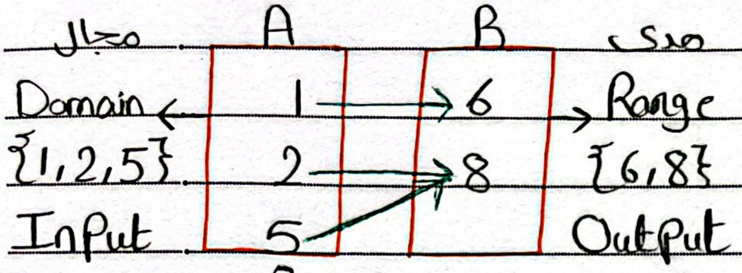
ملخص

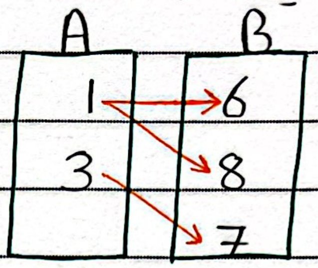
تفاضل وتكامل1 (الكامل المادة)

إعداد : رهف نوفل

Functions

* Function : $f: A \longrightarrow B$ كل عنصر في المجال يرتبط بعنصر واحد فقط في المدى





Function

$f(1) = 6$

$f(2) = 8$

$f(5) = 8$

Not Function

relation

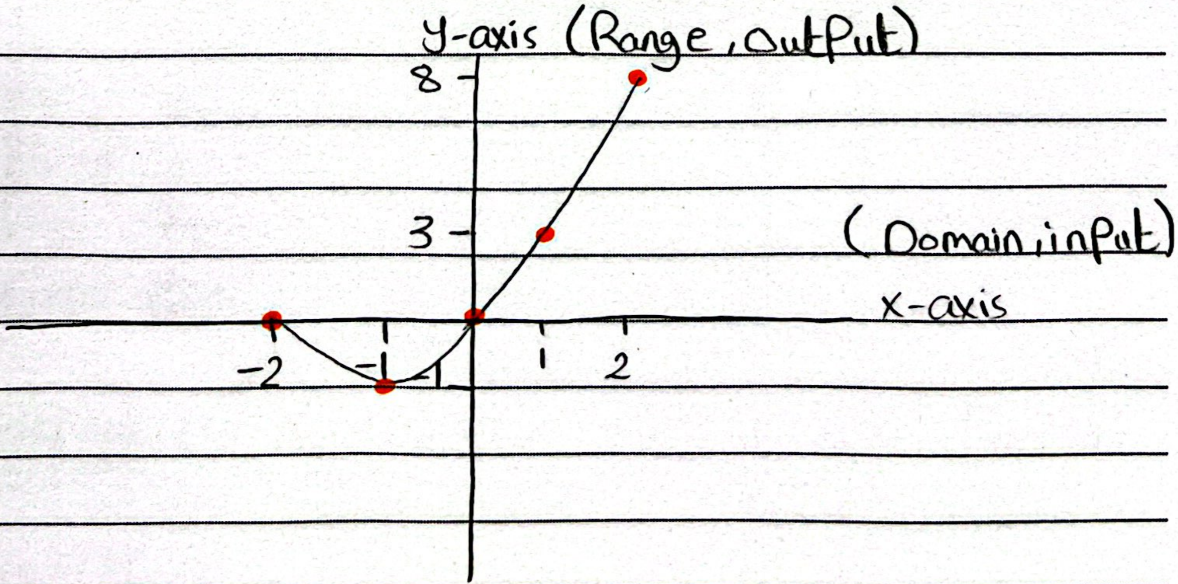
Ex: $f(x) = x^2 + 2x$ Graph

x	0	1	-1	2	-2
y = f(x)	0	3	-1	8	0

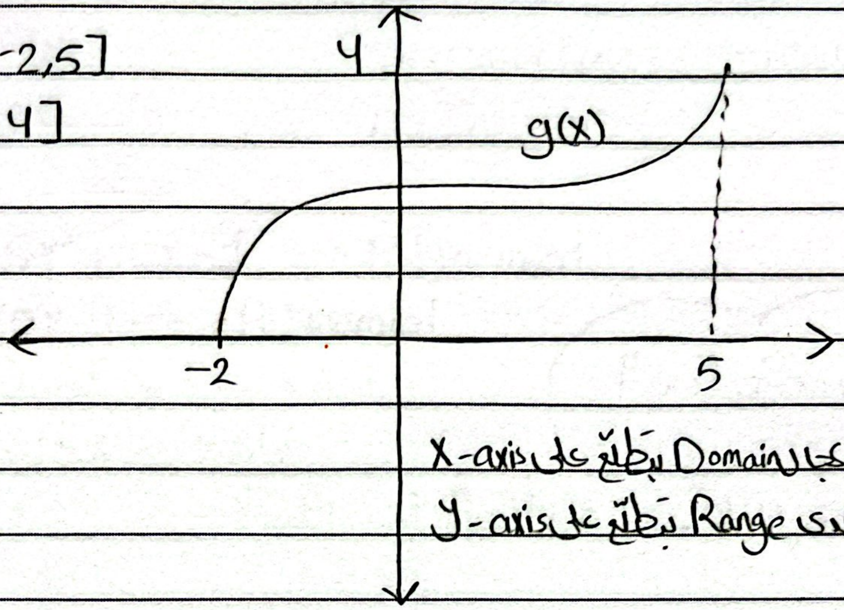
$f(1) = 1^2 + 2(1) = 3$

$f(-1) = (-1)^2 + 2(-1) = -1$

$(x,y) \Rightarrow (0,0), (1,3), (-1,-1), (2,8), (-2,0)$

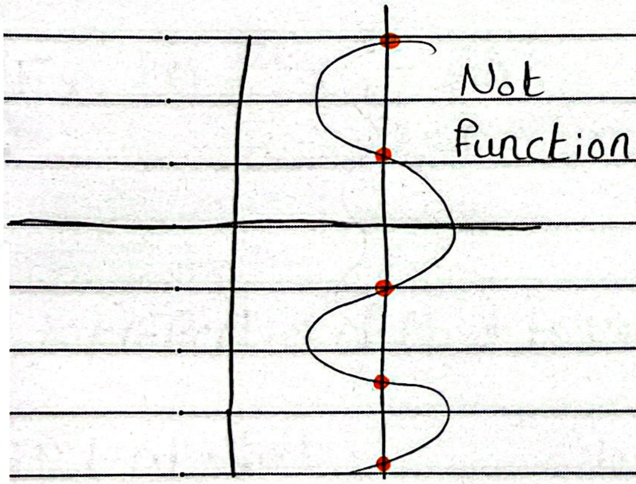


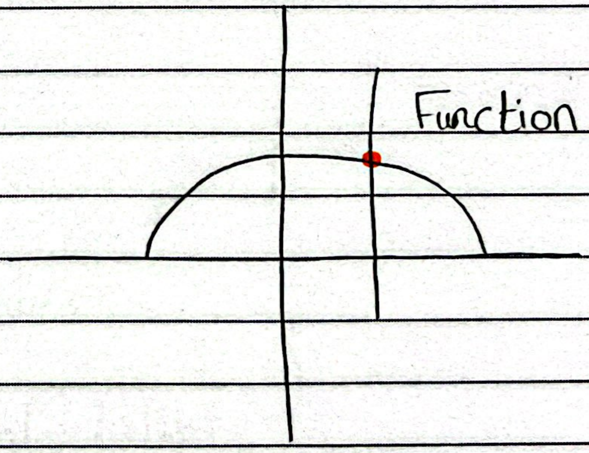
Domain: [-2,5]
Range: [0,4]

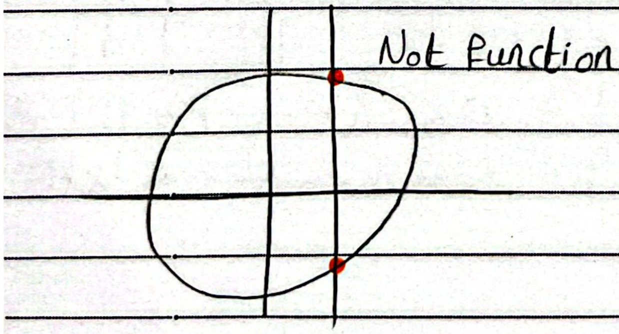


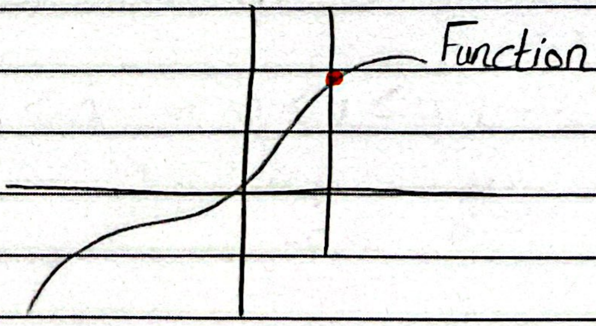
*لما بدي أحدد المجال Domain بطّلع على X-axis
لما بدي أحدد المدى Range بطّلع على Y-axis

*Vertical line test اختبار الخط العمودي :
↳ بحدد منه إذا الرسمة اقتران ولا علاقة ← الوظيفة









Types of Functions   Chapter 1

*Polynomials كثيرات الحدود : $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

$a_n, a_{n-1}, a_{n-2}, \cdots, a_0 \in \mathbb{R}$ (real number)

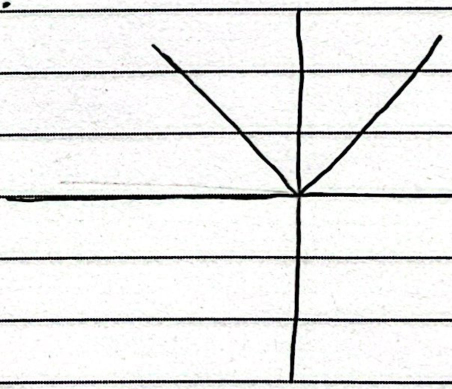
n: Integer

- $f(x) = x^3 + 5x^2 + 10x - 1$ → Polynomial
- $f(x) = x^2 + 10x - 1$ → Polynomial
- $f(x) = x^{\boxed{\frac{1}{5}}} + 10x^2 + 1$ → Not
- $f(x) = x^{\boxed{-2}} + 10x$ → Not
- $f(x) = \frac{2x+1}{x^2+1}$ → Rational function الاقتران النسبي

$f(x) = \sqrt{3x+1}$ → root function

*Absolute Value Function:

$|f(x)|$



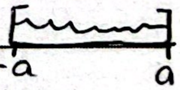
Ex: $|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$

*Properties of Absolute Value:

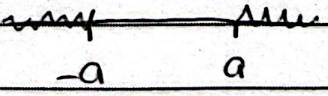
1) $|-a| = |a|$ 2) $|ab| = |a||b|$

3) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ 4) $|a+b| \leq |a| + |b|$

5) $|x| = a \longrightarrow x = \pm a$ 6) $|x| \leq a \longrightarrow -a \leq x \leq a$



7) $|x| \geq a \longrightarrow x \geq a$ OR $x \leq -a$



Types of Functions

Chapter 1

Ex : Find the Value of X :

$$|2X-1| < 7$$

$$-7 < 2X-1 < 7$$

$$-6 < 2X < 8$$

$$-3 < X < 4$$

$$(-3,4)$$

Ex : $f(x) = |8-2X|$ Picewise Function

$$8-2X=0$$

$$X=4$$

$$\underline{+\,+\,+\,+ \qquad -\,-\,-\,-}$$

$$(8-2x) \quad 4 \quad (2x-8)$$

$$f(x) = \begin{cases} 8-2X, & X \leq 4 \\ 2X-8, & X > 4 \end{cases}$$

Domain Rules

Chapter 1

*Domain : $y = \underset{\curvearrowright}{f(x)}$ input $\mathbb{R}$ (Real numbers)
المجال

1) $f(x) = x^3 + 2x^2 + 5x - 1$ المجال تبع ال Polynomial دائماً $\mathbb{R}$
$D_f = \mathbb{R}$

2) $g(x) = |\text{Polynomial}| \Longrightarrow D_f = \mathbb{R}$
$g(x) = |x^2 + 5x + 1| \Longrightarrow D_g = \mathbb{R}$

3) $g(x) = \dfrac{10}{x-5 \to \neq 0} \Longrightarrow D_f = \mathbb{R} - \{5\}$

4) $g(x) = \sqrt[n]{\boxed{f(x)}}$, n even 2, 4, 6, ----
$\to \geq 0$

$g(x) = \sqrt{x-7}$ $x - 7 \boxed{\geq 0}$
$D_f = [7, \infty)$ $x - 7 = 0$
$x = 7$

$- - - [+ + +$
7

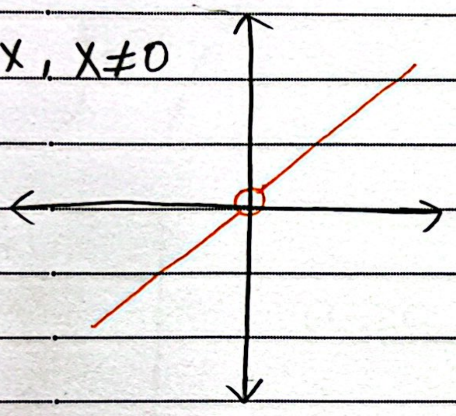
5) $g(x) = \sqrt[n]{f(x)}$, n odd 3, 5, 7, ----
$D_g = D_f$

$$g(x) = \sqrt[3]{\boxed{\frac{1}{x+5}}} \rightarrow \text{Domain}\left(\frac{1}{x+5}\right) \rightsquigarrow D_g = \mathbb{R} - \{-5\}$$

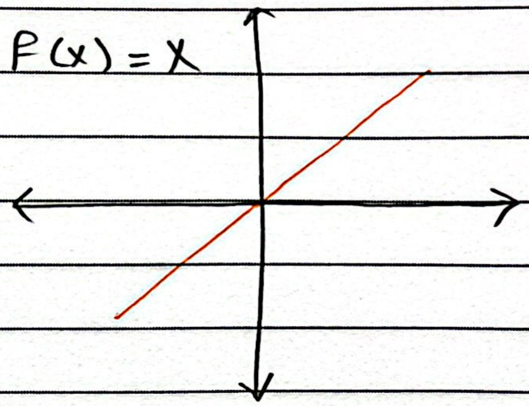
6) $f(x) = \frac{x^2}{x} \Rightarrow x = 0 \Rightarrow \mathbb{R} - \{0\}$

ما تختصر وتغيّر شكل الاقتران قبل ما توجد ال Domain

$f(x) = x, \ x \neq 0$



$f(x) = x$

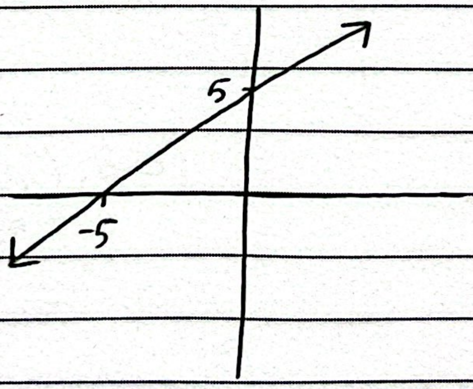


* Range : $y = f(x) \longrightarrow$ input

المدى $\hookrightarrow$ output

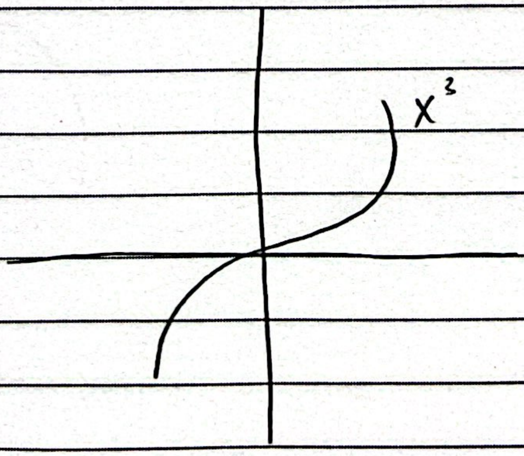
1) $f(x) = x+5$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

