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Complex numbers :-

$$Z = x + iy \rightarrow \text{Im } Z$$



Re Z

* Addition of two

• complex numbers :-

$$\text{Let } Z_1 = 8 + 3i$$

$$Z_2 = 9 - 2i$$

$$Z_1 + Z_2 = (8 + 3i) + (9 - 2i) = 17 + i$$

* Multiplication is defined by :-

$$\begin{aligned} Z_1 Z_2 &= (8 + 3i)(9 - 2i) = 72 - 16i + 27i - 6i^2 \\ &= 72 + 11i + 6 \\ &= 78 + 11i \end{aligned}$$

* Subtraction of two complex :-

$$\begin{aligned} Z_1 - Z_2 &= (8 + 3i) - (9 - 2i) \\ &= -1 + 5i \end{aligned}$$

$$= 0.02 \times \text{total sales}$$

* Division is defined by:-

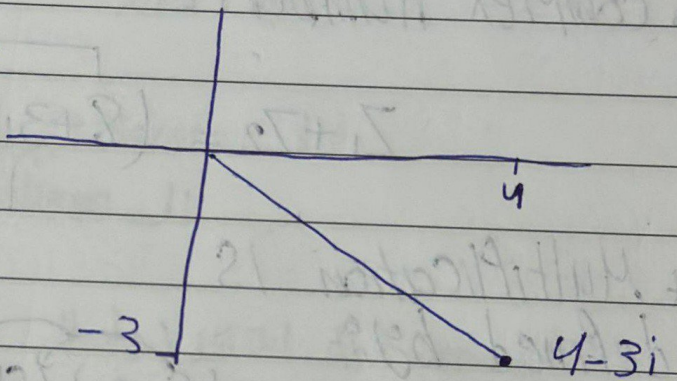
$$\frac{Z_1}{Z_2} = \frac{8+3i}{9-2i} \times \frac{9+2i}{9+2i} = \frac{72+16i+27i+6i^2}{81+18i-18i-4i^2}$$

-6
54(-1)
=+4

$$= \frac{66+43i}{85} = \frac{66}{85} + \frac{43i}{85}$$

* Complex Plane:-

$$Z = 4-3i$$



$$* Z = X + iy$$

$$* \bar{Z} = X - iy \rightarrow \text{conjugate}$$

1st $Z = 5+i$, $Z = 10$, $Z = 7i$

$$X = 5$$

$$X = 10$$

$$X = 0$$

$$Y = 1$$

$$Y = 0$$

$$Y = 7$$

$$\bar{Z} = 5-i$$

$$\bar{Z} = Z = 10$$

$$\bar{Z} = -7i$$

* Polar Form of complex number

$$Z = x + iy = r e^{i\theta}, \text{ Polar Form}$$

complex number

* $r = \text{modulus} = |Z| = \text{Modulus} = \text{magnitude} = \sqrt{x^2 + y^2}$

$$x = r \cos \theta$$

$$\theta = \tan^{-1} \frac{y}{x} \quad y = r \sin \theta$$

Let $Z = 1 + i$ Find $r = \sqrt{1^2 + 1^2} = \sqrt{2}$

Polar Form? $\theta = \tan^{-1} 1 = \pi/4$

sol $Z = \sqrt{2} e^{i\pi/4}$

Let $Z = \sqrt{2} e^{i\pi/4}$

$$x = r \cos \theta = \sqrt{2} \cos \pi/4$$

Find the cartesian?

$$= 1$$

$$y = r \sin \theta = 1$$

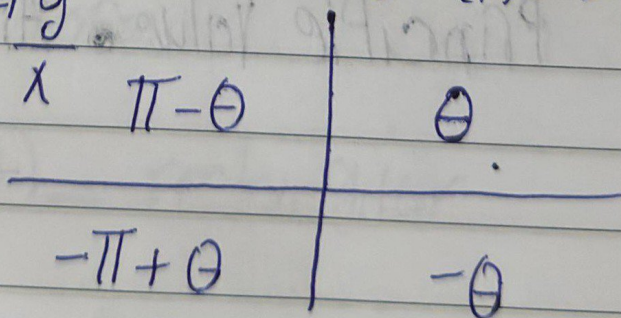
sol $Z = 1 + i$

* Euler Form $e^{i\theta} = \cos \theta + i \sin \theta$

How find the angle??

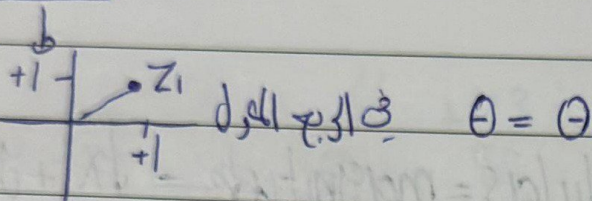
$$\theta = \tan^{-1} \frac{y}{x}$$

Complex plane



Find θ for the following the complex:-

• $Z_1 = 1 + i$ sol. $\theta = \tan^{-1} \frac{1}{1} = \pi/4$



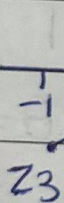
• $Z_2 = -1 + i$ $\theta = \tan^{-1} \frac{-1}{1} = \pi/4$

في الزاوية

$\theta = \pi - \theta$

$= \pi - \frac{\pi}{4} \rightarrow \theta = 3\pi/4$

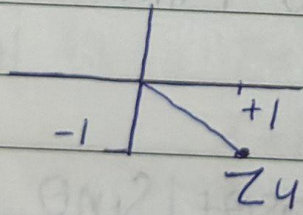
• $Z_3 = -1 - i$ $\theta = \tan^{-1} \frac{-1}{-1} = \pi/4$



$\theta = -\pi + \theta$

$= -\pi + \pi/4 = -3\pi/4$

• $Z_4 = 1 - i$ $\theta = \tan^{-1} \frac{1}{-1} = \pi/4$



$\theta = -\theta$

$\theta = -\pi/4$

* Principle value • $\text{Arg}(z) = \theta = \tan^{-1} y/x$

$-\pi \leq \text{Arg} \leq \pi$

$$\arg(z) = \text{Arg}(z) \pm 2\pi n \quad \begin{matrix} \nearrow \\ \pm 1, \pm 2, \dots \end{matrix}$$

Let $z = 1+i$

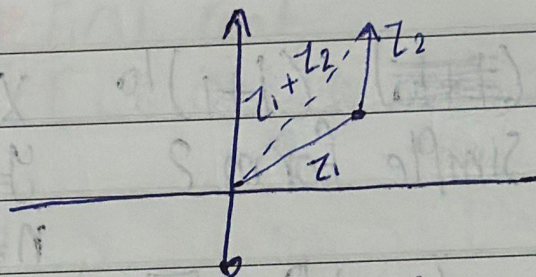
Find $\text{Arg}(z)$ and $\arg(z)$:- $\text{Arg} = \theta = \tan^{-1} \frac{1}{1}$
 $= \pi/4$

$$\arg = \text{Arg} \pm 2\pi n$$

$$= \pi/4 \pm 2\pi n$$

* Triangle Inequality :-

$$|z_1 + z_2| \leq |z_1| + |z_2|$$



* Multiplication :-

$$z_1 = r_1 e^{i\theta_1} \quad z_2 = r_2 e^{i\theta_2}$$

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)} = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$z_1 / z_2 = r_1 / r_2 e^{i(\theta_1 - \theta_2)} = r_1 / r_2 [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

