اللجنة الأكاديمية لقسم الهندسة المدنية
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دفتر
رياضيات تطبيقية (كامل المادة)

13.2) polar form of complex numbers, powers & roots:

$z = x + iy \Rightarrow re^{i\theta}$

① $r = |z| \Rightarrow$ reduis = moduluse = norm = magnitude $= \sqrt{x^2+y^2}$

② $\theta = \tan^{-1}\left[\frac{y}{x}\right]$

$x = r\cos\theta \quad , \quad y = r\sin\theta$

Ex: let $z_1 = 1+i$, Find the polar form? [cartesian ⇒ polar]

$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \Rightarrow \left[z = \sqrt{2}\, e^{i\frac{\pi}{4}}\right]$

Ex: let $z = \underbrace{\sqrt{2}}_{r}\, e^{i\underbrace{\frac{\pi}{4}}_{\theta}}$ Find the $x+iy$ form ??

$x = \sqrt{2}\cos\frac{\pi}{4} = 1 \quad , \quad y = \sqrt{2}\sin\frac{\pi}{4} = 1 \Rightarrow [z = 1+i]$

• Eular formula

$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

• Principle value of complex numbers:

$Arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$-\pi \leq Arg(z) \leq \pi$

$arg(z) = Arg(z) \pm 2\pi n$

$n = \pm 1, \pm 2, \ldots$

* How to find the angle θ ?

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

• مهم جداً التأكد من الربع الموجود فيه

Ex: Find θ for the following complex numbers:

$\pi - \theta$	θ
$-\pi + \theta$	$-\theta$

① $z_1 = 1+i \; (1,1) \Rightarrow \tan^{-1}(1) = \frac{\pi}{4}$ الأول

$-\frac{3\pi}{4} \Leftarrow$ ② $z_2 = -1-i \; (-1,-1) \rightarrow \tan^{-1}\left(\frac{-1}{-1}\right) = -\pi + \frac{\pi}{4}$

الثالث ③ $z_3 = 1-i \; (1,-1) \rightarrow \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$ الرابع

④ $z_4 = -1+i \; (-1,1) \rightarrow \frac{3\pi}{4}$ الثاني

No.

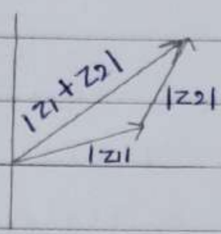
Ex: let $z_1 = 1+i$ find $Arg(z)$ & $arg(z)$?

$Arg(z) = \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$arg(z) = \frac{\pi}{4} \pm 2\pi n$, $n = \pm 1, \pm 2 \ldots$

- **Triangle Inequality**

$|z_1+z_2| \leq |z_1| + |z_2|$



- **Rules:**

[1] $|z_1 z_2| = |z_1||z_2|$

[2] $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

- **Power of complex numbers:**

$(1+i)^2 = 1+2i+i^2 = 2i$

$[z]^n = [x+iy]^n = [re^{i\theta}]^n$

n = integer $= \pm 1, \pm 2, \ldots$

$= r^n e^{i\theta n}$

$= r^n [\cos(\theta n) + i\sin(\theta n)]$

Ex: Find $(1+i)^{10}$ in simple form?

$= r^{10} e^{i\theta 10} \rightarrow r = \sqrt{2}, \theta = \frac{\pi}{4}$

$= (\sqrt{2})^{10} e^{i\frac{10\pi}{4}} = 32 e^{i\frac{5\pi}{2}}$ $\quad = 2^{\frac{10}{2}} = 2^5$

$= 32[\cos(\frac{5\pi}{2}) + i\sin(\frac{5\pi}{2})]$

$= 32i$

- **Multiplication & division of complex numbers:**

let $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

① $z_1 \cdot z_2 = r_1 \cdot r_2\, e^{i(\theta_1+\theta_2)} \Rightarrow$ (جمع)

$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$

② $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \Rightarrow$ (طرح)

$= \frac{r_1}{r_2} [\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_1)]$

Ex: let $z_1 = 2e^{i\frac{\pi}{4}}$, $z_2 = 3e^{i\frac{\pi}{2}}$

Find $z_1 \cdot z_2$ & $\frac{z_1}{z_2}$

[1] $z_1 \cdot z_2 = 6e^{i(\frac{3\pi}{4})}$

$= 6[\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})]$

[2] $\frac{z_1}{z_2} = \frac{2}{3} e^{i(-\pi/4)}$

$= \frac{2}{3}[\cos(-\frac{\pi}{4}) + i\sin(-\frac{\pi}{4})]$

$= \frac{2}{3}[\cos(\frac{\pi}{4}) - i\sin(\frac{\pi}{4})]$

- **Demovieve formula:**

$$[e^{i\theta}]^n = [e^{i\theta}]^n$$
$$e^{i\theta n} = [e^{i\theta}]^n$$
$$\cos(\theta n) + i\sin(\theta n) = [\cos\theta + i\sin\theta]^n$$

- $\theta_{deg} = \theta_{rad} \cdot \frac{180}{\pi}$
- $\theta_{rad} = \theta_{deg} \cdot \frac{\pi}{180}$

Ex: Find $(-1+i)^{-3}$ in simpleform?

$$(-1+i)^{-3} = r^n e^{i\theta n} \qquad r = \sqrt{2}, \theta = \frac{3\pi}{4}$$
$$= (\sqrt{2})^{-3} e^{i(-\frac{9\pi}{4})}$$
$$= 2^{-\frac{3}{2}} \left[\cos\left(-\frac{9\pi}{4}\right) + i\sin\left(-\frac{9\pi}{4}\right)\right]$$

- **Root of complex numbers:**

$$\sqrt[n]{z} = \sqrt[n]{(x+iy)}$$
$$= \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi K}{n}\right) + i\sin\left(\frac{\theta + 2\pi K}{2}\right)\right]$$

- $r = \sqrt{x^2 + y^2}$
- n = integer value
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
- $K = 0, 1, 2 \cdots (n-1)$

(الجذر الأول صفر)
(الجذر الثاني 1)

Ex: Find $\sqrt[3]{1}$ in Simple form.

$\Rightarrow r = \sqrt{1} = 1, \ \theta = \tan^{-1}\left(\frac{0}{1}\right) = 0, \quad n = 3, \ K = 0, 1, 2$

- For $K=0$ the 1^{st} root:

$$= 1[\cos(0) + i\sin(0)] = 1$$

- For $K=1$ the 2^{nd} root

$$= 1\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

- For $K=2$ the 3^{ed} root

$$= 1\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Solve:

① $\sqrt[3]{-i}$ ② $\sqrt[2]{1-i}$ ③ $(1-i)^{-10}$

No.

• Rules

1. $|Re(z)| \leq |z|$
2. $|IM(z)| \leq |z|$
3. $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$

13.5) Exponential Form of complex numbers:

• $e^z = e^{x+iy}$

$= e^x \cdot e^{iy}$

$= e^x \left[\cos y + i \sin y\right]$

$= \underbrace{e^x \cos y}_{Real} + \underbrace{i e^x \sin y}_{IM}$

• $|e^z| = \sqrt{x^2+y^2}$

$= \sqrt{(e^x \cos y)^2 + (e^x \sin y)^2}$

$= \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y}$

$= e^x \sqrt{\cos^2 y + \sin^2 y} = e^x$

• $\theta = \tan^{-1}\left(\frac{e^x \sin y}{e^x \cos y}\right) =$

$\tan^{-1}(\tan y) = y$

• $\arg(e^z) = y \pm 2\pi n \quad , n = \pm 1, \pm 2 \dots$

• $[e^z]' = e^z$

• $e^{z_1} e^{z_2} = e^{z_1+z_2}$

• $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

• $e^{i\pi}$, $e^{2\pi i}$

$e^{i\pi} = \cos\pi + i\sin\pi = -1$

$e^{2i\pi} = \cos 2\pi + i \sin 2\pi = 1$

Ex: Write $e^{1.4-0.6i}$ in simple form & Find the magnitude & θ.

$e^{1.4} \cdot e^{-0.6i} = e^{1.4}\left(\cos(-0.6) + i\sin(-0.6)\right)$

$= e^{1.4}\left(\cos(0.6) - i\sin(0.6)\right) \rightarrow$ ملاحظة وانتباه عند ادخال الزاوية على الآلة الحاسبة لايجاد الجواب النهائي التحويل من rad ـــ deg. أو العكس.

$\rightarrow |e^z| = e^{1.4}$

$\rightarrow \theta = -0.6$

$\rightarrow \arg(z) = -0.6 \pm 2\pi n$

No.

13.6) Trigonomitric & hyperbolic functions:

- $e^{iz} = \cos z + i \sin z \quad \ldots ①$
- $e^{-iz} = \cos(-z) + i\sin(-z)$

$= \cos z - i\sin z \quad \ldots ②$

① + ② $\Rightarrow$

$2\cos(z) + 0 = e^{iz} + e^{-iz}$

- $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2} \quad \ldots ⊛$

① − ② $\Rightarrow$

$e^{iz} - e^{-iz} = 0 + 2i\sin z$

- $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} \quad \ldots ⊛$

- $\tan(z) = \dfrac{\sin z}{\cos z} = \dfrac{e^{iz} - e^{iz}}{i(e^{iz} + e^{-iz})} \quad \ldots ⊛$

- $\cot(z) = \dfrac{1}{\tan(z)} = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{iz}} \quad \ldots ⊛$

- $\sec(z) = \dfrac{1}{\cos(z)} = \dfrac{2}{e^{iz} + e^{-iz}} \quad \ldots ⊛$

- $\csc(z) = \dfrac{1}{\sin(z)} = \dfrac{2i}{e^{iz} - e^{-iz}} \quad \ldots ⊛$

→ $[\cos(z)]' = -\sin(z)$

→ $[\sin(z)]' = \cos(z)$

→ $[\tan(z)]' = \sec^2(z)$

- $\cos(z) = \cos(x + iy)$

$= \cos x \cdot \cosh y - i \sin x \sinh y$

Ex: Find $\cos(\pi + 3i)$ in simple form.