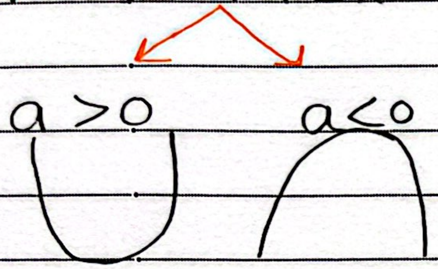


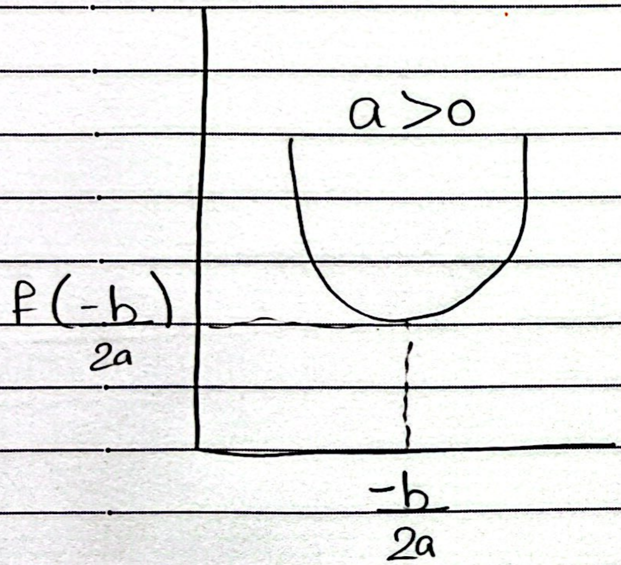
2) $f(x) = x^3$

$f : \mathbb{R} \longrightarrow \mathbb{R}$

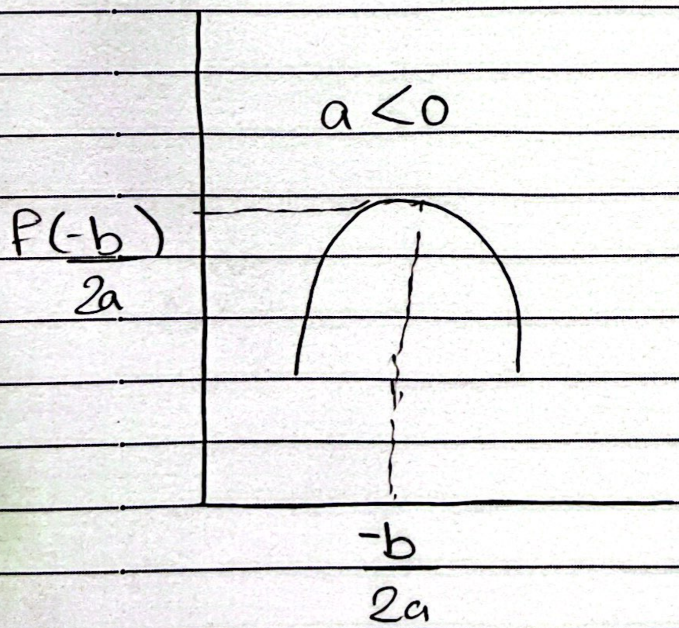


3) $f(x) = ax^2+bx+c$





Range $[f(\frac{-b}{2a}), \infty)$



Range $(-\infty, f(\frac{-b}{2a})]$

The range of Function

Ex: Range $f(x) = x^2+6x-1$

$$\frac{-b}{2a} = \frac{-6}{2} = -3$$

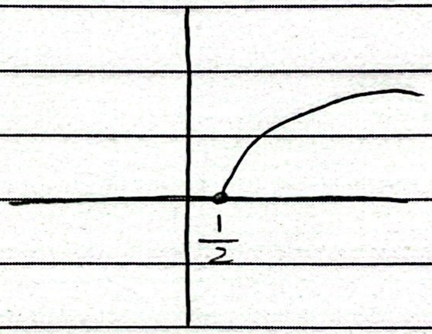
$$f(-3) = (-3)^2+6(-3)-1$$

$$= 9-18-1 = -10$$

Range : $[-10, \infty)$

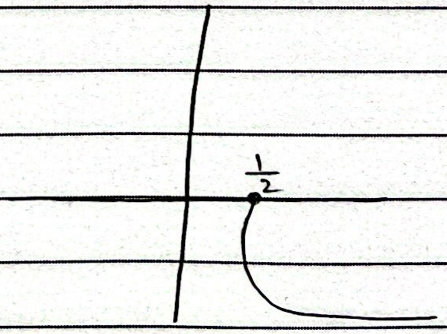
* $f(x) = \sqrt{2x-1}$

Range $[0, \infty)$



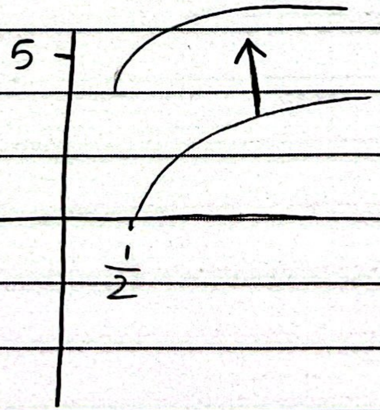
* $f(x) = \ominus\sqrt{2x-1}$

Range $(-\infty, 0]$



* $f(x) = \sqrt{2x-1} + 5$

Range $[5, \infty)$



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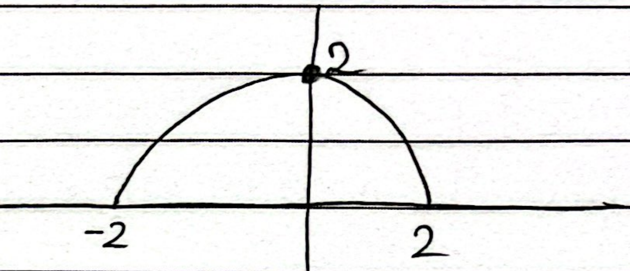
The range of Function

Chapter 1

* $f(x) = \sqrt{4-x^2}$

Df: $[-2,2]$

Range $[0,2]$



$f(x) = \sqrt{a-x^2}$, $a \in \mathbb{R}$

Domain $[-\sqrt{a}, \sqrt{a}]$

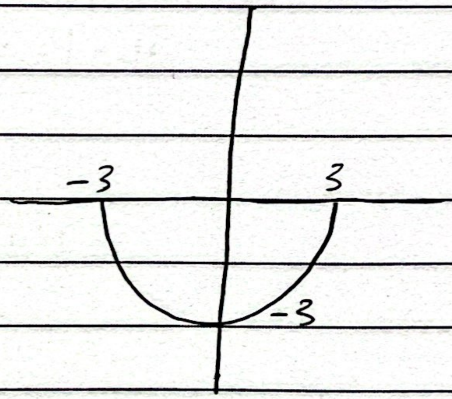
Range $[0, \sqrt{a}]$

* $f(x) = \bigcirc\!\!\!\!-\,\sqrt{a-x^2}$

$a = 9$

Range $[-3,0]$

Domain: $9 - x^2 \geq 0 \Rightarrow [-3,3]$



New functions from old

chapter 1

* f, g

$\to (f+g)(x) = f(x) + g(x)$

$\to (f-g)(x) = f(x) - g(x)$

$\to (fg)(x) = f(x)g(x)$

$\to \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \; g(x) \neq 0$

$\}$ 6 functions

$\to f+g, \; f-g$

$g-f, \; gf$

$\frac{f}{g}, \; \frac{g}{f}$

Ex: $f(x) = 1 - \sqrt{x-2}, \; g(x) = x-4$

1) $(f+g)(7) = f(7) + g(7)$

$= 1 - \sqrt{5} + 3 = 4 - \sqrt{5}$

2) $(f-g)(x) = f(x) - g(x)$

$= 1 - \sqrt{x-2} - x + 4$

$= 5 - \sqrt{x-2} - x$

3) $\left(\frac{f}{g}\right)(x) = \frac{1 - \sqrt{x-2}}{x-4}, \; x \neq 4$

* $f, g \Rightarrow \text{Domain}(f+g, f-g, fg) = D_f \cap D_g$

تقاطع (الأشياء المشتركة)

$\text{Domain}\left(\frac{f}{g}\right) = D_f \cap D_g - \{x : g(x) = 0\}$

Ex: $f(x) = 1 - \sqrt{x-2}, \; g(x) = x-4$

1) Domain (fg) :

$D_f = [2, \infty)$

$D_g = \mathbb{R}$

$D(fg) = D_f \cap D_g = [2, \infty) \cap \mathbb{R}$

$= [2, \infty)$

2) Domain $(\frac{g}{f})(x) = D_f \cap D_g - \{x : f(x)=0\}$

$f(x)=0$

$1-\sqrt{x-2}=0$

$(1)^2=(\sqrt{x-2})^2$

$1=x-2$

$x=3$

$[2,\infty)-\{3\}$

$=[2,3)\cup(3,\infty)$

Ex: Find the domain :

1) $f(x)=\sqrt{\frac{x^2-4}{x-4}} \longrightarrow \geq 0$

$\frac{x^2-4}{x-4}\geq 0$

$\boxed{D_f=[-2,2]\cup(4,\infty)}$

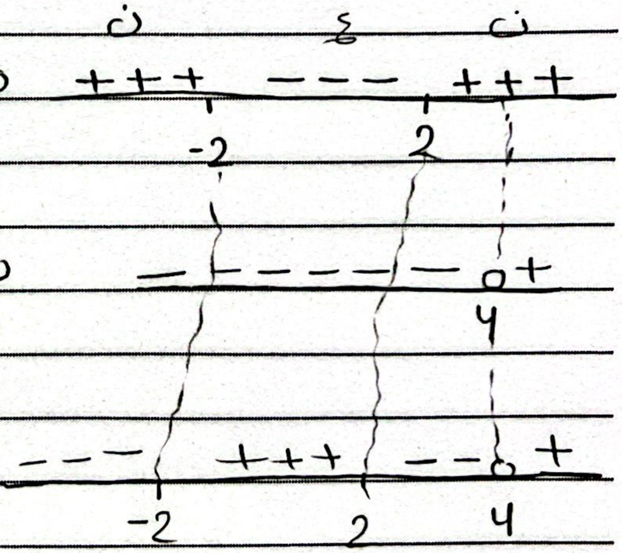
$x^2-4=0$

$x=\pm 2$

$x-4=0$

$x=4$

$\frac{x^2-4}{x-4}$



2) $f(x)=\sqrt{|x-1|-4}+\sqrt{\frac{2x-1}{3-|x|}}$

$\sqrt{|x-1|-4} \longrightarrow |x-1|-4\geq 0 \longrightarrow |x-1|\geq 4$

$x-1\geq 4$ or $x-1\leq 4 \longrightarrow x\geq 5,\ x\leq -3$

$[5,\infty)\cup(-\infty,3]$

New functions from old   Chapter 1

$\sqrt{2x-1} \longrightarrow 2x-1 \geq 0$

$3-|x| \qquad x \geq \frac{1}{2} \quad [\frac{1}{2}, \infty)$

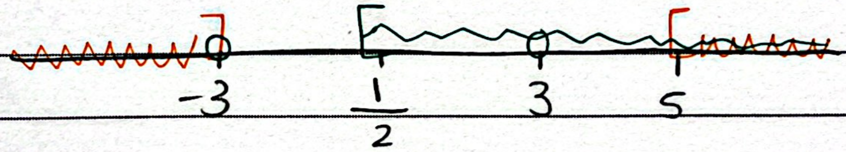
$\longrightarrow \mathbb{R} \quad \hookrightarrow [\frac{1}{2}, \infty) \cap \mathbb{R} - \{x : 3-|x| = 0\}$

$[\frac{1}{2}, \infty) \qquad 3-|x| = 0$

$3 = |x|$

$[\frac{1}{2}, \infty) - \{\pm 3\} \qquad x = \pm 3$

$[5, \infty) \cup (-\infty, -3] \bigcap [\frac{1}{2}, \infty) - \{\pm 3\}$

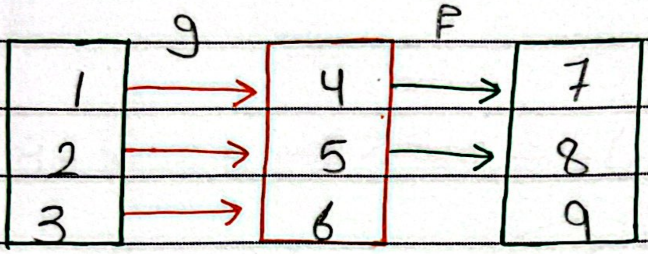


الجواب النهائي $\boxed{D_f = [5, \infty)}$

Composition of Functions

Chapter 1

$$* f, g \begin{cases} \rightarrow (f\circ g)(x) = \underset{\textcircled{2}}{\underline{f}}(\underset{\textcircled{1}}{\underline{g(x)}}) \\ \rightarrow (g\circ f)(x) = \underset{\textcircled{2}}{\underline{g}}(\underset{\textcircled{1}}{\underline{f(x)}}) \end{cases}$$



$$(f\circ g)(1) = f(g(1)) = f(4) = 7$$
$$(f\circ g)(2) = f(g(2)) = f(5) = 8$$
$$\underset{\substack{\downarrow \\ \notin D_{fog}}}{(f\circ g)(3)} = f(g(3)) = \underset{\substack{\downarrow \\ \notin D_f}}{f(6)} = \text{undefined}$$

$$D_{fog} = \{1, 2\} \quad , D_f = \{4, 5\}$$
$$D_g = \{1, 2, 3\}$$
$$D_{fog} = \{x \in D_g \ \& \ g(x) \in D_f\}$$
$$D_{g\circ f} = \{x \in D_f \ \& \ f(x) \in D_g\}$$

$$* \ f(x) = x^2 - 1 \qquad g(x) = \sqrt{3-x}$$

1) $(f\circ g)(-1) = f(g(-1)) = f(\sqrt{3--1}) = f(2) = 2^2 - 1 = 3$

2) $(g\circ f)(x) = g(f(x)) = g(x^2-1) = \sqrt{3-(x^2-1)} = \sqrt{4-x^2}$

3) $(f\circ g)(x) = f(g(x)) = f(\sqrt{3-x}) = (\sqrt{3-x})^2 - 1 = 3 - x - 1 = 2 - x$

$$g\circ f \neq f\circ g$$

Composition of Functions

Chapter 1

$g \circ f \neq f \circ g$

D_f , D_g , $D_{f \circ g}$, $D_{g \circ f}$?

1) $D_f = \mathbb{R}$

2) $D_g = 3 - x \geq 0 \Rightarrow 3 \geq x \longrightarrow (-\infty, 3]$

3) $D_{f \circ g} = \{x \in D_g \,\&\, \sqrt{3-x} \in \mathbb{R}\}$ True

$D_{f \circ g} = (-\infty, 3]$

4) $D_{g \circ f} = \{x \in D_f \,\&\, f(x) \in D_g\}$

$= \{x \in \mathbb{R} \,\&\, x^2 - 1 \in (-\infty, 3]\}$

$= \{x \in \mathbb{R} \cap x \in [-2, 2]\}$

$D_{g \circ f} = [-2, 2]$

$x^2 - 1 \in (-\infty, 3]$

$x^2 - 1 \leq 3$

$x^2 - 4 \leq 0$

$x^2 - 4 = 0$

$x = \pm 2$

++ ---- +++

-2 2

$x \in [-2, 2]$

Ex: $f(x) = \dfrac{1+x}{1-x}$, $g(x) = \dfrac{x}{1-x}$, Find domain $f \circ g$ & $g \circ f$?

$D_f = \mathbb{R} - \{1\}$, $D_g = \mathbb{R} - \{1\}$

$D_{f \circ g} = \{x \in D_g \,\&\, g(x) \in D_f\}$

$= \{x \in \mathbb{R} - \{1\} \,\&\, \dfrac{x}{1-x} \in \mathbb{R} - \{1\}\}$

$= \{x \in \mathbb{R} - \{1\}\} \bigcap x \in \mathbb{R} - \{\tfrac{1}{2}\}$

$D_{f \circ g} = \mathbb{R} - \{1, \tfrac{1}{2}\}$

$\dfrac{x}{1-x} \in \mathbb{R} - \{1\}$

$\dfrac{x}{1-x} \neq 1$

$\dfrac{x}{1+x} = 1$

$x = \dfrac{1}{2} \rightarrow x \in \mathbb{R} - \{\tfrac{1}{2}\}$

Composition of Functions

Chapter 1

$D_{g \circ f} = \{x \in D_f \ \& \ f(x) \in D_g\}$

$= \{x \in \mathbb{R} - \{1\} \ \& \ \frac{1+x}{1-x} \in \mathbb{R} - \{1\}\}$

$= \{x \in \mathbb{R} - \{1\} \cap x \in \mathbb{R} - \{0\}\}$

$D_{g \circ f} = \mathbb{R} - \{1, 0\}$

$\frac{1+x}{1-x} \in \mathbb{R} - \{1\}$

$\frac{1+x}{1-x} \neq 1$

$\frac{1+x}{1-x} = 1$

$x = 0$

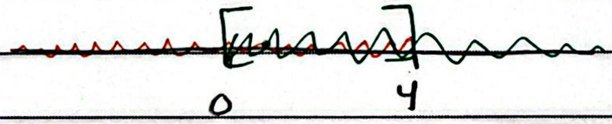
$x \in \mathbb{R} - \{0\}$

Ex: Find the domain $f(x) = \sqrt{2 - \sqrt{x}}$?

$x \geq 0 \ \& \ 2 - \sqrt{x} \geq 0$

$\downarrow$

$[0, \infty) \cap x \in (-\infty, 4]$



$D_f = [0, 4]$ input $\sqrt{2 - \sqrt{x}}$

$2 - \sqrt{x} \geq 0$

$2 - \sqrt{x} = 0$

$2 = \sqrt{x}$

$4 = x$

+++ ---

4

Ex: $(f \circ g)(x) = x^2 + 6x + 6$

$g(x) = x + 1$, find $f(x)$?

$f(g(x)) = x^2 + 6x + 6$

$f(x+1) = x^2 + 6x + 6$

$f(y) = (y-1)^2 + 6(y-1) + 6$

Find $f(2) = 13$

Composition of functions

Chapter 1

Ex : $(f \circ g)(x) = 3x^2 + 3x + 2$

$f(x) = 3x + 5$, find $g(x)$?