Let $z_1 = 2e^{i\pi/4}$, $z_2 = 3e^{i\pi/2}$ Find $z_1 z_2$ and z_1 / z_2

$$z_1 z_2 = 6e^{i3\pi/4} \rightarrow \text{Polar Form}$$

$$6(\cos(3\pi/4) + i \sin(3\pi/4)) \rightarrow \text{rectangular}$$

* Power complex numbers:-

$$(1+i)^2 = 1 + 2i + i^2 = 2i$$

$$[z]^n = [x+iy]^n = (r e^{i\theta})^n \quad n \rightarrow \text{integer } \pm 1, \pm 2, \dots$$

$$\sqrt{x^2+y^2} = r^n (\cos n\theta + i \sin n\theta)$$

Find ~~(1+i)~~ $(1+i)^{10}$ in simple form? $x=1$ $y=1$ $r = \sqrt{1^2+1^2} = \sqrt{2}$ $\theta = \tan^{-1} 1/1 = \pi/4$

$$\begin{aligned} (1+i)^{10} &= (\sqrt{2} e^{i\pi/4})^{10} = (\sqrt{2})^{10} e^{10(i\pi/4)} \\ &= 32 e^{5\pi/2} = 32 e^{i\pi/2} \end{aligned}$$

$\frac{5\pi}{2} - \frac{\pi}{2}$

* Roots of complex numbers :-

$$\sqrt[n]{z} = \sqrt[n]{x+iy} = \sqrt[n]{r} \left[\cos \frac{\theta+2\pi k}{n} + i \sin \frac{\theta+2\pi k}{n} \right]$$

$k: 0, 1, 2, \dots, n-1$

find $\sqrt[3]{1}$ in Simple Form^o

$$x=1$$

$$y=0 \quad = \sqrt[n]{r} \left[\cos \frac{\theta + 2\pi k}{n} + i \sin \frac{\theta + 2\pi k}{n} \right]$$

$$n=3$$

$$r = \sqrt{1^2 + 0^2} = 1$$

$$\theta = \tan^{-1} 0/1 = 0$$

$$k = 0, 1, 2$$

k=0 First root

$$= \sqrt[3]{1} \left[\cos \frac{0 + 2\pi(0)}{3} + i \sin \frac{0 + 2\pi(0)}{3} \right]$$

$$\text{**k=1** } \rightarrow \sqrt[3]{1} \left[\cos \frac{0 + 2\pi(1)}{3} + i \sin \frac{0 + 2\pi(1)}{3} \right]$$

$$= \frac{-1}{2} + i \frac{\sqrt{3}}{2}$$

$$\text{**k=2** } \rightarrow \frac{-1}{2} - i \frac{\sqrt{3}}{2}$$

* exponential of complex numbers:

$$z = x + iy$$

$$e^z = e^{x+iy} = e^x e^{iy} = e^x [\cos(y) + i \sin(y)]$$

$$\rightarrow e^x \cos(y) + i e^x \sin(y) \rightarrow \begin{matrix} \text{Re} \\ \text{Im} \end{matrix}$$

$$|e^z| = \sqrt{x^2 + y^2} = \sqrt{(e^x \cos(y))^2 + (e^x \sin(y))^2}$$

$$= \sqrt{e^{2x} \cos^2 y + e^{2x} \sin^2 y} = e^x \sqrt{\underbrace{\cos^2 y + \sin^2 y}_1} = e^x$$

(*) Principle Value ($\text{Arg}(z)$)

$$\text{Arg}(e^z) = \theta = \tan^{-1} \left[\frac{y}{x} \right] = \tan^{-1} \left[\frac{e^x \sin y}{e^x \cos y} \right]$$

$$= \tan^{-1} [\tan(y)] = y$$

$$\arg(z) = y \pm 2\pi n \rightarrow \pm 1, \pm 2, \dots$$

note:-

① $[e^z]' = e^z$ ② $[e^z]$

② $e^{z_1} e^{z_2} = e^{z_1 + z_2}$

③ $\frac{e^{z_1}}{e^{z_2}} = e^{(z_1 - z_2)}$

ex: write $e^{1.4 - 0.6i}$ in simple form and find the Magnitud and θ ??

$\rightarrow e^{1.4 - 0.6i} = e^{1.4} e^{-0.6i} = e^{1.4} [\cos -0.6 + i \sin -0.6]$
 $= e^{1.4} \cos 0.6 - i e^{1.4} \sin 0.6$

Radian

$\theta_{\text{rad}} = \theta_{\text{deg}} \times \frac{\pi}{180} \rightarrow \theta_{\text{deg}} = \theta_{\text{rad}} \times \frac{180}{\pi}$

Magnitud:-

$|e^z| = |e^{1.4 - 0.6i}| = e^{1.4} = x$

$\theta = \tan^{-1} y = -0.6$

$\arg = -0.6 \pm 2\pi n$

* Trigonometric and hyperbolic Functions:-

• $e^{iz} = \cos(z) + i \sin(z) \dots$ ①

• $e^{-iz} = \cos(-z) + i \sin(-z), \cos z - i \sin z \dots$ ②

① + ② $e^{iz} + e^{-iz} = 2 \cos(z) + 0$

$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

Zoom

①-②

$$e^{iz} - e^{-iz} = 0 + 2i \sin z \quad \left| \sin z = \frac{e^{iz} - e^{-iz}}{2i} \right|$$

$$(*) \quad \tan z = \frac{\sin(z)}{\cos(z)} = \frac{e^{iz} - e^{-iz}}{2i} \div \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$(*) \quad \cot z = \frac{1}{\tan z} = \frac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{-iz}}$$

$$(*) \quad \sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}$$

$$(*) \quad \csc z = 1/\sin z = 2i / (e^{iz} - e^{-iz})$$

$$(*) \quad [\cos(z)]' = -\sin(z) \quad (*) \quad [\sin(z)]' = \cos(z)$$

$$(*) \quad [\tanh(z)]' = \operatorname{sech}^2(z)$$

$$(*) \quad \cos(z) = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y$$

ex: Find $\cos(\pi+3i)$ in simple form? sol: $\cos(\pi) \cosh(3) - i \sin(\pi) \sinh(3)$
 $= -\cosh(3) + i(0)$

* $\sin(z) = \sin(x+iy) = \sin(x) \cosh(y) + i \cos(x) \sinh(y)$

ex: Find $\sin(\pi+3i)$ in simple form? sol: $\sin(\pi) \cosh(3) + i \cos(\pi) \sinh(3)$
 $= -i \sinh(3)$

(*) $\tan(z) = \tan(x+iy)$

Find $\tan(\pi+3i)$ in simple form? sol: $= \frac{\sin(\pi+3i)}{\cos(\pi+3i)} = \frac{-i \sinh(3)}{-\cosh(3)}$
 $= i \tanh(3)$

Rad $\rightarrow$ deg

* ملاحظات

• $\cos^2(z) + \sin^2(z) = 1$

• $\cosh^2(x) = 1 + \sinh^2(x)$

• $|\cos(z)|^2 = \cos^2(x) + \sinh^2(y)$

• $|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$

• $\sin(z_1) \cos(z_2) = 1/2 [\sin(z_1+z_2) + \sin(z_1-z_2)]$

• $\cos(z_1 \pm z_2) = \cos(z_1) \cos(z_2) \mp \sin(z_1) \sin(z_2)$

• $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$

3. المثلثات المثلثية المثلثية

$$\rightarrow |Z| = 1 \quad (X+iy) = 1X + i^2y = \underbrace{1X}_{\text{Re}} + \underbrace{-y}_{\text{Im}}$$

$$\frac{Z}{1} = \frac{(X+iy)}{0+1} * \frac{0-1}{0-i} = -iX - i^2(y)$$

$$= \underbrace{-iy}_{\text{Im}} + \underbrace{y}_{\text{Re}}$$

find:-

$$1) \sin(\pi + 5i)$$

$$2) \cos(2\pi + 3i)$$

$$3) e^{i(2\pi + \pi i)}$$

*

* hyperbolic Function^o.