$= \cos\pi \cosh 3 - (\sin\pi \sinh 3)i$

$= -\cosh 3$

- $\sin(z) = \sin(x+iy)$

$= \sin x \cosh y + (\cos x \sinh y)i$

Ex: Find $\sin(\pi+3i)$ in simple form.

$= \sin\pi \cosh 3 + (\cos\pi \sinh 3).i$

$= -(\sinh 3)i$

- Ex: $\tan(\pi+3i) = \dfrac{\sin(\pi+3i)}{\cos(\pi+3i)}$

$= \dfrac{-(\sinh 3)i}{-\cosh 3}$

$= i\tanh 3$

- $\cos^2(z) + \sin^2(z) = 1$
- $\cosh^2(z) = 1 + \sinh^2(z)$
- $|\cos z|^2 = \cos^2 x + \sinh^2 y$
- $|\sin z|^2 = \sin^2 x + \sinh^2 y$
- $\sin(z_1)\cos(z_2) = \frac{1}{2}[\sin(z_1+z_2) + \sin(z_1+z_2)]$
- $\cos(z_1 \pm z_2) = \cos z_1 \cos(z_2) \mp \sin(z_1)\sin(z_2)$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$
- Hyperbolic Functions:

$\cosh(z) = \dfrac{e^z + e^{-z}}{2}$, $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$, $\tanh(z) = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$

$\text{sech}(z) = \dfrac{2}{e^z + e^{-z}}$, $\text{csch}(z) = \dfrac{2}{e^z - e^{-z}}$, $\coth(z) = \dfrac{e^z + e^{-z}}{e^z - e^{-z}}$

* $[\cosh(z)]' = \sinh(z)$.
* $[\sinh(z)]' = \cosh(z)$.

* Complex Trigonomitric & hyperbolic Function are related:

① $\cosh(iz) = \cos(z)$.

② $\sinh(iz) = i\sin(z)$,

③ $\cos(iz) = \cosh(z)$

④ $\sin(iz) = i\sinh(z)$.

- $\cosh(z) = \cosh(x+iy)$

$= \cosh x \cos y + i \sinh x \sin y$

Ex: $\cosh(3+\pi i)$

$= \cosh 3 \cos\pi + i \sinh 3 \sin\pi$

$= -\cosh 3$

- $\sinh(z) = \sinh(x+iy)$

$= \sinh x \cos y + i\cosh x \sin y$

Ex: Find $\sinh(3+\pi i)$

$= \sinh 3 \cos\pi + i\cosh 3 \sin\pi$

$= -\sinh 3$

Ex: $\tanh(3+\pi i)$

$= \dfrac{-\sinh 3}{-\cosh 3} = \tanh 3$.

- $\cosh(z_1+z_2) =$

$\cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$

- $\sinh(z_1+z_2)$

$\sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2)$

Ex: Find $\operatorname{sech}(3+4i)$ in simple form

$$= \frac{1}{\cosh(3+4i)} = \frac{1}{\cosh 3\cos 4 + i\sinh 3\sin 4}$$

الضرب بالمرافق

$$= \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} \quad \ldots\ldots$$ نكمل الحل

Solve:

① $e^{5+2\pi i}$

② Find r & θ for $e^{2\pi+5i}$

③ $\cot(2\pi+3\pi i)$?

④ $\operatorname{sech}(2\pi+\pi i)$?

13.7] logarithm – General power & principle value:

$z = x + iy = re^{i\theta}$

$\ln|z| = \ln|x+iy| = \ln|re^{i\theta}|$

$= \ln r + \ln e^{i\theta}$

$= \ln r + i\theta\,[\ln e]$ $= 1$

$= \ln r + i\theta$

$r = \sqrt{x^2+y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right) \pm 2\pi n$

Ex: Find $\ln[3-4i] = ?$

$x = 3,\ y = -4$

$r = 5,\ \theta = -0.93$

$\ln[3-4i] = \ln r + i\theta$

$= \ln 5 + (-0.93 + 2\pi n)i$

No.

* Principle value of $\ln(z)$ is $Ln(z)$

[1] $Ln(z) = \ln(z) + i\,Arg(z)$

[2] $\ln(z) = L(z) \pm 2\pi n i$

At $n=0$

$\ln(z) = Ln(z)$

- $\ln(z_1 z_2) = \ln z_1 + \ln z_2$
- $\ln\left(\frac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$
- $[\ln(z)]' = \frac{1}{z}$

Ex: Find $Ln(3-4i)$

$1.61 + i[-0.93]$

* General power:

$[z]^c = e^{\ln[z]^c}$

$= e^{c\ln z}$

$\Rightarrow c = \pm 1, \pm 2 \cdots$

$c = 2/3, 3/4, \cdots$

$c = 1/2, 1/3, \cdots$

$c = 1+i, x+iy$.

Ex: Find $(i)^i$ in simple Form.

$(i)^i = e^{\ln(i)^i}$

$= e^{i\ln i} \rightarrow \ln i = \ln r + i\theta$

$\Downarrow = \cancel{\ln 1} + i\left[\frac{\pi}{2} + \frac{2\pi n}{}\right]$

$r = 1$

$\theta = \frac{\pi}{2}$

$(i)^i = e^{i[i[\pi/2 + 2\pi n]]}$

$= e^{-\pi/2 \pm 2\pi n}$

Ex: Find $(1+i)^{2-i}$ in simple Form:

Sol:

$(1+i)^{2-i} = e^{\ln(1+i)^{2-i}}$

$= e^{(2-i)\ln(1+i)} \rightarrow$

$= e^{(2-i)[\ln\sqrt{2} + i\pi/4 + 2\pi n i]}$

$\ln(1+i) = \ln r + i\theta$

$\Downarrow$ $r = \sqrt{2}$ $\quad = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi n\right)$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

تكملة السؤال.

* principle value for General power: $[n=0]$

Ex: Find the principle value For $(i)^i$?

$= e^{-\pi/2}$

[New chapter]

No.

7.1 Matrix, vectors addition & scalar Multiplication:

Matrix ⇒ is a rectangular array of numbers or function which we will enclose in brackets.

$$A = \begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} x & xy \\ \sin x & \cos y \end{bmatrix}$$

الارقام ⇐

elements, entery

* linear system:

$4x_1 + 6x_2 + 9x_3 = 6$

$6x_1 + 0x_2 - 2x_3 = 20$

$5x_1 - 8x_2 + x_3 = 10$

1. $Ax = b$

$$\begin{bmatrix} 4 & 6 & 9 \\ 6 & 0 & -2 \\ 5 & -8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 20 \\ 10 \end{bmatrix}$$

2. Augmenting Matrix $\tilde{A} = [A \vdots b]$

$$\tilde{A} = \left[\begin{array}{ccc|c} 4 & 6 & 9 & 6 \\ 6 & 0 & -2 & 20 \\ 5 & -8 & 1 & 10 \end{array}\right]$$

$$x = \begin{bmatrix} 3 \\ 1/2 \\ -1 \end{bmatrix}$$

smile for life

* For Matrix $A = [a_{jk}]$ has $m \times n$ Matrix size of $A = m \times n$

Row ↲ ↳ Column.

* square Matrix if Row & column are equal.

⊛ If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Main diagond element $= a_{11}, a_{22}, a_{33}$

⊛ Vectors: is a Matrix with one Row OR One Column.

$a = [a_1 \quad a_2 \quad a_3] \rightarrow$ size 1×3

$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \rightarrow$ size 3×1

⊛ equality of Matrix:

Def: Two matrix $A = [a_{jk}]$ & $B = [b_{jk}]$ are equal iff

① have same size $m \times n$

② corresponding element are equal, if $A \neq B$ then A & B are different

[1] do not have same size

[2] correspounding element are not equal.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$C = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 4 & 6 \end{bmatrix}$

[1] A & B are equal — Same size & elements

[2] $A \neq C$ — different element $1 \neq -1$

[3] $A \neq D$ — different size

No.

Addition of Matrix:

Def: the sum of Two matrix $A[a_{jk}]$ & $B=[b_{jk}]$ of same size is written $A+B=[a_{jk}+b_{jk}]$ & obtained by adding the corresponding element.

note: Matrix with different size cannot added.

Ex: let ⇒

$$A = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

1. $A+B = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}_{2\times3}$

2. $A-B = \begin{bmatrix} -9 & 7 & 3 \\ -3 & 0 & 2 \end{bmatrix}_{2\times 3}$

3. $A+C$ = can not be added (different size).

Scalar Multiplication:

Def: The product of any $m\times n$ Matrix $A=[a_{jk}]$ & any scalar (C) written CA & it's $m\times n$ Matrix $CA=[Ca_{jk}]$ obtained by multipling each element of A by C.