$f(g(x)) = 3x^2 + 3x + 2$

$3g(x) + 5 = 3x^2 + 3x + 2$

$3g(x) = 3x^2 + 3x + 3$

$g(x) = x^2 + x + 1$

find $g(2) = ?$

$g(2) = 4 + 2 + 1 = 7$

Even and Odd Functions

$f(-x)$

$f(-x) = f(x)$

Even

Ex: $f(x) = x^2$ $\boxed{\text{even}}$

$f(-x) = (-x)^2 = x^2 = f(x)$

$f(-2) = f(2) = 4$

$|x|$, $x^2, x^4, x^6, \dots$

$f(-x) = -f(x)$

Odd

Ex: $f(x) = x^3$

$f(-x) = (-x)^3 = -x^3 = -f(x)$

$f(-2) = -8$

$f(2) = 8$

$x, x^3, x^5, x^7, \dots$

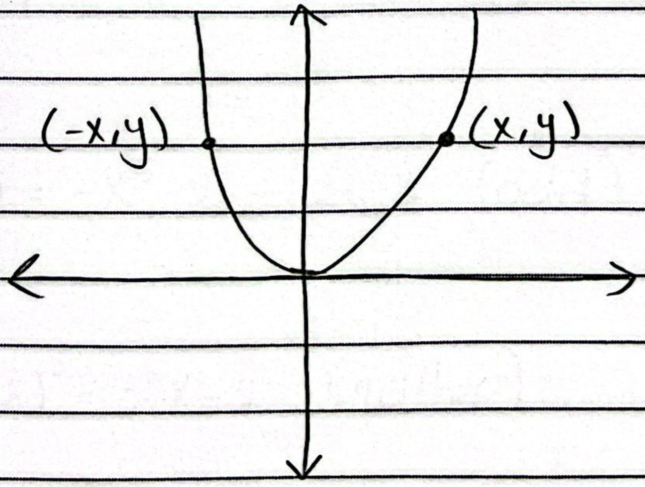
$f(-x) \neq f(x)$

$f(-x) \neq -f(x)$

Not even

Not odd

* Even Function:



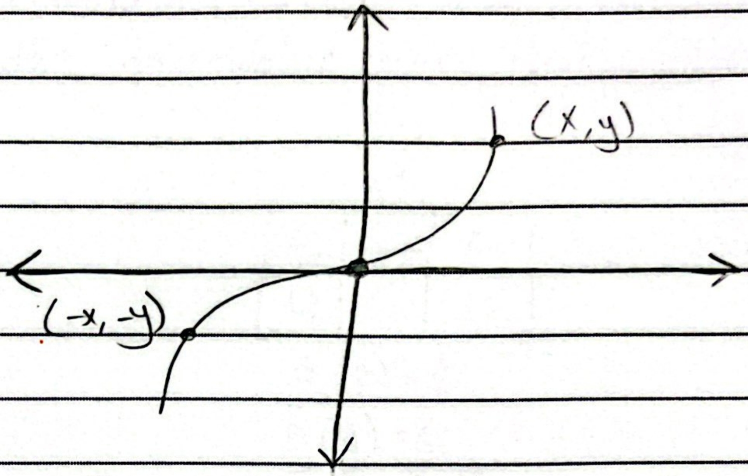
Symmetry about y-axis

تماثل حول محور الصادات

Even and Odd function

Chapter 1

* Odd function :



Symmetry about the origin

تماثل حول نقطة الأصل

Ex : even , odd , neither

1) $f(x) = 1 - x^4$

$f(-x) = 1 - (-x)^4 = 1 - x^4 = f(x)$ (even)

2) $f(x) = x^5 + x$

$f(-x) = (-x)^5 + (-x) = -x^5 - x = -f(x)$ (odd)

3) $f(x) = 2x - x^2$

$f(-x) = 2(-x) - (-x)^2 = -2x - x^2$ (neither)

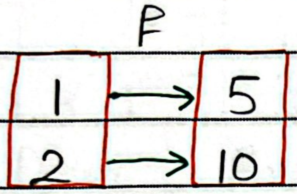
4) $f(x) = \frac{x^5 + x}{1 - x^4} = \frac{\text{odd}}{\text{Even}} = \text{Odd}$

	+	−
÷ *	Even	Odd
+ Even	E^+	O^-
− Odd	O^-	E^+

One to one Function

Chapter 1

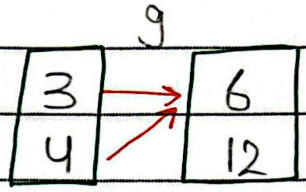
* One to one function : A function f is called a 1-1 function if it never takes on the same value twice. that is $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$



$f(1) = 5$

$f(2) = 10$

1-1



$g(3) = 6$

$g(4) = 6$

$3 \neq 4$

Not 1-1

* $f(x) = x^2$

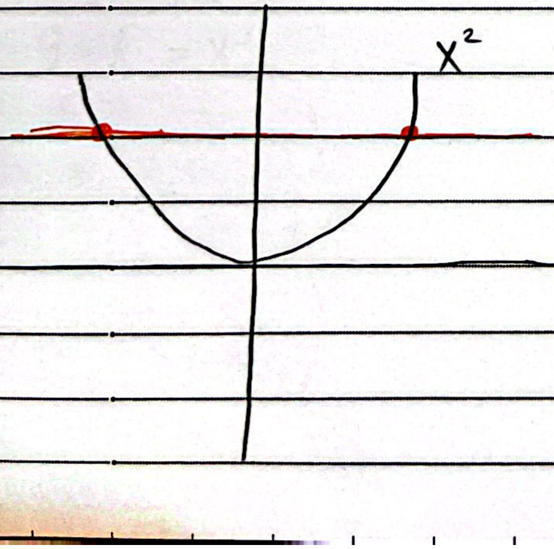
$f(-2) = f(2) = 4$

$2 \neq -2$

Not 1-1

* إذا أعطاني رسمة وكيف بدي أعرف إذا هي 1-1 ؟ بطريقة Horizontal line test

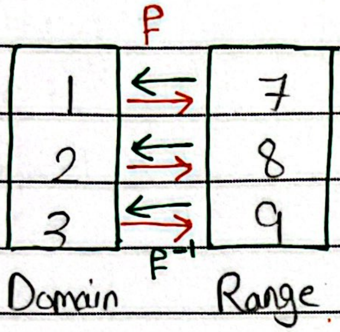
* Horizontal line test : اختبار الخط الأفقي



Not 1-1

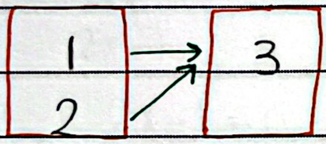
* لما ترسمي الخط لازم يقطع نقطة واحدة عشان نحكي إنها 1-1

*Inverse functions:



$f(1) = 7$ $f^{-1}(7) = 1$
$f(2) = 8$ $f^{-1}(8) = 2$
$f(3) = 9$ $f^{-1}(9) = 3$

$D_f = \{1,2,3\}$, Range $\{7,8,9\}$
$D_{f^{-1}} = \{7,8,9\}$, Range $\{1,2,3\}$
* نظرية : $D_{f^{-1}} = \text{Range } f$, $D_f = \text{Range } f^{-1}$



$g(1) = 3$ Not 1-1
$g(2) = 3$ لا يوجد g^{-1}

* نظرية : A function f has an inverse if is 1-1

*Find $f^{-1}(x)$:

1) $f(x) = 5x^3 + 7$

$y = 5x^3 + 7$

$y - 7 = 5x^3$

$\frac{y-7}{5} = x^3$

$x = \left(\frac{y-7}{5}\right)^{\frac{1}{3}}$

$f^{-1}(x) = \left(\frac{x-7}{5}\right)^{\frac{1}{3}}$

* كيف أوجد f^{-1} ؟ 1) بخلي x موضع القانون
2) بحط مكان ال x ← f^{-1}
ومكان y ← x

Ex: $f^{-1}(x) = ?$

$f(x) = \sqrt[5]{2x-1}$

$y = (2x-1)^{\frac{1}{5}}$

$y^5 = 2x-1$

$y^5+1 = 2x$

$x = \frac{y^5+1}{2}$

$f^{-1}(x) = \frac{x^5+1}{2}$

Ex: $f(x) = \frac{3x+5}{2x-1}$

1) $D_f = ?$ $2x-1=0 \longrightarrow x = \frac{1}{2} \longrightarrow D_f = \mathbb{R} - \{\frac{1}{2}\}$

2) $f^{-1}(x) = ?$ $y = \frac{3x+5}{2x-1}$

$2xy - y = 3x+5$

$2xy - 3x = 5+y$

$x(2y-3) = 5+y$

$x = \frac{5+y}{2y-3}$

$f^{-1}(x) = \frac{5+x}{2x-3}$

3) Range f = ? Domain f^{-1}

$2x-3 \rightarrow x = \frac{3}{2} \longrightarrow D_{f^{-1}} = \mathbb{R} - \{\frac{3}{2}\}$

Inverse functions

Chapter 1

* Restricting Domain الاقتطاع من المجال

$f(x)=x^2$, find $f^{-1}(x)$ X (x^2 not 1-1)

$f(x)=x^2$, $x\leq 0$, find $f^{-1}(x)$ ✓

$y=x^2$

$\sqrt{y}=-x \longrightarrow x=-\sqrt{y} \longrightarrow f^{-1}(x)=-\sqrt{x}$

Ex: $f(x)=3x^2+6x-6$, $x\geq -1$

$y=3x^2+6x-6$

$\frac{y}{3}=x^2+2x-2$

$\left(\frac{b}{2}\right)^2=\left(\frac{2}{2}\right)^2=1$

$\frac{y}{3}+1=(x^2+2x+1)-2$

$\frac{y}{3}+1=(x+1)^2-2$

$\frac{y}{3}+3=(x+1)^2$

$\sqrt{\frac{y}{3}+3}=x+1$

$\sqrt{\frac{y}{3}+3}-1=x$

$f^{-1}(x)=\sqrt{\frac{x}{3}+3}-1$

Ex: $f(x)=x^3+5x-2$

$4=x^3+5x-2$

$x^3+5x-6=0$

$\Rightarrow x=1$

$f^{-1}(4)=1$, $f(1)=4$

$\pm 1, \pm 2, \pm 3, \pm 6$

$1^3+5(1)-6=0$

Inverse function

Chapter 1

* نظرية : $(f \circ f^{-1})(x) = x \ , \forall x \in D_{f^{-1}}$

$(f^{-1} \circ f)(x) = x \ , \forall x \in D_f$

Ex: Determine whether f & g are inverse functions

$f(x) = x^3 + 3x^2 + 3x + 1 \ , g(x) = x^{\frac{1}{3}} - 1$

$f(x) = (x+1)^3$

$(f \circ g)(x) = x$?

$(g \circ f)(x) = x$

$(f \circ g)(x) = f(g(x)) = f(x^{\frac{1}{3}} - 1) = (x^{\frac{1}{3}} - 1 + 1)^3 = x$ ✓

$(g \circ f)(x) = g(f(x)) = g((x+1)^3) = ((x+1)^3)^3 - 1 = x + 1 - 1 = x$ ✓

f, g are inverse functions

$f(x) = y \longrightarrow f^{-1}(x) = g(x)$

Exponential functions

Chapter 1

*Exponential functions الاقتران الأسي:

$f(x) = b^x \quad , b > 0 \text{ and } b \neq 1$

Ex: $f(x) = 2^x$

x	2^x	
0	$2^0 = 1$	$(0,1)$
2	$2^2 = 4$	$(2,4)$
-1	$2^{-1} = \frac{1}{2}$	$(-1, \frac{1}{2})$
$\frac{1}{3}$	$2^{\frac{1}{3}}$	$(\frac{1}{3}, \sqrt[3]{2})$

* The Natural exponential function: الاقتران الأسي الطبيعي

$f(x) = e^x \quad , \quad e \cong 2.718\ldots$

Domain $\mathbb{R}$, Range $(0, \infty)$

$f(2) = e^2$

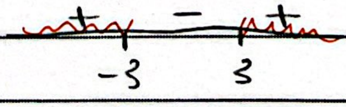
$f(0) = e^0 = 1$

⇒ Find the domain:

1) $f(x) = 2^{\sqrt{x^2 9}}$

$x^2 - 9 \geq 0 \Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3$

$D_f = (-\infty, -3] \cup [3, \infty)$



2) $f(x) = \dfrac{1}{e^x - e^{2x}}$

$e^x - e^{2x} = 0 \Rightarrow e^x(1 - e^x) = 0 \Rightarrow 1 = e^x \Rightarrow x = 0 , D_f = \mathbb{R} - \{0\}$

* Logarithmic Functions الاقترانات اللوغاريتمية :

$F(x) = \log_b x \quad , b>0 \text{ and } b \neq 1$

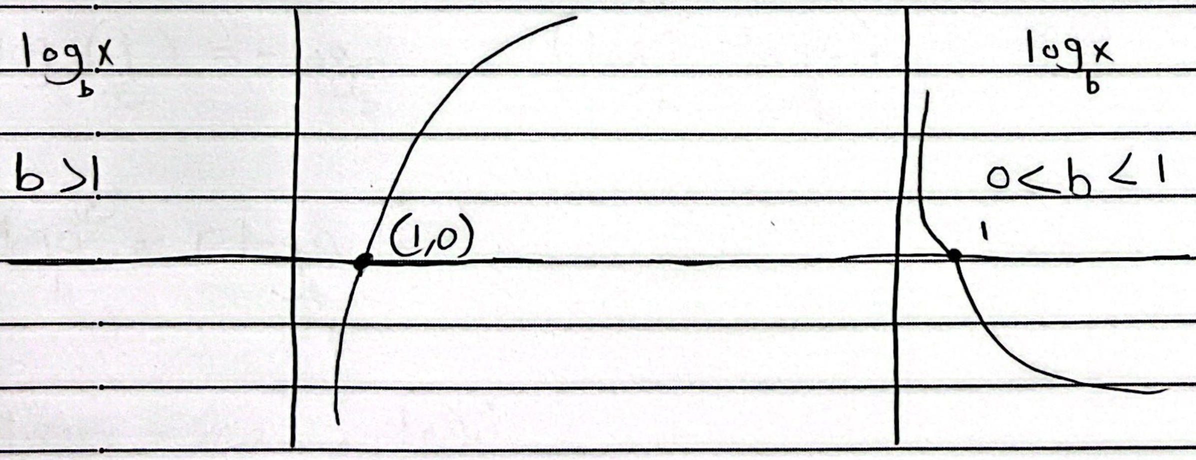
Ex: $\log_2 8 = 3 \quad (2^3 = 8)$

$\log_9 3 = \frac{1}{2} \quad (9^{\frac{1}{2}} = \sqrt{9} = 3)$

$\log_{10} \frac{1}{1000} = -3 \quad \left(10^{-3} = \frac{1}{1000}\right)$

$\log_{12} 12 = 1 \quad (12^1 = 12) \qquad \{\log_b b = 1\}$

$\log_2 1 = 0 \quad (2^0 = 1) \qquad \{\log_b 1 = 0\}$



Domain : $(0, \infty)$

Range : $\mathbb{R}$

Logarithmic Functions

Chapter 1

* The Natural log Function

$$\log_e x = \ln x \quad : (0,\infty) \longrightarrow \mathbb{R} \ , \ e \cong 2.718\ldots$$

Note: $\log_{10} x = \log x$

* Algebric Properties of log خصائص جبرية للوغاريتم :

$b > 0 \ , b \neq 1 \quad , \ a, c > 0 \quad r \in \mathbb{R}$

1) $\log_b (ac) = \log_b a + \log_b c$

2) $\log_b \left(\frac{a}{c}\right) = \log_b a - \log_b c$

3) $\log_b \left(\frac{1}{c}\right) = -\log_b c \qquad \left(\overset{0}{\cancel{\log_b 1}} - \log_b c\right)$

4) $\log_b a^{\boxed{r}} = r \log_b a$

5) $\log_b a = \dfrac{\ln a}{\ln b} \ , \ \dfrac{\log_c a}{\log_c b}$

Ex: $\log_{1000} 100 = \dfrac{\log 100}{\log 1000} = \dfrac{2}{3}$

Scanned with CamScanner

logarithmic Function

Chapter 1

6) $\log_b b = 1$

8) $\log_b 1 = 0$

7) $\ln e = 1$

9) $\ln 1 = 0$

Ex: Simplify تبسيط :

1) $\log\left(\frac{xy^5}{\sqrt{z}}\right) = \log(xy^5) - \log\sqrt{z} = \log x + 5\log y - \frac{1}{2}\log z$

Ex: Find the exact Value احسب القيمة الفعلية :

1) $\left(\log_2 6 - \log_2 15\right) + \log_2 20$

$\log_2 \frac{6}{15} + \log_2 20$

$\log_2\left(\frac{6}{\cancel{15}_{3}} \times \overset{4}{\cancel{20}}\right) = \log_2 \frac{24}{3} = \log_2 8 = 3 \quad (2^3 = 8)$

Ex: Find the domain : أوجد المجال

1) $f(x) = \ln(x-9)$

$x - 9 > 0$

$x > 9$

$D_f = (9, \infty)$

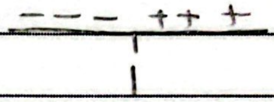
logarithmic Function

2) $F(x) = \ln(\ln x)$

$x > 0$ and $\ln x > 0 \longrightarrow \ln x = 0 \rightarrow x = 1$

$\downarrow \qquad \downarrow$

$(0,\infty) \cap (1,\infty)$

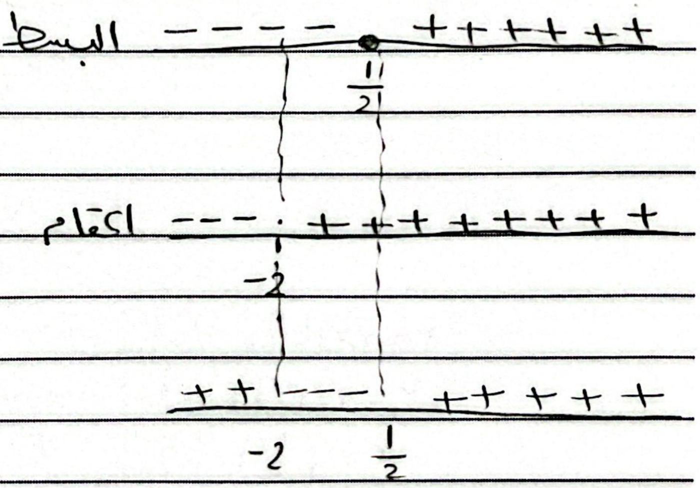


$D_f: (1,\infty)$

3) $F(x) = \log_7\left(\frac{4x-2}{2+x}\right)$

$\frac{4x-2}{2+x} > 0$

$(-\infty,-2)\cup(\frac{1}{2},\infty)$



Comparing exponential

Chapter 1

$f(x) = b^x \quad \boxed{\mathbb{R}} \longrightarrow \boxed{(0,\infty)}$

$g(x) = \log_b x \quad \boxed{(0,\infty)} \longrightarrow \boxed{\mathbb{R}}$

*نظرية : b^x, $\log_b x$ are inverse functions

Ex: Find $f^{-1}(x)$:

1) $f(x) = 5^x \Longrightarrow f^{-1}(x) = \log_5 x$

2) $g(x) = \log_7 x \Longrightarrow g^{-1}(x) = 7^x$

3) $f(x) = e^x \Longrightarrow f^{-1}(x) = \ln x$

$(f \circ f^{-1})(x) = x$

$f(x) = b^x$, $f^{-1}(x) = \log_b x$

$f(f^{-1}(x)) = x$

$f(\log_b x) = x$

$b^{\log_b x} = x$

$e^{\ln x} = x$

$(f^{-1} \circ f)(x) = x$

$f^{-1}(f(x)) = x$

$f^{-1}(b^x) = x$

$\log_b b^x = x$

$\downarrow$

$\ln e^x = x$

Ex: $5^{\log_5 x} = x$

$\ln e^2 = 2$

$\log_9 9^x = x$

$e^{-2\ln 5} = e^{\ln 5^{-2}} = 5^{-2} = \frac{1}{25}$

Ex: Find domain & Range $f(x) = \frac{e^x - 1}{e^x + 3}$

$D_f = e^x + 3 = 0 \longrightarrow e^x = -3 \times (0, \infty)$

$D_f = \mathbb{R}$

$y = \frac{e^x - 1}{e^x + 3}$

$ye^x + 3y = e^x - 1$

$ye^x - e^x = -1 - 3y$

$e^x(y - 1) = -1 - 3y$

$e^x = \frac{-1 - 3y}{y - 1}$

$\ln e^x = \ln\left(\frac{-1 - 3y}{y - 1}\right)$

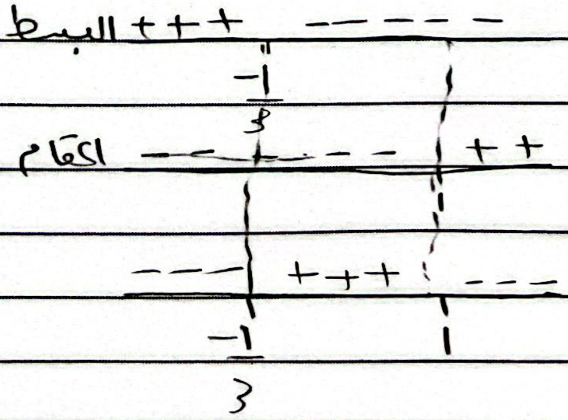
$x = \ln\left(\frac{-1 - 3y}{y - 1}\right) \longrightarrow f^{-1}(x) = \ln\left(\frac{-1 - 3x}{x - 1}\right)$

