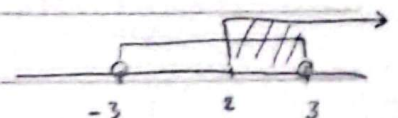
$$9-x^2 = 0$$

$$\downarrow$$

$$x = \pm 3$$

$$[-3, 3]$$

$$x = \pm 3$$



cont on $(2, 3)$

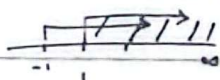
⑧ $f(x) = |\cos x|$ $g(x) = \cos x$
cont every where on $\mathbb{R}$

⑨ $f(x) = \sin\left(\frac{2\pi}{x-\pi}\right)$ $\mathbb{R} - \{\pi\}$
cont on $\mathbb{R} - \{\pi\}$

⑩ $f(x) = \frac{\log(x+1)}{x-1} = \frac{\log(x+1)}{\log(x-1)}$

البدا : $x+1 > 0 \Rightarrow x > -1$ $(-1, \infty)$

النهاية : $x-1 > 0 \Rightarrow x > 1$ $(1, \infty)$
 $x-1=1 \Rightarrow x=2$



cont on $(1, \infty) - \{2\}$

e.g) the discont points of the function

$f(x) = \frac{x^2 + 2x - 3}{(x-2)(x^2+1)}$ $\leftarrow$ م (2), 1, -1

$x-2=0$, $x^2-1=0$

$x=2$

$x=\pm 1$

$\therefore$ discont points $x \in \{\pm 1, 2\}$

e.g) $f(x) = \begin{cases} a(\tan^{-1}x + 2), & x < 0 \\ 2e^{bx} + 1, & 0 \leq x \leq 3 \\ \ln(x-2) + x^2, & x > 3 \end{cases}$

find a, b such that $f(x)$ is cont?

$f(x)$ cont $\Rightarrow f(x)$ cont at $x=0$

$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$

$2e^0 + 1 = a(\tan^{-1}0 + 2)$

$3 = 2a$

$a = \frac{3}{2}$

$f(x)$ cont $\Rightarrow f(x)$ cont at $x=3$

$\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x)$

$\ln 1 + 9 = 2e^{3b} + 1$

$9 = 2e^{3b} + 1$

$2e^{3b} = 8 \Rightarrow e^{3b} = 4$

$3b = \ln 4$

$b = \frac{1}{3} \ln 4$

or $b = \ln \sqrt[3]{4}$

* Thm: if $f(x)$ is cont, then

$$\lim_{x \rightarrow c} f(g(x)) = f(\lim_{x \rightarrow c} g(x))$$

($x \rightarrow c$, $x \rightarrow c^+$, $x \rightarrow c^-$)

$x \rightarrow \infty$, $x \rightarrow -\infty$)

e.g) $\lim_{x \rightarrow 0} e^{\sin x} = e$

$= e^{\lim_{x \rightarrow 0} \sin x} = e^0 = 1$

e.g) $\lim_{x \rightarrow \infty} \ln \left(\frac{x^6 - 1}{x^6 + 1} \right)$

$= \ln \lim_{x \rightarrow \infty} \frac{x^6 - 1}{x^6 + 1} = \ln 1 = 0$

e.g) $\lim_{x \rightarrow 3} \sqrt{\frac{3x-9}{2x^2-18}} = \sqrt{\lim_{x \rightarrow 3} \frac{3x-9}{2x^2-18}}$

$\lim_{x \rightarrow 3} \frac{3(x-3)}{2(x-3)(x+3)} = \lim_{x \rightarrow 3} \frac{3}{2(x+3)}$

$= \sqrt{\lim_{x \rightarrow 3} \frac{3}{2(x+3)}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

e.g) $\lim_{x \rightarrow \infty} \sin(\tan^{-1} x)$

$\sin(\lim_{x \rightarrow \infty} \tan^{-1} x) = \sin \frac{\pi}{2} = 1$

e.g) $\lim_{x \rightarrow 0^+} e^{-\frac{2}{x}} = e^{\lim_{x \rightarrow 0^+} -\frac{2}{x}} = e^{-\infty} = 0$

* limits involving trig functions

note that:

* $\lim_{x \rightarrow a} \sin x = \sin a$, $\lim_{x \rightarrow a} \cos x = \cos a$

since $\sin x, \cos x$ are cont everywhere.

* $\lim_{x \rightarrow \infty} \sin x = \text{d.n.e}$, $\lim_{x \rightarrow \infty} \cos x = \text{d.n.e}$

* $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

* $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \frac{a}{b}$

* $\lim_{x \rightarrow 0} \frac{bx}{\sin ax} = \frac{b}{a}$

* $\lim_{x \rightarrow 0} \frac{\tan ax}{bx} = \frac{a}{b}$

* $\lim_{x \rightarrow 0} \frac{bx}{\tan ax} = \frac{b}{a}$

* $\lim_{x \rightarrow 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$

$$\text{e.g.) } \lim_{x \rightarrow 0} \frac{\sin 3x}{\tan 5x} = \frac{3}{5}$$

$$\text{e.g.) } \lim_{x \rightarrow 0} x^2 \csc x = \lim_{x \rightarrow 0} \frac{x^2}{\sin x}$$

$$= \lim_{x \rightarrow 0} \frac{x}{\sin x} \cdot x$$

$$= 1 \cdot 0 = 0$$

$$\text{e.g.) } \lim_{x \rightarrow 0} \frac{2x + \sin x}{\tan x + 10x}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{2x}{x} + \frac{\sin x}{x}}{\frac{\tan x}{x} + \frac{10x}{x}} = \frac{2+1}{1+10} = \frac{3}{11}$$

$$\text{e.g.) } \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin(x - \frac{\pi}{2})}{(x - \frac{\pi}{2})}$$

$$x - \frac{\pi}{2} = y$$

$$\begin{aligned} x &\rightarrow \frac{\pi}{2} \\ y &\rightarrow 0 \end{aligned}$$

$$\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1$$

$$\text{e.g.) } \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} \times \frac{1 + \cos x}{1 + \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{1 - \cos^2 x}{x(2)}$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin^2 x}{x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin x}{x} \cdot \sin x$$

$$= \frac{1}{2} (1)(0) = 0$$

$$\text{e.g.) if } f(x) = \begin{cases} \frac{\sin(kx)}{x}, & x < 0 \\ 3x + 2k^2, & x \geq 0 \end{cases}$$

is cont find k?

$f(x)$ cont at $x=0$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x)$$

$$\lim_{x \rightarrow 0^+} 3x + 2k^2 = \lim_{x \rightarrow 0^-} \frac{\sin(kx)}{x}$$

$$2k^2 = k$$

$$2k^2 - k = 0$$

$$k(2k-1) = 0$$

$$k=0, k=\frac{1}{2}$$

$$\text{e.g.) if } \lim_{x \rightarrow 0} f(x) = 4 \text{ then } \lim_{x \rightarrow 0} \frac{\tan 8x}{x f(4x)}$$

$$4x \rightarrow y \Rightarrow x = \frac{y}{4} \quad \begin{matrix} x \rightarrow 0 \\ y \rightarrow 0 \end{matrix}$$

$$4 \lim_{y \rightarrow 0} \frac{\tan(8 \cdot \frac{y}{4})}{\frac{y}{4} f(\frac{y}{4})}$$

$$= 4 \cdot \frac{\lim_{y \rightarrow 0} \tan 2y}{y f(y)} = 4(2) \cdot \frac{1}{4} = 2$$

e.g) if $\lim_{x \rightarrow 0} \frac{\sin(2ax)}{\sin(8x)} = \frac{14}{8}$ then $a = ??$

$$\frac{2a}{8} = \frac{14}{8}$$

$$2a = 14 \Rightarrow \boxed{a = 7}$$

* Squeezing Thrm :

suppose that $g(x) \leq f(x) \leq h(x)$

for all x also suppose that $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$$

then $\lim_{x \rightarrow c} f(x) = L$ $x \rightarrow c, x \rightarrow c^+, x \rightarrow c^-$
 $x \rightarrow \infty, x \rightarrow -\infty$

e.g) $\lim_{x \rightarrow \infty} \frac{\sin x}{x} = \frac{\text{d.n.e}}{\infty}$

$$-1 \leq \sin x \leq 1$$

$$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$$

$$\lim_{x \rightarrow \infty} \frac{-1}{x} = 0$$

$$\lim_{x \rightarrow \infty} \frac{1}{x} = 0$$

= 0 by squeezing Thrm

e.g) $\lim_{x \rightarrow 0} x^4 \cos\left(\frac{2}{x}\right) = 0$ d.n.e

$$-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$$

$$-x^4 \leq x^4 \cos\left(\frac{2}{x}\right) \leq x^4$$

$$\lim_{x \rightarrow 0} -x^4 = 0$$

$$\lim_{x \rightarrow 0} x^4 = 0$$

= 0 by squeezing Thrm.

e.g) $\lim_{x \rightarrow \infty} e^{-3x} \sin^2 x = 0$ d.n.e

$$0 \leq \sin^2 x \leq 1$$

$$0 \leq e^{-3x} \sin^2 x \leq e^{-3x}$$

$$\lim_{x \rightarrow \infty} 0 = 0$$

$$\lim_{x \rightarrow \infty} e^{-3x} = 0$$

= 0 by squeezing Thrm.

e.g) if $3x \leq f(x) \leq x^5 + 2$ for all $x \in (-2, 2)$

find $\lim_{x \rightarrow 1} f(x) = ?$

$$3x \leq f(x) \leq x^5 + 2$$

$$\lim_{x \rightarrow 1} 3x = 3$$

$$\lim_{x \rightarrow 1} x^5 + 2 = 3$$

$$\lim_{x \rightarrow 1} f(x) = 3$$

$$\text{e.g.) } \lim_{x \rightarrow \infty} \frac{2x + \sin 2x}{x + \cos x} = \lim_{x \rightarrow \infty} \frac{\frac{2x}{x} + \frac{\sin 2x}{x}}{\frac{x}{x} + \frac{\cos x}{x}}$$

$$= \lim_{x \rightarrow \infty} 2 + \lim_{x \rightarrow \infty} \frac{\sin 2x}{x}$$

squeezing

$$\lim_{x \rightarrow \infty} \frac{1}{x} < \frac{\sin 2x}{x} < \lim_{x \rightarrow \infty} \frac{1}{x}$$

$$= 0$$

$$1 + \lim_{x \rightarrow \infty} \frac{\cos x}{x} \rightarrow 0$$

$$\lim_{x \rightarrow \infty} \frac{-1}{x} < \frac{\cos x}{x} < \frac{1}{x}$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \quad 0 \quad 0$$

$$= 2$$

e.g.) find $f'(x)$

$$\textcircled{1} f(x) = x^3 - 6x^2 + 5$$

$$f'(x) = 3x^2 - 12x$$

$$\textcircled{2} f(x) = 3x^5 - x^{\frac{1}{3}} + x^{-1}$$

$$f'(x) = 15x^4 - \frac{1}{3}x^{-\frac{2}{3}} - x^{-2}$$

$$\textcircled{3} f(x) = \frac{1}{3}x^{\frac{1}{2}} - \frac{5}{x^4} - 5x^{-4}$$

$$f'(x) = \frac{1}{6}x^{-\frac{1}{2}} + 20x^{-5}$$

* Differentiability - $\rightarrow$ $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

• Defn: for $f(x)$ we define the first derivative at $x=a$ by:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

• Rules of derivative:

• Remark: ① if $f(x) = c \Rightarrow f'(x) = 0$

② if $f(x) = x^n, n \in \mathbb{R} \Rightarrow f'(x) = nx^{n-1}$

③ if $f(x), g(x)$ exist $\Rightarrow (f \pm g)'(x) = f'(x) \pm g'(x)$

• Remark: if $f'(x), g'(x)$ exist, then,

$$\textcircled{1} (fg)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$$\textcircled{2} [(f(x))^n]' = n(f(x))^{n-1} \cdot f'(x)$$

$$\textcircled{3} \left(\frac{f}{g}\right)'(x) = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$$

e.g.) find $\frac{dy}{dx}$

$$\textcircled{1} y = (x^2 + 3x)^4 \cdot (2x^2 - 5)$$

$$y' = (x^2 + 3x)^4 (4x) + (2x^2 - 5) 4(x^2 + 3x)^3 (2x + 3)$$

$$\textcircled{2} y = \left(\frac{x^2 - 1}{x + 3}\right)^5$$

$$y' = 5 \left(\frac{x^2 - 1}{x + 3}\right)^4 \left(\frac{(x + 3)(2x) - (x^2 - 1)(1)}{(x + 3)^2}\right)$$

• note that:

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① if $f'(a)$ exist then $f(x)$ cont at $x=a$

② if $f(x)$ discont at $x=a$ then $f'(a)$ not exist

e.g) if $f(x) = \begin{cases} x^2 \cos(\frac{2}{x}) & , x \neq 0 \\ 5 & , x = 0 \end{cases}$

is $f'(x)$ exist? "differentiable"

$f(0) = 5, \lim_{x \rightarrow 0} x^2 \cos(\frac{2}{x})$

$-1 < \cos(\frac{2}{x}) < 1$
 $-x^2 < x^2 \cos(\frac{2}{x}) < x^2$
 $\lim_{x \rightarrow 0} -x^2 = 0 \quad \lim_{x \rightarrow 0} x^2 = 0$

$\lim_{x \rightarrow 0} f(x) = 0 \neq f(0) \therefore$ discont at $x=0$

$f'(x)$ d.n.e
 غير قابل للاشتقاق

e.g) let $f(x) = \begin{cases} 5x-1 & , x > 1 \\ x^2+9 & , x \leq 1 \end{cases}$

is $f'(x)$ exist, find $f'(x), f'(2), f'(0)$

~~$\lim_{x \rightarrow 1} f(x) = 4$~~

$\lim_{x \rightarrow 1^+} f(x) = 4, \lim_{x \rightarrow 1^-} f(x) = 10 \therefore$ discont at $x=1$

$\therefore f'(x) = \begin{cases} 5 & , x > 1 \\ 2x & , x < 1 \\ \text{d.n.e} & , x = 1 \end{cases}$