Ex: let $A = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9 & -4.5 \end{bmatrix}$

Find 1. $-A$ 2. $\frac{10}{9}A$ 3. $0A$

$$-A = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9 & 4.5 \end{bmatrix}$$

$$\frac{10}{9}A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}$$

$$0A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

No.

- **Rules of addition:**

1. $A+B = B+A$
2. $[A+B]+C = A+[B+C] = A+B+C$
3. $A+0=A$
4. $A+(-A)=0$

- **Rules of scalar Multiplication:**

1. $C[A+B] = CA+CB$
2. $(C+K)A = CA+KA$
3. $C[KA] = [CK]A$
4. $1A = A$

7.2) Matrix Multiplication:

Def: The product $C=AB$ of an $(m \times n)$ Matrix $A=[a_{jk}]$ times an $(r \times p)$ Matrix $B=[b_{jk}]$ is definied iff $n=r$ & the $m \times p$ matrix $C=[c_{jk}]$ with element

$$c_{jk} = \sum_{l=1}^{n} a_{jl} b_{lk} = a_{j1} b_{1k} + \dots + a_{jn} b_{nk}$$

(صف ضرب عامود واجمع)

$j = 1, 2, \dots m$

$k = 1, 2 \dots p$

$$\underbrace{\underset{m \times n}{A} \; \underset{r \times p}{B}}_{\text{equal}} = \underset{m \times p}{C}.$$

Ex: let $A = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix}$

find AB?

$A \times B = C \longrightarrow 3 \times 4$

$\underbrace{3 \times 3 \quad 3 \times 4}_{\text{equal}}$

$$B = \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$C_{11} = (3)(2) + 5(5) + (-1)(9)$

صف من A ضرب عامود من B واجمع

No.

Ex: let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$

AB = ??

$$AB = \begin{bmatrix} a_{11}(b_{11}) + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

* $\underset{(2\times2)}{\begin{bmatrix} 4 & 2 \\ 1 & 8 \end{bmatrix}} \underset{(2\times1)}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}} = \underset{(2\times1)}{\begin{bmatrix} 22 \\ 43 \end{bmatrix}}$

* $\underset{(1\times3)}{\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}} \underset{(3\times1)}{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}} = \underset{(1\times1)}{\begin{bmatrix} 19 \end{bmatrix}}$

* $\underset{(3\times1)}{\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}} \underset{(1\times3)}{\begin{bmatrix} 3 & 6 & 1 \end{bmatrix}} = \begin{bmatrix} 3 & 6 & 1 \\ 6 & 12 & 2 \\ 12 & 24 & 4 \end{bmatrix} (3\times3)$

$\therefore$ $AB \neq BA$ (علاقة غير تبادلية)

• Rules ⇒
① $(KA)B = K(AB)$
② $A(BC) = (AB)C$
③ $(A+B)C = AC+BC$
④ $C(A+B) = CA+CB$

$AC \neq CA$ ⇐ توزيع

* product in terms of Row & columns vector (طريقة ثانية)

Ex: let $\underset{(3\times3)}{A} = \begin{bmatrix} — a_1 — \\ — a_2 — \\ — a_3 — \end{bmatrix}$, $\underset{(3\times4)}{B} = \begin{bmatrix} | & | & | & | \\ b_1 & b_2 & b_3 & b_4 \\ | & | & | & | \end{bmatrix}$

$$\Rightarrow \underset{(3\times4)}{AB} = \begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \end{bmatrix}$$

- Parrallel processing of product:

$$AB = A \begin{bmatrix} | & | & \cdots & | \\ b_1 & b_2 & & b_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & \cdots & | \\ Ab_1 & Ab_2 & & Ab_n \\ | & | & & | \end{bmatrix}$$

Ex: $\underset{(2\times2)}{\begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix}} \underset{(2\times3)}{\begin{bmatrix} 3 & 0 & 7 \\ -1 & 4 & 6 \end{bmatrix}} = \underset{(2\times3)}{\begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & -23 \end{bmatrix}}$

$$Ab_1 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ -17 \end{bmatrix}$$

$$Ab_2 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$Ab_3 = \begin{bmatrix} 4 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 6 \end{bmatrix} = \begin{bmatrix} 34 \\ 23 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 4 & 34 \\ -17 & 8 & 23 \end{bmatrix}$$

* Transposition :

the transepose of $(m \times n)$ Matrix $A = [a_{jk}]$ in $(n \times m)$ Matrix A^T (A transpose) that has the first Row of A as its first column & second Row of A as its second column & so on.

[1] $A = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix}_{2\times3} \Rightarrow A^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}_{3\times2}$

[2] $A = [6\ 2\ 3]_{1\times3} \Rightarrow A^T = \begin{bmatrix} 6 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$

Rules:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(CA)^T = CA^T$
4. $(AB)^T = B^T A^T$

- $(ABC)^T = C^T B^T A^T$

معكوسة ←

- symmetric Matrix if $A^T = A$
- skew-symmetric Matrix $A^T = -A$

Ex: let $A = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}_{3\times3}$

$A^T = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$

$A = A^T \Rightarrow A$ is symmetric

Ex: let $A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$

$A^T = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 2 \\ -3 & -2 & 0 \end{bmatrix}$

$-A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \Rightarrow -A = A^T$ (A is skew-symmetric)

27/10

No.

* upper triangular Matrix :

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} , \quad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

* lower triangular Matrix :

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

* diagonal Matrix D :

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* Scalar Matrix S :

$$S = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

∴ c = Scalar (ثابت).

* Unite Matrix OR Identity Matrix :

$$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ,$$

$$I_{3\times3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[AI = IA = A \right]$$

7.3) Linear system of eq. Gauss-elimination.

let linear system with - m - eq - n - unknowns

$$a_{11}x_1 + \_\_\_ + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \_\_\_ a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{1m}x_1 + \_\_\_ + a_{mn}x_n = b_m .$$

No. ______________

1. linear system ⇒ $x_1 x_2 \_\_ x_3$ of Order 1
2. for linear system with all (b_j) are zero ⇒ [homogeneouse L.S]
3. for linear system with at least one (b_j) are not zero ⇒ [Non – Homogeneouse L.S]

* For linear system $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\tilde{A} = [A \,\vdots\, b]$$

* for linear system $Ax = b$ has three possible Causes:

1. prencisely on solution if the line Intersect.
2. Infinitly Many soln. if the lines coincide
3. No- soln if the lines are parrallel

* Gauss-elimination & back soln:

$2x_1 + 5x_2 = 2$ ⇒ $Ax = b$

$+13x_2 = -26$

$$\begin{bmatrix} 2 & 5 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

↳ $x_2 = -2$

نعوضها بالمعادلة الأولى

$x_1 = 6$

Ex: $2x_1 + 5x_2 = 2$

$-4x_1 + 3x_2 = -30$

find x_1 & x_2 ?

$\boxed{1}$ الطريقة الأولى (حذف وتعويض)

$[2x_1 + 5x_2 = 2] \times 2$

$-4x_1 + 3x_2 = -30$

↳ $13x_2 = -26$

$x_2 = -2$

↓ $x_1 = 6$

smi)e

No.

[2] الطريقة الثانية

Using G-E

① $AX + b$

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

② $\tilde{A} = [A \,\vdots\, b]$

$$\left[\begin{array}{cc|c} 2 & 5 & 2 \\ -4 & 3 & -30 \end{array}\right] \Rightarrow \text{pivot}$$

$$= \left[\begin{array}{cc|c} 2 & 5 & 2 \\ 0 & 13 & -26 \end{array}\right] \rightarrow R_2 + 2$$

$2x_1 + 5x_2 = 2$

$+13x_2 = -26$

$\hookrightarrow x_1 = 6, \; x_2 = -2$

EX: With one solution $\Rightarrow$ find x_1, x_2, x_3

$x_1 - x_2 + x_3 = 0$

$-x_1 + x_2 - x_3 = 0$

$+10x_2 + 25x_3 = 90$

$20x_1 + 10x_2 = 80$

① $AX = b$

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}$$

② $\tilde{A} = [A \,\vdots\, b]$

$$\begin{array}{r} \text{pivot} \Leftarrow \\ \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]\begin{array}{l} \\ \Rightarrow R_2 + R_1 \\ \text{ثابت} \\ \Rightarrow R_4 - 20R_1 \end{array}$$

$$\begin{array}{r} \\ \text{pivot} \rightarrow \\ \\ \\ \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \Rightarrow R_3 - 3R_2 \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 - x_2 + x_3 = 0$

$10x_2 + 25x_3 = 90$

$-95x_3 = -190$

$\Leftarrow [x_3 = 2, \; x_2 = 4, \; x_1 = 2]$

No.

* elementary Raw operation:

① Inter change of Two Raws
② Addition of constant multiple to on Row to another
③ Multiplication of one Row by non-zero constant

has no-effect on the soln.

Note ⇒

① over determined linear system if it has more eq. than unknown
② determined L.S if the number of eq. equal number of unknowns
③ underdetermined L.S if the system has fewer eq. than unknowns
④ consistent L.S if it has at least one soln.
⑤ In-consistent L.S if it has no-soln

① $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

② $\begin{bmatrix} 5a_{11} & 5a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5b_1 \\ b_{12} \end{bmatrix}$

③ $\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$

1+2+3 ⇒ Has soln.

smi)e for life

No.

Ex: with infinitly Many soln:

$3X_1 + 2X_2 + 2X_3 - 5X_4 = 8$

$0.6X_1 + 1.5X_2 + 1.5X_3 - 5.4X_4 = 2.7$

$1.2X_1 - 0.3X_2 - 0.3X_3 + 2.4X_4 = 2.1$

Find X_1, X_2, X_3, X_4 ?