$$\bullet \cosh(z) = \frac{e^z + e^{-z}}{2}$$

$$\bullet \sinh(z) = \frac{e^z - e^{-z}}{2}$$

$$\bullet [\cosh(z)]' = \sinh(z) \quad \bullet [\sinh(z)]' = \cosh(z)$$

$$\bullet [\tanh(z)]' = \sinh(z) / \cosh(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$$

$$\bullet \coth(z) = 1 / \tanh(z) = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\bullet \operatorname{sech}(z) = 1 / \cosh(z) = 2 / e^z + e^{-z}$$

$$\bullet \operatorname{csch}(z) = 1 / \sinh(z) = 2 / e^z - e^{-z}$$

complex trigonometric and hyperbolic function are related^o:

$$1) \cosh(iz) = \cos(z) \quad 2) \sinh(iz) = i \sin(z)$$

$$3) \cos(iz) = \cosh(z) \quad 3) \sin(iz) = i \sinh(z)$$

$$(*) \cosh(z) = \cosh(x+iy) = \cosh(x) \cos(y) + i \sinh(x) \sin(y)$$

Ex^o Find $\cosh(3+i\pi)$ in simple form?

$$\text{sol} = \cosh(3) \cos(\pi) + i \sinh(3) \sin(\pi)$$

$$= -\cosh(3)$$

Red

Bo * Deg
 $\frac{180}{\pi}$ $\rightarrow$ د. الج. 3

$$* \sinh(z) = \sinh(x+iy) = \sinh(x) \cos(y) + i \cosh(x) \sin(y)$$

Ex: Find $\sinh(3+\pi i)$ in simple form?

$$= \sinh(3) \cos(\pi) + i \cosh(3) \sin(\pi) = -\sinh(3)$$

Ex: $\tanh(3+\pi i)$ in simple form?

$$= \frac{\sinh(3+\pi i)}{\cosh(3+\pi i)} = \tanh(3)$$

$$*) \cosh(z_1+z_2) = \cosh(z_1) \cosh(z_2) + \sinh(z_1) \sinh(z_2)$$

$$*) \sinh(z_1+z_2) = \sinh(z_1) \cosh(z_2) + \cosh(z_1) \sinh(z_2)$$

• هذه المبرهنات التي راعى في كتابها

$$z^c = e^{\ln(z)c} = e^{c \ln(z)} = e^{c \ln(x+iy)} \\ = e^{c \ln(x+iy)}$$

Ex: Find $(i)^i$ in simple form?

$$(i)^i = e^{\ln(i)^i} = e^{i \ln(i)}$$

$$\ln(i) = \ln r + i\theta = \ln(i) + i [\pi/2 \pm 2\pi n] \\ = e^{i [i (\pi/2 \pm 2\pi n)]} = e^{-\pi/2 \pm 2\pi n}$$

Principle value of $(i)^i$? $\rightarrow e^{-\pi/2}$

note: For principle value $\boxed{n=0}$

Ex: Find $(1+i)^{2-i}$ in simple form? ~~(1+i)~~ $(1+i)^{2-i}$

$$e^{\ln(1+i)^{(2-i)}} = e^{(2-i) \ln(1+i)}$$

$$= e^{(2-i) [\ln r + i\theta]} = e^{(2-i) [\ln \sqrt{2} + i (\pi/4 \pm 2\pi n)]}$$

$$= e^{x_3 + iy_3} = e^{x_3} e^{iy_3} = e^{x_3} [\cos y_3 + i \sin y_3] \\ = \underbrace{e^{x_3} \cos y_3}_{\text{Re}} + i \underbrace{e^{x_3} \sin y_3}_{\text{Im}}$$

* solve:

$$1) \ln[-1+3i]$$

$$2) \ln[-1+3i]$$

$$3) (i)^{3i}$$

4) Principle value $(i)^{\pi}$

• Matrix, Vectors, Addition and Scalar Multiplication :-

→ Matrix is a rectangular array of numbers or functions which we will enclose in bracket

$$A = \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 10 \end{bmatrix}$$

↙
element entry

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad B = \begin{bmatrix} x & xy \\ \cos y & \sin x \end{bmatrix}$$

* Linear system $AX = b$

$$4x_1 + 6x_2 + 9x_3 = 6$$

$$6x_1 + 0x_2 - 2x_3 = 20$$

$$5x_1 - 8x_2 + x_3 = 10$$

$$AX = b$$

$$\begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 10 \end{bmatrix}$$

• Augmenting Matrix of $A = \tilde{A}$

$$\tilde{A} = [A | b]$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{array} \right]$$

$$X = \begin{bmatrix} 3 \\ 1/2 \\ -1 \end{bmatrix}$$

• For matrix $A = [a_{jk}]$ has
matrix size of $A = m \times n$
Row $\swarrow$ Column

• when number of Rows equal numbers of column the matrix is square.

• Main diagonal of matrix A is $a_{11}, a_{22}, a_{33}, \dots, a_{nn}$

→ Vectors: is a matrix with one Row or one column

$$a = [a_1, a_2, a_3] \rightarrow 1 \times 3$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow 3 \times 1$$

Def: equality of matrix:

two Matrix $A = [a_{jk}]$ and $B = [b_{jk}]$ are equal iff

- 1) have the same size $m \times n$
- 2) corresponding element

if $A \neq B$ then A, B are different

- 1) do not have same size
- 2) corresponding element are not equal

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad D = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$$

1) $A = B$

2) $A \neq D$

3) $A \neq C$

let $A = \begin{bmatrix} 4 & 8 \\ 6 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2x & 2y \\ 2z & 2 \end{bmatrix}$

Find x, y, z if $A = B$?

$$4 = 2x \rightarrow x = 2$$

$$8 = 2y \rightarrow y = 4$$

$$6 = 2z \rightarrow z = 3$$

Def: addition of matrix:

The sum of two matrix $A = [a_{jk}]$ and $B = [b_{jk}]$ of same size is written $A+B = [a_{jk} + b_{jk}]$ and obtained by adding the corresponding elements.

Note: Matrix with different size cannot be add

Ex: let $A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

1) $A+B =$

$$\begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}$$

2) $A-B =$

$$\begin{bmatrix} -9 & 7 & 3 \\ -3 & 0 & 2 \end{bmatrix}$$

3) $A+C =$ Can not be added

Def: scalar Multiplication:

The product of any $m \times n$ Matrix $A = [a_{jk}]$ and any scalar C written CA and is an $m \times n$ Matrix $CA = [C a_{jk}]$ obtained by multiplying each element of A by C .