Domain f^{-1} = Range f

$\frac{-1 - 3x}{x - 1} > 0$

$D_{f^{-1}}: \left(\frac{-1}{3}, 1\right)$

$= R_f$



Solving exponential equation

* Solve :

1) $2^{x-5}=3 \Rightarrow \log_2 2^{x-5}=\log_2 3 \Rightarrow x-5=\log_2 3 \Rightarrow x=\log_2 3+5$

2) $\ln(x+1)=5$

$e^{\ln x+1}=e^5$

$x+1=e^5$

$x=e^5-1$

3) $\log x^2+\log x=30$

$2\log x+\log x=30$

$3\log x=30$

$\log x=10$

$x=10^{10}$

4) $e^{2x}-e^x=6$

$e^{2x}-e^x-6=0$ $\boxed{e^x=y}$

$y^2-y-6=0$

$(y-3)(y+2)=0$

$y=3\ ,\ y=-2$

$e^x=3\ ,\ e^x=-2$

$\boxed{x=\ln 3}$ $\quad \downarrow \times$

Solving exponential equation Chapter 1

5) $(x^2-1)(x-5)x^3\log_x 2x = 0$

$x^2-1=0$

$x = \overset{\times}{1}, \overset{\times}{-1}$

$x-5=0$

$x = \overset{\checkmark}{5}$

$x^3=0$

$x = \overset{\times}{0}$

$\log_x 2x = 0 \longrightarrow 2x=1 \longrightarrow x=\frac{1}{2}$

$\log_x 2x$

$x=1 \longrightarrow \log 2$ X

$x=-1 \longrightarrow \log_{-2}{}^{1}$ X

$x=5 \longrightarrow \log 10$ ✓

$x=0 \longrightarrow \log_0 \overset{5}{0}$ X

$x=\frac{1}{2} \longrightarrow \log\frac{1}{2}$

6) $\ln x + \ln(x-1) = 1$

$\ln x(x-1) = 1$

$e^{\ln x(x-1)} = e^1$

$x^2 - x - e = 0$

$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$x = \frac{1 \pm \sqrt{(1)-4(1)(-e)}}{2} = \frac{1 \pm \sqrt{1+4e}}{2}$

ما ناخذها

Solving exponential equation

Chapter 1

7) $\log_2 3x + \log_4 9x^2 = 4$

$\log_b a = \frac{\log_c a}{\log_c b}$

$\log_2 3x + \frac{\log_2 9x^2}{\log_2 4} = 4$

$\log_2 3x + \frac{\log_2 9x^2}{2} = 4$

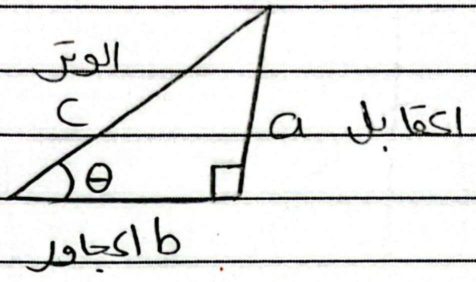
$\log_2 3x + \frac{1}{2} \log_2 9x^2 = 4$

$\log_2 3x + \log_2 (9x^2)^{\frac{1}{2}} = 4$

$= \log_2 3x + \log_2 3x = 4$

$= \log_2 (3x)(3x) = 4 \Rightarrow \log_2 9x^2 = 4 \Rightarrow 9x^2 = 2^4 \Rightarrow x = \pm \frac{4}{3}$

* Trigonometric Functions :



$$C^2 = a^2 + b^2$$

$$\sin\theta = \frac{a}{c} \rightarrow \csc\theta = \frac{1}{\sin\theta}$$

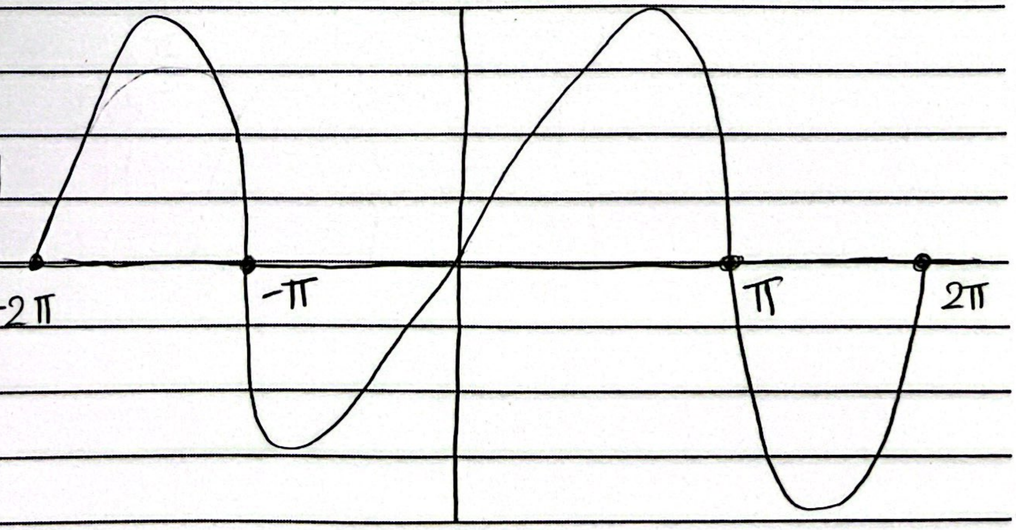
$$\cos\theta = \frac{b}{c} \rightarrow \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{a}{b} \rightarrow \cot\theta = \frac{1}{\tan\theta}$$

degrees	0°	30°	45°	60°	90°	180°	270°	360°
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0	1

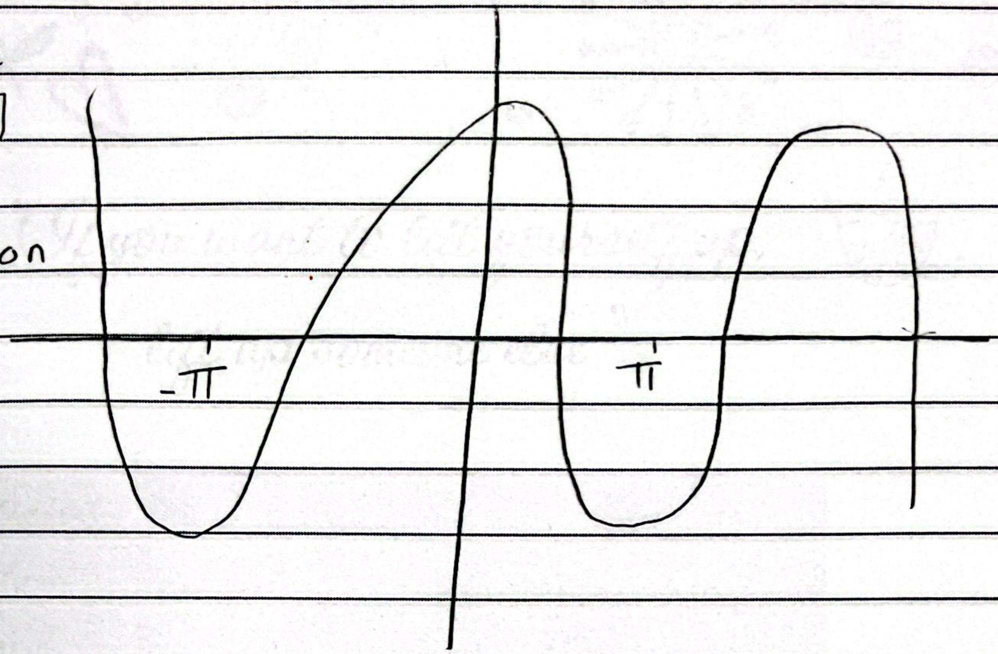
* Sin x :

Domain = $\mathbb{R}$
Range = $[-1,1]$
Not 1 - 1
Odd function



* Cos x :

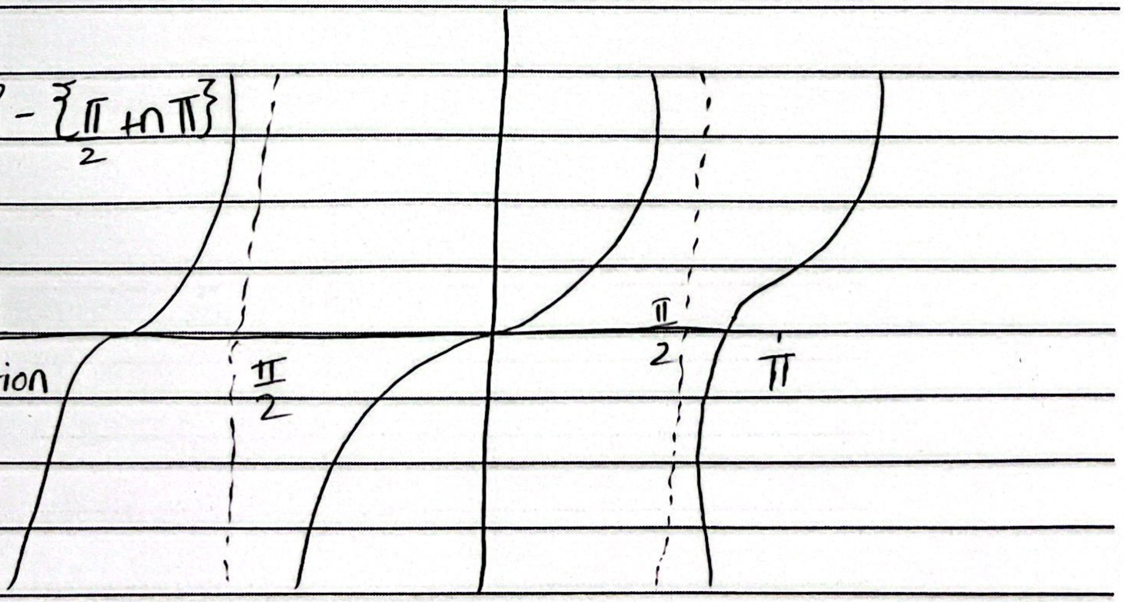
Domain = ℝ
Range [-1, 1]
Not 1 - 1
even function



* Tan x :

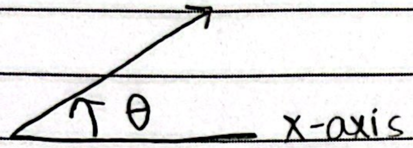
Domain = ℝ - $\{\frac{\pi}{2} + n\pi\}$

Range = ℝ
Not 1 - 1
Odd function

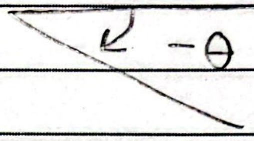


Trigonometric function

Chapter 1

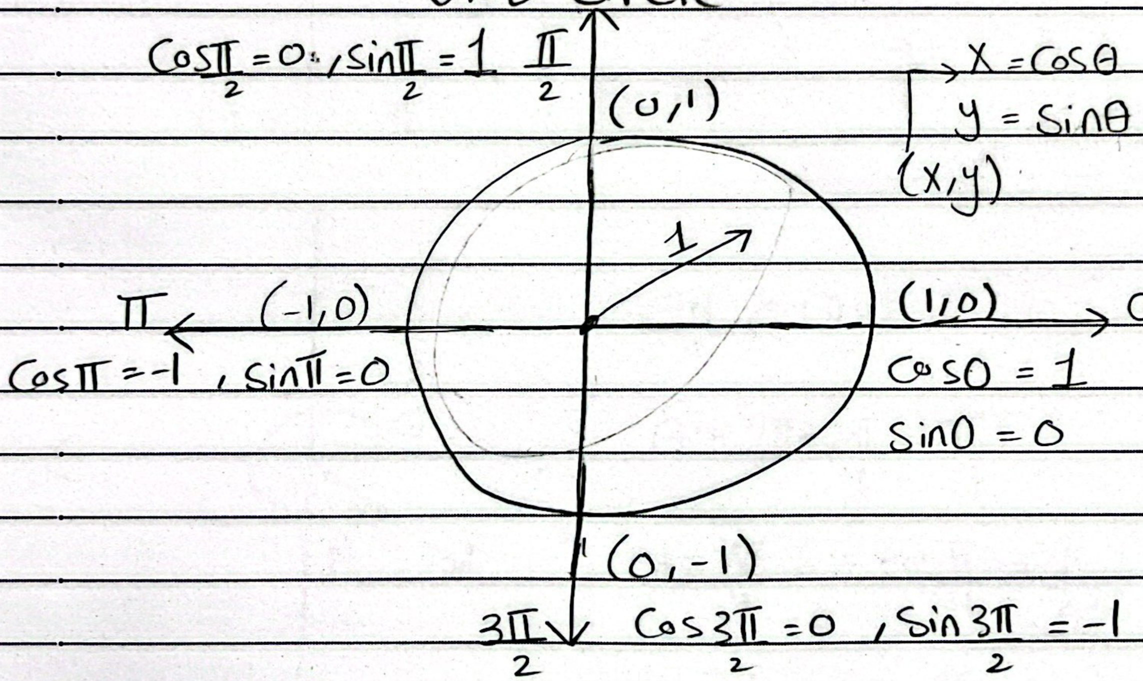


Caunter clockwise
(positive angle)

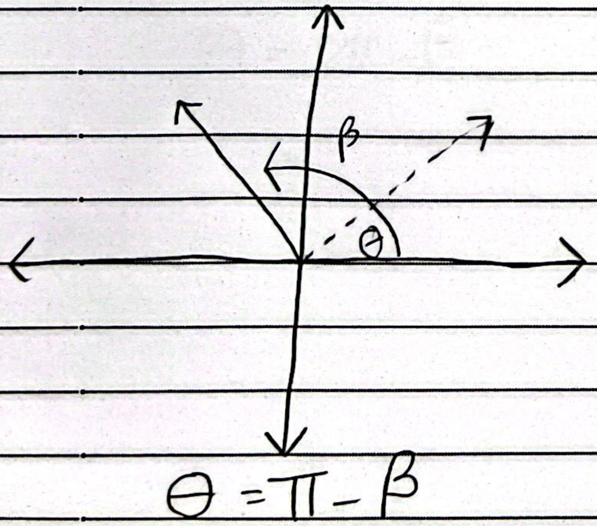


Clockwise
(negative angle)

Unit Circle



1)



$\theta = \pi - \beta$

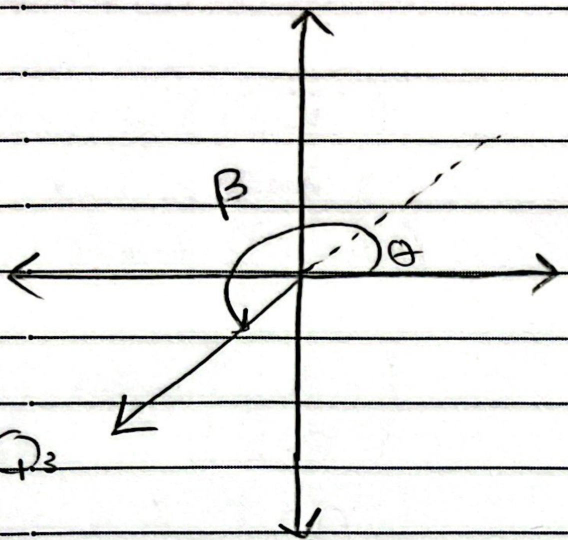
*Ex : Find $\sin\frac{5\pi}{6}$, $\cos\frac{5\pi}{6}$

$\frac{5\pi}{6} = \frac{5*180}{6} = 150°$

$\theta = \pi - \frac{5\pi}{6} = \frac{\pi}{6}$

$\sin\frac{5\pi}{6} = \sin\frac{\pi}{6}$, $\cos\frac{5\pi}{6} = -\cos\frac{\pi}{6}$

2)



$$\frac{4\pi}{3} = 240^\circ \in Q_3$$

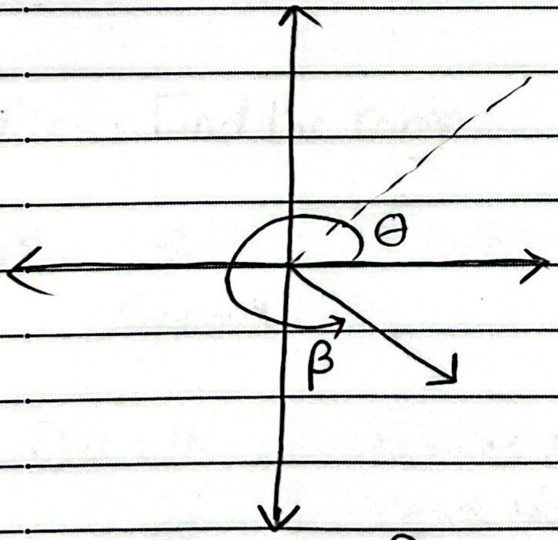
$$\theta = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$$

$$\sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

$$\cos\frac{4\pi}{3} = -\cos\frac{\pi}{3} = -\frac{1}{2}$$

$$\theta = \beta - \pi$$

3)



$$\frac{7\pi}{4} \in Q_4$$

$$\theta = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$$

$$\cos\frac{7\pi}{4} = +\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\sin\frac{7\pi}{4} = -\sin\frac{\pi}{4} = \frac{-1}{\sqrt{2}}$$

$$\theta = 2\pi - \beta$$

Trigonometric Function

Examples : Find domain :

1) $f(x) = \sin(\sqrt{x-5})$

$x-5 \geqslant 0 \Rightarrow x \geqslant 5$

$D_f = [5, \infty)$

2) $f(x) = \cos\left(\dfrac{1}{x-7}\right)$

$x - 7 = 0$

$x = 7$

$D_f = \mathbb{R} - \{7\}$

Examples: Find the range

1) $f(x) = \dfrac{3}{5+\cos x}$

$D_f = \mathbb{R}$ $\quad 5+\cos x = 0$

$\cos x \neq -5$

$[-1, 1]$

$-1 \leqslant \cos x \leqslant 1 \quad (+5)$

$4 \leqslant 5+\cos x \leqslant 6$

$\dfrac{1}{4} \geqslant \dfrac{1}{5+\cos x} \geqslant \dfrac{1}{6}$

$\dfrac{3}{4} \geqslant \dfrac{3}{5+\cos x} \geqslant \dfrac{3}{6}$

$\dfrac{3}{4} \geqslant f(x) \geqslant \dfrac{1}{2}$

Range : $[\frac{1}{2}, \frac{3}{4}]$

Inverse Trigonometric Function

Chapter 1

$$\left.\begin{array}{l} 1)\ \sin^{-1}\frac{1}{2} = \frac{\pi}{6} \\ 2)\ \sin^{-1}\frac{1}{\sqrt{2}} = \frac{\pi}{4} \\ 3)\ \sin^{-1}\frac{-1}{2} = -\sin^{-1}\frac{1}{2} = \frac{-\pi}{6} \end{array}\right\} \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\left.\begin{array}{l} 4)\ \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} \\ 5)\ \cos^{-1}(0) = \frac{\pi}{2} \\ 6)\ \cos^{-1}\left(\frac{-1}{2}\right) = \pi - \cos^{-1}\frac{1}{2} = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \end{array}\right\} \rightarrow [0, \pi]$$

$$\left.\begin{array}{l} 7)\ \tan^{-1}(1) = \frac{\pi}{4} \\ 8)\ \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \\ 9)\ \tan^{-1}(-1) = -\tan^{-1}(1) = \frac{-\pi}{4} \end{array}\right\} \rightarrow \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

Ex: Find the domain $f(x) = \sin^{-1}(2x+1)$?