① $Ax = b$

$$\begin{bmatrix} 3 & 2 & 2 & -5 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 2.7 \\ 2.1 \end{bmatrix}$$

② $\tilde{A} = [A \vdots b]$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right] \rightarrow \text{pivot}$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \Rightarrow \text{pivot} \rightarrow R_2 - 0.2R_1 \\ \rightarrow R_3 - 0.4R_1 \end{array}$$

$$= \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \rightarrow R_3 + R_2 \end{array}$$

$3X_1 + 2X_2 + 2X_3 - 5X_4 = 8$

$1.1X_2 + 1.1X_3 - 4.4X_4 = 1.1$

$X_2 + X_3 = 1 + 4X_4$

$3X_1 + 2[1 + 4X_4] - 5X_4 = 8$

$X_1 = 2 - X_4$ $\quad X_3 = \text{arbitary}$

$X_2 = 1 + 4X_4 - X_3$ $\quad X_4 = \text{arbitary}$

No.

ex: with no - soln

$3x_1 + 2x_2 + x_3 = 3$
$2x_1 + x_2 + x_3 = 0$
$6x_1 + 2x_2 + 4x_3 = 6$

Find x_1, x_2, x_3

① $Ax = b$

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

② $\tilde{A} = [A \vdots b]$

pivot $\leftarrow$
$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] = \text{pivot} \rightarrow \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$R_3 - 2R_1$ $\quad$ $R_2 - \frac{2}{3} R_1$

$$= \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right] \rightarrow R_3 - 6R_2$$

$3x_1 + 2x_2 + x_3 = 3$

$-\frac{1}{3} x_2 + \frac{1}{3} x_3 = -2$

$0 = 12 \Rightarrow$ false - statement, no - soln

* Row echelon form & information from it :

let L.S $Ax = b$ argmented Matrix $\tilde{A} = [A \vdots b]$ of the Gauss - elimination.

$$[R \vdots F] = \left[\begin{array}{cccc|c} r_{11} & r_{12} & \cdots & r_{1n} & f_1 \\ & r_{22} & \cdots & r_{2n} & f_2 \\ & & & & \vdots \\ & & r_{rr} & \cdots\ r_{rn} & f_r \\ & & & & \vdots \\ & & & & f_m \end{array}\right]$$

No.

① $r_{11} \neq 0$

② all element in the tringle & rectangle is zero

ⓐ no-soln if $r<m$ & at least one number $f_r, \dots f_m$ is not zero

$$[R \,|\, F] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 0 & -\frac{1}{3} & \frac{1}{3} & -2 \\ 0 & 0 & 0 & 12 \end{array}\right]$$

$r_{11} \neq 0 = 3$

$r_{23} = \frac{1}{3} = 0.$ $r_{rn} \left< \begin{array}{l} r=2 \\ n=3 \end{array} \right.$

$f_m = f_3 = 12$

$m = 3$

No-Soln.

ⓑ One-soln: if $r=n$ & all numbers $f_r \dots f_m$ are zero

$$[R \,|\, F] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r_{11} = 1 \neq 0$

$r_{rn} = r_{33} = -95$

$r=3 \quad n=3$

$f_m = f_4 = 0$

$(m=4)$

ⓒ infinitly Many soln if $r<m$ & all numbers $f_{r+1} \dots f_m$ are zero

$$[R \,|\, f] = \left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r_{11} = 3 \neq 0$

$r_{rn} = r_{24} = -4.4 \left< \begin{array}{l} r=2 \\ n=4 \end{array} \right.$

$f_m = f_3 = 0 \Rightarrow m=3$

No.

7.4) linear Indep. of vector & Rank of Matrix

let : $a_1, a_2 \dots a_m$ are m-vectors

$c_1, c_2 \dots c_m$ are m-scalar

then $c_1 a_1 + c_2 a_2 + \dots + c_m a_m = 0$ eq (1)

(1) If eq (1) holds for all (c_j) are zero then a's are linearly Indep.

(2) If eq (1) holds for at least one (c_j) isn't zero a's are linearly dep.

Ex: let $a_1 = [3 \ 0 \ 2 \ 2]$

$a_2 = [-6 \ 42 \ 24 \ 54]$

$a_3 = [21 \ -21 \ 0 \ -15]$

are a_1, a_2, a_3 linearly dep ??

$3c_1 - 6c_2 + 21c_3 = 0$

$0c_1 + 42c_2 - 21c_3 = 0$

$2c_1 + 24c_2 + 0c_3 = 0$

$2c_1 + 54c_2 - 15c_3 = -1$ $\Rightarrow$ $c_1 = 6, c_2 = 1/2$, $c_3 = -1$

$\therefore$ linearly dep.

* Rank of A : is the Max number of linearly Indep. Row vector of A

Ex: let $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ find Rank A ? pivot

$$= \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{matrix} \\ \rightarrow R_2 + 2R_1 \\ \rightarrow R_3 - 7R_1 \end{matrix} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Rank = 2)

عدد الصفوف يلي ما صارت صفر

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 7 \end{bmatrix}$

Rank = 1 Rank = 3

Theorem : Row equivelent Matrix:

① Matrix A_1 is a row equivelent to Matrix A_2 if A_1 can be obtained from A_2 by elementary Row operation.

② Row-equivelent Matrix has same Rank.

Theorem :

Consider p-vector that each have n-component then, these vectors are linearly Indep. if Matrix formed with these vectors as Row vector has Rank P, however these vectors are linearly dep. if Matrix formed has Rank less than P

⊛ Rank (A) = Rank (A^T).

Theorem:

consider p-vector each have n-component if $n < p$ then these vectors are linearly dep.

Ex: let p_1 [1,2] ⇒ are p's linearly dep ??
p_2 [3,4]
p_3 [5,6]

Soln : $p = 3$, $n = 2$ ⇒ $n < p$ ∴ linearly dep.

Theorem: Soln of linear system existance & uniqueness for $Ax = b$ with n-unknown , m-eq.

$$\tilde{A} = [A \vdots b]$$

① Existence : linear system has soln iff [Rank(A) = Rank$(\tilde{A})$]

② uniqueness : linear system has one soln iff [Rank(A) = Rank$(\tilde{A}) = n$]

③ infinitly Many soln : iff [Rank$(A) < n$, Rank$(\tilde{A}) < n$]

④ No-soln if : [Rank$(A) \neq$ Rank$(\tilde{A})$]

No.

7.6) Second & third order det. let:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad D = \det[A] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$= a_{11}a_{22} - a_{12}a_{21}$$

Ex: let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ find D ??

$D = (1)(3) - (2)(1) = 1$

* Cramer Rules: for second order linear system

$Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}$$

$$D_1 = \begin{bmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{bmatrix} = b_1a_{22} - a_{12}b_2$$

$$D_2 = \begin{bmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{bmatrix} = a_{11}b_2 - b_1a_{21}$$

Ex: let $4x_1 + 3x_2 = 12$, find x_1, x_2 ??

$2x_1 + 5x_2 = -8$

① $Ax = b$

$$\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D} = \frac{\begin{bmatrix} 12 & 3 \\ -8 & 5 \end{bmatrix}}{\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}}$$

$$x_2 = \frac{D_2}{D} = \frac{\begin{bmatrix} 4 & 12 \\ 2 & -8 \end{bmatrix}}{\begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}}$$

$$x_1 = \frac{12(5) - (3)(-8)}{4(5) - (3)(2)} = \frac{84}{14} = 6$$

$$x_2 = \frac{4(-8) - (12)(2)}{4(5) - (3)(2)} = \frac{-56}{14} = -4$$

No.

* Third order det:

For linear system $Ax = b$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

$$D = \det[A] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= + a_{11} \begin{bmatrix} a_{22} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} - a_{12} \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$$

$$= a_{11}[a_{22}a_{33} - a_{23}a_{31}] - a_{12}[a_{21}a_{33} - a_{23}a_{31}] + a_{13}[a_{21}a_{32} - a_{22}a_{31}]$$

Ex: let $A = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{bmatrix}$, find $\mathrm{Det}[A]$??

$$= 1 \begin{bmatrix} 6 & 4 \\ 0 & 2 \end{bmatrix} - 3 \begin{bmatrix} 2 & 4 \\ -1 & 2 \end{bmatrix} + 0 \begin{bmatrix} 2 & 6 \\ -1 & 0 \end{bmatrix} = -12$$

$$D_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}, \quad D_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}$$

$$D_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

No.

let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

* The Minore Matrix of A is M :

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} M_{22} & M_{23} \\ M_{32} & M_{33} \end{bmatrix} = M_{22}M_{33} - M_{23}M_{32}$$

$$M_{12} = \begin{bmatrix} M_{21} & M_{23} \\ M_{31} & M_{33} \end{bmatrix}$$

$\vdots$

$$M_{32} = \begin{bmatrix} M_{11} & M_{13} \\ M_{21} & M_{33} \end{bmatrix} , \quad M_{33} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

* The Cofactor Matrix of A is C :-

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \Rightarrow \left[C_{jk} = (-1)^{j+k} M_{jk} \right]$$

$$C_{11} = (-1)^2 M_{11} = M_{11}$$

$$C_{12} = (-1)^3 M_{12} = -M_{12}$$

$$C = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}$$

No.