Ex: let

$$A = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9 & -4.5 \end{bmatrix}$$

Find 1) $-A$

2) $\frac{1}{9}A$

3) $0A$

$$1) -A = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9 & 4.5 \end{bmatrix}$$

$$2) \frac{1}{9}A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$$

Rule for Matrix addition:

$$0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \neq 0$$

1) $A+B = B+A$

2) $[A+B]+C = A+[B+C] = A+B+C$

3) $A+0=A$ 4) $A+(-A)=0$

Matrix multiplication

Def: The product $C=AB$ of an $m \times n$ Matrix

$A=[a_{jk}]$ times $r \times p$ Matrix $B=[d_{jk}]$

is defined iff $\boxed{n=r}$ and the $M \times P$ Matrix $C=[c_{jk}]$ with elements

$$c_{jk} = \sum_{i=1}^n a_{ji} b_{ik} = a_{j1} b_{1k} + \dots + a_{jn} b_{nk}$$

$$j = 1, 2, \dots, n$$

$$k = 1, 2, \dots, p$$

$$C = AB \quad \begin{matrix} m \times p & p \times n & m \times n \\ & \downarrow & \\ & \text{equal} & \end{matrix}$$

Ex: let $A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 6 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$

And $AB?$

$$AB = C \quad \begin{matrix} 3 \times 3 & 3 \times 4 = 3 \times 4 \\ \uparrow & \uparrow \end{matrix}$$

$$C_{11} = 3(2) + (5)(5) + (-1)(9) = 22$$

$$C_{12} = 3(-2) + 5(0) + (-1)(-4) = -2$$

⋮

$$C_{33} = -6(3) + -3(7) + 2(1) = -37$$

$$C_{34} = -28$$

$$1) \begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 22 \\ 43 \end{bmatrix}$$

$\begin{matrix} 2 \times 2 & 2 \times 1 & 2 \times 1 \end{matrix}$

$$2) \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 19 \end{bmatrix}$$

$\begin{matrix} 1 \times 3 & 3 \times 1 & 1 \times 1 \end{matrix}$

$$3) \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix}$$

$3 \times 1 \quad 1 \times 3 = \quad 3 \times 3$

$$AB \neq BA$$

Rules

- 1) $(kA)B = k(AB)$
- 2) $(AB)C = A(BC)$
- 3) $(A+B)C = AC+BC$
- 4) $C(A+B) = CA+CB$

* Production in terms of Rows and Columns
vectors

$$A \ 3 \times 3 = \begin{bmatrix} \text{---} & a_1 & \text{---} \\ \text{---} & a_2 & \text{---} \\ \text{---} & a_3 & \text{---} \end{bmatrix}$$

$$B \ 3 \times 4 = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ | & | & | & | \\ | & | & | & | \end{bmatrix}$$

And $A B$ $A = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix}$

$AB = \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}$ $AB_1 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 17 \end{bmatrix}$

$AB_2 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

Ex: dot $A = \begin{bmatrix} 5 & 8 & 1 \\ 4 & 0 & 0 \end{bmatrix}^{2 \times 3}$

$A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 6 \end{bmatrix}^{3 \times 2}$

Ex: dot $A = [6 \ 2 \ 3]_{1 \times 3}$

$A^T = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}^{3 \times 1}$

* Rules:

- 1) $(A^T)^T = A$
- 2) $(A+B)^T = A^T + B^T$
- 3) $[CA]^T = CA^T$
- 4) $(AB)^T = B^T A^T$

Symmetric Matrix
 * If $A^T = A$

Skew-symmetric Matrix
 iff $A^T = -A$

1) upper triangular Matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

2) lower triangular Matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

* diagonal Matrix D :- * Scalar Matrix S :-

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad S = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix}$$

$C = \text{any scalar}$

* Unit Matrix or Identity Matrix

$$I_{2 \times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_{3 \times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AI = IA = A$$

Linear system of equations:

Gauss - elimination

Let linear system with m -equation
and n -unknowns

$$a_{11}x_1 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \dots + a_{2n}x_n = b_2$$

$$\vdots$$
$$a_{m1}x_1 + \dots + a_{mn}x_n = b_m$$

1) linear system $\Rightarrow x_1 x_2 \dots x_n$ of order 1.

2) For linear system with all b_j are
zero $\Rightarrow$ called homogeneous linear system

3) For linear system with at least one (b_j)
is not zero called non-homogeneous
L. system

Linear system $AX=b$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

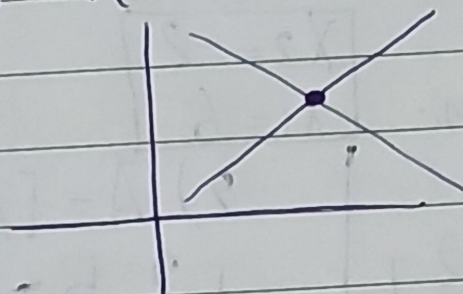
$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

* Augmented Matrix

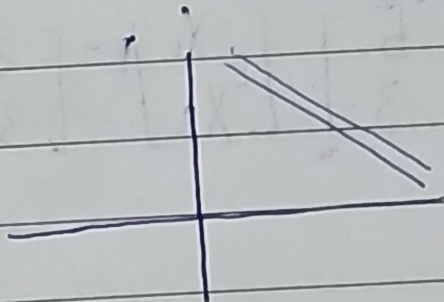
$$\tilde{A} = [A : b]$$

for $AX=b$ has the Possible Cases:

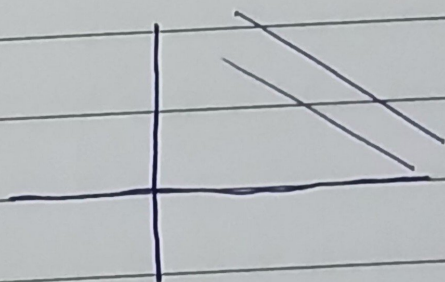
1) Exactly one solution iff ~~the~~ lines intersect.



2) Infinitely many solutions
if the lines coincide.



3) no-solution if the lines
are parallel



Gauss-elimination and Back substitution:-

$$2x_1 + 5x_2 = 2 \quad x_2 = -2 \rightarrow -26/13$$

$$\text{det} \quad +13x_2 = -26 \quad x_1 = 6 \rightarrow \frac{1}{2}(2 - 5x_2) = 6$$

$$\begin{bmatrix} 2 & 5 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

$$\text{Ex: } 2x_1 + 5x_2 = 2 \quad \dots \text{eq(1)}$$

$$-4x_1 + 3x_2 = -30 \quad \dots \text{eq(2)}$$

Find x_1, x_2 ??

$$\text{eq(2)} + 2\text{eq(1)}$$

$$0x_1 + 13x_2 = -26$$

$$\boxed{\begin{matrix} x_2 = -2 \\ x_1 = 6 \end{matrix}}$$

Using Gauss-elimination:-

$$1) \quad AX = b$$

$$2) \quad A = [A|b]$$

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 5 & : & 2 \\ -4 & 3 & : & -30 \end{bmatrix} \rightarrow \begin{matrix} R_1 \\ R_2 \end{matrix}$$

$$= \begin{bmatrix} 2 & 5 & : & 2 \\ 0 & 13 & : & -26 \end{bmatrix} \rightarrow R_1 + 2R_2$$

$$\begin{matrix} 2x_1 + 5x_2 = 2 & x_2 = -2 \\ +13x_2 = -26 & \rightarrow x_1 = 6 \end{matrix}$$

Ex: with one Solution

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ +10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80 \end{aligned}$$

Find x_1 x_2 x_3 ?

Using G-E

1) $AX = b$

$$= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ -80 \end{bmatrix}$$

2) $\tilde{A} = [A: b]$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array} \right] \begin{array}{l} \text{pivot} \\ \rightarrow R_2 + R_1 \\ \rightarrow R_4 + 20R_1 \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array} \right] \begin{array}{l} \text{pivot} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow R_3 - 3R_2$$

$$\begin{aligned} 1x_1 - 1x_2 + 1x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ -95x_3 &= -190 \end{aligned}$$

$$x_3 = 2$$

$$x_2 = 4$$

$$x_1 = 2$$

$$x = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$$

Elementary Row operations for matrix:-

1) Interchange of two Rows.