$f'(2) = 5, f'(0) = 0$

e.g) let $f(x) = \begin{cases} x^3+2x & , x \leq 1 \\ 5x^2-2 & , x > 1 \end{cases}$

find $f'(1), f'(3)$

~~$\lim_{x \rightarrow 1} f(x) = 3$~~
 $\lim_{x \rightarrow 1^+} f(x) = 3, \lim_{x \rightarrow 1^-} f(x) = 3$
 $\therefore$ cont at $x=1$

$f'(x) = \begin{cases} 3x^2+2 & , x < 1 \\ 10x & , x > 1 \\ \text{d.n.e} & , x = 1 \end{cases}$

$f'(1^+) = 10, f'(1^-) = 5 \therefore$ d.n.e

$f'(3) = 30$

• Remark

$$\text{[1]} \frac{d}{dx} (\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\text{[2]} \frac{d}{dx} (\sqrt{f(x)}) = \frac{f'(x)}{2\sqrt{f(x)}}$$

e.g) find $\frac{dy}{dx}$

$$\text{① } y = \cos(2x^2 - 1) + \sin 5x$$

$$y' = -4x \sin(2x^2 - 1) + 5 \cos 5x$$

$$\text{② } y = \sin^3(\cos(x^2 + 1)) = [\sin(\cos(x^2 + 1))]^3$$

$$y' = 3 [\sin^2(\cos(x^2 + 1))]^2 \cdot 2x \sin(x^2 + 1) \cdot \cos(\cos(x^2 + 1))$$

$$\text{③ } y = \sec^3(\tan^2(2x + 5))$$

$$y' = 3 \sec^2(\tan^2(2x + 5)) \cdot 2 \tan(2x + 5) \cdot 2 \sec^2(2x + 5)$$

$$= 3 \sec^2(\tan^2(2x + 5)) \cdot \sec(\tan^2(2x + 5)) \cdot \tan(\tan^2(2x + 5)) \cdot 2 \tan(2x + 5) = \sec^2(2x + 5) \cdot 2$$

$$\text{④ } y = x^2 \sin^4(2x - 5)$$

$$y' = x^2 (4 \sin^3(2x - 5) \cos(2x - 5) \cdot 2) + 2x \sin^4(2x - 5)$$

$$y' = x^2 (4 \sin(2x - 5) \cos(2x - 5) \cdot 2) + 2x \sin^4(2x - 5)$$

$$\text{⑤ } y = \sqrt{1 + \sin 2x}$$

$$y' = \frac{2 \cos 2x}{2 \sqrt{1 + \sin 2x}} = \frac{\cos 2x}{\sqrt{1 + \sin 2x}}$$

e.g) find $\frac{dy}{dx}$

$$\text{① } y = \sqrt{x^2 - 3x + 1}$$

$$y' = \frac{2x - 3}{2\sqrt{x^2 - 3x + 1}}$$

$$\text{② } y = \sqrt[5]{x - x^3 + 1} \Rightarrow y = (x - x^3 + 1)^{\frac{1}{5}}$$

$$y' = \frac{1}{5} (x - x^3 + 1)^{-\frac{4}{5}} (1 - 3x^2)$$

• Remark $f(x) \rightarrow f'(x)$

$$\sin(g(x)) \rightarrow g'(x) \cos(g(x))$$

$$\cos(g(x)) \rightarrow -g'(x) \sin(g(x))$$

$$\tan(g(x)) \rightarrow g'(x) \sec^2(g(x))$$

$$\cot(g(x)) \rightarrow -g'(x) \csc^2(g(x))$$

$$\sec(g(x)) \rightarrow g'(x) \sec(g(x)) \tan(g(x))$$

$$\csc(g(x)) \rightarrow -g'(x) \csc(g(x)) \cot(g(x))$$

e.g) $f(x) = -\sin^2(x) + (\sin 2x)$
find $f'(\frac{\pi}{4}) = ?$

$$f'(x) = -2 \sin(x) \cos(x) + \sin 2x$$

$$f'(\frac{\pi}{4}) = -\sin(2x) + \sin 2$$

$$= -\sin(\frac{2\pi}{4}) + \sin 2$$

$$= -1 + \sin 2$$

e.g) if $f(x) = 3 \tan x - 2 \csc x$ find $f'(\frac{\pi}{3})$

$$f'(x) = 3 \sec^2 x + 2 \csc x \cot x$$

~~$f'(x) = 3 \sec^2 x + 2 \csc x \cot x$~~

$$f'(x) = \frac{3}{\cos^2 x} + 2 \frac{1}{\cos x} \cdot \frac{1}{\tan x}$$

$$f'(\frac{\pi}{3}) = \frac{3}{(\frac{1}{2})^2} + 2 \frac{1}{(\frac{1}{2})} \cdot \frac{1}{\sqrt{3}}$$

$$= 12 + 4 \cdot \frac{1}{\sqrt{3}} = 12 + \frac{4}{\sqrt{3}}$$

• chain Rule $\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

① if $y = f(t)$, $t = g(x)$ then

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$② (f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

e.g) let $y = t^3 + 4t^{10}$, $t = \sin x^2$ find $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$$

$$= (3t^2 + 40t^9) \cdot (2x \cos x^2)$$

$$= 3 \sin^2 x^2 + 40 \sin^9 x^2 \cdot 2x \cos x^2$$

e.g) if $f(1) = 2$, $f'(3) = 5$, $f'(1) = 1$, $g(1) = 3$, $g'(1) = -1$ find

$$① (f \circ g)'(1) = f'(g(1)) \cdot g'(1)$$

$$= f'(3) \cdot (-1)$$

$$= 5(-1) = -5$$

$$② (3f^2 + 4g^3)'(1) = \cancel{6f'(1)} + 12g'(1)$$

$$= \cancel{6} + 12 = 18$$

$$= 6f(1)f'(1) + 12g'(1)g'(1)$$

$$= 12 + -108 = -96$$

e.g) if $y = f(2x-3)$ such that $f(1)=2$

find $\frac{dy}{dx} \Big|_{x=2}$

$$\frac{dy}{dx} = f'(2x-3) \cdot 2 \Big|_{x=2}$$

$$= f'(1) \cdot 2 = 4$$

e.g) if $f(x^3-1) = \frac{3x^2}{x^2+1}$ find $f'(7)$

$$f'(x^3-1) \cdot 3x^2 = \frac{(x^2+1)(6x) - (3x^2)(2x)}{(x^2+1)^2}$$

$x^3-1=7 \leftarrow$
 $x^3=8$
 $x=2$

$$f'(7) \cdot 12 = \frac{12}{25}$$

$$f'(7) = \frac{1}{25}$$

• Implicit Diff - get 3.6.21

For $f(x,y)=C$ we use implicit diff
to find $\frac{dy}{dx}$

e.g) find $\frac{dy}{dx}$:

$$\text{① } y^3 + xy^2 = 5 - x^2 + y$$

$$3y^2y' + x(2yy') + y^2 = -2x + y'$$

$$3y^2y' + 2xyy' + y^2 = -2x + y'$$

$$3y^2y' + 2xyy' - y' = -y^2 - 2x$$

$$y'(3y^2 + 2xy - 1) = -y^2 - 2x$$

$$y' = \frac{-y^2 - 2x}{(3y^2 + 2xy - 1)}$$

e.g) if $\frac{d}{dx} [f(2x+4)] = 4x+2$

find $\frac{d}{dx} [f(x)] \Big|_{x=4} f'(4)$

$$\frac{d}{dx} [f(2x+4)] = 4x+2$$

$2x+4=4$
 $x=0$

$$f'(4) \cdot 2 = 4(0) + 2$$

$$f'(4) = 1$$

② $\sin^2(xy) = x - y^2 + 3$

$$2 \sin(xy) \cdot xy' + y = 1 - 2yy'$$

$$2 \sin(xy) + y - 1 = -xy' - 2yy'$$

$$y' = \frac{2 \sin(xy) + y - 1}{(-x - 2y)}$$

$$\boxed{2} \sin^2(xy) = x - y^2 + 3$$

$$2 \sin(xy) \cos(xy) (xy' + y) = 1 - 2yy'$$

$$xy' 2 \sin(xy) \cos(xy) + 2y \sin(xy) \cos(xy) = 1 - 2yy'$$

$$2y \sin(xy) \cos(xy) - 1 = -2y'x \sin(xy) \cos(xy) - 2yy'$$

$$y' = \frac{2y \sin(xy) \cos(xy) - 1}{-2x \sin(xy) \cos(xy) - 2y}$$

e.g) For $xy^2 + (x+y)^3 = 5x + 2y - 2$

find $\frac{dy}{dx} \Big|_{(1, -1)}$

$$x(2yy') + y^2 + 3(x+y)^2(1+y') = 5 + 2y'$$

$$-2y' + 1 = 5 + 2y'$$

$$1 - 5 = 4y'$$

$$-4 = 4y'$$

$$\boxed{y' = -1}$$

• Diff of log and exponential function

$$f(x) \longrightarrow f'(x)$$

$$\log_a(g(x)) \longrightarrow \frac{g'(x)}{g(x) \ln a}$$

$$\ln(g(x)) \longrightarrow \frac{g'(x)}{g(x)}$$

$$a^{g(x)} \longrightarrow a^{g(x)} \cdot g'(x) \cdot \ln a$$

$$e^{g(x)} \longrightarrow e^{g(x)} \cdot g'(x)$$

e.g)

$$\textcircled{1} y = \log_3(x^2 + 5) - x \cdot 4^{x^2}$$

$$y' = \frac{2x}{(x^2 + 5) \ln 3} - x(4^{x^2} (2x) \ln 4) - 4^{x^2}$$

$$\textcircled{2} \ln(x^2 + y^2) - 5y^3 = xe^y + 3$$

$$\frac{2x + 2yy'}{(x^2 + y^2)} - 15y^2y' = xe^y y' + e^y$$

$$\frac{2x + 2yy'}{(x^2 + y^2)} = y'xe^y + e^y + 15y'y^2$$

$$2x + 2yy' = (x^2 + y^2)g'xe^y + (x^2 + y^2)e^y + (x^2 + y^2)15y'y^2$$

2x

$$2x - e^y(x^2 + y^2) = y'xe^y(x^2 + y^2) + 15y'y^2(x^2 + y^2)$$

$$\textcircled{6} \text{ if } f(x) = \cos(\log_2 x) \text{ then } f'(x) = ??$$

$$y' = \frac{2x - e^y(x^2 + y^2)}{xe^y(x^2 + y^2) + 15y^2(x^2 + y^2)}$$

$$f'(x) = -\sin(\log_2 x) \cdot \frac{1}{x \ln 2}$$

$$\textcircled{3} \frac{d}{dx} [\tan^3 \sqrt{1+e^x}]$$

$$\textcircled{7} \frac{d}{dx} \left[\ln \left(\frac{\sqrt{x+1} x^5}{(7+x)^3} \right) \right]$$

$$y' = 3 \tan^2 \sqrt{1+e^x} \cdot \sec^2 \sqrt{1+e^x} \cdot \frac{e^x}{2\sqrt{1+e^x}}$$

$$y' = 3 \left(\ln \sqrt{x+1} + 5 \ln x - 3 \ln(7+x) \right)$$

$$\textcircled{4} \text{ if } f(x) = \log_{(x+3)}(x+1) \text{ then } f'(0) = ??$$

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$$y' = 3 \left[\frac{1}{2\sqrt{x+1}} + \frac{5}{x} - \frac{3}{7+x} \right]$$

$$y' = \frac{3}{2\sqrt{x+1}} + \frac{15}{x} - \frac{9}{7+x}$$

$$\textcircled{8} \frac{d}{dx} [\pi^{3x+1}] =$$

$$y' = \pi^{3x+1} \cdot 3 \cdot \ln \pi$$

$$f(x) = \frac{\ln(x+1)}{\ln(x+3)}$$

$$f'(x) = \frac{\ln(x+3) \left(\frac{1}{x+1} \right) - \ln(x+1) \left(\frac{1}{x+3} \right)}{(\ln(x+3))^2}$$

$$\textcircled{9} \text{ if } \ln_y = \ln^{\sin x} \text{ find } \frac{dy}{dx}$$

$$f'(0) = \frac{\ln 3 - \ln(1) \left(\frac{1}{3} \right)}{(\ln(3))^2} = \frac{1}{\ln(3)}$$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \sin \frac{1}{x} + \ln x \cdot \cos x$$

$$y' = y \left[\frac{\sin x}{x} + \cos x \ln x \right]$$

$$y' = x^{\sin x} \left[\frac{\sin x}{x} + \cos x \ln x \right]$$

e.g)	x	$f''(x)$	$f'(x)$	x	g(x)	$g'(x)$
	1	3	0	1	2	7
	2	1	4	2	3	8
	3	5	6	3	1	0

③ if $h(x) = g(x) \cos\left(\frac{\pi}{4}x\right)$ then $h'(2) =$

$$h'(x) = g(x) \left(-\sin\left(\frac{\pi}{4}x\right) \frac{\pi}{4}\right) + g'(x) \cos\left(\frac{\pi}{4}x\right)$$

$$h'(2) = g(2) \left(-\frac{\pi}{4} \sin\left(\frac{\pi}{2}\right)\right) + g'(2) \cos\frac{\pi}{2}$$

$$= (3) \left(-\frac{\pi}{4}\right) + (8)(0)$$

$$h'(x) = -\frac{3\pi}{4}$$

① if $h(x) = f(g(x))$ then $h'(1) =$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$h'(1) = f'(g(1)) \cdot g'(1)$$

$$= f'(2) (7)$$

$$= (4)(7)$$

$$h'(1) = 28$$

④ if $h(x) = \frac{4^x}{f(x)}$ then $h'(1) =$

$$h'(x) = \frac{f(x) 4^x \ln 4 - 4^x f'(x)}{(f(x))^2}$$

$$h'(1) = \frac{f(1)(4 \ln 4) - 4 f'(1)}{(f(1))^2}$$

$$= \frac{12 \ln 4 - 0}{9} = \frac{4 \ln 4}{3}$$

$$h'(1) = \frac{4 \ln 4}{3}$$

② if $h(x) = \sqrt{3+f(x)}$ then $h'(2) =$

$$h'(x) = \frac{f'(x)}{2\sqrt{3+f(x)}}$$

$$h'(2) = \frac{f'(2)}{2\sqrt{3+f(2)}} = \frac{4}{2\sqrt{3+1}}$$

$$h'(2) = \frac{4}{4}$$

$$h'(2) = 1$$

⑤ if $h(x) = x^{g(x)}$ then $h'(3) =$

$$\ln h(x) = g(x) \ln x$$

$$\frac{h'(x)}{h(x)} = g(x) \frac{1}{x} + \ln x \cdot g'(x)$$

$$\frac{h'(3)}{h(3)} = g(3) \frac{1}{3} + \ln 3 \cdot g'(3)$$

$$\frac{h'(3)}{3} = (1) \left(\frac{1}{3} \right) + \ln 3 (0)$$

$$h'(3) = 1$$

$$h(3) = 3^{g(3)}$$

$$h(3) = 3^1$$

$$h(3) = 3$$

• Differentiation of inverse function :-

if $y = f(x)$ is one to one, the $f^{-1}(x)$ exist

$$\text{and } \frac{d}{dx} [f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))}$$

e.g) if $f(x) = x^3 + 3x + 2$, find $(f^{-1})'(6)$

$$① f^{-1}(6) = 1$$

$$② f'(x) = 3x^2 + 3$$

$$③ (f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{6}$$

e.g) if $f(x) = 3e^{2x-4} + x^3$, find $(f^{-1})'(11) = ??$

$$① f^{-1}(11) = 2$$

$$② f'(x) = 6e^{2x-4} + 3x^2$$

$$③ (f^{-1})'(11) = \frac{1}{f'(2)} = \frac{1}{18}$$

• Diff of inverse trigonometric functions.