Theorem: (a) Interchange of two Rows Multiply the value of determined by [-1]

let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $|A| = 1$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$ $|B| = -1$

(b) addition of multiple of Row to another Row doesn't alter the value of determinte

let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $|A| = 1$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $|B| = 1$

(c) multiplication of Row by non-zero constant [C] multiply the value of det by [C]

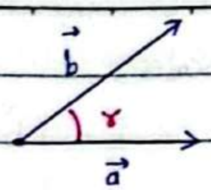
(d) $\det[CA] = C^n \det[A]$, n = number of Row in A

EX: let $A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix}$ find $|A|$, $|3A|$??

$|A| = 4-2 = 2$, $|3A| = \begin{bmatrix} 3 & 6 \\ 3 & 12 \end{bmatrix} = 3(12) - (3)(6) = 18$

$|3A| = 3^2 \det[A] = 9(2) = 18$

* 9.2 : Dot product



$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\gamma$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

- $\vec{a}\cdot\vec{b} = 0$ if $\vec{a} = 0$ OR $\vec{b} = 0$
- for $\vec{a} = [a_1, a_2, a_3]$
  $\vec{b} = [b_1, b_2, b_3]$

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$

Theorem: $\vec{a}, \vec{b}$ are orthogonal if $\gamma = 90, \pi/2$ so $\vec{a}\cdot\vec{b} = 0$

Ex: let $\vec{a} = [1,2,0]$, $\vec{b} = [3,-2,1]$ find γ between $\vec{a}, \vec{b}$?

$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\gamma$

$(1)(3) + (2)(-2) + (0)(1) = \sqrt{5}\,\sqrt{14}\cos\gamma$

$\gamma = 96.865^\circ$

Rules: ① $[q_1\vec{a} + q_2\vec{b}]\cdot\vec{c} = q_1\vec{a}\vec{c} + q_2\vec{b}\vec{c}$

② $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

③ $\vec{a}\cdot\vec{a} \geq 0$

④ $\vec{a}\cdot\vec{a} = 0$, if $\vec{a} = 0$

⑤ $[\vec{a}+\vec{b}]\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

⑥ $|\vec{a}\cdot\vec{b}| \leq |\vec{a}|\,|\vec{b}|$
↳ (cauchy-schwarz inequality)

⑦ $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$
↳ Trainglure inequality

⑧ $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left[|\vec{a}|^2 + |\vec{b}|^2\right]$
↳ parallegram equality

Notes:

① $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

② $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{i} = \hat{k}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = \hat{i}\cdot\hat{k} = 0$

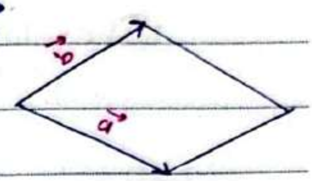
(9.3) Cross product

$$\vec{v} = \vec{a} \times \vec{b}$$

① if $\vec{a} = 0$ OR $\vec{b} = 0$ then $\vec{v} = \vec{a} \times \vec{b} = 0$

② if $\vec{a} \neq 0$, $\vec{b} \neq 0$ then $|\vec{v}| = |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \gamma$

③ $|\vec{b}|$: Area of parallegram contralled by $\vec{a}, \vec{b}$



④ if (γ) between $\vec{a}, \vec{b}$ is zero OR 180 same OR opposite direction

$\sin(0) = \sin(180) = 0$, $\vec{v} = \vec{a} \times \vec{b} = 0$

if $\vec{a} \neq 0$, $\vec{b} \neq 0$ ⑤ $\vec{v} = \vec{a} \times \vec{b}$ then $\vec{v}$ is perpendecular to plane formed by $\vec{a}, \vec{b}$

⑥ $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

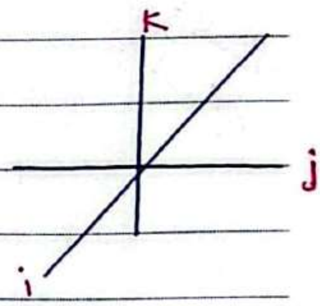
$$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i [a_2 b_3 - a_3 b_2] - j [a_1 b_3 - a_3 b_1] + k [a_1 b_2 - a_2 b_1]$$

Ex: if $\vec{a} = [1,1,0]$, $\vec{b} = [3,0,0]$ find $\vec{v} = \vec{a} \times \vec{b}$?

$$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = -3k = [0,0,-3]$$



Notes: ① $i \times j = k$, $j \times k = i$, $k \times i = j$

② $j \times i = -k$, $k \times j = -i$, $i \times k = -j$

Rules:

① $(l\vec{a}) \times \vec{b} = l(\vec{a} \times \vec{b}) = \vec{a} \times (l\vec{b})$

② $\vec{a} \times [\vec{b} + \vec{c}] = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

③ $[\vec{a} + \vec{b}] \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

الترتيب مهم

No.

④ $\vec{a} \times \vec{b} = -[\vec{b} \times \vec{a}]$

⑤ $[\vec{a} \times \vec{b}] \times \vec{c} \neq \vec{a} \times [\vec{b} \times \vec{c}]$.الأولوية للي بين أقواس

⊛ Scalar Triple product:

Def: for vector $\vec{a}, \vec{b}, \vec{c}$

$$(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot [\vec{b} \times \vec{c}] = [\vec{a} \times \vec{b}] \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$|(\vec{a}\ \vec{b}\ \vec{c})|$ = volume of Box

* **Theorem:** $\vec{a}, \vec{b}, \vec{c}$ are linearly independent if $(\vec{a}\ \vec{b}\ \vec{c}) \neq 0$

if $(\vec{a}\ \vec{b}\ \vec{c}) = 0$ linearly dependent.

Ex: let $\vec{a} = [2,0,3]$, $\vec{b} = [0,4,1]$, $\vec{c} = [5,6,0]$ find the volume of Box.

$$|(\vec{a}\ \vec{b}\ \vec{c})| = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{bmatrix} = |-72| = 72 \text{ unit.}$$

• Volume of tetrahedrom $= \frac{1}{6}$ Volume of Box

في السؤال السابق $\Rightarrow$ $\frac{1}{6}(72) = 12$ unit.

(9.4) vector + Scalar function

① vector function $\vec{v}$ that dep. on point p.

$$\vec{v}(p) = [v_1(p), v_2(p), v_3(p)]$$

ex: ① tangent vector ② normal vector

② Scalar function (f) that dep. on point p.

$$f(p) = f(x,y,z)$$

ex: ① Temp. ② pressure.

$$\Rightarrow \vec{v}(x,y,z) = xyz\,\hat{i} + x^2y\,j + z^2xy\,k$$

$$f(x,y,z) = x^2y + z^2xy + z^2$$

• Dervative of vector function :- the vector function is differentable iff.

① convergences $\lim_{n\to\infty} |\vec{a}_n - \vec{a}| = 0$

② continuity $\lim_{t\to t_0} \vec{v}(t) = \vec{v}(t_0)$

$$\vec{v} = [v_1(t), v_2(t), v_3(t)]$$

$$\vec{v}' = [v_1'(t), v_2'(t), v_3'(t)]$$

$$\vec{v}'' = [v_1''(t), v_2''(t), v_3''(t)]$$

ex: let $\vec{v}(t) = [t, t^2, 0]$ find $\vec{v}'(t)$ & $\vec{v}''(t)$?

$$\vec{v}'(t) = [1, 2t, 0]$$

$$\vec{v}''(t) = [0, 2, 0]$$

No.

Rules :

① $(c\vec{v})' = c\vec{v}'$

② $(\vec{u}+\vec{v})' = \vec{u}' + \vec{v}'$

③ $(\vec{u}\cdot\vec{v})' = \vec{u}'\vec{v} + \vec{u}\cdot\vec{v}'$

④ $(\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

⑤ $(\vec{u}\ \vec{v}\ \vec{w})' = (\vec{u}'\ \vec{v}\ \vec{w}) + (\vec{u}\ \vec{v}'\ \vec{w}) + (\vec{u}\ \vec{v}\ \vec{w}')$

* partial dervitave :-

$\vec{v}(t) = [\ v_1(t)\ ,\ v_2(t)\ ,\ v_3(t)\ ]$

$$\frac{\partial \vec{v}(t)}{\partial t_m} = \left[\frac{\partial v_1(t)}{\partial t_m} , \frac{\partial v_2(t)}{\partial t_m} , \frac{\partial v_3(t)}{\partial t_m} \right]$$

$$\frac{\partial^2 \vec{v}(t)}{\partial t_1 \partial t_m} = \left[\frac{\partial^2 v_1(t)}{\partial t_1 \partial t_m} , \frac{\partial^2 v_2(t)}{\partial t_1 \partial t_m} , \frac{\partial^2 v_3(t)}{\partial t_1 \partial t_m} \right]$$

Ex: let $\vec{r}\,(t_1, t_2) = a\cos t_1\, i + a\sin t_1\, j + t_2 K$

find $\frac{\partial \vec{r}}{\partial t_1}$, $\frac{\partial \vec{r}}{\partial t_2}$, $\frac{\partial^2 \vec{r}}{\partial t_1^2}$??

① $\frac{\partial \vec{r}}{\partial t_1} = -a\sin t_1\, i + a\cos t_1\, j + 0K$

② $\frac{\partial \vec{r}}{\partial t_2} = 0i + 0j + 1K$

③ $\frac{\partial^2 \vec{r}}{\partial t_1^2} = -a\cos t_1\, i - a\sin t_1\, j + 0K$

(9.7) gradent of scalar field directional derivative :-

$$\text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} , \frac{\partial f}{\partial z} \right]$$

$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$$

∇ = nablaf

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} K$$

No.