2) addition of constant Multiple of one row to another row

3) Multiplication of a row by a nonzero constant C.

has no-effect on the solution

EX:-

$$1) \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$2) \begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$$

1 + 2 has same solution

- 1) one determined linear system:
if it has more equation than unknowns
- 2) determined linear system:
if the number of equation equal number of unknowns.
- 3) under determined linear system
if it has fewer equation than unknowns
- 4) consistent L.S $\Rightarrow$ if it has at least one solution
- 5) IV consistent if it has no-solution

Ex: with infinitely many solutions:

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 - 5x_4 &= 8 \\ 0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 &= 2.7 \\ 1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 &= 2.1 \end{aligned}$$

Find x_1, x_2, x_3, x_4 ??

using G-E

1) $AX=b$

$$\begin{bmatrix} 3 & 2 & 2 & -5 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2.7 \\ 2.1 \end{bmatrix}$$

2) $\tilde{A} = [A|b]$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array} \right] \rightarrow \text{Pivot}$$

$$\text{Pivot} \rightarrow \left[\begin{array}{ccc|c} 3 & 2 & 2 & -5 \\ 0 & 1.1 & 1.1 & -4.4 \\ 0 & -1.1 & -1.1 & 4.4 \end{array} \right] \begin{array}{l} \rightarrow R_2 - 0.2R_1 \\ \rightarrow R_3 - 0.4R_1 \end{array}$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 2 & -5 \\ 0 & 1.1 & 1.1 & -4.4 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow R_3 - R_2$$

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 - 5x_4 &= 8 \\ 1.1x_2 + 1.1x_3 - 4.4x_4 &= 1.1 \end{aligned}$$

$$x_2 + x_3 - 4x_4 = 1$$

$$x_2 + x_3 = 1 + 4x_4$$

$$\begin{aligned} 3x_1 + 2[1 + 4x_4] - 5x_4 &= 8 \\ 3x_1 + 2 + 8x_4 - 5x_4 &= 8 \\ 3x_1 + 3x_4 &= 6 \end{aligned}$$

$$x_1 = 2 - x_4$$

$$x_2 = 1 + 4x_4 - x_3$$

$$x_3 = \text{arbitrary}$$

$$x_4 = \text{arbitrary}$$

$$\lambda = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

infinitely many solution

EX: No-solution

$$3x_1 + 2x_2 + x_3 = 3$$

$$2x_1 + x_2 + x_3 = 0$$

$$6x_1 + 2x_2 + 4x_3 = 6$$

Find x_1 x_2 x_3 ?

1) $AX=b$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

2) $A=[A:b]$

$$\begin{bmatrix} 3 & 2 & 1 & : & 3 \\ 2 & 1 & 1 & : & 0 \\ 6 & 2 & 4 & : & 6 \end{bmatrix} \rightarrow \text{Pivot}$$

$$= \begin{bmatrix} -3 & 2 & 1 & : & 3 \\ 0 & -1/3 & 1/3 & : & -2 \\ 0 & -2 & 2 & : & 0 \end{bmatrix} \rightarrow R_2 - 2/3 R_1$$

$$= \begin{bmatrix} 3 & 2 & 1 & : & 3 \\ 0 & -1/3 & 1/3 & : & -2 \\ 0 & 0 & 0 & : & 2 \end{bmatrix} \rightarrow R_3 - 6R_2$$

$$3x_1 + 2x_2 + x_3 = 3$$

$$-1/3x_2 + 1/3x_3 = -2$$

$$0 = 12$$

False statement
has no solution

* Row-echelon form and information from it:

1) linear system $AX=b$

2) $A=[A:b]$

3) after doing G-E $\rightarrow [R:P]$

$$[P:P] = \begin{bmatrix} V_{11} & V_{12} & \dots & V_{1n} & P_1 \\ & V_{22} & \dots & V_{2n} & P_2 \\ & & \ddots & V_{rr} \rightarrow V_{rn} & P_r \\ & & & & P_{r+1} \\ & & & & \vdots \\ & & & & P_m \end{bmatrix}$$

1) $V_{ii} \neq 0$

2) $V \leq m$

3) all element in the triangle + rectangle is zero

1) No-solution:-

i.f. $r < m$ and at least one number $P_{r+1} \dots P_m$ is not zero

$V_{11} = 3 \neq 0$

$V_{rn} = V_{23} = 1/3$

$r = 2$

$n = 3$

$P_m = P_3 = 12$

$m = 3$

no-solution

one solution:

i.f. $r = n$ all number $P_1 \dots P_m$ are zero

$$[R:P] = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 0 & 0 & 25 & 90 \\ 0 & 0 & -95 & -140 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

infinitely many solution
 i.e. P_m and all numbers
 $P_{r+1} \dots P_m$ are zero

$$[A: P] = \begin{bmatrix} 3 & 2 & 2 & -5 & : & 8 \\ 0 & 1 & 1 & -4 & : & 1 \\ 0 & 0 & 0 & 0 & : & 0 \end{bmatrix}$$

$$r = 3 \neq 0 \quad r = 2$$

$$r = m = 2 \quad -4.4 \leq n = 4$$

$$P_3 = P_m = 0 \quad m = 3$$

* Linear Independence Rank of matrix

Def: let $C_1, C_2, \dots, C_m$ are m vectors

$C_1, C_2, \dots, C_m$ are m -scalar

then $C_1 C_1 + C_2 C_2 + \dots + C_m C_m = 0 \rightarrow \text{eq (1)}$

1) if eq (1) hold for all C 's are zero then

System [linearly indep]

2) if eq (1) hold for at least one C 's are not zero then system is [linearly dep]

$$\text{def: } C_1 = \begin{bmatrix} 3 & 0 & 2 & 2 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} -6 & 4 & 2 & 5 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 2 & -2 & 0 & -1 \end{bmatrix}$$

are C_1, C_2, C_3 linearly dep??

$$C_1 a_1 + C_2 a_2 + C_3 a_3 = 0$$

$$C_1 [3 \ 0 \ 2 \ 2] + C_2 [-6 \ 42 \ 24 \ 54] + C_3 [21 \ -21 \ 0 \ -15] = 0$$

$$3C_1 - 6C_2 + 21C_3 = 0$$

$$0C_1 + 42C_2 - 21C_3 = 0$$

$$2C_1 + 24C_2 + 0C_3 = 0$$

$$2C_1 + 54C_2 - 15C_3 = 0$$

Using G-E $C_1 = 6 \quad C_2 = -1/2 \quad C_3 = -1$

a_1, a_2, a_3 are linearly dep

Rank A :

is the Maximum number of linearly Indep Row v2 col of A .

Ex: let $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ Find Rank A ?

Using G-E:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -24 \end{bmatrix} \begin{array}{l} \rightarrow \text{Pivot} \\ \rightarrow R_2 + 2R_1 \\ \rightarrow R_3 - 7R_1 \end{array}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow R_3 + 1/2 R_2$$

Rank of $A = 2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Rank} = 1$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{Rank} = 3$$

Row-equivalent Matrix:-

1) Matrix A_1 is Row equivalent to Matrix A_2 if A_1 can be obtained from A_2 by elementary Row operation

2) Row-equivalent Matrix have same Rank.

3)

Theremo:-

Consider P -vectors that each have n -component then these vectors are linearly Indep.

if matrix formed with these vectors as Row vector has Rank = P , however these vector are linearly dep if Matrix has Rank less than P .

~~Theorem~~: Rank in terms of column vector
 The Rank of matrix A equals the
 Maximum number of linearly indep
 column vector of A.