$$① \frac{d}{dx} [\sin^{-1}(g(x))] = \frac{g'(x)}{\sqrt{1-g^2(x)}}$$

$$② \frac{d}{dx} [\cos^{-1}(g(x))] = \frac{-g'(x)}{\sqrt{1-g^2(x)}}$$

$$③ \frac{d}{dx} [\tan^{-1}(g(x))] = \frac{g'(x)}{1+g^2(x)}$$

$$④ \frac{d}{dx} [\cot^{-1}(g(x))] = \frac{-g'(x)}{1+g^2(x)}$$

$$⑤ \frac{d}{dx} [\sec^{-1}(g(x))] = \frac{g'(x)}{|g(x)| \sqrt{g^2(x)-1}}$$

$$⑥ \frac{d}{dx} [\csc^{-1}(g(x))] = \frac{-g'(x)}{|g(x)| \sqrt{g^2(x)-1}}$$

e.g) find $\frac{dy}{dx}$

$$[1] y = \tan^{-1} x + \sin^{-1} x$$

$$y' = \frac{1}{1+x^2} + \frac{1}{\sqrt{1-x^2}}$$

$$[2] y = \sin^{-1}(2x) + 3 \cos^{-1} x^2$$

$$y' = \frac{2}{\sqrt{1-4x^2}} + 3 \frac{2x}{\sqrt{1-x^4}}$$

$$y' = \frac{2}{\sqrt{1-4x^2}} - \frac{6x}{\sqrt{1-x^4}}$$

$$[3] y = \sec^{-1}(\ln x)$$

$$y' = \frac{\frac{1}{x}}{|\ln x| \sqrt{(\ln x)^2 + 1}}$$

$$[4] y = \pi^{\cot^{-1}(5x)}$$

$$y' = \pi^{\cot^{-1}(5x)} \cdot \frac{-5}{25x^2 + 1} \ln \pi$$

$$[5] y = \sin^{-1}(\cos(5x)) + \tan^{-1}(e^{3x+7})$$

$$y' = \frac{-5 \sin 5x}{\sqrt{1-\cos^2(5x)}} + \frac{3e^{3x+7}}{(e^{3x+7})^2 + 1}$$

$$[6] x^3 + x \tan^{-1} y = e^y$$

$$3x^2 + x \left(\frac{y'}{1+y^2} \right) + \tan^{-1} y = y' e^y$$

$$3x^2 + \tan^{-1} y = y' e^y - \frac{x y'}{1+y^2}$$

$$y' = \frac{3x^2 + \tan^{-1} y}{e^y - \frac{x}{1+y^2}}$$

e.g) if $f(x) = e^{2 \tan^{-1} x}$ then $f'(0) = ??$

$$f'(x) = e^{2 \tan^{-1} x} \cdot 2 \left(\frac{1}{1+x^2} \right)$$

$$f'(0) = e^0 \cdot 2(1)$$

$$f'(0) = 2$$

• slope and tangent line

• Remark: for any curve $y = f(x)$

① the slope of the tangent line at $x_0 = f'(x_0)$

② the equation of the tangent line at $x_0 =$

$$y - y_0 = m(x - x_0)$$

$$m = \text{slope} = f'(x_0), \quad y_0 = f(x_0)$$

e.g) Let $f(x) = x^3 - 3x^2 + 1$

① find the slope at $(1, -1)$

$$\text{slope} = f'(x) = 3x^2 - 6x \quad \leftarrow [x=1]$$

$$f'(1) = -3$$

② find the equation of tangent line at $(1, -1)$

$$① m = f'(1) = -3$$

$$② y - y_0 = m(x - x_0)$$

$$y + 1 = -3(x - 1)$$

$$y + 1 = -3x + 3$$

$$\boxed{y = -3x + 2}$$

e.g) let $f(x) = \frac{3 \cos 2x}{1 + \sin x}$ find the

equation of tangent line at $x=0$
 $y=3$

$$① m = f'(x) = \frac{(1 + \sin x)(-3(2)(\sin 2x) - (3 \cos 2x))}{(1 + \sin x)^2}$$

$$= f'(0) = \frac{(1+0)(0) - 3}{(1+0)^2} = -3$$

$$f'(0) = -3$$

$$\therefore y - 3 = -3(x - 0)$$

$$\boxed{y = -3x + 3}$$

e.g) find the slope of the tangent line

to the curve: $2x^2y + y^2 + \cos\left(\frac{\pi x}{2}\right) = 3$
at $(1, 1)$

$$2x^2y' + 4xy + 2yy' + \left(\frac{\pi}{2}\right)(-\sin\left(\frac{\pi x}{2}\right)) = 0$$

$$2y' + 4 + 2y' - \frac{\pi}{2} = 0$$

$$4y' + 4 - \frac{\pi}{2} = 0$$

$$\frac{4y'}{4} = \frac{-4 + \frac{\pi}{2}}{4}$$

$$\boxed{y' = -1 + \frac{\pi}{8}} \leftarrow \text{slope}$$

e.g) find the equation of tangent line for
 $f(x) = x e^x + \ln(x^2+1) - 1$ at $x=0$
 $y = -1$

$$① f'(x) = x e^x + e^x + \frac{2x}{x^2+1} \quad \leftarrow x=0$$

$$= 1 + 0$$

$$\boxed{f'(0) = 1} \quad \leftarrow \text{slope}$$

$$② y+1 = 1(x-0)$$

~~$$y = x - 1$$~~

$$\boxed{y = x - 1}$$

e.g) find the equation of tangent line for:
 $x^2 y^2 - 2x = 4 - 4y$ at $(2, -2)$

$$① x^2(2yy') + y^2(2x) - 2 = -4y'$$

$$4(-4)y' + (4)(4) - 2 = -4y'$$

$$-16y' + 14 = -4y'$$

$$14 = 12y'$$

$$\boxed{y' = \frac{14}{12}}$$

$$② y+2 = \frac{14}{12}(x-2)$$

$$y+2 = \frac{14}{12}x - \frac{7}{3}$$

$$\boxed{y = \frac{14}{12}x - \frac{7}{3} - 2}$$

e.g) find the equation of tangent line
of $y = f^{-1}(x)$ at $x=3$ where $f(x) = x^3 - 5$
 $\& f'(x) = 3x^2$

$$① x_0 = 3$$

$$② y_0 = f^{-1}(3) = 2$$

$$③ (f^{-1})'(3) = \frac{1}{f'(f(3))} = \frac{1}{f'(2)} = \frac{1}{12}$$

$$\therefore y - 2 = \frac{1}{12}(x - 3)$$

• Remark:

① if the tangent is horizontal then slope = 0

② if $f(x) \parallel g(x)$, then $m_f = m_g \Rightarrow$ parallel

③ if $f(x) \perp g(x)$, then $m_f \cdot m_g = -1$

e.g) find x on $f(x) = x^2 + 1$ where $f(x)$ is parallel to $y = x$ if exist.

$$① f'(x) = \frac{(x-1)(2x) - (x^2+1)}{(x-1)^2} = 1$$

~~$$② y'$$~~
$$(x-1)(2x) - (x^2+1) = (x-1)^2$$

$$2x^2 - 2x - x^2 - 1 = x^2 - 2x + 1$$

$$x^2 - 2x - 1 = x^2 - 2x + 1$$

$$-1 \neq 1 \quad \text{False}$$

$$\text{No solu} \Rightarrow \text{No } x$$

e.g) let $f(x) = \ln(x-4x^2)$ the $f(x)$ has a horizontal tangent line at $x =$

$$\frac{m=0}{\text{horizontal}}$$

$$f'(x) = 0$$

$$0 = \frac{1-8x}{x-4x^2}$$

$$1-8x = 0$$

$$\boxed{x = \frac{1}{8}}$$

e.g) the points of the tangent lines to the curve $x^2 + y^2 = 4$ are horizontal are:

$$2x + 2y \frac{dy}{dx} = 0$$

$$2x = 0 \Rightarrow \boxed{x=0} \rightarrow \boxed{y=\pm 2}$$

$\therefore (0, 2), (0, -2)$ horizontal line

e.g) find the slope of the normal tangent line for $f(x) = xe^{-x}$ at $x=0$?

$$\frac{-1}{\text{slope}}$$

$$(1) f'(x) = -xe^{-x} + e^{-x} \leftarrow \boxed{x=0}$$

$$f'(0) = 1$$

$$\therefore \text{slope of the normal} = \frac{-1}{f'(0)} = \frac{-1}{1} = -1$$

المسألة الأولى

① if $\lim_{x \rightarrow \infty} \frac{\sin 7x}{\sin(2bx)} = \frac{7}{8}$, then $b = ?$

$$\frac{7}{2b} = \frac{7}{8} \Rightarrow 2b = 8 \Rightarrow \boxed{b = 4}$$

② let $y = -3 \tan x \csc x$, $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$

$$y = -3 \frac{\sin x}{\cos x} \cdot \frac{1}{\sin x} \Rightarrow y = \frac{-3}{\cos x}$$

$$y' = \frac{-3(-\sin x)}{\cos^2 x} \quad \leftarrow x = \frac{\pi}{4}$$

$$y' = \frac{\frac{3}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2} = \frac{6}{\sqrt{2}} = 3\sqrt{2}$$

③ $\lim_{x \rightarrow -\infty} \frac{\sqrt{16x^2 + 4} - 5}{-x + 5}$

$$= \lim_{x \rightarrow -\infty} \frac{\sqrt{16x^2}}{-x} = \lim_{x \rightarrow -\infty} \frac{4|x|}{-x} =$$

لأن $x \rightarrow -\infty$ $\frac{4(-x)}{-x} = 4$

④ let $f(4) = 4$, $f'(4) = 4$, $f''(4) = 3$

then $\left(\frac{f'}{f}\right)'(4) = ??$

$$= \frac{f(4)f''(4) - f'(4)f'(4)}{[f(4)]^2} = \frac{-4}{16} = -\frac{1}{4}$$

⑤ $f(x) = \sin^2 x + (\sin 2)x$ the $f'\left(\frac{\pi}{4}\right) =$

$$f'(x) = 2 \sin x \cos x + \sin 2 \quad \leftarrow x = \frac{\pi}{4}$$

$$f'\left(\frac{\pi}{4}\right) = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \sin 2$$

$$f'\left(\frac{\pi}{4}\right) = 1 + \sin 2$$

⑥ let $\frac{d}{dx} [f(2x+4)] = 4x+4$

then $\frac{d}{dx} (f(x)) \big|_{x=4} = ?? = f'(4)$

$$\boxed{x=0}$$

$$f'(2x+4) \cdot 2 = 4x+4$$

$$f'(4) \cdot 2 = 4$$

$$\boxed{f'(4) = 2}$$

⑦ if $f(x) = \cos(\log_2 x)$, $f'(x) = ??$

$$f'(x) = -\sin(\log_2 x) \cdot \frac{1}{x \ln 2}$$

⑧ $f(x) = \cos^{-1}(\tan 2x)$, $f'(x) = ??$

$$f'(x) = \frac{-2 \sec^2 2x}{\sqrt{1 - (\tan 2x)^2}}$$

9) $e^{2 \tan^{-1} x}$, $f'(0) = ??$

$$f'(x) = e^{2 \tan^{-1} x} \cdot \frac{2}{1+x^2}$$

$$f'(0) = e^{2 \tan^{-1}(0)} \cdot \frac{2}{1+0} = 1 \cdot 2 = 2$$

12) $\lim_{x \rightarrow 1} \frac{1}{x-1} + \frac{1}{x^2-3x+2}$

$$\lim_{x \rightarrow 1} \frac{1 \cdot (x-2)}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)}$$

$$= \lim_{x \rightarrow 1} \frac{(x-2)}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)}$$

$$= \lim_{x \rightarrow 1} \frac{\cancel{x-1}}{(\cancel{x-1})(x-2)} = \lim_{x \rightarrow 1} \frac{1}{(x-2)}$$

$$= \frac{1}{-1} = -1$$

10) slope of the tangent line:

$$2x^2y + y^2 + \cos\left(\frac{\pi x}{2}\right) = 3 \quad \text{at } (1,1)$$

$$2x^2y' + 4xy + 2yy' - \frac{\pi \sin(\frac{\pi x}{2})}{2} = 0 \quad \leftarrow \text{at } (1,1)$$

$$2y' + 4 + 2y' - \frac{\pi}{2} = 0$$

$$\frac{4y'}{4} = \frac{-4 + \frac{\pi}{2}}{4}$$

$$y' = \frac{\pi}{8} - 1$$

13) $\lim_{x \rightarrow 0^+} \tan^{-1}(\ln x)$

$$\tan^{-1}\left(\lim_{x \rightarrow 0^+} \ln x\right)$$

$$= \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

14) $\lim_{x \rightarrow 0} \frac{\sqrt{ax+b} - 2}{x} = 1$, find a, b

$$\frac{0}{0} \Rightarrow \sqrt{b} - 2 = 0 \Rightarrow \sqrt{b} = 2 \Rightarrow \boxed{b=4}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{ax+4} - 2}{x} = 1$$

$$\frac{(ax+4) - 4}{x(4)} = 1$$

$$\frac{ax}{4} = 1 \Rightarrow \boxed{a=4}$$

11) The points at which the tangent

line to the curve $x^2 + y^2 = 9$ on

horizontal $\therefore 2x + 2yy' = 0$

$$y' = 0$$

$$\boxed{x=0}$$

points

$$y^2 = 9$$

$$(0, 3)$$

$$\boxed{y = \pm 3}$$

$$(0, +3)$$

$$\frac{0}{0} \mid \frac{0}{0}$$

$$\frac{0}{0} \mid \frac{0}{0}$$

• L'H Rule

• Main Idea: for $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{0}{0}, \frac{\pm\infty}{\pm\infty}, \frac{\pm\infty}{-\infty}, \frac{-\infty}{\pm\infty}$

$$= \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

e.g) $\lim_{x \rightarrow \infty} \frac{x}{e^x} = \frac{\infty}{\infty}$ L'H rule

$$= \frac{1}{e^x} = \frac{1}{\infty} = 0$$

e.g) $\lim_{x \rightarrow \infty} \frac{\ln(3x)}{\ln(2x)} = \frac{\infty}{\infty}$