Ex: let $f(x,y,z) = 2y^3+4xz+3x$ find $\nabla f \big|_{(1,1,1)}$?

Soln:

$\nabla f = [0+4z+3 \; , \; 6y^2+0+0 \; , \; 0+4x+0]$

$= [4z+3 \; , \; 6y^2 \; , \; 4x] = [7 \; , \; 6 \; , \; 4]$

⊛ directional dervitave :

- $D_{\vec{a}} f$ = direction dervitive of $f(x,y,z)$ in the direction of $\vec{a}$.

$D_{\vec{a}} f = \frac{\vec{a}}{|\vec{a}|} \cdot \nabla f$

* $\vec{a} = [a_1, a_2, a_3]$

* $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

* $\bullet$ Dot product.

* $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

Ex: find the directional drvative of $f(x,y,z) = 2x^2+3y^2+z^2$ at point $P(2,1,3)$ in the direction of $\vec{a} = [1,0,-2]$.

Soln:

$D_{\vec{a}} f = \frac{[1,0,-2]}{\sqrt{5}} \cdot [4x \; , \; 6y \; , \; 2z] = \frac{[1,0,-2]}{\sqrt{5}} \cdot [8,6,6]$

$= \frac{-4}{\sqrt{5}}$

⊛ Laplace - equation :-

$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

∇^2 = nable square $= \Delta$

Ex: let $f(x,y,z) = 4[x^2+y^2]-z^2$ find $\nabla^2 f$??

$\nabla^2 f = \Delta f$

$\nabla^2 f = 8+8+(-2) = 14$

smile for life

Ex: Find eigen value + eigen vector for

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

① $\det [A - \lambda I] = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda = 5, -3, -3$$

② for $\lambda = 5$ the x is:

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tilde{A} = [A \,\vdots\, b]$

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right] \Rightarrow \text{pivot}$$

$$\Leftarrow \begin{array}{r} \\ R_2 + \frac{2}{7} R_1 \\ R_3 - \frac{1}{7} R_1 \end{array} \left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 0 & -24/7 & -48/7 & 0 \\ 0 & -16/7 & -32/7 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 0 & \frac{-24}{7} & -48/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow R_3 - 2/3 R_2$$

$$7x_1 + 2x_2 - 3x_3 = 0$$

$$-24/7\, x_2 - \frac{48}{7} x_3 = 0$$

$\boxed{} \Rightarrow$ (معادلة ناقصة)

$$x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

③ for $\lambda = -3$ the x is:

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tilde{A} = [A \,\vdots\, b]$

$$\text{pivot} \leftarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \to R_2 - 2R_1 \\ \to R_3 + R_1 \end{array}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$\boxed{}$
$\boxed{}$ معادلتين ناقصات

$$x = \begin{bmatrix} \vdots \end{bmatrix}$$

No.

Ex: Find eigen value + eigen vector for

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\det [A - \lambda I] = 0$

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0$$

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm\sqrt{-1} = \pm i$$

- for $\lambda = -i$ the x is:

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

- for $\lambda = i$ the x is:

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

(8.3) Symmetric, skew-symmetric & orthogonal Matrix

Def: a real square Matrix $A = [a_{jk}]$ is called

1. symmetric if $A^T = A$
2. skew-symmetric if $A^T = -A$
3. orthogonal if $A^T = A^{-1}$

Ex: ① symmetric

$$A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

② skew-symmetric

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

③ Orthogonal

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

No.

* for any square Matrix (A)

① symmetric Matrix $(R) = \frac{1}{2}\left[A + A^T\right]$

② Skew-symmetric Matrix $(S) = \frac{1}{2}\left[A - A^T\right]$

Ex: let $A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$, find (R, S)

$A^T = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} 9 & 3.5 & 3.5 \\ 3.5 & 3 & -2 \\ 3.5 & -2 & 3 \end{bmatrix}$

$S = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6 \\ 1.5 & 6 & 0 \end{bmatrix}$

* Theorem:

① The eigen value of symmetric Matrix are Real

② The eigen value of skew-symmetric Matrix are pure imaginary or zero

↳ $(z = iy)$

Ex: Find eigen value of $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

$\det \left[A - \lambda I\right] x = 0$

$\begin{bmatrix} -\lambda & 9 & -12 \\ -9 & -\lambda & 20 \\ 12 & -20 & -\lambda \end{bmatrix}$

$-\lambda^3 - 625\lambda = 0$

$-\lambda\left[\lambda^2 + 625\right] = 0$

$\lambda = 0, \pm 25i$

No. ________

* Orthogonal Matrix :

EX: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = A^{-1}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} C^T$

$= \frac{1}{1} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Theorem :

① The determinent of an orthogonal Matrix is (1) OR (-1)

② The eigen value of orthogonal Matrix A are Real or [complex conjegate] in pairs & have [absolute value] of (1)

↳ $z = x + yi \Rightarrow \bar{z} = x - yi$ ↳ $|z| = \sqrt{x^2 + y^2} = 1$

EX: find det. + eigen value of

$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$ → ① $\det[A] = -1$

② $\det[A - \lambda I] = 0$

$\begin{bmatrix} 2/3 - \lambda & 1/3 & 2/3 \\ -2/3 & +2/3 - \lambda & 1/3 \\ 1/3 & 2/3 & -2/3 - \lambda \end{bmatrix} = 0$

$-\lambda^3 + \frac{2}{3}\lambda^2 + \frac{2}{3}\lambda - 1 = 0$ ⇒ الحل بستخدام القسمة التركيبية.

$\lambda = -1 , \frac{5 + i\sqrt{11}}{6} , \frac{5 - i\sqrt{11}}{6}$

$|z| = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2} = 1$

19/11

No.

Similar Matrix, similarity transformation let $(n\times n)$ Matrix $\hat{A}$ is called similar to $(n\times n)$ Matrix A iff

$$\hat{A} = P^{-1}AP \Rightarrow P = \text{non singular Matrix}$$

ملاحظة: الضرب عملية غير تبديلية و
اذاً لا يجوز ضرب $P^{-1}P$ يعني
AI لان الترتيب مهم

Theorem:

① If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigen value as A.

② If x is an eigen value of A, then $y = P^{-1}x$ is the eigen vector of $\hat{A}$ corresponding to same eigen value

$$\begin{tikzcd}
A \arrow[r, "\hat{A} = P^{-1}AP"] \arrow[d] & \hat{A} \arrow[d] \\
\lambda \arrow[r, "="] \arrow[d] & \lambda \arrow[d] \\
x \arrow[r, "y = P^{-1}x"] & y
\end{tikzcd}$$

EX: let $A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$ & $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, find eigen value & eigen vector for A and $\hat{A}$.

① $\det[A - \lambda I] = 0$

$$\begin{bmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{bmatrix} = 0$$

$$(6-\lambda)(-1-\lambda) + 12 = 0$$

$$\lambda = 2, 3$$

* for $\lambda = 2$ x is:

$$\begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

* for $\lambda = 3$ x is:

$$\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{A} = P^{-1}AP$$

$$= \frac{1}{1}\begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$\Rightarrow$ تكملة

No.

$\hat{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$

$\det[\hat{A} - \lambda I] = 0$

$\begin{bmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0$

$(3-\lambda)(2-\lambda) - 0 = 0$

$\lambda = 2, 3$

$y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

* for $\lambda = 2$ the (y) is

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

* for $\lambda = 3$ the (y) is

$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

- $y_1 = p^{-1} x_1$

$= \frac{1}{1} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $y_2 = p^{-1} x_2$

$= \frac{1}{1} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Theorem: diagonalization of Matrix if an $(n \times n)$ Matrix A has abasis of eigen vectors then

$\underline{\underline{D}} = X^{-1} A X$

↳ diagonal Matrix ⇒ $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

with eigen vector of A as element on main diagonal

$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$

No.

x = eigen vector of A as column vector.

$$D^2 = D \times D = x^{-1} A \underbrace{x \cdot x^{-1}}_{I} A x$$
$$= x^{-1} A I A x$$
$$= x^{-1} A^2 x$$

$$\left[D^m = x^{-1} A^m x \right]$$

Ex: Find D for A?

$$A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

$$D = x^{-1} A x$$

① $\det [A - \lambda I] = 0$

$$\begin{bmatrix} 7.3-\lambda & 0.2 & -3.7 \\ -11.5 & 1-\lambda & 5.5 \\ 17.7 & 1.8 & -9.3-\lambda \end{bmatrix}$$

$$7.3-\lambda \begin{bmatrix} 1-\lambda & 5.5 \\ 1.8 & -9.3-\lambda \end{bmatrix} - 0.2 \begin{bmatrix} -11.5 & 5.5 \\ 17.7 & -9.3-\lambda \end{bmatrix} + -3.7 \begin{bmatrix} -11.5 & 1-\lambda \\ 17.7 & 1.8 \end{bmatrix}$$

$\hookrightarrow$ $-\lambda^3 - \lambda^2 + 12\lambda = 0$

$-\lambda [\lambda^2 + \lambda - 12] = 0$

$\lambda = 0, 3, -4$

- for $\lambda = 0$ the x is:

$$x = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

- for $\lambda = 3$ the x is:

$$x = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

- for $\lambda = -4$ the x is:

$$x = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \Rightarrow x^{-1} = \frac{1}{|x|} C^T = \begin{bmatrix} -0.7 & 0.2 & 0.2 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

No.