Theorem:- linearly dep of vectors consider
 P -vectors each have n -component i.e.
 $n < P$ then these vectors are linearly

EX: let $P_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are P_5 linearly dep?
 $P_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $P = 5$ $n < P$
 $P_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ $n = 2$ $2 < 5$
 $P_4 = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ $P_5 \Rightarrow$ linearly dep
 $P_5 = \begin{bmatrix} 9 \\ 10 \end{bmatrix}$

(*) $\text{Rank}(A) = \text{Rank}(A^T)$

(*) Theorem: solution of linear system $AX = b$ is unique.

let $AX = b$ has m -equation
 n -unknowns

~~let~~ $A = [A_i b]$

1) Existence: linear system has solution
 then it is consistent i.e. A
 $\text{Rank } A = \text{Rank } A$

2) Uniqueness: linear system has one-solution
iff $\text{Rank}(A) = \text{Rank}(\tilde{A}) = n$

3) Infinitely many solution iff

$$\text{Rank}(A) < n$$

$$\text{Rank}(\tilde{A}) < n$$

4) no-solution:

linear system is inconsistent

$$\text{iff } \text{Rank}(A) \neq \text{Rank}(\tilde{A})$$

Second and third order determinants

$$\det A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad D = \det[A] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \\ = a_{11}a_{22} - a_{12}a_{21}$$

Cramer Rule:

For second order linear system

$$AX = b$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D}$$

$$x_2 = \frac{D_2}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{21} \end{vmatrix} = b_1 a_{22} - a_{12} b_2$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix} = a_{11} b_2 - b_1 a_{21}$$

EX: let $4x_1 + 3x_2 = 12$

$2x_1 + 5x_2 = -8$

Find x_1, x_2 ??

$$AX = b$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = 20 - 6 = 14 \quad D_1 = \begin{bmatrix} 12 & 3 \\ -8 & 5 \end{bmatrix} = 60 - 24 = 36$$

$$D_2 = \begin{bmatrix} 4 & 12 \\ 2 & -8 \end{bmatrix} = -32 - 24 = -56$$

$$X_1 = D_1/D = 36/14 = 6 \quad X_2 = D_2/D = -56/14 = -4$$

* Third order determinant:
for linear system $AX=b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$X_1 = D_1/D \quad X_2 = D_2/D \quad X_3 = D_3/D$$

$$D = \det [A] = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D = +a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix}$$

$$+ a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= +a_{11} [a_{22}a_{33} - a_{23}a_{32}] - a_{12} [a_{21}a_{33} - a_{23}a_{31}] + a_{13} [a_{21}a_{32} - a_{22}a_{31}]$$

$$D_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$D_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

$$D_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

EX: Find D for A if $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$

$$D = +1 \begin{bmatrix} 6 & 4 \\ 0 & 2 \end{bmatrix} - 3 \begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix} + 0 \begin{bmatrix} 2 & 6 \\ -1 & 0 \end{bmatrix}$$

$$= 1 [12] - 3 [4+4] = -12$$

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The Minor Matrix of A is M_0 -

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \quad M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = a_{22}a_{33} - a_{23}a_{32}$$

$$M_{12} = \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} \dots \dots M_{32} = \begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$$

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

The Cofactor Matrix of A is C_0

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \quad C_{jk} = (-1)^{j+k} M_{jk}$$

$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

Theorem:-

- 1) Interchange of two rows Multiply the value of determinant by $[-1]$.

2) addition of multiple of Row to another row does not alter the value of determinant.

3) multiplication of row by non-zero constant $[C]$ multiplies the value of determinant by $[C]$

$$4) \det [CA] = C^n \det [A]$$

$\downarrow$
 $n = \text{number of row in } A.$

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad |A| = 1 - 0 = 1$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \quad |B| = 0 - 1 = -1$$

$$C = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \quad |C| = 2$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad |A| = 3 - 2 = 1$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \rightarrow R_2 - R_1$$

$$|D| = 1 - 0 = 1$$

$$|cA| = c^n |A|$$

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \quad |A| = 3 - 2 = 1$$

$$|3A| = \begin{vmatrix} 3 & 6 \\ 3 & 9 \end{vmatrix} = 3(9) - 3(6) = 3[3] = 9$$

$$|3A| = 3^2 |A| = 9(1) = 9$$

7.6 Second and third order determinant

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad D = \det[A] = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Cramer Rule:-

For second order linear system

$$AX = b$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 = D_1/D \quad x_2 = D_2/D$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - a_{12} b_2$$

$$D2 = \begin{bmatrix} a_{11} & b_1 \\ a_{22} & b_2 \end{bmatrix} = a_{11}b_2 - b_1a_{21}$$

$$EX = \begin{vmatrix} 1 & 1 \\ 4 & 3 \end{vmatrix} \quad 4x_1 + 3x_2 = 12 \quad \text{Find } x_1, x_2?$$

$$2x_1 + 5x_2 = -8$$

$$AX = b$$

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = 20 - 6 = 14$$

$$D_2 = \begin{bmatrix} 4 & 12 \\ 2 & -8 \end{bmatrix} = -32 - 24 = -56$$

$$x_1 = D_1/D = 84/14 = 6$$

$$x_2 = D_2/D = -56/14 = -4$$

7.6] second and third order determinant:-

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad D = \det[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \\ = a_{11}a_{22} - a_{12}a_{21}$$

Cramer Rule:-

For second order linear system

$$Ax = b \quad \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 = D_1/D, \quad x_2 = D_2/D$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix} = b_1 a_{22} - a_{12} b_2$$

$$D_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix} = a_{11} b_2 - b_1 a_{21}$$

The Minor Matrix of
A is M

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} = a_{32}a_{33} - a_{23}a_{32}$$

$$M_{12} = \begin{bmatrix} a_{31} & a_{23} \\ a_{31} & a_{23} \end{bmatrix} \dots \dots M_{32} = \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{23} \end{bmatrix}$$

$$M_{33} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

The Cofactor Matrix of A is C.

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \quad C_{jk} = (-1)^{j+k} M_{jk}$$

$$\begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

Theorems:

- 1) Interchange of two rows multiplies the value of determinant by $[-1]$.
- 2) Addition of multiple of row to another row does not alter the value of determinant.

3) Multiplication of by non-zero constant
[C] multiplies the value of determinant by [C]

$$4) \det [cA] = c^n \det [A]$$

n : number of rows in A .

Ex: 10

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

Find x_1, x_2 ?

$$AX=b \quad \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = 20 - 6 = 14$$

$$D_1 = \begin{bmatrix} 12 & 3 \\ -8 & 5 \end{bmatrix} = 60 + 24 = 84$$

$$D_2 = \begin{bmatrix} 4 & 12 \\ 2 & -8 \end{bmatrix} = -32 - 24 = -56$$

$$x_1 = D_1 / D = 84 / 14 = 6$$

$$x_2 = \frac{D_2}{D} = \frac{-56}{14} = -4$$

evaluation of determinant by reduction to triangular from Gauss elimination:

$$D = \begin{vmatrix} 2 & 0 & -4 & 6 \\ 4 & 5 & 1 & 0 \\ 0 & 2 & 6 & -1 \\ -3 & 8 & 9 & 1 \end{vmatrix} \rightarrow \text{Pivot}$$

$$\begin{array}{l} = \\ \text{Pivot} \rightarrow \end{array} \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 2 & 6 & -1 \\ 0 & 8 & 3 & 10 \end{vmatrix} \begin{array}{l} \\ \rightarrow R_2 - 2R_1 \\ \rightarrow R_4 + 1.5R_1 \end{array}$$

$$\begin{array}{l} = \\ \text{Pivot} \rightarrow \end{array} \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & -11.4 & 29.2 \end{vmatrix} \begin{array}{l} \\ \rightarrow R_3 - 0.4R_2 \\ \rightarrow R_4 - 1.6R_2 \end{array}$$

$$= \begin{vmatrix} 2 & 0 & -4 & 6 \\ 0 & 5 & 9 & -12 \\ 0 & 0 & 2.4 & 3.8 \\ 0 & 0 & 0 & 47.25 \end{vmatrix} \rightarrow R_4 + 4.75R_3$$

$$D = [2] [5] [2.4] [47.25] = 1134$$

Theorems:

1) Interchange of two columns
Multiply the value of determinant
by $[-1]$.