L'H $\lim_{x \rightarrow \infty} \frac{\frac{3}{3x}}{\frac{2}{2x}} = \lim_{x \rightarrow \infty} 1 = 1$

e.g) $\lim_{x \rightarrow 0} \frac{1 - \cos 4x}{2x^2} = \frac{0}{0}$

L'H $= \lim_{x \rightarrow 0} \frac{4 \sin 4x}{4x} = \frac{0}{0}$

$= \lim_{x \rightarrow 0} \frac{4 \cos 4x}{1} = 4$

e.g) $\lim_{x \rightarrow 0} \frac{5x}{\tan^{-1} x} = \frac{0}{0}$

L'H $\lim_{x \rightarrow 0} \frac{5}{\frac{1}{1+x^2}} = \lim_{x \rightarrow 0} 5 + 5x^2 = 5$

e.g) $\lim_{x \rightarrow 0^+} \frac{1 - \ln x}{e^{\frac{1}{x}}} = \frac{1 - \infty}{\infty} = \frac{-\infty}{\infty}$

L'H $\lim_{x \rightarrow 0^+} \frac{-\frac{1}{x}}{e^{\frac{1}{x}} \cdot \frac{1}{x^2}} = \lim_{x \rightarrow 0^+} \frac{-1}{x} \cdot \frac{x^2}{e^{\frac{1}{x}}}$

$$= \lim_{x \rightarrow 0^+} \frac{x}{e^{\frac{1}{x}}} = \frac{0}{\infty} = 0$$

e.g) $\lim_{x \rightarrow 0^+} \frac{\ln(\sin x)}{\ln(\tan x)} = \frac{-\infty}{-\infty}$

L'H $\lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}} = \lim_{x \rightarrow 0^+} \frac{\cos x}{\sin x} \cdot \frac{\tan x}{\sec^2 x}$

$= \lim_{x \rightarrow 0^+} \cos^2 x = 1$

L'H Rule $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$

Not L'H Rule:

* case 1 :- $0 \cdot \infty$

$\lim_{x \rightarrow a} f(x) \cdot g(x) = 0 \cdot \infty = \lim_{x \rightarrow a} \frac{f(x)}{1/g(x)} = \frac{0}{1/\infty} = \frac{0}{0}$ L'H

or $= \lim_{x \rightarrow a} \frac{g(x)}{1/f(x)} = \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$ L'H

NOTEBOOK

$$\text{e.g.) } \lim_{x \rightarrow \infty} x e^{-x} = \infty \cdot 0$$

ويعتبر أن e^{-x} هو الصغرى
الأسفلية التي اقرب إلى 0

$$\lim_{x \rightarrow \infty} \frac{x}{e^x} = \frac{\infty}{\infty}$$

$$\text{L'H } \lim_{x \rightarrow \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

$$\text{e.g.) } \lim_{x \rightarrow \infty} x \sin\left(\frac{\pi}{x}\right) = \infty \cdot 0$$

نقلب ولدينا صفر
لنرى نؤول الصغرى
الأسفلية

$$\lim_{x \rightarrow \infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\frac{1}{x}} = \frac{0}{0}$$

$$\text{L'H } \lim_{x \rightarrow \infty} \frac{-\frac{\pi}{x^2} \cos\left(\frac{\pi}{x}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x \rightarrow \infty} \frac{-\pi \cos\left(\frac{\pi}{x}\right)}{1} = -\pi$$

$$= \pi$$

$$\text{e.g.) } \lim_{x \rightarrow \frac{\pi}{4}^+} (1 - \tan x) \sec 2x = 0 \cdot \infty$$

$$\frac{1}{\cos 2x} = \frac{1}{0}$$

$$\lim_{x \rightarrow \frac{\pi}{4}^+} \frac{(1 - \tan x)}{\frac{1}{\sec 2x}} = \lim_{x \rightarrow \frac{\pi}{4}^+} \frac{(1 - \tan x)}{\cos 2x}$$

$$= \text{L'H } \lim_{x \rightarrow \frac{\pi}{4}^+} \frac{\sec^2 x}{-2 \sin x} = \lim_{x \rightarrow \frac{\pi}{4}^+} \frac{1}{2 \cos^2 x \cdot \sin x}$$

$$= 1$$

L'H Rule

• Case (2) $\infty - \infty$ / $-\infty - \infty$

$$\text{e.g.) } \lim_{x \rightarrow 0^+} \frac{1}{x} - \frac{1}{\sin x} = \infty - \infty$$

$$\lim_{x \rightarrow 0^+} \frac{\sin x}{x \sin x} - \frac{x}{x \sin x}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x} = \frac{0}{0}$$

$$\text{L'H } \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{x \cos x + \sin x} = \frac{0}{0}$$

$$\text{L'H } \lim_{x \rightarrow 0^+} \frac{-\sin x}{-x \sin x + \cos x + \cos x}$$

$$\lim_{x \rightarrow 0^+} \frac{-\sin x}{-x \sin x + 2 \cos x} = \frac{0}{0+1+1} = \frac{0}{2} = 0$$

$$\text{e.g.) } \lim_{x \rightarrow 0^+} \frac{1}{x} - \frac{1}{e^x - 1} = \infty - \infty$$

نوجد صغرى

$$\lim_{x \rightarrow 0^+} \frac{e^x - 1 - x}{x(e^x - 1)} = \frac{0}{0}$$

$$\text{L'H } \lim_{x \rightarrow 0^+} \frac{e^x - 1}{x e^x + e^x - 1} = \frac{0}{0}$$

$$\text{L'H } \lim_{x \rightarrow 0^+} \frac{e^x}{x e^x + e^x + e^x} = \frac{1}{0+1+1} = \frac{1}{2}$$

$$\text{e.g.) } \lim_{x \rightarrow 0^+} \frac{1}{x} - \frac{1}{x^2} = \infty - \infty$$

$$\lim_{x \rightarrow 0^+} \frac{x-1}{x^2} = \frac{-1}{0^+} = -\infty$$

$$\frac{0}{0} \rightarrow -$$

$$\text{e.g.) } \lim_{x \rightarrow 0^+} \frac{1}{\ln(x+1)} - \frac{1}{x} = \infty - \infty$$

$$\lim_{x \rightarrow 0^+} \frac{x - \ln(x+1)}{x \ln(x+1)} = \frac{0}{0}$$

$$\text{L'H } \lim_{x \rightarrow 0^+} \frac{1 - \frac{1}{x+1}}{x \left(\frac{1}{x+1} \right) + \ln(x+1)} = \frac{\frac{x+1-1}{x+1}}{\frac{x + (x+1)\ln(x+1)}{(x+1)}}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x+1} \cdot \frac{x+1}{x + (x+1)\ln(x+1)}$$

$$= \lim_{x \rightarrow 0^+} \frac{x}{x + (x+1)\ln(x+1)} = \frac{0}{0}$$

$$\text{L'H } \lim_{x \rightarrow 0^+} \frac{1}{1 + (x+1)\frac{1}{x+1} + \ln(x+1)}$$

$$= \lim_{x \rightarrow 0^+} \frac{1}{1+1+\ln(x+1)} = \frac{1}{1+1+0} = \frac{1}{2}$$

$$\text{e.g.) } \lim_{x \rightarrow \infty} 2x - \ln(x^2+1) = \infty - \infty$$

$$\lim_{x \rightarrow \infty} \ln e^{2x} - \ln(x^2+1)$$

$$= \lim_{x \rightarrow \infty} \ln \left(\frac{e^{2x}}{x^2+1} \right)$$

$$= \ln \left(\lim_{x \rightarrow \infty} \frac{e^{2x}}{x^2+1} = \frac{\infty}{\infty} \right) = \ln \frac{\infty}{\infty}$$

$$= \ln \left(\text{L'H } \lim_{x \rightarrow \infty} \frac{2e^{2x}}{2x} \right)$$

$$= \ln \left(\lim_{x \rightarrow \infty} \frac{e^{2x}}{x} = \frac{\infty}{\infty} \right)$$

$$= \ln \left(\text{L'H } \lim_{x \rightarrow \infty} \frac{2e^{2x}}{1} \right) = \ln \infty =$$

• case 3: $0^0 / \infty^0 / 1^\infty$

$$\lim_{x \rightarrow a} (f(x))^{g(x)} \quad \text{① } y = (f(x))^{g(x)}$$

$$\text{② } \ln y = \ln(f(x))^{g(x)}$$

$$\text{③ } \ln y = g(x) \ln(f(x))$$

$$\text{④ } \lim_{x \rightarrow a} \ln y = \lim_{x \rightarrow a} g(x) \ln(f(x))$$

e.g) $\lim_{x \rightarrow 0^+} (1 + \sin x)^{\frac{1}{x}} = 1^{\infty}$

$$y = (1 + \sin x)^{\frac{1}{x}}$$

lim
L

$$\ln y = \frac{1}{x} \ln(1 + \sin x)$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} \frac{\ln(1 + \sin x)}{x} \leftarrow \infty \cdot 0$$

$$\ln y = \frac{L'H}{\lim_{x \rightarrow 0^+} \frac{\cos x}{1 + \sin x}}$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\cos x}{1 + \sin x} \cdot \frac{1}{1}$$

$$\ln y = \frac{1}{e}$$

$$\boxed{y = e^1} \leftarrow \lim_{x \rightarrow 0^+} (1 + \sin x)^{\frac{1}{x}}$$

e.g) $\lim_{x \rightarrow 0^+} (e^{2x} - 1)^x = 0^0$

$$y = (e^{2x} - 1)^x$$

$$\ln y = x \ln(e^{2x} - 1)$$

$$\lim_{x \rightarrow 0^+} \ln y = \lim_{x \rightarrow 0^+} x \ln(e^{2x} - 1) \leftarrow \begin{matrix} \text{indeterminate} \\ 0 \cdot -\infty \end{matrix}$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{\ln(e^{2x} - 1)}{\frac{1}{x}}$$

$$\ln y = \frac{L'H}{\lim_{x \rightarrow 0^+} \frac{\frac{2e^{2x}}{e^{2x} - 1}}{-\frac{1}{x^2}}}$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{2e^{2x}}{e^{2x} - 1} \cdot -x^2 = \infty \cdot 0$$

$$\ln y = \lim_{x \rightarrow 0^+} \frac{-2x^2 e^{2x}}{e^{2x} - 1}$$

$$\ln y = \frac{L'H}{\lim_{x \rightarrow 0^+} \frac{-4x^2 e^{2x} - 4x e^{2x}}{2e^{2x}}} \rightarrow \frac{0+0}{2} = 0$$

$$\ln y = \frac{0}{e} \Rightarrow \boxed{y = 1}$$

e.g) $\lim_{x \rightarrow \infty} x^{\frac{-1}{\ln x}} = \infty^0$

$$y = x^{\frac{-1}{\ln x}}$$

$$\ln y = \frac{-1}{\ln x} \ln x$$

$$\lim_{x \rightarrow \infty} \ln y = \lim_{x \rightarrow \infty} \frac{-1}{\ln x} \ln x$$

$$\ln y = \frac{-1}{e}$$

$$\boxed{y = \frac{1}{e}}$$

• Rule: $\lim_{x \rightarrow \infty} \left(1 + \frac{n}{x}\right)^x = e^n$

e.g) $\lim_{x \rightarrow \infty} \left(1 + \frac{3}{x}\right)^x = e^3$

e.g) $\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^{4x} = \left(\lim_{x \rightarrow \infty} \left(1 - \frac{2}{x}\right)^x\right)^4$
 $= (\bar{e}^2)^4 = e^{-8}$

e.g) $\lim_{x \rightarrow \infty} \left(\frac{x+1}{x-1}\right)^x = \lim_{x \rightarrow \infty} \left(\frac{x(1+\frac{1}{x})}{x(1-\frac{1}{x})}\right)^x$

$= \lim_{x \rightarrow \infty} \left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)^x = \frac{e^1}{e^{-1}} = e^2$

Maximum and Minimum value

• Defn: for a function $f(x)$ defined on a set " S " and $c \in S$.

[1] $f(c)$ is absolute Max for $f(x)$ if:

$$f(c) \geq f(x) \text{ for all } x \in S.$$

[2] $f(c)$ is absolute min for $f(x)$ if:

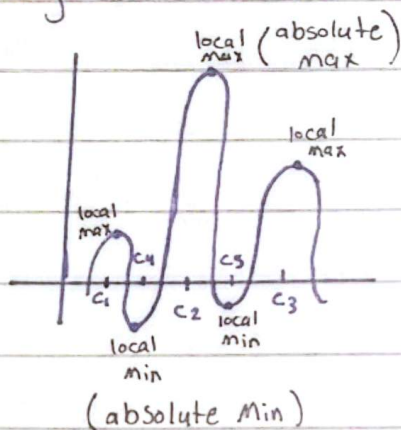
$$f(c) \leq f(x) \text{ for all } x \in S.$$

• Defn: [1] $f(c)$ is a local Max for $f(x)$ if:

$f(c) \geq f(x)$ for all x in some open interval containing (c)

[2] $f(c)$ is a local Min for $f(x)$ if:

$f(c) \leq f(x)$ for all x in some open interval containing (c) .