* $D = X^{-1} A X$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Trace of $A = a_{11} + a_{22} + a_{33}$

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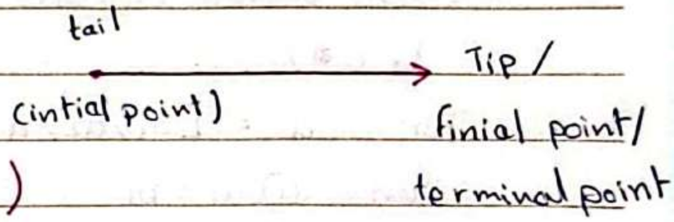
New chapter:

9.1) vector in 2-space & 3-space

① **Scalar:** quantity that is determined by it's magnitude like [time, length]

② **vector:** quantity that is determined by magnitude & direction like [force]

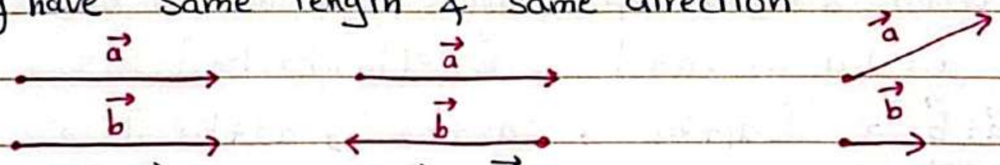
For vector $a \equiv \vec{a}$



($|\vec{a}|$ = length = norm = Magnitude)

if $|\vec{a}| = 1$ $\quad \vec{a}$ is unit vector

Def: Two vectors $\vec{a}$ & $\vec{b}$ are equal so $\vec{a} = \vec{b}$ iff they have same length & same direction



$\vec{a} = \vec{b}$ $\qquad$ $\vec{a} = -\vec{b}$ $\qquad$ $\vec{a} \neq \vec{b}$

Component of vectors:

for vector $\vec{a}$ has initial point $p: (x_1, y_1, z_1)$ & terminal point $Q: (x_2, y_2, z_2)$

$$a_1 = x_2 - x_1 \quad , \quad a_2 = y_2 - y_1 \quad , \quad a_3 = z_2 - z_1$$

$$\vec{a} = [a_1, a_2, a_3] \quad , \quad |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

Ex: for vector $\vec{a}$ with intial point $p: (4,0,2)$ & terminal point $Q: (6,-1,2)$ find vector component + $|\vec{a}|$?

$a_1 = 6-4 = 2$ $\qquad$ $\vec{a} = (2,-1,0)$

$a_2 = -1-0 = -1$ $\qquad$ $|\vec{a}| = \sqrt{4+1+0} = \sqrt{5}$

$a_3 = 2-2 = 0$

No.

* position vector ($\vec{r}$) at point A: (x, y, z) is a vector with origin (0,0,0) as initial point & A as terminal point.

(0,0,0) ⟶ (x,y,z)

* zero vector :- has length zero & no-direction
↳ (●)

For $\vec{a} = [a_1, a_2, a_3]$ & $\vec{b} = [b_1, b_2, b_3]$ if $\vec{a} = \vec{b}$

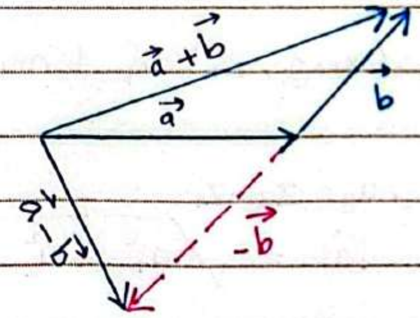
then ① $a_1 = b_1$
② $a_2 = b_2$
③ $a_3 = b_3$

* vector addition & scalar multiplication for

$\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

$\vec{a} + \vec{b} = [a_1 + b_1 , a_2 + b_2 , a_3 + b_3]$

$\vec{a} - \vec{b} = [a_1 - b_1 , a_2 - b_2 , a_3 - b_3]$



* Rules :

① $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

② $[\vec{u} + \vec{v}] + \vec{w} = \vec{u} + [\vec{v} + \vec{w}]$

③ $\vec{a} + 0 = \vec{a}$

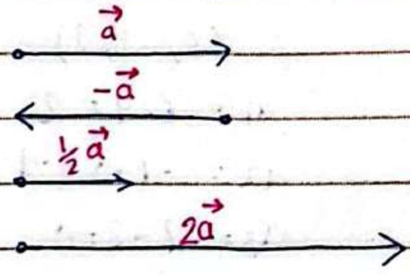
④ $\vec{a} + (-\vec{a}) =$ zero

* Scalar multiplication:

$\vec{a} = [a_1, a_2, a_3]$

$C\vec{a} = [Ca_1, Ca_2, Ca_3]$

C: Scalar



No.

Rules:

1. $c[\vec{a}+\vec{b}] = c\vec{a} + c\vec{b}$
2. $(c+k)\vec{a} = c\vec{a} + k\vec{a}$
3. $c[k\vec{a}] = (ck)\vec{a}$
4. $1\vec{a} = \vec{a}$
5. $0\vec{a} = 0$
6. $-1\vec{a} = -\vec{a}$

Ex: let $\vec{a} = [4, 0, 1]$ & $\vec{b} = [2, -5, \frac{1}{3}]$

1. $-\vec{a} = [-4, 0, -1]$
2. $7\vec{a} = [28, 0, 7]$
3. $\vec{a} + \vec{b} = [6, -5, \frac{4}{3}]$
4. $2[\vec{a} - \vec{b}] = [4, 10, \frac{4}{3}]$

Unit vector: $\hat{i}, \hat{j}, \hat{k}$

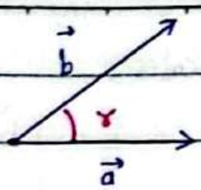
for any vector $\vec{a} = [a_1, a_2, a_3]$ we can write it as

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\hat{i} = [1, 0, 0]$, $\hat{j} = [0, 1, 0]$, $\hat{k} = [0, 0, 1]$

Ex:

$\vec{a} = [0, 0, -3] \Rightarrow \vec{a} = -3\hat{k}$



* 9.2 : Dot product

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\gamma$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

- $\vec{a}\cdot\vec{b} = 0$ if $\vec{a}=0$ OR $\vec{b}=0$
- for $\vec{a} = [a_1, a_2, a_3]$
  $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Theorem: $\vec{a}, \vec{b}$ are orthogonal if $\gamma = 90, \pi/2$ so $\vec{a}\cdot\vec{b}=0$

Ex: let $\vec{a} = [1,2,0]$, $\vec{b} = [3,-2,1]$ find γ between $\vec{a}, \vec{b}$?

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\gamma$$

$$(1)(3) + (2)(-2) + (0)(1) = \sqrt{5}\,\sqrt{14}\cos\gamma$$

$$\gamma = 96.865^\circ$$

Rules: ① $[q_1\vec{a} + q_2\vec{b}]\cdot\vec{c} = q_1\vec{a}\vec{c} + q_2\vec{b}\vec{c}$

② $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

③ $\vec{a}\cdot\vec{a} \geq 0$

④ $\vec{a}\cdot\vec{a} = 0$, if $\vec{a}=0$

⑤ $[\vec{a}+\vec{b}]\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

⑥ $|\vec{a}\cdot\vec{b}| \leq |\vec{a}|\,|\vec{b}|$
↳ (cauchy-schwarz inequality)

⑦ $|\vec{a}+\vec{b}| \leq |\vec{a}| + |\vec{b}|$
↳ Trainglure inequality

⑧ $|\vec{a}+\vec{b}|^2 + |\vec{a}-\vec{b}|^2 = 2\left[|\vec{a}|^2 + |\vec{b}|^2\right]$
↳ parallegram equality

Notes:

① $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

② $i\cdot j = j\cdot i = k\cdot j = j\cdot k = k\cdot i = i\cdot k = 0$

No.

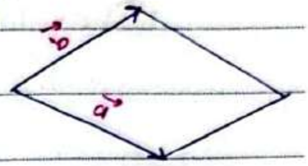
(9.3) Cross product

$$\vec{v} = \vec{a} \times \vec{b}$$

① if $\vec{a} = 0$ OR $\vec{b} = 0$ then $\vec{v} = \vec{a} \times \vec{b} = 0$

② if $\vec{a} \neq 0$, $\vec{b} \neq 0$ then $|\vec{v}| = |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\gamma$

③ $|\vec{b}|$: Area of parallegram contralled by $\vec{a}, \vec{b}$



④ if (γ) between $\vec{a}, \vec{b}$ is zero OR 180 same OR opposite direction

$\sin(0) = \sin(180) = 0$, $\vec{v} = \vec{a} \times \vec{b} = 0$

if $\vec{a} \neq 0$, $\vec{b} \neq 0$ ⑤ $\vec{v} = \vec{a} \times \vec{b}$ then $\vec{v}$ is perpendecular to plane formed by $\vec{a}, \vec{b}$

⑥ $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

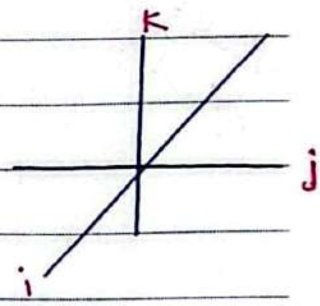
$$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & K \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + K \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i\,[a_2 b_3 - a_3 b_2] - j\,[a_1 b_3 - a_3 b_1] + K\,[a_1 b_2 - a_2 b_1]$$

Ex: if $\vec{a} = [1,1,0]$, $\vec{b} = [3,0,0]$ find $\vec{v} = \vec{a} \times \vec{b}$?

$$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = -3K = [0,0,-3]$$



Notes: ① $i \times j = K$, $j \times K = i$, $K \times i = j$

② $j \times i = -K$, $K \times j = -i$, $i \times K = -j$

Rules:

① $(L\vec{a}) \times \vec{b} = L(\vec{a} \times \vec{b}) = \vec{a} \times (L\vec{b})$

② $\vec{a} \times [\vec{b} + \vec{c}] = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

③ $[\vec{a} + \vec{b}] \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

الترتيب مهم

No.