2) Addition of multiple of column
to another column does not alter
the value of determinant.

3) Multiplication of column by non-zero
constant $[C]$ Multiplies the value of
det by $[C]$

$$4) \det [A] = \det [A^T]$$

5) Zero Row or zero column result in
the value of determinant to zero

6) Proportional Rows or Columns
render the value of det to zero

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \quad |A| = 0$$

7.8 Inverse of Matrix : Gauss Jordan
Elimination

→ For square Matrix $n \times n$ A its
inverse is A^{-1}

$$A^{-1}A = AA^{-1} = I$$

note , 1) if A has invers then A is
non-singular

2) if A has no-invers then A is
Singular

For system $AX=b$ we can find X using A^{-1}

$$AX=b$$

$$A^{-1}AX = A^{-1}b$$

$$IX = A^{-1}b$$

$$X = A^{-1}b$$

Theorem: the inverse A^{-1} of an $n \times n$ Matrix A exist iff

1) $\text{Rank}(A) = n$

2) $\det[A] \neq 0$

3) A is non-singular

* if $\text{Rank}(A) < n$ then

→ A is singular

→ A has no-inverse

Finding inverse of Matrix by Gauss-elimination

Ex Find A^{-1} if $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$

1) $\tilde{A} = [A : I]$

$\begin{bmatrix} -1 & 1 & 2 & | & 1 & 0 & 0 \\ 3 & -1 & 1 & | & 0 & 1 & 0 \\ -1 & 3 & 4 & | & 0 & 0 & 1 \end{bmatrix} \rightarrow \text{Pivot}$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array} \right] \begin{array}{l} \\ \rightarrow R_2 + 3R_1 \\ \rightarrow R_3 - R_1 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} -1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 7 & 3 & 1 & 0 \\ 0 & 0 & -5 & -4 & -1 & 1 \end{array} \right] \rightarrow R_3 - R_2$$

$$= \left[\begin{array}{ccc|ccc} 0 & -1 & -2 & -1 & 0 & 0 \\ 0 & 1 & 3.5 & 1.5 & 0.5 & 0 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right] \begin{array}{l} \rightarrow -R_1 \\ \rightarrow 0.5 R_2 \\ \text{Pivot} \rightarrow -0.2 R_3 \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0.6 & 0.4 & -0.4 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right] \begin{array}{l} \rightarrow R_1 + 2R_3 \\ \text{Pivot} \rightarrow R_2 - 3.5R_3 \\ \end{array}$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.7 & 0.2 & 0.3 \\ 0 & 1 & 0 & -1.3 & -0.2 & 0.7 \\ 0 & 0 & 1 & 0.8 & 0.2 & -0.2 \end{array} \right] \rightarrow R_1 + R_2$$

$$\rightarrow [I : A^{-1}]$$

$$A^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Theorem: inverse of Matrix by determinat
The inverse of non singular $n \times n$
Matrix $A = [a_{jk}]$ is given by

$$A^{-1} = \frac{1}{|A|} C^T$$

$$j+k \quad A \rightarrow M \rightarrow C \rightarrow C^T$$

$$C_{jk} = (-1)^{j+k} M_{jk}$$

Ex 0 let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ Find A^{-1} ?

$$A^{-1} = \frac{1}{|A|} C^T$$

$$|A| = a_{11}a_{22} - a_{12}a_{21}$$

$$C^T = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Ex : let $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$ find A^{-1} ?

$$A^{-1} = \frac{1}{|A|} C^T = \frac{1}{10} \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.1 \\ -0.2 & 0.3 \end{bmatrix}$$

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $A^{-1} = \frac{1}{|A|} C^T$

$$|A| = +a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$C^T \rightarrow C \rightarrow M \rightarrow A$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$C_{11} = (-1)^2 M_{11} = M_{11} = \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix}$$

Exo: let $A = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ Find A^{-1} ?

$$A^{-1} = \frac{1}{|A|} C^T$$

$$|A| = -1 \begin{bmatrix} -1 & 1 \\ 3 & 4 \end{bmatrix} - 1 \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix} + 2 \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

$$= 10$$

$$C^t = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$C_{11} = (-1)^2 M_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7$$

$$C_{12} = - \begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13$$

$$C_{13} = (-1)^4 \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8 \quad C_{21} = (-1)^3 \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2$$

$$C_{22} = (-1)^4 \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2 \quad C_{23} = (-1)^5 \begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2$$

$$C_{31} = (-1)^4 \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3 \quad C_{32} = (-1)^3 \begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7$$

$$C_{33} = (-1)^6 \begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -2$$

$$C^T = \begin{bmatrix} -7 & 2 & 3 \\ -13 & -2 & 7 \\ 8 & 2 & -2 \end{bmatrix} \quad A = \frac{1}{10} C^T \begin{bmatrix} 0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

Ex: let $A = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Find A^{-1}

$$A^{-1} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rules: 1) $[AC]^{-1} = C^{-1}A^{-1}$

2) $[A^{-1}]^T = A^T$

3) $(A^{-1})^T = (A^T)^{-1}$

* Un Usual Properties of Matrix Multiplication

1) $AB \neq BA$

2) $AB=0 \rightarrow$ It does not generally imply

$A=0$ or $B=0$

Ex: let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}_{2 \times 2} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2 \times 2}$$

$\xrightarrow{\text{equal}}$

Theorem: determinant of product of Matrix
For any $n \times n$ Matrix A, B

$$\det [AB] = \det [BA] = \det [A] \det [B]$$

Ex: let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ Find $[AB]?$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \quad 7 - 8 = -1$$

$$|AB| = |A| |B| = -1 \times 1 = -1$$

adjoint Matrix:-

$$1) \text{adj}[A] = C^T$$

$$2) [A] [\text{adj}(A)] = [\text{adj}(A)] [A] = \det(A) I$$

$$A \rightarrow M \rightarrow C \rightarrow C^T$$

scalar matrix = $S = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \end{bmatrix}$

scalar

2) $AC = AD$ it does not generally imply that $C = D$

Theorems on Cancellation laws

let A, B, C be $n \times n$ matrix then

a) if $\text{rank}(A) = n$ and $AB = AC$ then $B = C$

b) if $\text{rank}(A) = n$ and $AB = AO$ then $B = O$

c) if $AB = O$ and $A \neq O, B \neq O$ then

$$\text{Rank}(A) < n, \text{Rank}(B) < n$$

d) if A is singular then BA, AB are singular

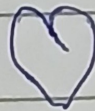
Theorem: determinant of Product of Matrix
For any $n \times n$ Matrix A, B
 $\det [AB] = \det [BA] = \det [A] \det [B]$

$$\text{Ex} = \det A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Find $\det [AB]$?

$$|AB| = \begin{bmatrix} 1 & 4 \\ 2 & 7 \end{bmatrix} \quad 7 - 8 = -1$$

$$|AB| = |A||B| = -1 \times 1 = -1$$



Find eigen value + eigen value for

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$1) \det (A - \lambda I) = 0 \quad \begin{vmatrix} 0 - \lambda & 1 \\ -1 & 0 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm \sqrt{-1} = \pm i$$