• Thm: if $f(x)$ is cont on $[a, b]$, then $f(x)$ attains Max & Min.

• Defn: A number $c \in \text{Domain } f(x)$ is called a critical number of $f(x)$ if, $f'(c) = 0$ or $f'(c)$ d.n.e.

• Thm: if $f(c)$ is a local extremum "Min or Max" then c must be a critical number for $f(x)$, but the converse is not true.

• Defn: A function $f(x)$ is strictly increasing for all $x_1 < x_2 \in \text{open interval } (I)$ then $f(x_1) < f(x_2)$.

• A function $f(x)$ is strictly decreasing for all $x_1 < x_2 \in \text{open interval } (I)$ then $f(x_1) > f(x_2)$.

• Thm: suppose $f(x)$ is differentiable on I , then:

[1] if $f'(x) > 0$ for all $x \in I$ then $f(x)$ increasing on I .

[2] if $f'(x) < 0$ for all $x \in I$ then $f(x)$ decreasing on I .

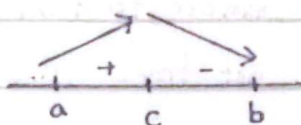
[3] if $f'(x) = 0$ for all $x \in I$ then $f(x)$ is Constant.

• First Derivative test:

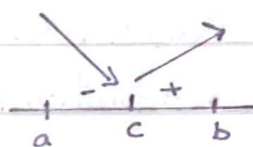
① let $f(x)$ is cont on $[a, b]$ and

② let $c \in [a, b]$ critical number then

① if $f'(x) > 0$ for all $x \in (a, c)$ & $f'(x) < 0$ for all $x \in (c, b)$ then $(c, f(c))$ local max.



② if $f'(x) < 0$ for all $x \in (a, c)$ & $f'(x) > 0$ for all $x \in (c, b)$ then $(c, f(c))$ local min



• Defn: for a function $f(x)$ that is diffble on I then:

① $f(x)$ is concave up on the interval I if $f'(x)$ increasing on I " $f''(x) > 0$ for all $x \in I$ "

② $f(x)$ is concave down on the interval I if $f'(x)$ decreasing on I " $f''(x) < 0$ for all $x \in I$ "

• Defn: suppose that $f(x)$ is cont on (a, b) and that graph changes concavity at point $(c, f(c))$, $c \in (a, b)$ is an inflection point.

e.g) let $f(x) = x^3 - 12x$, Domain $[0, 4]$

1. find the increasing and decreasing domain.

2. find the extrema point.

3. find x such that $f(x)$ has critical points.

4. find the domains of concavity.

5. find the inflection point.

① Dom: $[0, 4]$

② $f'(x) = 3x^2 - 12$

critical $f'(x) = 0 \rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

critical number $x \in \{2, 0, 4\}$. $2 \in \text{Dom}$, $-2 \notin \text{Dom}$

③ critical point $(2, f(2))$ $(4, f(4))$ $(0, f(0))$

increasing $(2, 4)$ decreasing $(0, 2)$ absolute min point $(2, f(2))$

absolute max at $x=4$

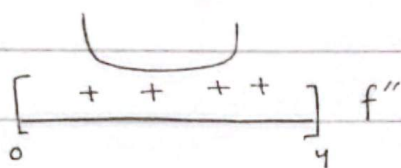
$$f(0) = 0$$

$$f(4) = 16$$

$(4, 16)$ absolute max

$$\rightarrow f''(x) = 6x$$

$$6x = 0 \Rightarrow x = 0 \in \text{Dom}$$



$\therefore$ Concave up $(0, 4)$

$\therefore$ there is no inflection point.

$$\text{e.g. } f(x) = x^3 - 6x^2 + 1 \rightarrow D \Rightarrow \mathbb{R}$$

ابسط كل شيء الى أبسط صورة
:w) compute all of the above.

$$f'(x) = 3x^2 - 12x$$

$$3x^2 - 12x = 0 \Rightarrow x(3x - 12) = 0$$

$$x = 0 \quad x = 4$$

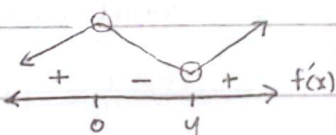
critical point : $(0, 1)$ / $(4, -31)$

$\rightarrow$ increasing $(-\infty, 0) \cup (4, \infty)$

$\rightarrow$ decreasing $(0, 4)$

$\rightarrow$ local min point $(4, -31)$

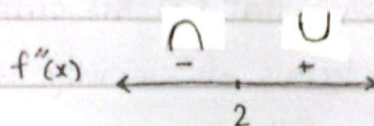
$\rightarrow$ local Max point $(0, 1)$



$$f''(x) = 6x - 12$$

$$6x - 12 = 0$$

$$x = 2$$



$\rightarrow$ concave down $(-\infty, 2)$

$\rightarrow$ concave up $(2, \infty)$

$\rightarrow$ inflection point $(2, -15)$

$$\text{e.g. let } f(x) = x e^{-x} \therefore \text{Dom: } \mathbb{R}$$

a) find the intervals of increasing and decreasing

b) find the extrema points.

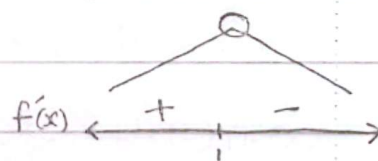
c) find x such that $f(x)$ has critical points.

$$f'(x) = -x e^{-x} + e^{-x}$$

$$-x e^{-x} + e^{-x} = 0 \Rightarrow e^{-x}(-x + 1) = 0$$

$$e^{-x} \neq 0 \quad x = 1 \in \mathbb{R}$$

$\rightarrow$ critical points $(1, f(1))$



$\rightarrow$ increasing $(-\infty, 1)$

$\rightarrow$ decreasing $(1, \infty)$

$\rightarrow$ absolute Max point $(1, f(1))$

d) find the interval of concavity, inflection point.

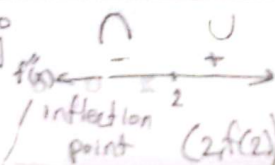
$$f''(x) = x e^{-x} + -e^{-x} + -e^{-x} \Rightarrow f''(x) = x e^{-x} - 2e^{-x}$$

$$x e^{-x} - 2e^{-x} = 0 \Rightarrow e^{-x}(x - 2) = 0$$

$$e^{-x} \neq 0 \quad x = 2$$

$\rightarrow$ concave up $(2, \infty)$

$\rightarrow$ concave down $(-\infty, 2)$

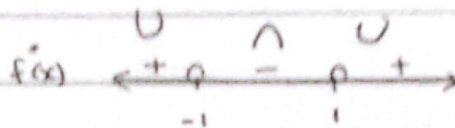


e.g) 1. find the interval of increasing and decreasing.

2. find the extrema points.

3. find x such that $f(x)$ has critical points

4. find the intervals of concavity inflection point for the following:-



→ concave up $(-\infty, -1) \cup (1, \infty)$

→ concave down $(-1, 1)$

there is no inflection point $\leftarrow R - \{\pm 1\}$

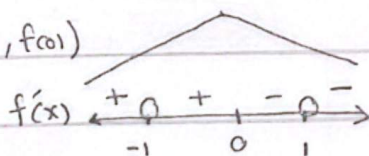
$$\textcircled{1} f(x) = \frac{x^2}{x^2-1} \quad \therefore \text{Dom } R - \{\pm 1\}$$

$$f'(x) = \frac{2x^3 - 2x - 2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

$$\frac{-2x}{(x^2-1)^2} = 0 \Rightarrow -2x = 0 \Rightarrow x = 0 \in \text{Dom}$$

$$f'(x) = \text{d.n.e} \Rightarrow (x^2-1)^2 = 0 \Rightarrow x = \pm 1 \notin \text{Dom}$$

→ critical point $(0, f(0))$



→ increasing $(-\infty, 0)$

→ decreasing $(0, \infty)$

→ absolute max point $(0, f(0))$ and local max

$$f''(x) = \frac{(x^2-1)(-2x^2+2+2x(4x))}{(x^2-1)^4} = \frac{-2x^2+2+8x^2}{(x^2-1)^3}$$

$$= \frac{(x^2-1)(6x^2+2)}{(x^2-1)^3} = \frac{6x^2+2}{(x^2-1)^2}$$

$$\frac{6x^2+2}{(x^2-1)^2} = 0 \quad 6x^2+2 \neq 0$$

$$\textcircled{2} f(x) = x^{\frac{2}{3}} - x \quad \text{Dom: } R$$

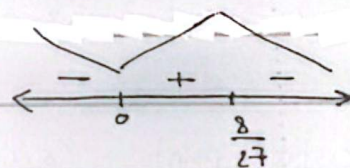
$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} - 1 = \frac{2}{3x^{\frac{1}{3}}} - \frac{1}{1} \cdot \frac{3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$$

$$f'(x) = \frac{2-3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$$

$$f'(x) = 0 \Rightarrow 2-3x^{\frac{1}{3}} = 0 \Rightarrow 3x^{\frac{1}{3}} = 2 \Rightarrow x^{\frac{1}{3}} = \frac{2}{3} \Rightarrow x = \frac{8}{27}$$

$$f'(x) = \text{d.n.e} \Rightarrow 3x^{\frac{1}{3}} = 0 \Rightarrow x = 0$$

→ critical point: $(\frac{8}{27}, f(\frac{8}{27})) / (0, f(0))$



→ increasing $(0, \frac{8}{27})$

→ decreasing $(-\infty, 0) \cup (\frac{8}{27}, \infty)$

→ local min point $(0, f(0))$

→ local max point $(\frac{8}{27}, f(\frac{8}{27}))$

$$f''(x) = \frac{-2}{9} x^{-\frac{4}{3}} = \frac{-2}{9x^{\frac{4}{3}}} \neq 0$$

→ concave down $(-\infty, \infty)$

→ there is no inflection point

3) $f(x) = \sin x - \cos x$, $\text{Dom} [0, \pi]$

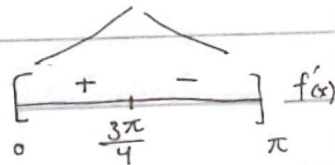
$$f'(x) = \cos x + \sin x$$

$$\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x$$

$$1 = -\tan x$$

$$x = \frac{3\pi}{4} \in \text{Dom}$$

→ critical point $(\frac{3\pi}{4}, f(\frac{3\pi}{4}))$



→ increasing $(0, \frac{3\pi}{4})$

→ decreasing $(\frac{3\pi}{4}, \pi)$

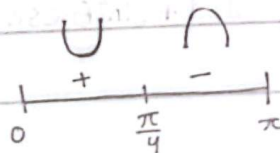
→ absolute max point $(\frac{3\pi}{4}, f(\frac{3\pi}{4}))$

$$f(0) = -1 \Rightarrow \text{absolute Min point } (0, -1)$$

$$f(\pi) = 1 \quad \times$$

$$f''(x) = -\sin x + \cos x = 0 \Rightarrow \tan x = 1$$

$$x = \frac{\pi}{4}$$



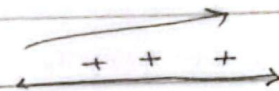
→ concave up $(0, \frac{\pi}{4})$

→ concave down $(\frac{\pi}{4}, \pi)$

→ inflection point $(\frac{\pi}{4}, f(\frac{\pi}{4}))$

4) $f(x) = \tan^{-1}(2x)$ $\text{Dom: } \mathbb{R}$

$$f'(x) = \frac{2}{1+4x^2} \neq 0 \neq \text{d.n.e}$$



→ there is no critical point

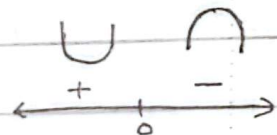
→ increasing $(-\infty, \infty)$

→ there is no decreasing

→ No extrema

$$f''(x) = \frac{-2(8x)}{(1+4x^2)^2} = \frac{-16x}{(1+4x^2)^2} = 0$$

$$x = 0$$



→ concave up $(-\infty, 0)$

→ concave down $(0, \infty)$

→ inflection point $(0, f(0))$

• Integration:

$$\textcircled{1} \int a \, dx = ax + C$$

$$\textcircled{2} \int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\textcircled{3} \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

$$\text{e.g.) } \int 3x - x^2 + 5 \, dx$$

$$\frac{3x^2}{2} - \frac{x^3}{3} + 5x + C$$

$$\text{e.g.) } \int 2x(3x-1)^2 \, dx$$

$$\int 2x(9x^2 - 6x + 1) \, dx$$

$$\int 18x^3 - 12x^2 + 2x \, dx$$

$$\frac{18x^4}{4} - \frac{12x^3}{3} + x^2 + C$$

$$\text{e.g.) } \int \frac{5x^3 + 4}{\sqrt{x}} \, dx$$

$$\int \frac{5x^3}{\sqrt{x}} + \frac{4}{\sqrt{x}} = 5x^3 \cdot x^{-\frac{1}{2}} + 4x^{-\frac{1}{2}}$$

$$= \int 5x^{\frac{5}{2}} + 4x^{\frac{1}{2}} \, dx$$

$$5x^{\frac{7}{2}} \left(\frac{2}{7}\right) + 4x^{\frac{3}{2}} (2) + C$$

$$= \frac{10}{7} x^{\frac{7}{2}} + 8x^{\frac{3}{2}} + C$$

• Remark:

$$1. \int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$$

$$2. \int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$$

$$3. \int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$$

$$4. \int \csc^2(ax+b) \, dx = -\frac{1}{a} \cot(ax+b) + C$$

$$5. \int \sec(ax+b) \tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$$

$$6. \int \csc(ax+b) \cot(ax+b) \, dx = -\frac{1}{a} \csc(ax+b) + C$$

$$\text{e.g.) } \int \cos(2x-3) \, dx = \frac{1}{2} \sin(2x-3) + C$$

$$\text{e.g.) } \int \frac{\sin 2x}{\cos x} \, dx = \int \frac{2 \sin x \cos x}{\cos x} \, dx$$