④ $\vec{a} \times \vec{b} = -[\vec{b} \times \vec{a}]$

⑤ $[\vec{a} \times \vec{b}] \times \vec{c} \neq \vec{a} \times [\vec{b} \times \vec{c}]$ الأولوية للي بين أقواس.

⊛ Scalar Triple product:

Def: for vector $\vec{a}, \vec{b}, \vec{c}$

$(\vec{a}\,\vec{b}\,\vec{c}) = \vec{a} \cdot [\vec{b} \times \vec{c}] = [\vec{a} \times \vec{b}] \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$|(\vec{a}\,\vec{b}\,\vec{c})|$ = volume of Box

* Theorem: $\vec{a}, \vec{b}, \vec{c}$ are linearly independent if $(\vec{a}\,\vec{b}\,\vec{c}) \neq 0$

if $(\vec{a}\,\vec{b}\,\vec{c}) = 0$ linearly dependent.

Ex: let $\vec{a} = [2,0,3]$, $\vec{b} = [0,4,1]$, $\vec{c} = [5,6,0]$ find the volume of Box.

$|(\vec{a}\,\vec{b}\,\vec{c})| = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{bmatrix} = |-72| = 72$ unit.

• Volume of tetrahedrom $= \frac{1}{6}$ Volume of Box

في السؤال السابق $\Rightarrow \frac{1}{6}(72) = 12$ unit.

(9.4) vector + Scalar function

① vector function $\vec{v}$ that dep. on point p.

$$\vec{v}(P) = [v_1(P), v_2(P), v_3(P)]$$

ex: ① tangent vector ② normal vector

② Scalar function (f) that dep. on point p.

$$f(p) = f(x, y, z)$$

ex: ① Temp. ② pressure.

$$\Rightarrow \vec{v}(x,y,z) = xyz\,\hat{i} + x^2y\,j + z^2xy\,k$$

$$f(x,y,z) = x^2y + z^2xy + z^2$$

- **Dervative of vector function** :- the vector function is differentable iff.

① convergences $\lim_{n\to\infty} |\vec{a}_n - \vec{a}| = 0$

② continuity $\lim_{t\to t_0} \vec{v}(t) = \vec{v}(t_0)$

$$\vec{v} = [v_1(t), v_2(t), v_3(t)]$$

$$\vec{v}' = [v_1'(t), v_2'(t), v_3'(t)]$$

$$\vec{v}'' = [v_1''(t), v_2''(t), v_3''(t)]$$

ex: let $\vec{v}(t) = [t, t^2, 0]$ find $\vec{v}'(t)$ & $\vec{v}''(t)$?

$$\vec{v}'(t) = [1, 2t, 0]$$

$$\vec{v}''(t) = [0, 2, 0]$$

No.

Rules :

① $(c\vec{v})' = c\vec{v}'$ ② $(\vec{u}+\vec{v})' = \vec{u}' + \vec{v}'$

③ $(\vec{u}\cdot\vec{v})' = \vec{u}'\vec{v} + \vec{u}\cdot\vec{v}'$ ④ $(\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

⑤ $(\vec{u}\ \vec{v}\ \vec{w})' = (\vec{u}'\ \vec{v}\ \vec{w}) + (\vec{u}\ \vec{v}'\ \vec{w}) + (\vec{u}\ \vec{v}\ \vec{w}')$

* **partial dervitave :-**

$\vec{v}(t) = [\ v_1(t)\ ,\ v_2(t)\ ,\ v_3(t)\ ]$

$$\frac{\partial \vec{v}(t)}{\partial t_m} = \left[\frac{\partial v_1(t)}{\partial t_m} , \frac{\partial v_2(t)}{\partial t_m} , \frac{\partial v_3(t)}{\partial t_m} \right]$$

$$\frac{\partial^2 \vec{v}(t)}{\partial t_1 \partial t_m} = \left[\frac{\partial^2 v_1(t)}{\partial t_1 \partial t_m} , \frac{\partial^2 v_2(t)}{\partial t_1 \partial t_m} , \frac{\partial^2 v_3(t)}{\partial t_1 \partial t_m} \right]$$

Ex: let $\vec{r}(t_1, t_2) = a\cos t_1\, i + a\sin t_1\, j + t_2 K$

find $\frac{\partial \vec{r}}{\partial t_1}$, $\frac{\partial \vec{r}}{\partial t_2}$, $\frac{\partial^2 \vec{r}}{\partial t_1^2}$??

① $\frac{\partial \vec{r}}{\partial t_1} = -a\sin t_1\, i + a\cos t_1\, j + 0K$

② $\frac{\partial \vec{r}}{\partial t_2} = 0i + 0j + 1K$

③ $\frac{\partial^2 \vec{r}}{\partial t_1^2} = -a\cos t_1\, i - a\sin t_1\, j + 0K$

(9.7) gradent of scalar field directional derivative :-

$$\text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} , \frac{\partial f}{\partial z} \right]$$

$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$$

∇ = nablaf

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} K$$

No.

Ex: let $f(x,y,z) = 2y^3+4xz+3x$ find $\nabla f \big|_{(1,1,1)}$?

Soln:

$$\nabla f = \left[0+4z+3 \,,\, 6y^2+0+0 \,,\, 0+4x+0\right]$$

$$= \left[4z+3 \,,\, 6y^2 \,,\, 4x\right] = \left[7 \,,\, 6 \,,\, 4\right]$$

(*) directional derivitave :

- $D_{\vec a} f$ = direction dervitive of $f(x,y,z)$ in the direction of $\vec a$.

$$D_{\vec a} f = \frac{\vec a}{|\vec a|} \cdot \nabla f$$

* $\vec a = [a_1, a_2, a_3]$

* $|\vec a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

* $\bullet$ Dot product.

* $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

Ex: find the directional drvative of $f(x,y,z) = 2x^2+3y^2+z^2$ at point $P(2,1,3)$ in the direction of $\vec a = [1,0,-2]$.

Soln:

$$D_{\vec a} f = \frac{[1,0,-2]}{\sqrt 5} \cdot [4x \,,\, 6y \,,\, 2z] = \frac{[1,0,-2]}{\sqrt 5} \cdot [8,6,6]$$

$$= \frac{-4}{\sqrt 5}$$

(*) Laplace - equation :-

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

∇^2 = nable square $= \Delta$

Ex: let $f(x,y,z) = 4[x^2+y^2]-z^2$ find $\nabla^2 f$??

$\nabla^2 f = \Delta f$

$$\nabla^2 f = 8+8+(-2) = 14$$

(9.8) divergence of vector field

let $\vec{v}(x,y,z) = [v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)]$

$$\text{div } \vec{v} = \nabla \cdot \vec{v} = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot [v_1, v_2, v_3]$$

$$= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Ex: let $\vec{v} = [3xz, 2xy, -yz^2]$ find div $\vec{v}$?

$$\text{div } \vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

$$= 3z + 2x - 2yx$$

(9.9) Curl of vector field

let $\vec{v}(x,y,z) = [v_1(x,y,z), v_2(x,y,z), v_3(x,y,z)]$

$$\nabla = \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]$$

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i\left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] - j\left[\frac{\partial v_3}{\partial x} - \frac{\partial v_1}{\partial z}\right] + k\left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right]$$

Ex: let $\vec{v} = [yz, 3zx, z]$ Find Curl $\vec{v}$?

$$\text{Curl } \vec{v} = \nabla \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 3zx & z \end{vmatrix}$$

$\Rightarrow$

No.

$= i[0-3x] - j[0-y] + k[3z-z]$
$= -3x\hat{i} + yj + 2z\,k$

Rules $\Rightarrow$

① $\nabla(fg) = f\nabla g + g\nabla f$

② $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

③ $\text{div}\,(f\vec{v}) = f\,\text{div}\,\vec{v} + \vec{v}\cdot\nabla f$

④ $\text{div}\,(f\nabla g) = f\nabla^2 g + \nabla f\cdot\nabla g$

⑤ $\nabla^2 f = \text{div}\,(\nabla f)$

⑥ $\nabla^2(fg) = g\nabla^2 f + 2\nabla f\cdot\nabla g + f\nabla^2 g$

⑦ $\text{Curl}\,(f\vec{v}) = \nabla f \times \vec{v} + f\,\text{curl}\,\vec{v}$

⑧ $\text{div}\,(\vec{u}\times\vec{v}) = \vec{v}\cdot\text{curl}\,\vec{u} - \vec{u}\cdot\text{curl}\,\vec{v}$

⑨ $\text{Curl}\,(\nabla f) = \vec{0}$

⑩ $\text{div}\,(\text{curl}\,\vec{v}) = 0$ (scalar).

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