For $\lambda = i$ the X is

$$[A - \lambda I] X = 0 \quad \begin{vmatrix} -i & 1 \\ -1 & -i \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$X = \begin{pmatrix} 1 \\ i \end{pmatrix} \quad iX_1 + 1X_2 = 0$$

For $\lambda = -i$ the X is

$$[A - \lambda I] X = 0$$

$$\begin{vmatrix} i & 1 \\ -1 & i \end{vmatrix} \begin{vmatrix} X_1 \\ X_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \quad iX_1 + 1X_2 = 0$$

$$X = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Therem^o

The transpose A^T of square Matrix A has the same eigen value as A

$$A^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad \det[A^T - \lambda I] = 0$$

$$\begin{vmatrix} 0-\lambda & -1 \\ 1 & 0-\lambda \end{vmatrix} = 0$$

$$\lambda^2 + 1 = 0 \quad \lambda = \pm i$$

Symmetric/symmetric and orthogonal Matrix^o

Def^o: Areal square matrix $A = [a_{jk}]$

is called^o:

1) Symmetric if $A^T = A$

2) skew-symmetric if $A^T = -A$

3) Orthogonal if $A^T = A^{-1}$

Ex^o: 1) Symmetric

$$A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

2) skew-symmetric

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

3) orthogonal

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

* For any real square Matrix A

1) Symmetric Matrix $(B) = 1/2 [A + A^T]$

2) skew-symmetric Matrix $(S) = 1/2 [A - A^T]$

Ex^o: let $A = \begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$ find B, S ?

$$A^T = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 4 \\ 5 & -8 & 3 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 4 \\ 5 & -8 & 3 \end{bmatrix}$$

$$R = \frac{1}{\sqrt{2}} [A + A^T] = \begin{bmatrix} 0 & 3.5 & 3.5 \\ 3.5 & 3 & -2 \\ 3.5 & -2 & 3 \end{bmatrix}$$

$$S = \frac{1}{\sqrt{2}} [A - A^T] = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6 \\ 1.5 & 6 & 0 \end{bmatrix}$$

Ex 8 Find det, Eigenvalue of

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix} \quad |A| = -1$$

$$\det |A - \lambda I| = 0$$

$$\begin{vmatrix} 2/3 - \lambda & 1/3 & 2/3 \\ -2/3 & 2/3 - \lambda & 1/3 \\ 1/3 & 2/3 & -2/3 - \lambda \end{vmatrix} = 0$$

$$= -\lambda^3 + 2/3 \lambda^2 + 2/3 \lambda - 1 = 0$$

$$\lambda = -1, \quad \frac{S + i\sqrt{11}}{6}, \quad \frac{S - i\sqrt{11}}{6}$$

$$|Z| = \sqrt{x^2 + y^2} = \sqrt{(5/6)^2 + (\sqrt{11}/6)^2} = 1$$

ARARAW

Orthogonal Matrix

Ex 10 Let $A = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

$$A^T = A^{-1}$$

$$A^{-1} = \frac{1}{|A|} A^T$$

$$A^T = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = \frac{1}{1} \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$$

Theorems

1) The determinant of Orthogonal Matrix has value of 1 or -1

2) The eigen value of Orthogonal Matrix A are Real or Complex conjugate and has a bsolnt. value of 1.

Theorems-

1) The eigen value of symmetric Matrix are Real

2) The eigen value of skew-symmetric matrix are Pure Imaginary or zero.

Sigma

Ex: let $A = \begin{pmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{pmatrix}$ Find Eigen λ

$$\det(A - \lambda I) = 0 \quad \begin{vmatrix} 0-\lambda & 9 & -12 \\ -9 & 0-\lambda & 20 \\ 12 & -20 & 0-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -\lambda & 9 & -12 \\ -9 & -\lambda & 20 \\ 12 & -20 & -\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -(-\lambda) & -1 & 20 \\ -20 & -\lambda & -9 \\ 12 & -1 & -\lambda \end{vmatrix} + (-12) \begin{vmatrix} -9 & 20 \\ 12 & -20 \end{vmatrix} = 0$$

$$-\lambda^3 - 625\lambda = 0 \quad -\lambda[\lambda^2 + 625] = 0 \quad \lambda = 0, \pm 25i$$

Theorem:

If an $n \times n$ Matrix A has n -distinct Eigen value then A has a basis of Eigen vector $x_1, x_2, \dots, x_n$ for $\mathbb{R}^n$

Ex: Find Eigen value + Eigen vector

for $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

1) $\det[A - \lambda I] = 0$

$$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$= \begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)(5-\lambda) - 9 = 0$$

$$\lambda^2 - 10\lambda + 16 = 0$$

$$(\lambda-2)(\lambda-8) = 0 \quad \lambda = 2, 8$$

for $\lambda = 2$ the x is:

$$[A - \lambda I]x = 0$$

$$\begin{vmatrix} 3 & 3 \\ 3 & 3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \quad 3x_1 + 3x_2 = 0 \quad x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

* for $\lambda = 8$ the x is:

$$[A - \lambda I]x = 0$$

$$\begin{vmatrix} -3 & 3 \\ 3 & -3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$-3x_1 + 3x_2 = 0$$

$$X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Similar Matrix + similarity transformation

let $n \times n$ matrix A is called similar

to $n \times n$ matrix A' if

$$A' = P^{-1}AP$$

where P is non-singular

Theorem:

1) if A is similar to A' , then A' has

the same eigen value as A .

2) if X is an eigen vector of A , then

$Y = P^{-1}X$ is the eigen vector of A'

Conversely, finding to same eigen value

$$A' = P^{-1}AP$$

$$A' = P^{-1}AP$$

$$A' = P^{-1}AP$$

$$A' = P^{-1}AP$$

$$X = P^{-1}X'Y$$

Ex: let $A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$

Find eigen value + eigen vector

of A and A' ?

for A'

$$\det [A - \lambda I] = 0$$

$$\begin{vmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(-1-\lambda) + 12 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0 \quad \lambda = 2, 3$$

For $\lambda = 2$ the X is:

$$[A - \lambda I]X = 0$$

$$\begin{vmatrix} 4 & -3 \\ 4 & -3 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \quad \text{where } \begin{vmatrix} 4 & -3 \\ 4 & -3 \end{vmatrix} X = 0$$

$$X = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

For $\lambda = 3$ the X is:

$$[A - \lambda I]X = 0$$

$$\begin{vmatrix} 3 & -3 \\ 4 & -4 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \quad \text{where } \begin{vmatrix} 3 & -3 \\ 4 & -4 \end{vmatrix} X = 0$$

$$\text{Sigma } X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$A = P^{-1}AP = \frac{1}{1} \begin{vmatrix} 4 & -3 \\ -1 & 1 \end{vmatrix} \begin{vmatrix} 6 & -3 \\ 4 & -1 \end{vmatrix} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 \\ 0 & 2 \end{vmatrix}$$

$$\det [A - 2I] = 0$$

$$\begin{vmatrix} 3-1 & 0 \\ 0 & 2-1 \end{vmatrix} = 0 \quad (3-1)(2-1) - 0 = 0$$

$$\lambda = 3, 2$$

For $\lambda = 2$ the y is?

$$(A - 2I)y = 0 \quad \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \begin{vmatrix} y_1 \\ y_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

$$y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

For $\lambda = 3$ the y is?

$$[A - 2I]y = 0$$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$y_1 = P^{-1}x_1 = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y_2 = P^{-1}x_2 = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$$

Sigma