$$\int 2 \sin x \, dx = -2 \cos x + C$$

$$\text{e.g.) } \int 1 + \sin^2 \theta \csc \theta \, d\theta =$$

$$\int 1 + \sin^2 \theta \cdot \frac{1}{\sin \theta} \, d\theta = \int 1 + \sin \theta \, d\theta$$

$$= \theta + (-\cos \theta) + C$$

$$\text{e.g.) } \int \sec(2x) (\sec(2x) + \tan(2x)) dx$$

$$= \int \sec^2(2x) + \sec(2x) \tan(2x) dx$$

$$= \frac{1}{2} \tan(2x) + \frac{1}{2} \sec(2x) + C$$

$$\text{e.g.) } \int \cos^2(4x) + \tan^2(5x) dx$$

$$\int \frac{1}{2} + \frac{1}{2} \cos(8x) + \sec^2(5x) - 1 dx$$

$$\frac{1}{2} x + \frac{1}{16} \sin(8x) + \frac{1}{5} \tan(5x) - x + C$$

$$\text{e.g.) } \int \frac{\sin 4x}{\cos^2 4x} dx = \int \frac{\sin 4x}{\cos 4x \cdot \cos 4x} dx$$

$$= \int \tan 4x \sec 4x dx = \frac{1}{4} \sec 4x + C$$

• Remark:

$$① \int \cosh(ax+b) dx = \frac{1}{a} \sinh$$

$$② \int \sinh(ax+b) dx = \frac{1}{a} \cosh(ax+b) + C$$

$$③ \int \operatorname{sech}^2(ax+b) dx = \frac{1}{a} \tanh(ax+b) + C$$

$$④ \int \operatorname{csch}^2(ax+b) dx = -\frac{1}{a} \coth(ax+b) + C$$

$$⑤ \int \operatorname{sech}(ax+b) \tanh(ax+b) dx = -\frac{1}{a} \operatorname{sech}(ax+b) + C$$

$$⑥ \int \operatorname{csch}(ax+b) \coth(ax+b) dx = -\frac{1}{a} \operatorname{csch}(ax+b) + C$$

$$\text{e.g.) } \int (\sin x + \cos x)^2 dx$$

$$\int \sin^2 x + 2 \sin x \cos x + \cos^2 x dx$$

$$\int 1 + \sin 2x dx$$

$$x + -\frac{1}{2} \cos 2x + C$$

$$\text{e.g.) } \int \cosh(3x-5) dx$$

$$= \frac{1}{3} \sinh(3x-5) + C$$

$$\text{e.g.) } \int \text{sech}^2(2x) dx$$

$$\frac{1}{2} \tanh(2x) + C$$

$$\text{e.g.) } \int 2^{3x+5} dx$$

$$\frac{2^{3x+5}}{3 \ln 2} + C$$

$$\text{e.g.) } \int \text{sech}(3x) \tanh(3x) dx$$

$$-\frac{1}{3} \text{sech}(3x) + C$$

$$\text{e.g.) } \int \frac{e^x + 3}{e^x} dx$$

$$\int \frac{e^x}{e^x} + \frac{3}{e^x} dx = \int 1 + 3e^{-x} dx$$

$$x + \frac{3e^{-x}}{-1} + C = x - 3e^{-x} + C$$

$$\text{e.g.) } \int \frac{3e^x}{\cosh x + \sinh x} dx = \int \frac{3e^x}{e^x} dx$$

$$= 3x + C$$

$$\text{e.g.) } \int 3^{3-2\log_3 x} dx$$

$$\int 3^3 \cdot 3^{-2\log_3 x} dx$$

$$\int 27 \cdot 3^{\log_3 x^{-2}} dx$$

$$\int 27 x^{-2} dx$$

$$= 27 \frac{x^{-1}}{-1} + C = -\frac{27}{x} + C$$

• Remark :

$$\textcircled{1} \int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C$$

$$\textcircled{2} \int a^{bx+c} dx = \frac{a^{bx+c}}{b \cdot \ln a} + C$$

$$\text{e.g.) } \int e^{2x-1} dx$$

$$\frac{e^{2x-1}}{2} + C$$

Maximum and Minimum value

• Defn: for a function $f(x)$ defined on a set " S " and $C \in S$.

① $f(c)$ is absolute Max for $f(x)$ if:

$$f(c) \geq f(x) \text{ for all } x \in S.$$

② $f(c)$ is absolute min for $f(x)$ if:

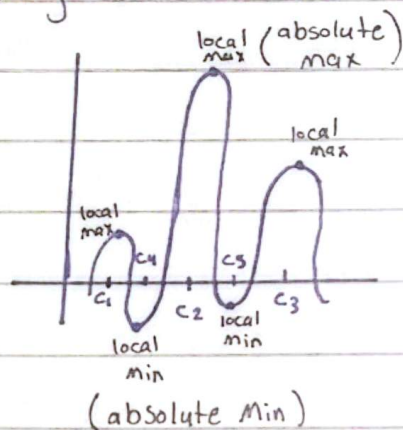
$$f(c) \leq f(x) \text{ for all } x \in S.$$

• Defn: ① $f(c)$ is a local Max for $f(x)$ if:

$f(c) \geq f(x)$ for all x in some open interval containing (c)

② $f(c)$ is a local Min for $f(x)$ if:

$f(c) \leq f(x)$ for all x in some open interval containing (c) .



• Thm: if $f(x)$ is cont on $[a, b]$, then $f(x)$ attains Max & Min.

• Defn: A number $C \in \text{Domain } f(x)$ is called a critical number of $f(x)$ if, $f'(c) = 0$ or $f'(c)$ d.n.e.

• Thm: if $f(c)$ is a local extremum "Min or Max" then C must be a critical number for $f(x)$, but the converse is not true.

• Defn: A function $f(x)$ is strictly increasing for all $x_1 < x_2 \in \text{open interval } (I)$ then $f(x_1) < f(x_2)$.

• A function $f(x)$ is strictly decreasing for all $x_1 < x_2 \in \text{open interval } (I)$ then $f(x_1) > f(x_2)$.

• Thm: suppose $f(x)$ is differentiable on I , then:

① if $f'(x) > 0$ for all $x \in I$ then $f(x)$ increasing on I .

② if $f'(x) < 0$ for all $x \in I$ then $f(x)$ decreasing on I .

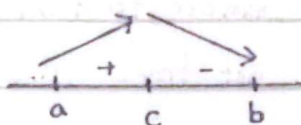
③ if $f'(x) = 0$ for all $x \in I$ then $f(x)$ is Constant.

• First Derivative test:

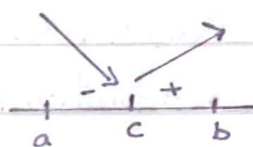
① let $f(x)$ is cont on $[a, b]$ and

② let $c \in [a, b]$ critical number then

① if $f'(x) > 0$ for all $x \in (a, c)$ & $f'(x) < 0$ for all $x \in (c, b)$ then $(c, f(c))$ local max.



② if $f'(x) < 0$ for all $x \in (a, c)$ & $f'(x) > 0$ for all $x \in (c, b)$ then $(c, f(c))$ local min



• Defn: for a function $f(x)$ that is diffble on I then:

① $f(x)$ is concave up on the interval I if $f'(x)$ increasing on I " $f''(x) > 0$ for all $x \in I$ "

② $f(x)$ is concave down on the interval I if $f'(x)$ decreasing on I " $f''(x) < 0$ for all $x \in I$ "

• Defn: suppose that $f(x)$ is cont on (a, b) and that graph changes concavity at point $(c, f(c))$, $c \in (a, b)$ is an inflection point.

e.g) let $f(x) = x^3 - 12x$, Domain $[0, 4]$

1. find the increasing and decreasing domain.

2. find the extrema point.

3. find x such that $f(x)$ has critical points.

4. find the domains of concavity.

5. find the inflection point.

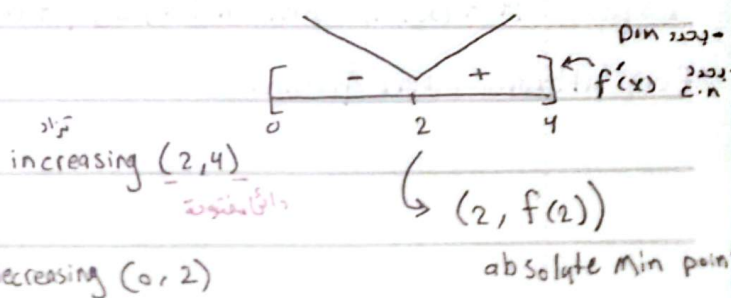
① Dom: $[0, 4]$

② $f'(x) = 3x^2 - 12$

critical $f'(x) = 0 \rightarrow 3x^2 - 12 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

critical number $x \in \{2, 0, 4\}$. $2 \in \text{Dom}$, $-2 \notin \text{Dom}$

③ critical point $(2, f(2))$ $(4, f(4))$ $(0, f(0))$



absolute max at $x=4$

$$f''(x) = 6x - 12$$

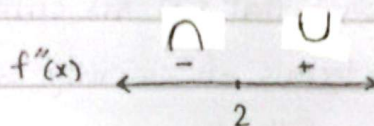
$$f(0) = 0$$

$$f(4) = 16$$

$(4, 16)$ absolute max

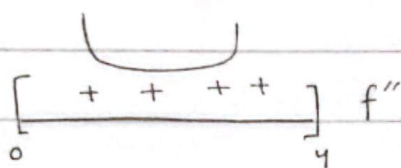
$$6x - 12 = 0$$

$$x = 2$$



$$\rightarrow f''(x) = 6x$$

$$6x = 0 \Rightarrow x = 0 \in \text{Dom}$$



$\therefore$ Concave up $(0, 4)$

$\therefore$ there is no inflection point.

$\rightarrow$ concave down $(-\infty, 2)$

$\rightarrow$ concave up $(2, \infty)$

$\rightarrow$ inflection point $(2, -15)$

e.g) $f(x) = x^3 - 6x^2 + 1 \rightarrow D \Rightarrow \mathbb{R}$

ابسط كل شيء الى أبسط
:w) compute all of the above.

$$f'(x) = 3x^2 - 12x$$

$$3x^2 - 12x = 0 \Rightarrow x(3x - 12) = 0$$

$$x = 0 \quad x = 4$$

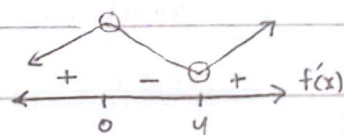
critical point : $(0, 1) / (4, -31)$

$\rightarrow$ increasing $(-\infty, 0) \cup (4, \infty)$

$\rightarrow$ decreasing $(0, 4)$

$\rightarrow$ local min point $(4, -31)$

$\rightarrow$ local max point $(0, 1)$



e.g) let $f(x) = x e^{-x} \therefore \text{Dom: } \mathbb{R}$

a) find the intervals of increasing and decreasing

b) find the extrema points.

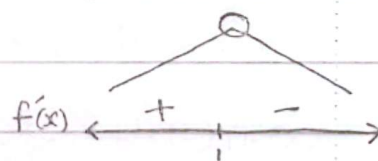
c) find x such that $f(x)$ has critical points.

$$f'(x) = -x e^{-x} + e^{-x}$$

$$-x e^{-x} + e^{-x} = 0 \Rightarrow e^{-x}(-x + 1) = 0$$

$$e^{-x} \neq 0 \quad x = 1 \in \mathbb{R}$$

$\rightarrow$ critical points $(1, f(1))$



$\rightarrow$ increasing $(-\infty, 1)$

$\rightarrow$ decreasing $(1, \infty)$

$\rightarrow$ absolute Max point $(1, f(1))$

d) find the interval of concavity, inflection point.

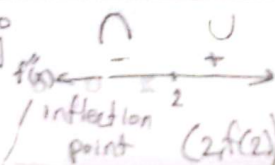
$$f''(x) = x e^{-x} + -e^{-x} + -e^{-x} \Rightarrow f''(x) = x e^{-x} - 2e^{-x}$$

$$x e^{-x} - 2e^{-x} = 0 \Rightarrow e^{-x}(x - 2) = 0$$

$$e^{-x} \neq 0 \quad x = 2$$

$\rightarrow$ concave up $(2, \infty)$

$\rightarrow$ concave down $(-\infty, 2)$

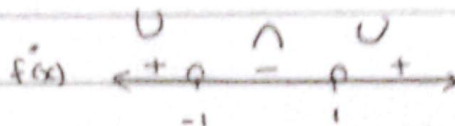


e.g) 1. find the interval of increasing and decreasing.

2. find the extrema points.