$\sin^{-1}x = [-1, 1]$

$$-1 \leq 2x + 1 \leq 1$$

$$-2 \leq 2x \leq 0$$

$$-1 \leq x \leq 0$$

$$D_f : [-1, 0]$$

Ex: Find the range $f(x) = \pi + |\tan^{-1}x|$

Range $\tan^{-1}x : \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$$\frac{-\pi}{2} < \tan^{-1}x < \frac{\pi}{2}$$

$$0 \leq |\tan^{-1}x| < \frac{\pi}{2}$$

$$\pi < \pi + |\tan^{-1}x| \leq \frac{\pi}{2} + \pi$$

Range $\left[\pi, \frac{3\pi}{2}\right)$

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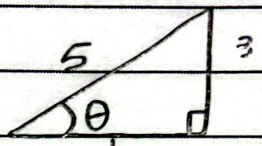
Inverse Trigonometric Function

Chapter 1

* Find the value of the following :

1) $\sec\left(\sin^{-1}\frac{3}{5}\right)$

$\sin^{-1}\frac{3}{5} = \theta \longrightarrow \frac{3}{5} = \sin\theta$

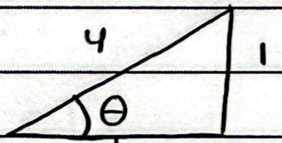


$\sqrt{25-9} = 4$

$\longrightarrow \sec\theta = \frac{5}{4}$

2) $\sin\left[2\sin^{-1}\frac{1}{4}\right]$

$\sin^{-1}\frac{1}{4} = \theta \longrightarrow \sin\theta = \frac{1}{4}$



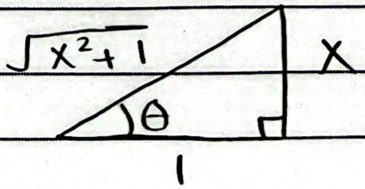
$\sqrt{16-1} = \sqrt{15}$

$\longrightarrow \sin[2\theta] = 2\sin\theta\cos\theta$

$= 2\left(\frac{1}{4}\right)\left(\frac{\sqrt{15}}{4}\right) = \frac{2\sqrt{15}}{16}$

3) $\cos(\tan^{-1}x)$

$\tan^{-1}x = \theta \longrightarrow \tan\theta = \frac{x}{1}$



$\longrightarrow \cos\theta = \frac{1}{\sqrt{x^2+1}}$

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Inverse Trigonometric function

Chapter 1

* ملاحظة : $(f\circ f^{-1})(x) = x$ and $(f^{-1}\circ f)(x) = x$

$f(x) = \sin x$ $\qquad f^{-1}(x) = \sin^{-1} x$

1) $\sin(\sin^{-1} x) = x \quad , \forall x \in [-1,1]$

2) $\sin^{-1}(\sin x) = x \quad , \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Ex : 1) $\sin(\sin^{-1}\frac{1}{3}) = \frac{1}{3} \qquad \frac{1}{3} \in [-1, 1]$

2) $\sin^{-1}(\sin\frac{\pi}{4}) = \frac{\pi}{4} \qquad \frac{\pi}{4} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

3) $\sin^{-1}(\sin\frac{2\pi}{3}) \qquad \frac{2\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\frac{2\pi}{3} \in Q_2 \longrightarrow Q_1 \quad \theta = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

$\to \sin^{-1}(\sin\frac{4\pi}{3}) = \frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

4) $\sin^{-1}(\sin\frac{4\pi}{3}) \qquad \frac{4\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\frac{4\pi}{3} \in Q_3 \longrightarrow Q_1$

$\theta = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$

$\to \sin^{-1}(-\sin\frac{\pi}{3}) = -\sin^{-1}(\sin\frac{\pi}{3}) = -\frac{\pi}{3} \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

Inverse trigonometric function

Chapter 1

* $f(x) = \cos x.$ $\quad , f^{-1}(x) = \cos^{-1} x$

$\cos(\cos^{-1} x) = x \qquad x \in [-1,1]$

$\cos^{-1}(\cos x) = x \qquad x \in [-1,1]$

Ex: 1) $\cos^{-1}(\cos \frac{2\pi}{3}) = \frac{2\pi}{3} \qquad , \frac{2\pi}{3} \in [0,\pi]$

2) $\cos^{-1}(\cos \frac{7\pi}{4})$

$\frac{7\pi}{4} \in Q_4 \longrightarrow Q_1$

$\theta = 2\pi - \frac{7\pi}{4} = \frac{\pi}{4}$

$\longrightarrow \cos^{-1}(\cos \frac{\pi}{4}) = \frac{\pi}{4} \in [0,\pi]$

3) $\cos^{-1}(\cos \frac{4\pi}{3})$

$\frac{4\pi}{3} \in Q_3 \longrightarrow Q_1$

$\theta = \frac{4\pi}{3} - \pi = \frac{\pi}{3}$

$\longrightarrow \cos^{-1}(-\cos \frac{\pi}{3}) = \pi - \cos^{-1}(\cos \frac{\pi}{3})$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3} \in [0,\pi]$

~~Ex~~: $f(x) = \tan x \qquad , f^{-1}(x) = \tan^{-1}(x)$

$\tan(\tan^{-1} x) = x \quad , \forall x \in \mathbb{R}$

$\tan^{-1}(\tan x) = x \quad , \forall x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

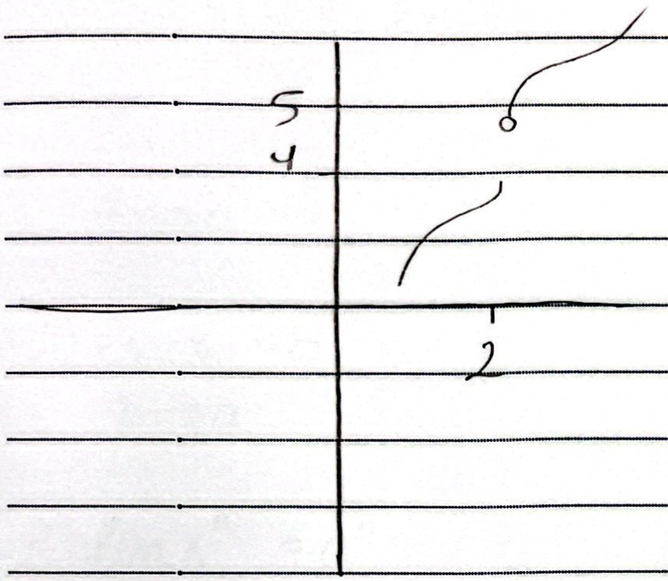
Ex: 1) $\tan^{-1}(\tan \frac{2\pi}{3}) \longrightarrow \tan^{-1}(-\tan \frac{\pi}{3}) = -\tan^{-1}(\tan \frac{\pi}{3}) = -\frac{\pi}{3}$

$\frac{2\pi}{3} \in Q_2 \longrightarrow Q_1$

$\theta = \pi - \frac{2\pi}{3} = \frac{\pi}{3}$

The limits of function

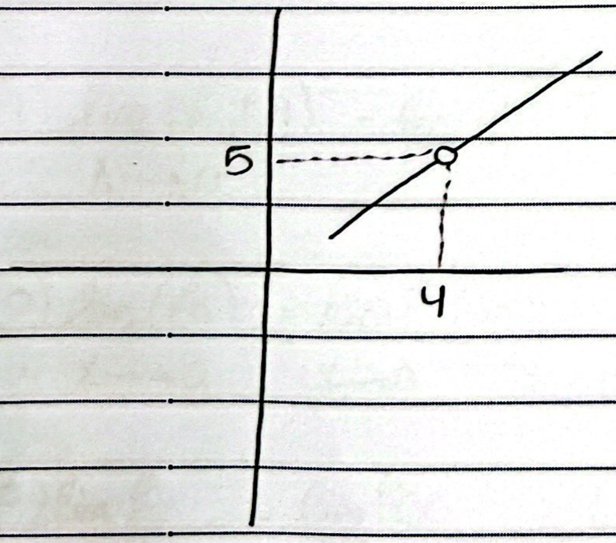
Chapter 2



$$\lim_{x\to 2} f(x) \begin{cases} \lim_{x\to 2^+} f(x) = 5 & \text{اليمين} \\ \lim_{x\to 2^-} f(x) = 4 & \text{اليسار} \end{cases} \neq$$

$\lim_{x\to 2} f(x)$ does not exist

* $\lim_{x\to a} f(x) = L \longleftrightarrow \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$



$$\lim_{x\to 4} f(x) \begin{cases} \lim_{x\to 4^+} f(x) = 5 \\ \lim_{x\to 4^-} f(x) = 5 \end{cases} =$$

$\lim_{x\to 4} f(x) = 5$

The limits of function

Chapter 2

* Calculating limits: $a, c \in \mathbb{R}$

1) $\lim_{x \to a} c = c$ $\qquad \lim_{x \to 7} 10 = 10$

2) $\lim_{x \to a} x = a$ $\qquad \lim_{x \to 9} x = 9$

3) $\lim_{x \to a} x^n = a^n$ $\qquad \lim_{x \to 4} x^2 = 4^2 = 16$, $n =$ عدد موجب

Thu: $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, $a, K \in \mathbb{R}$

1) $\lim_{x \to a} (f \pm g) = \lim f(x) \mp \lim g(x) = L \pm M$ (توزيع)

2) $\lim_{x \to a} (fg) = \lim_{x \to a} f \lim_{x \to a} g = LM$

3) $\lim_{x \to a} \frac{f}{g} = \frac{\lim f(x)}{\lim g(x)} = \frac{L}{M}$, $M \neq 0$

4) $\lim_{x \to a} Kf(x) = \lim_{x \to a} K \lim_{x \to a} f(x) = KL$

5) $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)} = \sqrt[n]{L}$ $\left(\begin{matrix} n\text{even} \\ L > 0 \end{matrix}\right)$

The limits of Function

Chapter 2

Ex:

1) $\lim_{x\to 2} x^2+5x-1 = 2^2+10-1 = 13$ *كثيرات الحدود دائما تعويض مباشر

* $\lim_{x\to a} P(x) = P(a)$ $P(x) = \text{Poly}$

2) $f(x) = \begin{cases} \sqrt{x^2-1} & , x \geq 1 \ (+) \\ \dfrac{5x^3+4}{x-3} & , x<1 \ (-) \end{cases}$

(+) يمين

(−) يسار

a) $\lim_{x\to 3} f(x) = \lim_{x\to 3} \sqrt{x^2-1} = \sqrt{9-1} = 8$ تعويض مباشر

b) $\lim_{x\to 1} f(x)$

$\longrightarrow \lim_{x\to 1^+} \sqrt{x^2-1} = 0$

$\longrightarrow \lim_{x\to 1^-} \dfrac{5x^3+4}{x-3} = \dfrac{9}{-2}$

$\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x) \longrightarrow \lim_{x\to 1} f(x)$ does not exist

Ex: $\lim_{x\to 2} f(x) = 3$, $\lim_{x\to 2} g(x) = 4$

Find:

The limits of function

Chapter 2

1) $\lim_{x\to 2} f(x) + 5g(x) = \lim_{x\to 2} f(x) + 5\lim_{x\to 2} g(x) = 3+5(4) = 23$

2) $\lim_{x\to 2} f(x) + \frac{g(x)}{f(x)} = \lim_{x\to 2} f(x) + \frac{\lim g(x)}{x\to 2\ f(x)} = 3+\frac{4}{3} = \frac{13}{3}$

Ex: Find k such that

$\lim_{x\to 1} f(x)$: exist موجودة

$$f(x) = \begin{cases} x^2+1, & x>1 \quad (+) \text{ اليمين} \\ 2kx+5, & x\leq 1 \quad (-) \text{ اليسار} \end{cases}$$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x)$

$\lim_{x\to 1^+} x^2+1 = \lim_{x\to 1^-} 2kx+5$

$2 = 2k+5$

$2k = -3$

$k = \frac{-3}{2}$

The limits of Function

Chapter 2

*Calculating limits : $\left(\frac{0}{0}\right)$

1) $\lim\limits_{x\to 2} \frac{x^2-4}{x^2+x-6} = \frac{2^2-4}{2^2+2-6} = \frac{0}{0}$ ⚠

$\lim \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x+3)} = \lim \frac{(x+2)}{x+3} = \frac{4}{5}$ ✓

2) $\lim\limits_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = \frac{0}{0}$ ⚠

$\lim\limits_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} \times \frac{\sqrt{x+1}+2}{\sqrt{x+1}+2}$ الضرب بالمرافق

$\lim\limits_{x\to 3} \frac{x+1-4}{(x-3)(\sqrt{x+1}+2)}$ (الأول^2 + الثاني^2) $= \frac{\cancel{x-3}}{\cancel{(x-3)}(\sqrt{x+1}+2)} = \frac{1}{\sqrt{x+1}+2} = \frac{1}{4}$

3) $\lim \frac{\frac{3}{x}-\frac{3}{2}}{x-2} = \frac{0}{0}$ ⚠

$\lim\limits_{x\to 2} \frac{\frac{6-3x}{2x}}{x-2} = \lim\limits_{x\to 2} \frac{\frac{3\overset{-1}{\cancel{(2-x)}}}{2x}}{\cancel{x-2}} = \lim\limits_{x\to 2} \frac{-3}{2x} = \frac{-3}{4}$

4) $\lim\limits_{x\to 2} \frac{|x-2|}{x-2} = f(x) = \begin{cases} \frac{x-2}{x-2}, x \geq 2 = 1 \\ x-2, x<2 = -1 \end{cases}$

$\underset{2-x}{- - -} \overset{2}{|} \underset{x-2}{+ + +}$

$2-x$

$\lim\limits_{x\to 2^+} \frac{|x-2|}{x-2} = 1$, $\lim\limits_{x\to 2^-} \frac{|x-2|}{x-2} = -1$

$\lim\limits_{x\to 2} \frac{|x-2|}{x-2}$ does not exist

The limits of function

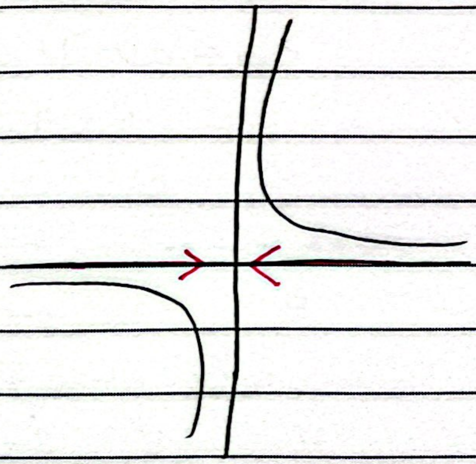
chapter 2

***Infinite limits :** $\frac{c}{0}$, $c \in \mathbb{R}$, $c \neq 0$

$\downarrow \quad \downarrow \quad \downarrow$

$+\infty \quad -\infty \quad -\infty , +\infty$

1) $\lim_{x \to 0} \frac{1}{x} = \frac{1}{0} \begin{matrix} \nearrow +\infty \\ \searrow -\infty \end{matrix}$



2) $\lim_{x \to 7} \frac{1}{(x-7)^2} = \frac{1}{0} = +\infty$

$+ + + + +$

7

3) $\lim_{x \to 5} \frac{-1}{(x-5)^2} = \frac{-1}{0} = -\infty$

4) $\lim_{x \to 3} \frac{-1}{x-3} = \frac{-1}{0} \begin{matrix} \nearrow -\infty \\ \searrow +\infty \end{matrix}$

Ex: $f(x) = \frac{2-x}{(x-4)(x+2)}$

Find:

The limits of function

Chapter 2

1) $\lim\limits_{x\to 2} f(x) = 0$ ✓

2) $\lim\limits_{x\to 4} f(x) = \frac{2-4}{0} = \frac{-2}{0}$ $\begin{cases} x \to 4^{+} \rightsquigarrow -\infty \\ x \to 4^{-} \rightsquigarrow +\infty \end{cases}$

$\lim\limits_{x\to -2} f(x) = \frac{4}{0}$ $\begin{cases} \xrightarrow{+} -\infty \\ \xrightarrow{-} +\infty \end{cases}$

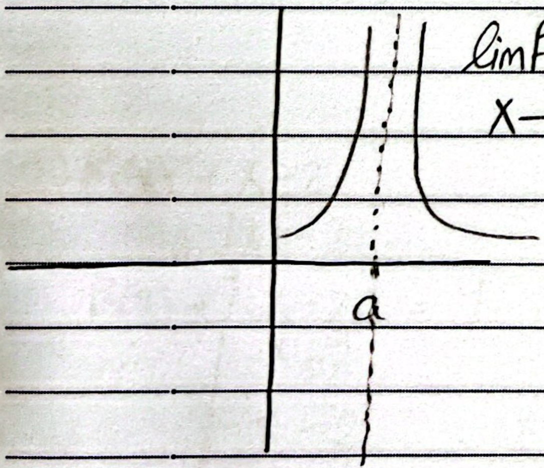
$+\,+\,+\,+ \quad \underset{2}{|} \quad -\,-\,-\,-$