3. find x such that $f(x)$ has critical points

4. find the intervals of concavity inflection point for the following:-



→ concave up $(-\infty, -1) \cup (1, \infty)$

→ concave down $(-1, 1)$

there is no inflection point ← $\mathbb{R} - \{\pm 1\}$

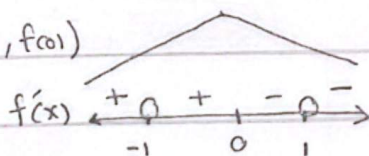
$$\textcircled{1} f(x) = \frac{x^2}{x^2-1} \quad \therefore \text{Dom } \mathbb{R} - \{\pm 1\}$$

$$f'(x) = \frac{2x^3 - 2x - 2x^3}{(x^2-1)^2} = \frac{-2x}{(x^2-1)^2}$$

$$\frac{-2x}{(x^2-1)^2} = 0 \Rightarrow -2x = 0 \Rightarrow x = 0 \in \text{Dom}$$

$$f'(x) = \text{d.n.e} \Rightarrow (x^2-1)^2 = 0 \Rightarrow x = \pm 1 \notin \text{Dom}$$

→ critical point $(0, f(0))$



→ increasing $(-\infty, 0)$

→ decreasing $(0, \infty)$

→ absolute max point $(0, f(0))$ and local max

$$f''(x) = \frac{(x^2-1)(-2x^2+2+2x(4x))}{(x^2-1)^4} = \frac{-2x^2+2+8x^2}{(x^2-1)^3}$$

$$= \frac{(x^2-1)(6x^2+2)}{(x^2-1)^3} = \frac{6x^2+2}{(x^2-1)^2}$$

$$\frac{6x^2+2}{(x^2-1)^2} = 0 \quad 6x^2+2 \neq 0$$

$$\textcircled{2} f(x) = x^{\frac{2}{3}} - x \quad \text{Dom: } \mathbb{R}$$

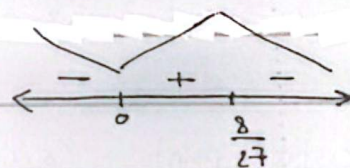
$$f'(x) = \frac{2}{3} x^{-\frac{1}{3}} - 1 = \frac{2}{3x^{\frac{1}{3}}} - \frac{1}{1} \cdot \frac{3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$$

$$f'(x) = \frac{2-3x^{\frac{1}{3}}}{3x^{\frac{1}{3}}}$$

$$f'(x) = 0 \Rightarrow 2-3x^{\frac{1}{3}} = 0 \Rightarrow 3x^{\frac{1}{3}} = 2 \Rightarrow x^{\frac{1}{3}} = \frac{2}{3} \Rightarrow x = \left(\frac{2}{3}\right)^3 = \frac{8}{27}$$

$$f'(x) = \text{d.n.e} \Rightarrow 3x^{\frac{1}{3}} = 0 \Rightarrow x = 0$$

→ critical point: $(\frac{8}{27}, f(\frac{8}{27})) / (0, f(0))$



→ increasing $(0, \frac{8}{27})$

→ decreasing $(-\infty, 0) \cup (\frac{8}{27}, \infty)$

→ local min point $(0, f(0))$

→ local max point $(\frac{8}{27}, f(\frac{8}{27}))$

$$f''(x) = \frac{-2}{9} x^{-\frac{4}{3}} = \frac{-2}{9x^{\frac{4}{3}}} \neq 0$$

→ concave down $(-\infty, \infty)$

→ there is no inflection point

3) $f(x) = \sin x - \cos x$, $\text{Dom} [0, \pi]$

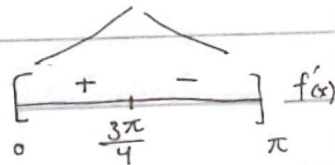
$$f'(x) = \cos x + \sin x$$

$$\cos x + \sin x = 0 \Rightarrow \cos x = -\sin x$$

$$1 = -\tan x$$

$$x = \frac{3\pi}{4} \in \text{Dom}$$

→ critical point $(\frac{3\pi}{4}, f(\frac{3\pi}{4}))$



→ increasing $(0, \frac{3\pi}{4})$

→ decreasing $(\frac{3\pi}{4}, \pi)$

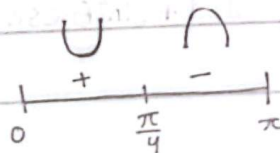
→ absolute max point $(\frac{3\pi}{4}, f(\frac{3\pi}{4}))$

$$f(0) = -1 \Rightarrow \text{absolute Min point } (0, -1)$$

$$f(\pi) = 1 \quad \times$$

$$f''(x) = -\sin x + \cos x = 0 \Rightarrow \tan x = 1$$

$$x = \frac{\pi}{4}$$



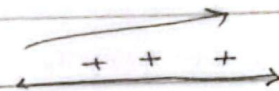
→ concave up $(0, \frac{\pi}{4})$

→ concave down $(\frac{\pi}{4}, \pi)$

→ inflection point $(\frac{\pi}{4}, f(\frac{\pi}{4}))$

4) $f(x) = \tan^{-1}(2x)$ $\text{Dom: } \mathbb{R}$

$$f'(x) = \frac{2}{1+4x^2} \neq 0 \neq \text{d.n.e}$$



→ there is no critical point

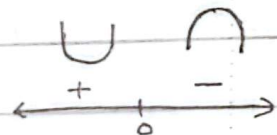
→ increasing $(-\infty, \infty)$

→ there is no decreasing

→ No extrema

$$f''(x) = \frac{-2(8x)}{(1+4x^2)^2} = \frac{-16x}{(1+4x^2)^2} = 0$$

$$x = 0$$



→ concave up $(-\infty, 0)$

→ concave down $(0, \infty)$

→ inflection point $(0, f(0))$

• Integration:

$$\textcircled{1} \int a \, dx = ax + C$$

$$\textcircled{2} \int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

$$\textcircled{3} \int f(x) \pm g(x) \, dx = \int f(x) \, dx \pm \int g(x) \, dx$$

e.g) $\int 3x - x^2 + 5 \, dx$
 $\frac{3x^2}{2} - \frac{x^3}{3} + 5x + C$

e.g) $\int 2x(3x-1)^2 \, dx$
 $\int 2x(9x^2 - 6x + 1) \, dx$

$\int 18x^3 - 12x^2 + 2x \, dx$
 $\frac{18x^4}{4} - \frac{12x^3}{3} + x^2 + C$

e.g) $\int \frac{5x^3 + 4}{\sqrt{x}} \, dx$

$$\int \frac{5x^3}{\sqrt{x}} + \frac{4}{\sqrt{x}} = 5x^3 \cdot x^{-\frac{1}{2}} + 4x^{-\frac{1}{2}}$$

$$= \int 5x^{\frac{5}{2}} + 4x^{\frac{1}{2}} \, dx$$

$$5x^{\frac{7}{2}} \left(\frac{2}{7}\right) + 4x^{\frac{3}{2}} (2) + C$$

$$= \frac{10}{7} x^{\frac{7}{2}} + 8x^{\frac{3}{2}} + C$$

• Remark:

$$1. \int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + C$$

$$2. \int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + C$$

$$3. \int \sec^2(ax+b) \, dx = \frac{1}{a} \tan(ax+b) + C$$

$$4. \int \csc^2(ax+b) \, dx = -\frac{1}{a} \cot(ax+b) + C$$

$$5. \int \sec(ax+b) \tan(ax+b) \, dx = \frac{1}{a} \sec(ax+b) + C$$

$$6. \int \csc(ax+b) \cot(ax+b) \, dx = -\frac{1}{a} \csc(ax+b) + C$$

e.g) $\int \cos(2x-3) \, dx = \frac{1}{2} \sin(2x-3) + C$

e.g) $\int \frac{\sin 2x}{\cos x} \, dx = \int \frac{2 \sin x \cos x}{\cos x} \, dx$

$$\int 2 \sin x \, dx = -2 \cos x + C$$

e.g) $\int 1 + \sin^2 \theta \csc \theta \, d\theta =$

$$\int 1 + \sin^2 \theta \cdot \frac{1}{\sin \theta} \, d\theta = \int 1 + \sin \theta \, d\theta$$

$$= \theta + (-\cos \theta) + C$$

$$\text{e.g.) } \int \sec(2x) (\sec(2x) + \tan(2x)) dx$$

$$= \int \sec^2(2x) + \sec(2x) \tan(2x) dx$$

$$= \frac{1}{2} \tan(2x) + \frac{1}{2} \sec(2x) + C$$

$$\text{e.g.) } \int \cos^2(4x) + \tan^2(5x) dx$$

$$\int \frac{1}{2} + \frac{1}{2} \cos(8x) + \sec^2(5x) - 1 dx$$

$$\frac{1}{2} x + \frac{1}{16} \sin(8x) + \frac{1}{5} \tan(5x) - x + C$$

$$\text{e.g.) } \int \frac{\sin 4x}{\cos^2 4x} dx = \int \frac{\sin 4x}{\cos 4x \cdot \cos 4x} dx$$

$$= \int \tan 4x \sec 4x dx = \frac{1}{4} \sec 4x + C$$

• Remark:

$$\textcircled{1} \int \cosh(ax+b) dx = \frac{1}{a} \sinh$$

$$\textcircled{2} \int \sinh(ax+b) dx = \frac{1}{a} \cosh(ax+b) + C$$

$$\textcircled{3} \int \operatorname{sech}^2(ax+b) dx = \frac{1}{a} \tanh(ax+b) + C$$

$$\textcircled{4} \int \operatorname{csch}^2(ax+b) dx = -\frac{1}{a} \coth(ax+b) + C$$

$$\textcircled{5} \int \operatorname{sech}(ax+b) \tanh(ax+b) dx = -\frac{1}{a} \operatorname{sech}(ax+b) + C$$

$$\textcircled{6} \int \operatorname{csch}(ax+b) \coth(ax+b) dx = -\frac{1}{a} \operatorname{csch}(ax+b) + C$$

$$\text{e.g.) } \int (\sin x + \cos x)^2 dx$$

$$\int \sin^2 x + 2 \sin x \cos x + \cos^2 x dx$$

$$\int 1 + \sin 2x dx$$

$$x - \frac{1}{2} \cos 2x + C$$

$$\text{e.g.) } \int \cosh(3x-5) dx$$

$$= \frac{1}{3} \sinh(3x-5) + C$$

$$\text{e.g.) } \int \text{sech}^2(2x) dx$$

$$\frac{1}{2} \tanh(2x) + C$$

$$\text{e.g.) } \int 2^{3x+5} dx$$

$$\frac{2^{3x+5}}{3 \ln 2} + C$$

$$\text{e.g.) } \int \text{sech}(3x) \tanh(3x) dx$$

$$-\frac{1}{3} \text{sech}(3x) + C$$

$$\text{e.g.) } \int \frac{e^x + 3}{e^x} dx$$

$$\int \frac{e^x}{e^x} + \frac{3}{e^x} dx = \int 1 + 3e^{-x} dx$$

$$x + \frac{3e^{-x}}{-1} + C = x - 3e^{-x} + C$$

$$\text{e.g.) } \int \frac{3e^x}{\cosh x + \sinh x} dx = \int \frac{3e^x}{e^x} dx$$

$$= 3x + C$$

$$\text{e.g.) } \int 3^{3-2\log_3 x} dx$$

$$\int 3^3 \cdot 3^{-2\log_3 x} dx$$

$$\int 27 \cdot 3^{\log_3 x^{-2}} dx$$

$$\int 27 x^{-2} dx$$

$$= 27 \frac{x^{-1}}{-1} + C = -\frac{27}{x} + C$$

• Remark :

$$\textcircled{1} \int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C$$

$$\textcircled{2} \int a^{bx+c} dx = \frac{a^{bx+c}}{b \cdot \ln a} + C$$

$$\text{e.g.) } \int e^{2x-1} dx$$

$$\frac{e^{2x-1}}{2} + C$$

• Remark :-

$$\textcircled{1} \int \frac{f(x)}{f(x)} dx = \ln |f(x)| + C$$

$$\textcircled{2} \int \frac{f(x)}{a^2 + (f(x))^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{f(x)}{a} \right) + C$$

* Example :

$$\textcircled{1} \int \frac{5x}{x^2+1} dx = \frac{5}{2} \ln |x^2+1| + C$$

$$\textcircled{2} \int \frac{\cos(3x)}{5 + \sin^2(3x)} dx = \frac{1}{3} \int \frac{\cos(3x)}{(\sqrt{5})^2 + (\sin(3x))^2} dx$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{\sin(3x)}{\sqrt{5}} \right) + C$$

$$\textcircled{3} \int \frac{e^x}{5 + e^{2x}} dx = \int \frac{e^x}{(\sqrt{5})^2 + (e^x)^2} dx$$

$$= \frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{e^x}{\sqrt{5}} \right) + C$$

$$\textcircled{4} \int \frac{5x}{3 + x^4} dx = \frac{5}{2} \int \frac{2x}{(\sqrt{3})^2 + (x^2)^2} dx$$

$$= \frac{5}{2} \cdot \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2}{\sqrt{3}} \right) + C$$

$$\textcircled{5} \int \frac{1}{e^x + 1} dx = \int \frac{1}{e^x(1 + e^{-x})} dx$$

$$= \int \frac{e^{-x}}{1 + e^{-x}} dx = -\ln |1 + e^{-x}| + C$$

$$\textcircled{6} \int \frac{1}{x^3 + x} dx \leftarrow \text{Partial Fractions}$$

$$= \int \frac{1}{x^3(1 + x^2)} dx = \frac{x^{-3}}{(1 + x^2)}$$

$$\int \frac{-2x^{-3}}{-2(1 + x^2)} dx = -\frac{1}{2} \ln |1 + x^2| + C$$

$$\textcircled{7} \int \frac{1}{x \ln x} dx = \int \frac{\frac{1}{x}}{\ln x} dx$$

$$= \ln |\ln x| + C$$

$$\textcircled{8} \int \frac{1}{(x^2+1) \tan^{-1} x} dx = \int \frac{\frac{1}{1+x^2}}{\tan^{-1} x} dx$$

$$= \ln |\tan^{-1} x|$$

* Remark:

$$1) \int \tan(ax+b) dx = -\frac{1}{a} \ln |\cos(ax+b)| + C$$

$$= \int \frac{\sin(ax+b)}{\cos(ax+b)} dx$$

$$2) \int \cot(ax+b) dx = \frac{1}{a} \ln |\sin(ax+b)| + C$$

$$3) \int \sec(ax+b) dx = \frac{1}{a} \ln |\sec(ax+b) + \tan(ax+b)| + C$$

$$\int \frac{\sec(ax+b) (\sec(ax+b) + \tan(ax+b))}{(\sec(ax+b) + \tan(ax+b))} dx$$

$$= \int \frac{\sec^2(ax+b) + \sec(ax+b) \tan(ax+b)}{\sec(ax+b) + \tan(ax+b)} dx$$

$$= \frac{1}{a} \ln |\sec(ax+b) + \tan(ax+b)| + C$$

$$4) \int \csc(ax+b) dx = -\frac{1}{a} \ln |\csc(ax+b) + \cot(ax+b)| + C$$

$$\text{e.g.) } \int \tan 5x dx = -\frac{1}{5} \ln |\cos 5x| + C$$

$$\text{e.g.) } \int \csc(5x-1) dx$$

$$= -\frac{1}{5} \ln |\csc(5x-1) + \cot(5x-1)| + C$$

$$\text{e.g.) } \int \frac{1}{1+\sin x} dx = \frac{1-\sin x}{1-\sin x}$$

$$\int \frac{1-\sin x}{1-\sin^2 x} = \int \frac{1-\sin x}{\cos^2 x} = \int \frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}$$

$$\int \sec^2 x - \tan x \sec x$$

$$= \tan x - \sec x + C$$

* Remark:

$$\cos^{-1} x + \sin^{-1} x = \frac{\pi}{2}$$

$$\text{e.g.) } \int \cos^{-1} x + \sin^{-1} x dx = \int \frac{\pi}{2} dx = \frac{\pi}{2} x + C$$