$+\,+\,+ \quad \underset{-2}{|} \quad -\,-\,-\,-\,-\,- \quad \underset{4}{|} \quad +\,+\,+$

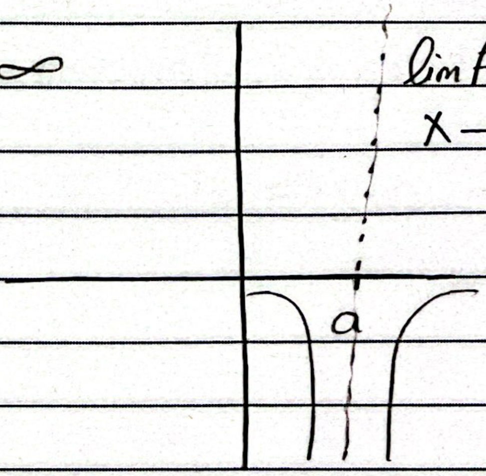
$f(x) \quad +\,+\,+ \quad \underset{-2}{|} \quad -\,-\,- \quad \underset{2}{|} \quad +\,+\,+ \quad \underset{4}{|} \quad -\,-\,-$

★ Vertical asymptotes خطوط التقارب العمودية :

$x = a$ V.asy



$\lim\limits_{x\to a} f(x) = +\infty$



$\lim\limits_{x\to a} f(x) = -\infty$

Vertical asymptotes

Chapter 2

Ex: Find the vertical asymptotes:

1) $f(x) = \frac{x^2-4}{x^2-2x}$

*لازم يطلع ∞+ أو ∞- يعني عدد / صفر

$x^2-2x=0$

$x(x-2)=0$

$x=0 \ , x=2$

$\lim_{x\to 0} \frac{x^2-4}{x^2-2x} = \frac{-4}{0} = \infty \longrightarrow x=0$ is vertical asymptotes

$\lim_{x\to 2} \frac{x^2-4}{x^2-2x} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{x\cancel{(x-2)}} = \lim_{x\to 2} \frac{x+2}{x} = \frac{4}{2} = 2$ X

* خطوات الحل:

١) نخرج أصفار المقام

٢) بنعوضها في lim (إذا كان الجواب ∞ أو ∞-) بيكون V.asy

٣) إذا كان الجواب عدد بيكون not V.asy

2) $f(x) = \frac{x-2}{|x|-2}$

$|x| \Rightarrow x=0$ − + 0

$$f(x) = \begin{cases} \frac{x-2}{x-2} = 1 \ , & x \geq 0 \\ \frac{x-2}{-\cancel{x}-2} \ , & x<0 \end{cases}$$

$-x-2=0 \longrightarrow x=-2$ V. asy

$|x|-2=0 \longrightarrow x=\pm 2$

$\lim_{x\to 2} f(x) \ , \ \lim_{x\to -2} f(x)$

Vertical asymptotes

Chapter 2

1) $f(x) = x^2+5x+7$ (not V.asy)

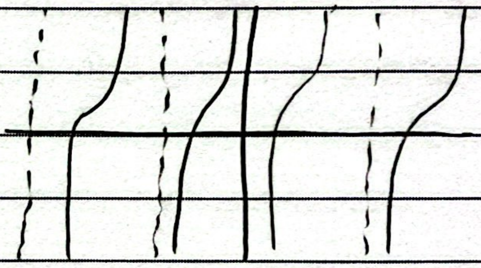
2) $f(x) = \frac{1}{x-7}$, $x=7$ (V.asy)

3) $f(x) = \frac{1}{(x-1)(x-2)}$, $x=1$, $x=2$ (V.asy)

4) $f(x) = \frac{5}{(x-3)(x)(x+4)}$, $x=3$, $x=0$, $x=-4$ (V.asy)

5) $f(x) = \tan x$

$x = \frac{\pi}{2} + n\pi$ (V.asy)



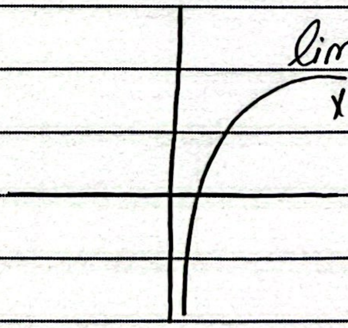
6) $f(x) = \log_b x$, $b>1$

$x=0$ (V.asy)

$f(x) = \ln x$

$x=0$ (V.asy)

$\lim_{x\to 0^+} \ln x = -\infty$



$\lim_{x\to 0^+} \log_b x = -\infty$

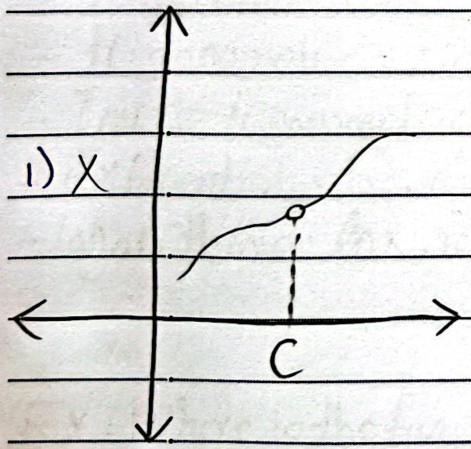
Continuity Chapter 2

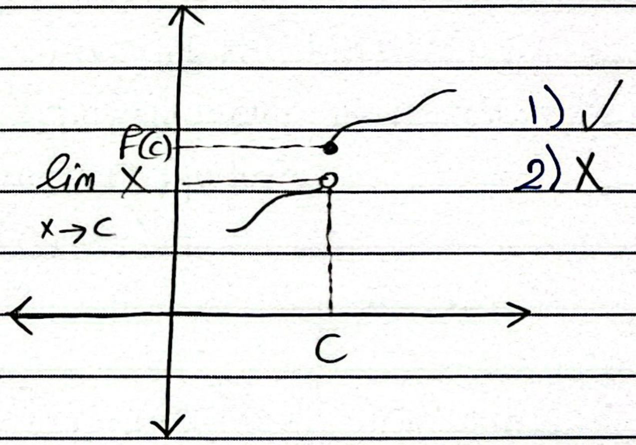
*Defintion : A function f is said to be continuous at $x = c$ Provided the following condituary are satisfied :

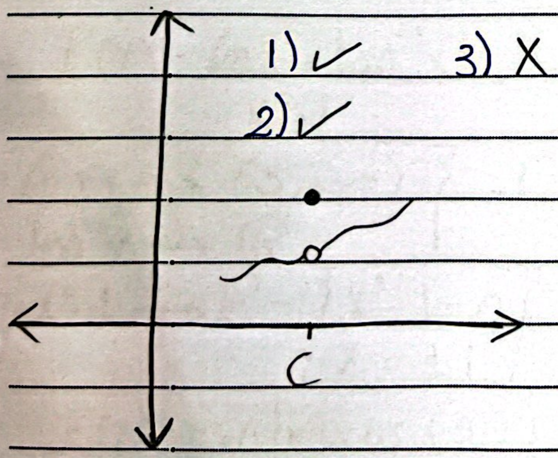
1) $f(c)$ is defined ($c \in D_f$)
2) $\lim\limits_{x \to c} f(x)$ exists ($\lim\limits_{x \to c^+} f = \lim\limits_{x \to c^-} f$)
3) $\lim\limits_{x \to c} f(x) = f(c)$

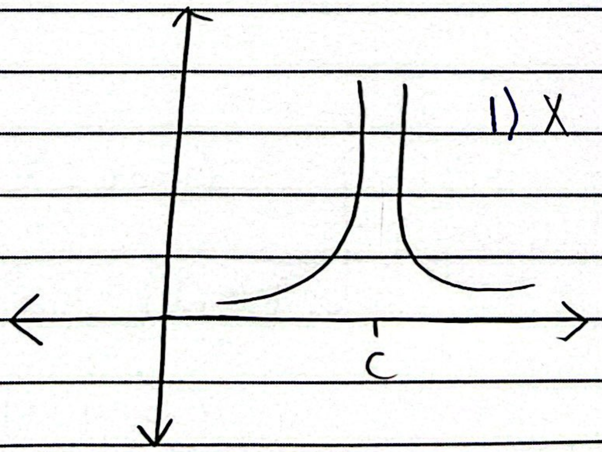
*إذا الشرط الثالث تحقق معناها الأول والثاني تحققوا
إذا الشرط الأول ما تحقق ما بنكمل ، إذا الشرط الثاني ما تحقق ما بنكمل

*ملاحظة : f is not continuous at $x = c$, f discontinuous at $x = c$









Continuity

* نظريات الاتصال :

1) If the functions f & g are continuous at $x=c$

a) $f \pm g$ is continuous at $x=c$

b) fg is continuous at $x=c$

c) $\frac{f}{g}$ is continuous at $x=c$, $g(c) \neq 0$

2) The following types of functions are continuous at every number in their domains

- Polynomials : x^2+5x+7 , x^3 , --- (everywhere)
- rational : $\frac{1}{x} \longrightarrow \mathbb{R}-\{0\}$
- root functions : $\sqrt{x}$ continuous $[0,\infty)$, $\sqrt[3]{x}$ continuous everywhere
- trigonometric : $\sin x$, $\cos x$ everywhere , $\tan x$ $\mathbb{R}-\{\frac{\pi}{2}+n\pi\}$
- Inverse trigonometric : $\tan^{-1}x$ continuous everywhere , $\sin^{-1}x$ $\cos^{-1}x$ $[-1,1]$
- exponential : b^x, e^x continuous everywhere
- logarithmic : $\ln x$, $\log_b x$ $(0,\infty)$

Ex : Where is the function $f(x)=\frac{\ln x+\tan^{-1}x}{x^2-1}$ continuous ?

$$f(x)=\frac{\ln x+\tan^{-1}x}{x^2-1}$$

$\ln x \longrightarrow (0,\infty)$ $\Big\}\rightarrow \cap\ (0,\infty)$

$\tan^{-1}x \longrightarrow \mathbb{R}$

$x^2-1 \longrightarrow \mathbb{R}-\left\{\begin{matrix} x^2-1=0 \\ x=\pm 1\end{matrix}\right\} = \mathbb{R}-\{1,-1\}$

$\Big\} \rightarrow \cap = (0,1)\cup(1,\infty)$

$f(x)=$ continuous $(0,1)\cup(1,\infty)$

Continuity

Chapter 2

Ex: $f(x) = \sin^{-1}(1+2x)$

D_f ? $\quad -1 \leq 1+2x \leq 1$

$-2 \leq 2x \leq 0$

$-1 \leq x \leq 0$

$D_f = [-1,0] \rightarrow f(x)$ continuous $[-1,0]$

Ex: $f(x) = \dfrac{1}{1+e^{\frac{1}{x}}}$

$-1 = e^{\frac{1}{x}}$ X

$\frac{1}{x} \rightarrow x = 0 \qquad D_f = \mathbb{R} - \{0\}$

$f(x)$ continuous $\mathbb{R} - \{0\}$

* Continuity On an interval : الاتصال على الفترة المغلقة

f continuous $[a,b]$ if

1) f continuous (a,b)

2) $\lim\limits_{x \to a^+} f(x) = f(a)$ (f continuous from right at $x = a$)

3) $\lim\limits_{x \to b^-} f(x) = f(b)$ (f continuous from left at $x = b$)

Ex: $f(x) = \begin{cases} x^2 - 1 & , 0 \leq x < 1 \\ 5 & , x = 1 \end{cases}$ Is f(x) continuous on the closed Interval $[0,1]$?

✓ Is f(x) continuous on $(0,1)$? Yes : $x^2 - 1$ Poly everywhere

Continuity

Chapter 2

✓ Is f(x) continuous at $x=0$ (right) ?

$$\lim_{x\to 0^+} f(x) \overset{?}{=} f(0)$$

$$\lim_{x\to 0^+} x^2-1 \overset{?}{=} 0-1$$

$$-1 = -1$$

X Is f(x) continuous at $x=1$? (left) ?

$$\lim_{x\to 1^-} f(x) \overset{?}{=} f(1)$$

$$\lim_{x\to 1^-} x^2-1 = 5$$

$$0 \neq 5$$

No , f(x) continuous $[0,1)$

* Continuity of Composition :

Theorem1 : F continuous everywhere , g continuous everywhere Then Fog , gof continuous everywhere

1) $f(x) = \sin x$, $g(x) = |x| \longrightarrow \sin|x|$, $|\sin x|$ continuous everywhere

2) $\underset{\downarrow}{\tan^{-1}}(e^x) \longrightarrow \tan^{-1}x$ continuous everywhere

continuous everywhere e^x continuous everywhere

Theorem 2 : If g continuous at $x=c$ & f(x) continuous at g(c) then Fog continuous at $x=c$

$$f\,\boxed{\left(\boxed{\underset{\underset{c}{\downarrow}}{g(x)}}\right)}$$

Fog continuous $\boxed{C}$

Continuity of Composition

Chapter 2

Ex: $f(x) = \frac{1}{x - \frac{1}{2}}$, $g(x) = \frac{1}{2x}$

f is continuous $\mathbb{R} - \{\frac{1}{2}\}$
g is continuous $\mathbb{R} - \{0\}$

1) Is $f \circ g$ continuous at $x=0$? discontinuous at $x=0$
Is g continuous at $x=0$? No

2) Is $f \circ g$ continuous at $x=1$?discontinuous at $x=1$
Is g continuous at $x=1$? yes
Is f continuous at $g(1) = \frac{1}{2}$? No

3) Is $f \circ g$ continuous at $x = \frac{1}{2}$?
Is g continuous at $x = \frac{1}{2}$? yes
I $f(x)$ continuous at $g(\frac{1}{2}) = 1$? yes
$f \circ g$ continuous at $x = \frac{1}{2}$

Theorem 3 : If $\lim_{x\to c} g(x) = L$ and $f(x)$ is continuous at L , Then

$$\lim_{x\to c} (f(g(x))) = \underset{\substack{\downarrow \\ \text{continuous at } L}}{f}\left(\lim_{x\to c} g(x)\right) = f(L)$$

1) $$\lim_{x\to 1} \sin^{-1}\left(\frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{1-\sqrt{x}}{1-x}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{\cancel{(1-\sqrt{x})}}{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}\right)$$

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1}{1+\sqrt{x}}\right) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

Continuity of Composition

Chapter 2

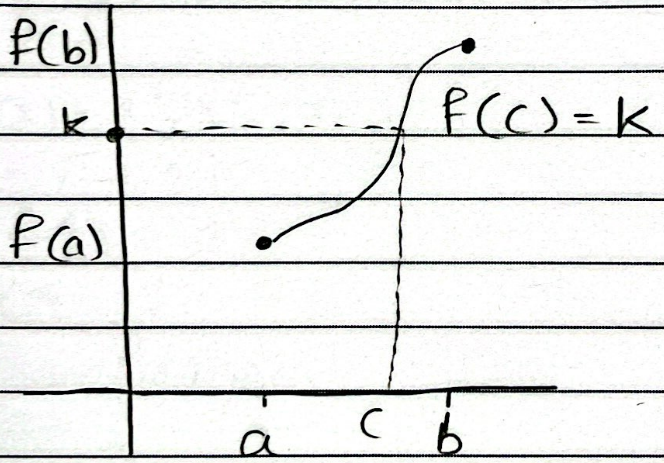
2) $\lim_{x\to1}\cos\left(\frac{x^2-1}{x-1}\right) = \cos\left(\lim_{x\to1}\frac{x^2-1}{x-1}\right) = \cos\left(\lim_{x\to1}\frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}\right)$

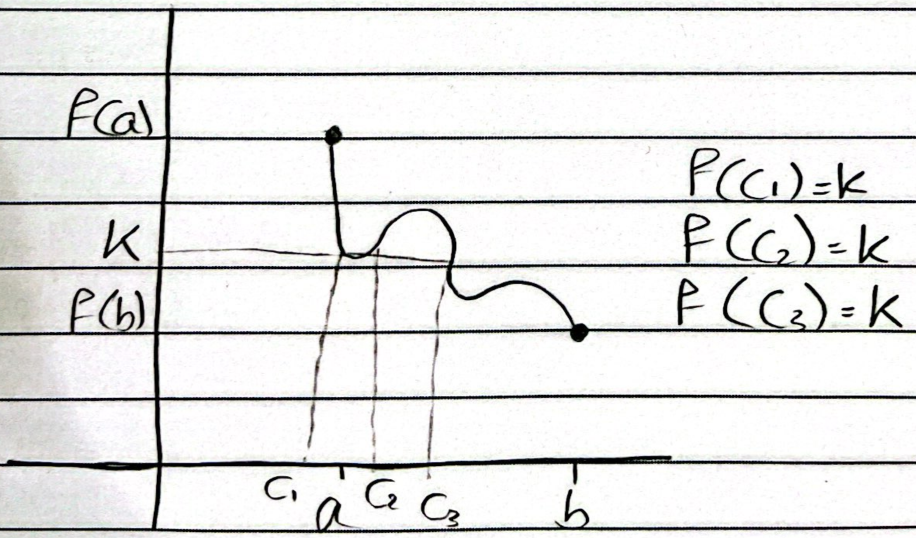
$= \cos\left(\lim_{x\to1} x+1\right) = \cos 2$

* The Intermediate Value Theorem: نظرية القيمة المتوسطة

الفترة المغلقة

If f is continuous on a closed Interval [a,b] and k any number between f(a) and f(b), Then there is at least one number c in the Interval [a,b] such that f(c) = K.





The intermediate Value theorem

Chapter 2

Ex: If $f(x) = x^5 + 7x^4 + 3$

Show that there is a number c such that $f(c) = 100$ in the interval $[0, 2]$

$f(0) = 3$

$f(2) = 147$

$3 < 100 < 147$

by IVT, c such that $f(c) = 100$

Ex: Show that $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[-1, 2]$

$f(x) = x^3 + x^2 - 2x - 1$

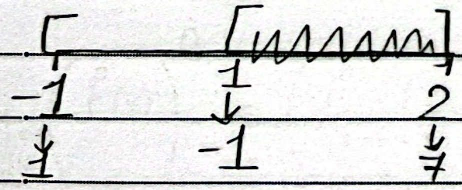
$[-1, 2]$

c root $f(x)$

$f(c) = 0$

$f(-1) = 1$

$f(2) = 7$



$f(1) = -1$

$-1 < 0 < 7$

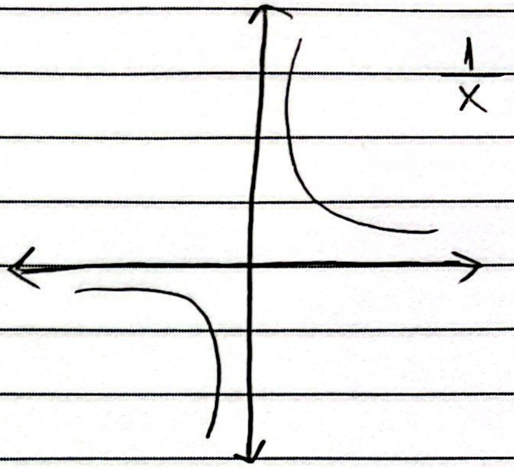
by IVT at least c

$f(c) = 0$

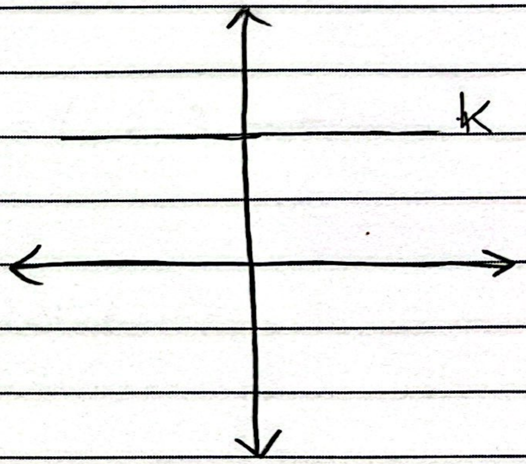
Limits at infinity $\begin{smallmatrix} x\to+\infty \\ x\to-\infty \end{smallmatrix}$ Chapter 2

1) $\lim\limits_{x\to+\infty} \frac{1}{x} = 0$

$\lim\limits_{x\to-\infty} \frac{1}{x} = 0$



2) $\lim\limits_{\substack{x\to+\infty \\ x\to-\infty}} K = K = K$ الجواب رح يكون



Ex: $\lim\limits_{x\to+\infty} \pi = \pi$, $\lim\limits_{x\to-\infty} 7 = 7$

3) $\lim\limits_{\substack{x\to+\infty \\ x\to-\infty}} \left(f(x)\right)^n = \left(\lim\limits_{\substack{x\to+\infty \\ x\to-\infty}} f(x)\right)^n$, n Positive integer 1, 2, 3, 4

Ex: $\lim\limits_{x\to+\infty} f(x) = 3$, find $\lim \left(f(x)\right)^4$?

$\left(\lim\limits_{x\to+\infty} f(x)\right)^4 = 3^4 = 81$

4) $\lim_{\substack{x \to +\infty \\ x \to -\infty}} k f(x) = k \lim_{\substack{x \to +\infty \\ x \to -\infty}} f(x)$

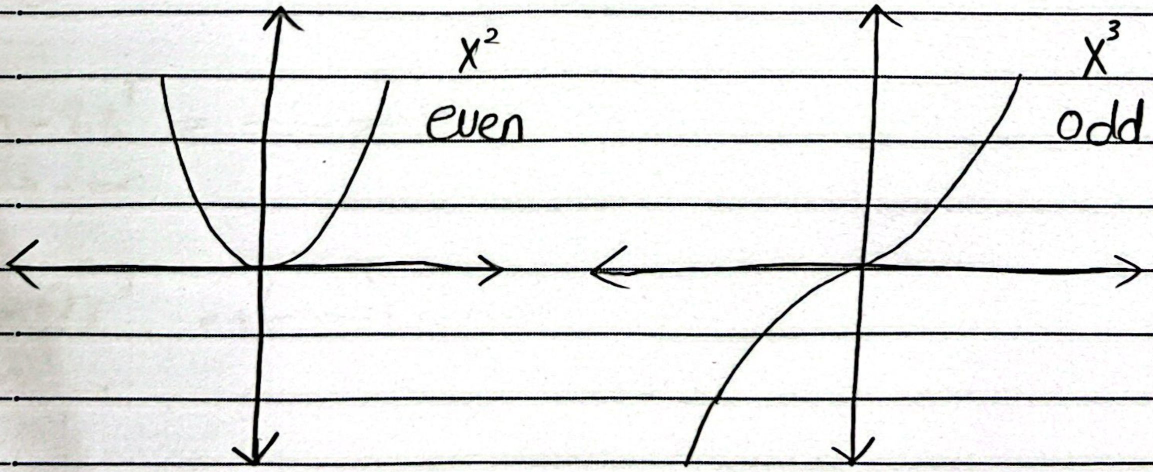
Ex: $\lim_{x \to -\infty} f(x) = 2$, $\lim_{x \to -\infty} 5 f(x)$?

$5 \lim_{x \to -\infty} f(x) = 5 * 2 = 10$

* Limit of x^n $x \to +\infty$, $x \to -\infty$

1) $\lim_{x \to +\infty} x^n = +\infty$, $n = 1, 2, 3, \ldots$

2) $\lim_{x \to -\infty} x^n = \begin{cases} +\infty & , n \text{ even } 2, 4, 6, \ldots \\ -\infty & , n \text{ odd } 1, 3, 5, \ldots \end{cases}$



$\lim_{x \to +\infty} x^2 = +\infty$

$\lim_{x \to -\infty} x^2 = +\infty$

$\lim_{x \to +\infty} x^3 = +\infty$

$\lim_{x \to -\infty} x^3 = -\infty$

Limits at infinity

Chapter 2

Ex: $\lim_{x\to+\infty} x^7 = +\infty$

$\lim_{x\to-\infty} x^7 = -\infty$

Ex: $\lim_{x\to+\infty} x^{10} = +\infty$

$\lim_{x\to-\infty} x^{10} = +\infty$

ركزنا بالإشارة وبس

$*\ \lim_{\substack{x\to+\infty \\ x\to-\infty}} kx^n = k \lim_{\substack{x\to+\infty \\ x\to-\infty}} x^n$ $\quad k:\ +,\ -$

$-(+\infty) = -\infty$

$-(-\infty) = +\infty$

1) $\lim_{x\to+\infty} -5x^2 = -(+\infty) = -\infty$

2) $\lim_{x\to-\infty} -8x^7 = -(-\infty) = +\infty$

3) $\lim_{x\to-\infty} 9x^7 = +(-\infty) = -\infty$

Limits of Polynomials   Chapter 2

$$*\lim_{\substack{x\to+\infty\\x\to-\infty}}(C_0+C_1x+C_2x^2+\cdots+C_nx^n)=\lim_{\substack{x\to+\infty\\x\to-\infty}}C_nx^n$$

Ex: $\lim_{x\to+\infty}7x^5-4x^2+10x^3+8x-19=\lim_{x\to+\infty}7x^5=+\infty$ بهمنا صاحب أكبر قوة

Ex: $\lim_{x\to-\infty}1-x^2-3x^3+4x^4-5x^5+10x^6=\lim_{x\to-\infty}10x^6=+\infty$

Ex: $\lim_{x\to-\infty}(-3x^3+2x^2+8)^4=\lim_{x\to-\infty}(-3x^3)^4=\lim_{x\to-\infty}81x^{12}=+\infty$

*Limits of rational functions $x\to+\infty$, $x\to-\infty$:

الطريقة السريعة :   بناخذ الكبير من البسط والكبير من المقام

1) $\lim_{x\to-\infty}\frac{3x+5}{6x-8}=\lim_{x\to-\infty}\frac{3x}{6x}=\lim_{x\to-\infty}\frac{1}{2}=\frac{1}{2}$

2) $\lim_{x\to+\infty}\frac{5x^3-2x^2+1}{1-3x}=\lim_{x\to+\infty}\frac{5x^3}{-3x}=\lim_{x\to+\infty}\frac{-5x^2}{3}=-\infty$

3) $\lim_{x\to-\infty}\frac{4x^2-x}{2x^3-5}=\lim_{x\to-\infty}\frac{4x^2}{2x^3}=\lim_{x\to-\infty}\frac{2}{x}=0$

Limits of Polynomials

Chapter 2

Ex: $\lim_{x\to-\infty}\frac{3x+5}{6x-8} = \frac{-\infty}{-\infty}$!

$$\lim_{x\to-\infty}\frac{\cancel{x}(3+\frac{5}{x})}{\cancel{x}(6-\frac{8}{x})} = \lim_{x\to-\infty}\frac{3+\frac{5}{x}}{6-\frac{8}{x}} = \frac{3}{6} = \frac{1}{2}$$

1) $\lim_{x\to+\infty}\sqrt{\frac{2x+1}{6x-1}} = \sqrt{\lim_{x\to+\infty}\frac{2x+1}{6x-1}} = \sqrt{\lim_{x\to+\infty}\frac{2x}{6x}} = \frac{1}{3}$

2) $\lim_{x\to-\infty}\frac{\sqrt{x^2+2}}{3x-6}$ $\quad\frac{+\infty}{-\infty}$!

$$\lim_{x\to-\infty}\frac{\sqrt{x^2(1+\frac{2}{x^2})}}{x(3-\frac{6}{x})} = \lim_{x\to-\infty}\frac{|x|\sqrt{1+\frac{2}{x^2}}}{x(3-\frac{6}{x})}$$

$|x|$

$\begin{array}{ccc} - & & + \\ \hline -x & 0 & x \end{array}$

$$= \lim_{x\to-\infty}\frac{-\cancel{x}\sqrt{1+\overset{zero}{\cancel{\frac{2}{x^2}}}}}{\cancel{x}(3-\cancel{\frac{6}{x}})_{zero}} = \frac{-1}{3}$$

3) $\lim_{x\to+\infty}\sqrt{x^6+5x^3}-x^3$ $\quad\infty-\infty$!

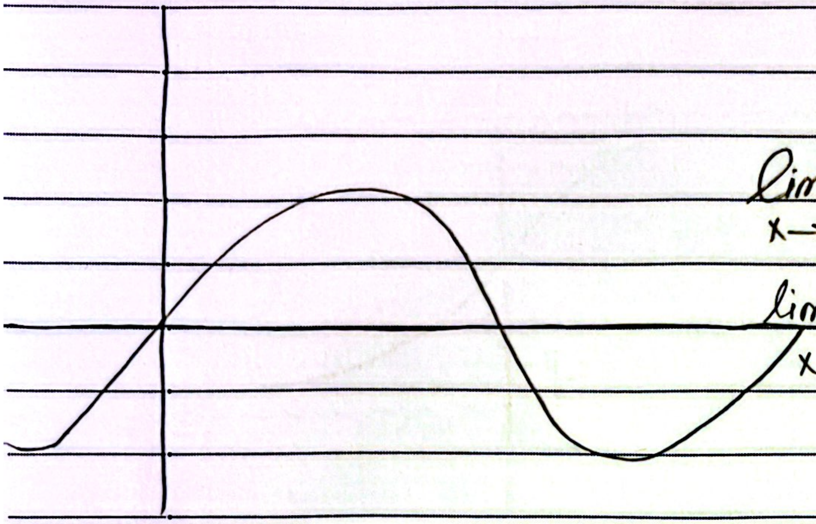
$$\lim_{x\to+\infty}\frac{\sqrt{x^6+5x^3}-x^3}{1}\times\frac{\sqrt{x^6+5x^3}+x^3}{\sqrt{x^6+5x^3}+x^3}$$

$$= \lim_{x\to+\infty}\frac{\cancel{x^6}+5x^3-\cancel{x^6}}{\sqrt{x^6+5x^3}+x^3} \quad \frac{\infty}{\infty} \quad = \lim_{x\to+\infty}\frac{5x^3}{\sqrt{x^6(1+\frac{5}{x^3})}+x^3}$$

$$= \lim_{x\to+\infty}\frac{5x^3}{|x^3|\sqrt{1+\frac{5}{x^3}}+x^3} = \lim\frac{5\cancel{x^3}}{\cancel{x^3}(\sqrt{1+\cancel{\frac{5}{x^3}}}+1)} = \frac{5}{\sqrt{1+0}+1} = \frac{5}{2}$$

Limits of Polynomials Chapter 2

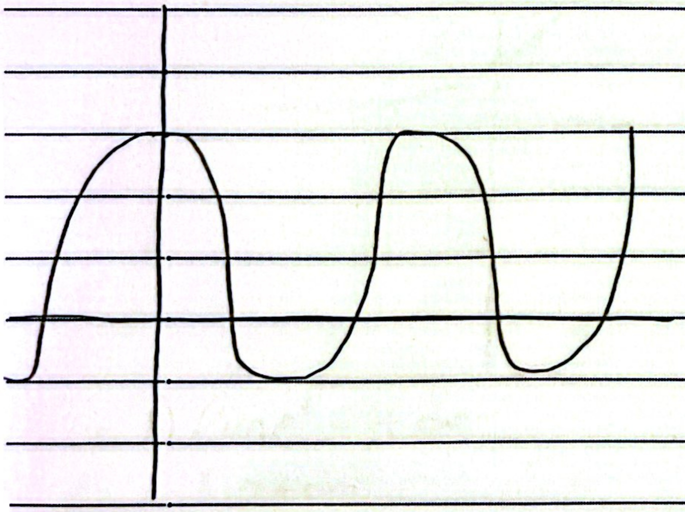
1) Sinx :



$\lim_{x\to+\infty} \text{Sin}x$: does not exist

$\lim_{x\to-\infty} \text{Sin}x$: does not exist

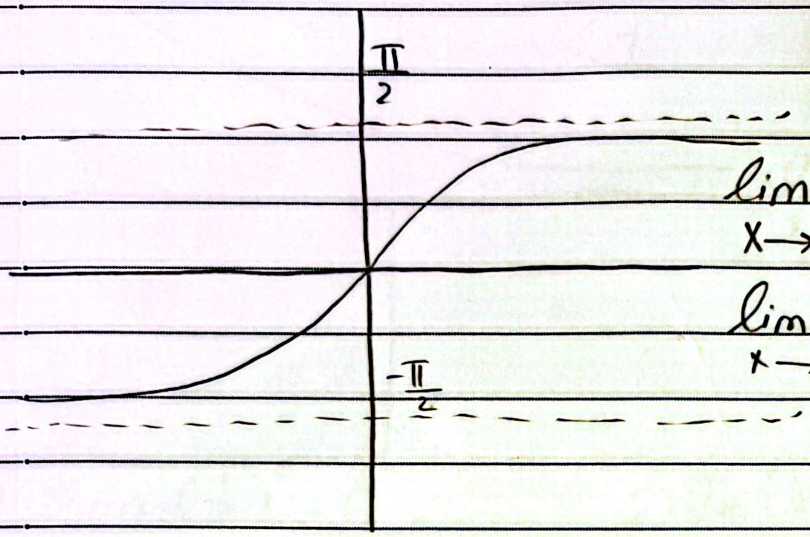
2) Cosx :



$\lim_{x\to+\infty} \text{Cos}x$: does not exist

$\lim_{x\to-\infty} \text{Cos}x$: does not exist

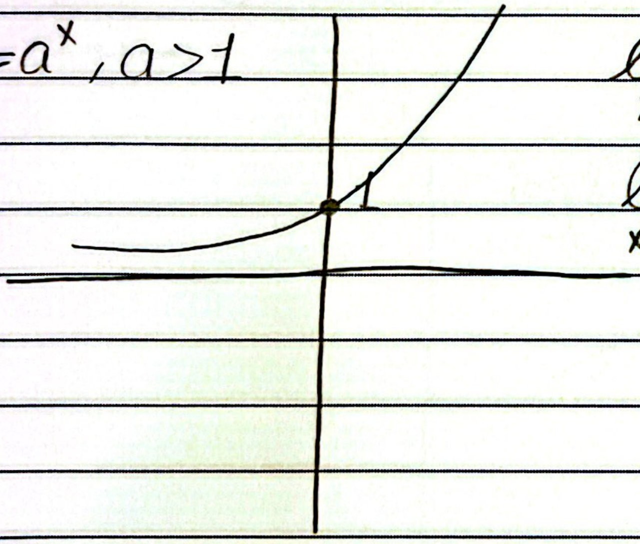
3) $\tan^{-1} x$



$\lim_{x \to +\infty} \tan^{-1} x : \frac{\pi}{2}$

$\lim_{x \to -\infty} \tan^{-1} x : -\frac{\pi}{2}$

4) $f(x) = a^x$, $a > 1$



$\lim_{x \to +\infty} a^x = +\infty$

$\lim_{x \to -\infty} a^x = 0$

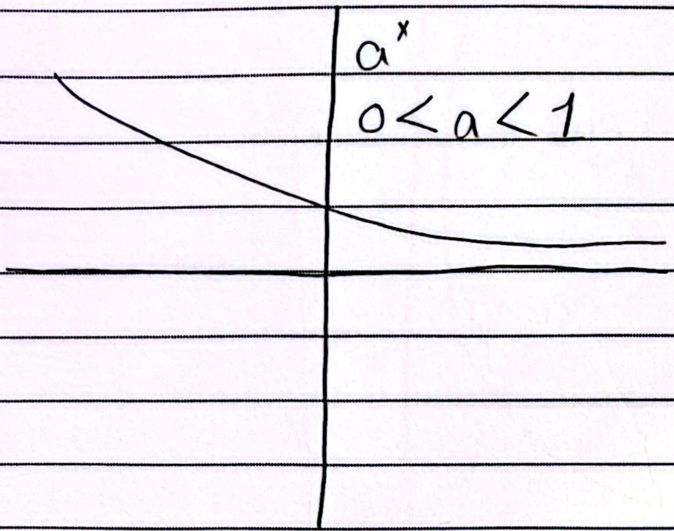
Ex: 1) $\lim_{x \to +\infty} e^x = +\infty$

2) $\lim_{x \to -\infty} e^x = 0$

3) $\lim_{x \to +\infty} 2^x = +\infty$

4) $\lim_{x \to -\infty} 5^x = 0$

5)