* Remark:

$$① \int \frac{f'(x)}{\sqrt{a^2 + f^2(x)}} dx = \sin^{-1} \left(\frac{f(x)}{a} \right) + C$$

$$= -\cos^{-1} \left(\frac{f(x)}{a} \right) + C$$

$$② \int \frac{f'(x)}{|f(x)| \sqrt{(f(x))^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \left(\frac{f(x)}{a} \right) + C$$

$$= -\frac{1}{a} \csc^{-1} \left(\frac{f(x)}{a} \right) + C$$

$$\text{e.g.) } \int \frac{2}{\sqrt{3-x^2}} dx = \int \frac{2}{\sqrt{(\sqrt{3})^2 - (x)^2}} dx$$

$$= 2 \sin^{-1} \left(\frac{x}{\sqrt{3}} \right) + C$$

$$\text{e.g.) } \int \frac{e^x}{\sqrt{4-e^{2x}}} dx = \int \frac{e^x}{\sqrt{(2)^2 - (e^x)^2}}$$

$$= \sin^{-1} \left(\frac{e^x}{2} \right) + C$$

$$\text{e.g.) } \int \frac{\cos(2x)}{\sin 2x \sqrt{\sin^2 2x - 1}} = \frac{1}{2} \int \frac{2 \cos 2x}{|\sin 2x| \sqrt{\sin^2 2x - 1}}$$

$$= \frac{1}{2} \cdot \frac{1}{1} \sec^{-1}(\sin 2x) + C$$

$$\text{e.g.) } \int \frac{\sin 2x}{\sqrt{5 - \cos^2 2x}} dx = \frac{1}{2} \int \frac{2 \sin 2x}{\sqrt{(\sqrt{5})^2 - (\cos 2x)^2}}$$

$$= -\frac{1}{2} \sin^{-1} \left(\frac{\cos 2x}{\sqrt{5}} \right) + C$$

29/12/2020

* Definite integrals :-

$$\textcircled{1} \int_0^1 (3x^2 - 1)^2 dx$$

$$\int_0^1 (9x^4 - 6x^2 + 1) dx$$

$$= \frac{9x^5}{5} - \frac{6x^3}{3} + x \Big|_0^1$$

$$= \left(\frac{9}{5} - 2 + 1 \right) - (0) = \frac{9}{5} - 1 = \frac{4}{5}$$

$$\textcircled{2} \int_{-1}^1 \frac{1}{1+x^2} dx = \tan^{-1} x \Big|_{-1}^1$$

$$\tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4} \right) = \frac{\pi}{2}$$

$$\textcircled{3} \int_1^{e^2} \frac{t}{1+t^2} dt = \frac{1}{2} \ln|1+t^2| \Big|_1^{e^2}$$

$$= \frac{1}{2} \left[\ln(1+e^4) - \ln(2) \right]$$

$$= \frac{1}{2} \ln \frac{1+e^4}{2}$$

$$= \ln \sqrt{\frac{1+e^4}{2}}$$

$$(4) \int_1^{e^2} \frac{t}{1+t^4} dt = \int_1^{e^2} \frac{t}{(1)^2 + (t^2)^2}$$

$$2 \int_1^{e^2} \frac{2t}{(1)^2 + (t^2)^2} = 2 \tan^{-1} \left(\frac{t^2}{1} \right) \Big|_1^{e^2}$$

خصائص التكامل
* properties of definite integral :-

$$(1) \int_a^a f(x) dx = 0$$

$$(2) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$(3) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\text{e.g.) } \int_{-5}^5 \frac{1}{\sqrt{x + \sin x}} dx = 0$$

$$\text{e.g.) if } \int_3^{2x-1} f(y) dy = 5ax^2 - 4 \text{ find } a$$

$$\text{sub } [x=2] : \int_3^3 f(y) dy = 5a(4) - 4$$

$$5a = 1 \Rightarrow a = \frac{1}{5}$$

$$\int_2^2 f(x) = 2 \quad \text{Date } \frac{2}{2} \int_1^2 f(x) = \frac{4}{2} \quad \text{No } \int_5^1 f(x) = 4$$

$$(3) \text{ if } \int_1^2 2f(x) dx = 4 \quad \int_{\frac{1}{3}}^1 3f(x) dx = 2$$

$$\text{find } \int_5^2 7f(x) = ?? \quad 7 \int_5^2 f(x) = ??$$

$$7 \int_5^2 f(x) = 7 \left(\int_5^1 f(x) dx + \int_1^2 f(x) dx \right)$$

$$= 7(4 + 2) = 7(6) = 42$$

$$\text{e.g.) } \int_{-1}^5 f(x) dx = 10, \quad \int_{-2}^{-1} 3f(x) dx = -9$$

$$\text{find } \int_5^{-2} f(x) dx = ??$$

$$= \int_5^{-1} f(x) dx + \int_{-1}^{-2} f(x) dx$$

$$= -10 + 3 = -7$$

$$\text{e.g.) } \int_{-1}^{+2} |2x-2| dx \quad 2x-2=0 \Rightarrow x=1$$

$$= \int_{-1}^1 (2x-2) dx + \int_1^2 (2x-2) dx = \left(x^2 - 2x \right) \Big|_{-1}^1 + \left(x^2 - 2x \right) \Big|_1^2$$

$$= [-1 - 3] + (0 - -1) = 4 + 1 = 5$$

* Remark :-

$$\text{e.g.) } \frac{d}{dx} \left(\int_{\cos(3x)}^x \sin t \, dt \right)$$

1) if $f(x)$ odd function ($f(-x) = -f(x)$)
then $\int_{-a}^a f(x) \, dx = 0$

$$\sin\left(\frac{x}{3}\right) \cdot \frac{1}{3} \cdot \ln 3 - \sin(\cos(3x)) \cdot (-3 \sin(3x))$$

2) if $f(x)$ even function ($f(-x) = f(x)$) then

e.g) find $f'(4)$ such that

$$\int_{-a}^a f(x) \, dx = 2 \int_0^a f(x) \, dx$$

$$f(x) = x \int_{16}^{x^2} \frac{t}{t^2+1} \, dt$$

e.g) $\int_{-3}^3 \frac{\sin x}{x^4+x^2+1} \, dx$ موجب ابيضاض
 $\int_{-3}^3 \text{odd} = 0$ odd = odd function

$f'(x) = x \left(\frac{x^2(2x)}{x^4+1} - 0 \right) + \int_{16}^{x^2} \frac{t}{t^2+1} \, dt \cdot 1$ اقرانيا

$$f'(4) = 4 \left(\frac{(4)^2(8)}{4^4+1} \right) + \int_{16}^{16} \frac{t}{t^2+1} \, dt$$

$$f'(4) = 1.99$$

* Fundamental Thm of calculus *

$$1) \frac{d}{dx} \left(\int f(x) \, dx \right) = f(x)$$

2) $\frac{d}{dx} \left(\int_{f(x)}^{g(x)} h(y) \, dy \right) =$ بر مسئله كمال

$$= h(g(x)) \cdot g'(x) - h(f(x)) \cdot f'(x)$$

* Integration by substitution

$$\int f(x) \cdot g(f(x)) dx$$

eg) $\int \frac{\cos^2(\ln x)}{x} dx$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$dx = x du$$

eg) $\int \frac{\cos^2 u}{x} dx =$

$$= \int \frac{1}{2} + \frac{1}{2} \cos 2u du$$

$$\frac{1}{2} u + \frac{1}{2} \left(\frac{1}{2}\right) \sin 2u +$$

$$= \frac{1}{2} \ln x + \frac{1}{4} \sin 2 \ln x$$

eg) $\int \frac{\sin x}{3} \cos x dx$

$$\int 3^u \cos x \cdot \frac{du}{\cos x}$$

$$= \frac{3^u}{\ln 3} + C$$

$$\ln 3$$

$$= \frac{3^{\sin x}}{\ln 3} + C$$

$$u = \sin x$$

$$du = \cos x dx$$

$$dx = \frac{du}{\cos x}$$

eg) $\int \frac{\sqrt{1+\tan x}}{\cos^2 x} dx$

نفرضه $u = 1 + \tan x$
الجزء

eg) $\int x^2 e^{x^3} dx$

$$\int x^2 e^u \frac{du}{3x^2}$$

$$\frac{1}{3} \int e^u du = \frac{1}{3} e^u$$

$$\frac{1}{3} e^{x^3} + C$$

$$u = x^3$$

$$du = 3x^2 dx$$

$$dx = \frac{du}{3x^2}$$

$$= \int \frac{(u)^{\frac{1}{2}}}{\cos^2 x} du$$

$$\frac{2}{3} u^{\frac{3}{2}} + C$$

$$\frac{2}{3} \sqrt{(1+\tan x)^3} + C$$

$$u = 1 + \tan x$$

$$du = \sec^2 x dx$$

$$dx = \frac{du}{\sec^2 x}$$

$$dx = \cos^2 x du$$

تقريب التفاضل والتكامل

e.g) $\int \left(1 + \frac{1}{x}\right)^5 \frac{1}{x^2} dx$

$$\int u^5 \frac{1}{x^2} - x^2 du$$

$$\frac{-u^6}{6} + C$$

$$\frac{-\left(1 + \frac{1}{x}\right)^6}{6} + C$$

$$\begin{aligned} u &= 1 + \frac{1}{x} \\ du &= -\frac{1}{x^2} dx \\ dx &= -x^2 du \end{aligned}$$

e.g) $\int \frac{e^{-x}}{\sqrt{1-e^{-x}}} dx$

$$\begin{aligned} u &= 1 - e^{-x} \\ du &= e^{-x} dx \\ dx &= \frac{du}{e^{-x}} \end{aligned}$$

$$\int \frac{e^{-x}}{\sqrt{u}} \frac{du}{e^{-x}}$$

$$\int u^{-\frac{1}{2}} du$$

$$= 2u^{\frac{1}{2}} + C$$

$$= 2\sqrt{u} + C$$

$$= 2\sqrt{1-e^{-x}} + C$$

e.g) $\int_0^1 x \sqrt{1-x} dx$

$$\begin{aligned} u &= 1-x \\ du &= -dx \\ x &= 1-u \end{aligned}$$

$$\int_0^1 x (u)^{\frac{1}{2}} - du$$

$$-\int_1^0 u^{\frac{1}{2}} - u^{\frac{3}{2}} du$$

$$-\left(\frac{2u^{\frac{3}{2}}}{3} - \frac{2u^{\frac{5}{2}}}{5}\right) \Big|_1^0 = -\left(0 - \left(\frac{2(1)^{\frac{3}{2}}}{3} - \frac{2(1)^{\frac{5}{2}}}{5}\right)\right)$$

$$= \frac{4}{15}$$

e.g) $\int \frac{\ln x}{x} dx$

تقريب التفاضل والتكامل

$$u = \ln x$$

$$\int \frac{u}{x} x du$$

$$du = \frac{1}{x} dx$$

$$dx = x du$$

$$\frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

e.g) $\int \frac{dx}{x \sqrt{1-(\ln x)^2}}$

f(x)

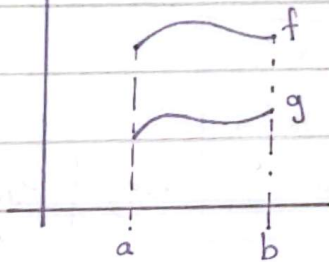
$$= \sin^{-1}(\ln x) + C$$

تقريب التفاضل والتكامل

* Area between two Curves *

* Thm: if f & g are cont functions on the interval $[a, b]$ such that $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region bounded above by $y = f(x)$ below by $y = g(x)$ on the left by $x = a$, on the right by $x = b$ is:

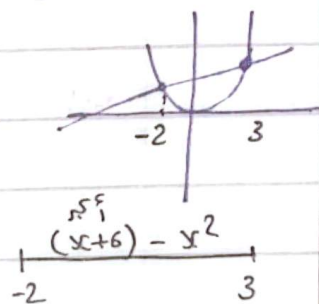
$$A = \int_a^b (f(x) - g(x)) dx$$



المنطقة المحيطة بالخط $f=g$ / من $x=a$ إلى $x=b$

المنطقة المحيطة بالخط $y=x^2$ - $y=x+6$ من $x=-2$ إلى $x=3$

e.g) find the area of region enclosed by $y = x^2$, $y = x + 6$?



$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

$$(x-3)(x+2)$$

$$\boxed{x=3} \quad \boxed{x=-2}$$

$$A = \int_{-2}^3 (x+6) - x^2 dx$$

مثال: إيجاد المساحة المحيطة بالخط $y=e^x$ و $y=e^{2x}$ من $x=0$ إلى $x=\ln 2$

e.g) find the area enclosed by $y = e^x$, $y = e^{2x}$, $x = 0$, $x = \ln 2$?

$$e^x = e^{2x}$$

$$e^x - e^{2x} = 0$$

$$e^x (e^x - 1) = 0$$

$$e^x \neq 0$$

$$\ln e^x = 1$$

$$\boxed{x=0}$$

$$x = \ln 2$$

$$\int_0^{\ln 2} (e^{2x} - e^x) dx$$

$$A = \int_0^{\ln 2} (e^{2x} - e^x) dx$$

e.g) find the area inclosed by

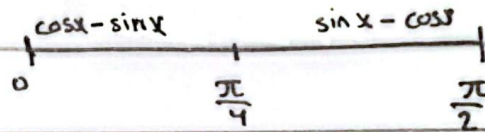
$$y = \sin x, y = \cos x, x=0, x=\frac{\pi}{2}$$

$$\sin x = \cos x$$

$$\frac{\sin x}{\cos x} = 1$$

$$\tan x = 1$$

$$\text{إذن } \left\{ x = \frac{\pi}{4} \right\}$$



$$A = \int_0^{\frac{\pi}{4}} (\cos x - \sin x) + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x - \cos x$$