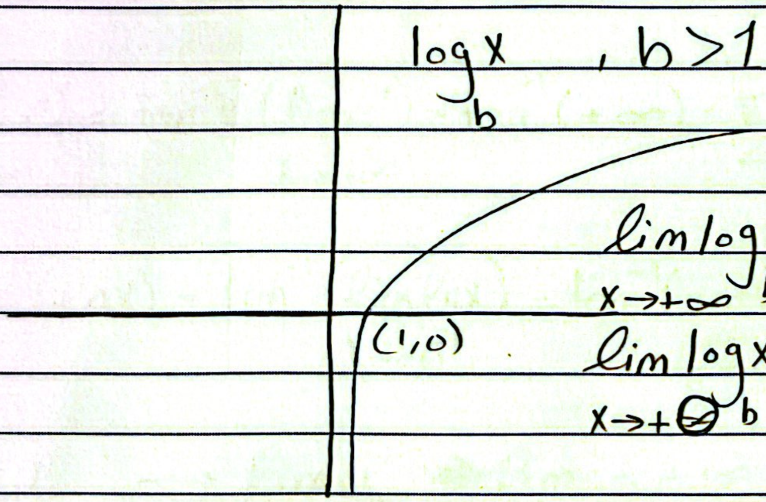
$\lim_{x\to+\infty} a^x = 0$

$\lim_{x\to-\infty} a^x = +\infty$

Ex. 1) $\lim_{x\to+\infty} \left(\frac{1}{2}\right)^x = 0$ 2) $\lim_{x\to-\infty} \left(\frac{1}{3}\right)^x = +\infty$

3) $\lim_{x\to+\infty} e^{-x} = 0$ 4) $\lim_{x\to-\infty} e^{-x} = +\infty$

6)



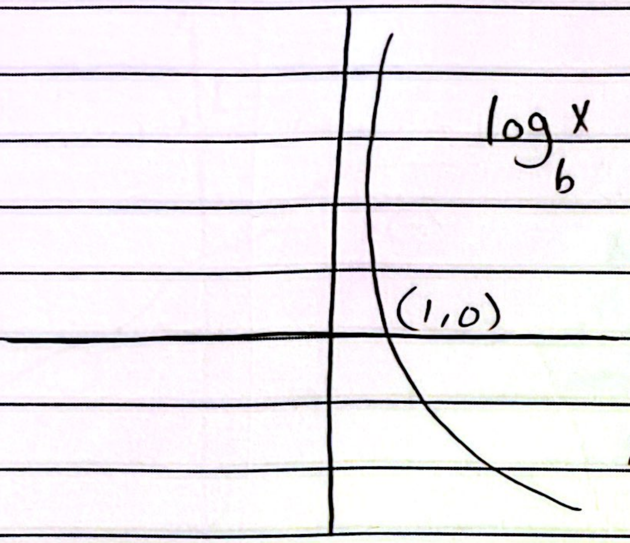
$\lim_{x\to+\infty} \log_b x = +\infty$

$\lim_{x\to 0^+} \log_b x = -\infty$

Ex: 1) $\lim_{x\to+\infty} \log_5 x = +\infty$ 2) $\lim_{x\to+\infty} \ln x = +\infty$

3) $\lim_{x\to 0^+} \ln x = -\infty$

7)



$\log_b x \ , \ 0<b<1$

$\lim\limits_{x\to+\infty} \log_b x = -\infty$

$\lim\limits_{x\to0^+} \log_b x = +\infty$

Ex: 1) $\lim\limits_{x\to+\infty} \log_{0.5} x = -\infty$ 2) $\lim\limits_{x\to0^+} \log_{\frac{1}{4}} x = +\infty$

Examples :

1) $\lim\limits_{x\to+\infty} \tan^{-1} e^x = \tan^{-1}x \left(\lim\limits_{x\to+\infty} e^x\right) = \tan^{-1}(+\infty) = \frac{\pi}{2}$

2) $\lim\limits_{x\to0^+} \tan^{-1}(\ln x) = \tan^{-1}\left(\lim\limits_{x\to0^+} \ln x\right) = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

3) $\lim\limits_{x\to0} \frac{1}{1+e^{\frac{1}{x}}} \longrightarrow \lim\limits_{x\to0} \frac{1}{x} \Rightarrow \lim\limits_{x\to0^+} = +\infty$ $\Rightarrow \lim\limits_{x\to0^-} = -\infty$ (d.n.e)

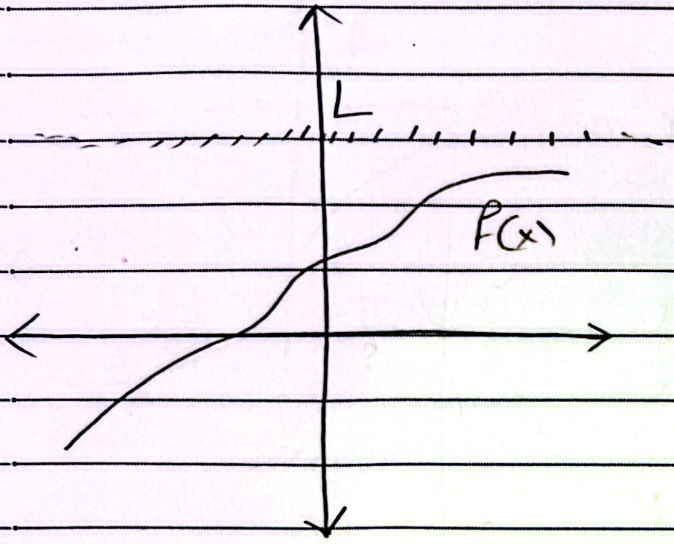
$\lim\limits_{x\to0^+} \frac{1}{1+e^{\frac{1}{x}}} = \lim\limits_{x\to0^+} \frac{1}{1+e^{+\infty}}$ $= \frac{1}{+\infty} = 0$

$\lim\limits_{x\to0^-} \frac{1}{1+e^{\frac{1}{x}}} = \frac{1}{1+e^{-\infty}}$ $= \frac{1}{1+0} = 1$

Horizontal asymptotes

Chapter 2

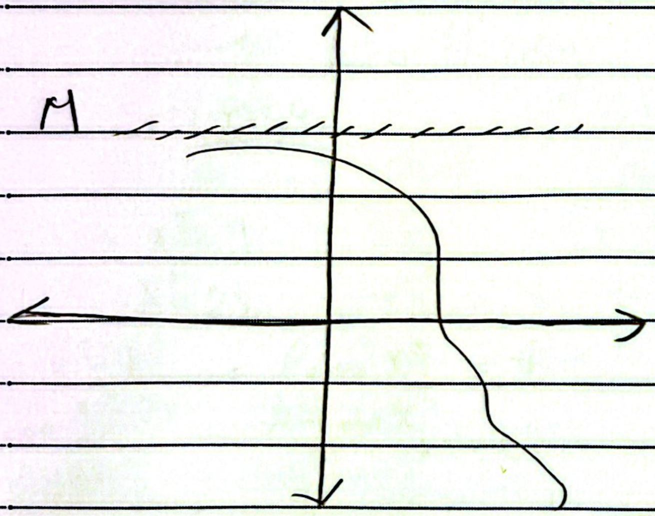
*



$y = L$ is H, asy

$\lim\limits_{x \to +\infty} f(x) = L$

*

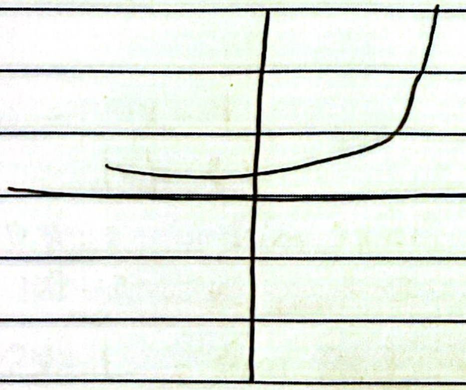


$y = M$ is H. asy

$\lim\limits_{x \to -\infty} f(x) = M$

*

$b^x, b > 1$

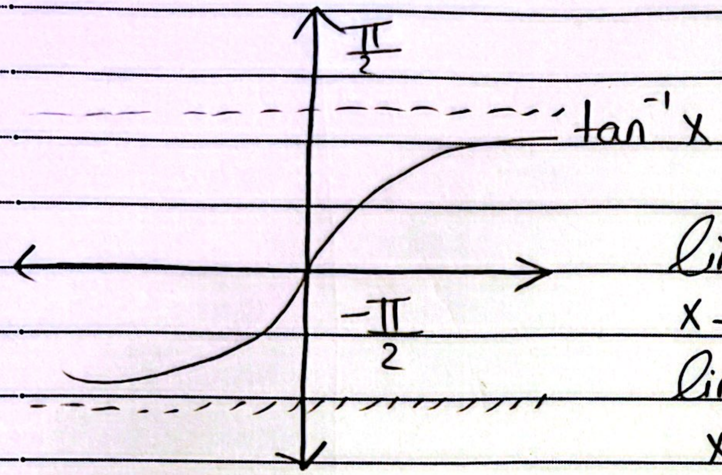


$\lim\limits_{x \to -\infty} b^x = 0$, $y = 0$ is H. asy

Horizontal asymptotes

Chapter 2

*



$$\lim_{x\to+\infty}\tan^{-1}x = \frac{\pi}{2}$$

$$\lim_{x\to-\infty}\tan^{-1}x = -\frac{\pi}{2}$$

$y=\frac{\pi}{2}$, $y=\frac{-\pi}{2}$ H.asy

* $F(x) = x^2+x+1$ No H.asy

* Find the H.asy

1) $F(x) = \frac{x^2-4}{x^2-2x}$

$$\lim_{x\to+\infty}\frac{x^2-4}{x^2-2x} = \lim_{x\to+\infty}\frac{x^2}{x^2} = 1$$

y=1 is H.asy

$$\lim_{x\to-\infty}\frac{x^2-4}{x^2-2x} = \lim_{x\to-\infty}\frac{x^2}{x^2} = 1$$

2) $F(x) = \frac{2x-1}{|x|-3}$

$$\lim_{x\to+\infty}\frac{2x-1}{|x|-3} = \lim_{x\to+\infty}\frac{2x-1}{x-3} = \lim_{x\to+\infty}\frac{2x}{x} = 2$$

$$\lim_{x\to-\infty}\frac{2x-1}{|x|-3} = \lim_{x\to-\infty}\frac{2x-1}{-x-3} = \lim_{x\to-\infty}\frac{2x}{-x} = -2$$

$y=2, y=-2$ are H.asy

Limit of the trigonometric

1) $\lim_{x\to\pi} \sin x = \sin\pi = 0$ (تعويض مباشر)

2) $\lim_{x\to\frac{\pi}{4}} \cos x = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

3) $\lim_{x\to\pi} \tan x = \tan\pi = 0$

4) $\lim_{x\to\frac{\pi}{2}} \tan x = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} = \frac{1}{0}$ $\begin{array}{ccccc} & +++ & & --- & \\ \hline 0 & Q_1 & \frac{\pi}{2} & Q_2 & \pi \end{array}$

$$\left.\begin{array}{l} \lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty \\ \lim_{x\to\frac{\pi}{2}^-} \tan x = +\infty \end{array}\right\} \rightarrow \text{does not exist}$$

5) $\lim_{x\to 2\pi^-} x\csc x = \lim_{x\to 2\pi^-} \frac{x}{\sin x} = \frac{2\pi}{0} =$ [illegible] $\begin{array}{cc} (----) & \\ \hline Q4 & 2\pi \end{array} = -\infty$

Thm: $\lim_{x\to 0} \frac{\sin x}{x} = 1$

1) $\lim_{x\to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

2) $\lim\limits_{x\to 0} \frac{ax}{\sin bx} = \frac{a}{b}$

3) $\lim \frac{\sin ax}{\sin bx} = \frac{a}{b}$

4) $\lim\limits_{x\to 0} \frac{\tan ax}{bx} = \frac{a}{b}$

5) $\lim\limits_{x\to 0} \frac{ax}{\tan bx} = \frac{a}{b}$

6) $\lim\limits_{x\to 0} \frac{\tan ax}{\tan bx} = \frac{a}{b}$

7) $\lim\limits_{x\to 0} \frac{\tan ax}{\sin bx} = \frac{a}{b}$

8) $\lim\limits_{x\to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$

Ex:

1) $\lim\limits_{x\to 0} \frac{\sin 7x}{5x} = \frac{7}{5}$

Limit of the trigonometric — Chapter 2

2) $\lim\limits_{x\to 0} \frac{\sin x}{\sin \pi x} = \frac{1}{\pi}$

3) $\lim\limits_{x\to 0} \frac{\tan 8x}{\sin 10x} = \frac{8}{10}$

4) $\lim\limits_{x\to 0} \frac{3x+\tan x}{\sin x + x} = \lim\limits_{x\to 0} \frac{x(3+\frac{\tan x}{x})}{x(\frac{\sin x}{x}+1)} = \frac{3+1}{1+1} = 2$

5) $\lim\limits_{x\to 0} \frac{1-\cos x}{x} = \lim\limits_{x\to 0} \frac{1-\cos x}{x} * \frac{1+\cos x}{1+\cos x}$

$= \lim\limits_{x\to 0} \frac{1-\cos^2 x}{x(1+\cos x)} = \lim\limits_{x\to 0} \frac{\sin^2 x}{x(1+\cos x)}$

$= \lim\limits_{x\to 0} \frac{\sin x}{x} * \frac{\sin x}{1+\cos x} = \lim\limits_{x\to 0} \frac{\sin x}{x} * \lim\limits_{x\to 0} \frac{\sin x}{1+\cos x}$

$= 1 * \frac{0}{1+1} = 0$

$$* \; f(x) = \begin{cases} 1+\sin x & , x<0 \\ \cos x & , 0<x\leq\pi \\ \sin x & , x>\pi \end{cases}$$

Find :

Limit of the trigonometric

Chapter 2

1) $\lim\limits_{x\to0} f(x) =$

$$\left.\begin{array}{l} \to \lim\limits_{x\to0^+} \cos x = 1 \\ \\ \to \lim\limits_{x\to0^-} 1+\sin x = 1 \end{array}\right] \to \lim_{x\to0} f(x) = 1$$

2) $\lim\limits_{x\to\pi} f(x) =$

$$\left.\begin{array}{l} \to \lim\limits_{x\to\pi^+} \sin x = \sin\pi = 0 \\ \\ \to \lim\limits_{x\to\pi^-} \cos x = \cos\pi = -1 \end{array}\right] \to \lim_{x\to\pi} f(x) = \text{d.n.e}$$

EX: Suppose that $\lim\limits_{x\to0} f(x) = 4$

Find $\lim\limits_{x\to0} \frac{\tan 8x}{x f(4x)}$

$y = 4x \longrightarrow x = \frac{y}{4}$

$f \to 0$

$y \to 0$

$$\lim_{y\to0} \frac{\tan 8\left(\frac{y}{4}\right)}{\frac{y}{4} f(y)} = \lim_{y\to0} \frac{4\tan 2y}{y f y} = 4 \lim \frac{\tan 2y}{y} * \frac{1}{f(y)}$$

$$= 4 \lim \frac{\tan 2y}{y} * \lim_{y\to0} \frac{1}{f y}$$

$\longleftarrow$

$$= (4)(2)\left(\frac{1}{4}\right) = 2$$

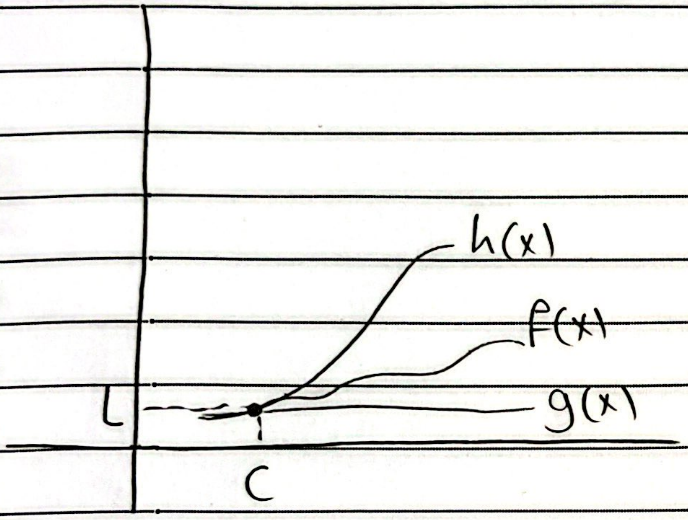
The squeeze Theorem

Chapter 2

* The squeeze Theorem نظرية الشطيرة : let f, g, h be Functions Such that $g(x) \le f(x) \le h(x)$

if $\lim_{x \to c} h(x) = \lim_{x \to c} g(x) = L$

Then $\lim_{x \to c} f(x) = L$



Ex: $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$ * d.n.e غير متصل

$g(x) \le x^2 \sin\frac{1}{x} \le h(x)$

$-1 \le \sin\frac{1}{x} \le 1$ * $x^2 > 0$

$-x^2 \le x^2 \sin\frac{1}{x} \le x^2$

$\lim_{x \to 0} \downarrow$ $\downarrow$ $\searrow \lim_{x \to 0}$

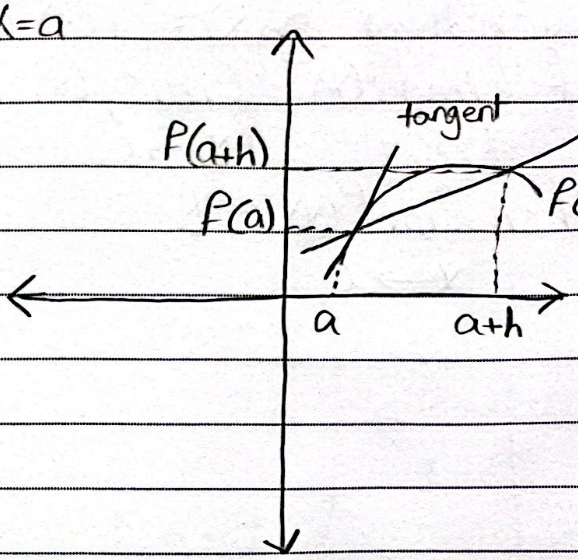
0 $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$ 0

by squeeze Thm

$f(x) \longrightarrow f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{d}{dx}\right|_{x=a}$

$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$

Slope of tangent $= f'(a)$



$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a}$

Ex: Find the derivative of a the function $f(x) = x^2$?

$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim \frac{x^2 + 2xh + h^2 - x^2}{h}$

$= \lim \frac{2xh + h^2}{h} \quad \frac{0}{0} \; ?$

$\lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h\to 0} 2x + h$

$f'(x) = 2x$

$f'(5) = 2(5) = 10$

*Differentiation Rules :

1) $\frac{d}{dx}(C)=0 \quad , C\in\mathbb{R}$

Ex: $f(x)=10 \longrightarrow f'(x)=0$

2) $\frac{d}{dx}(x^n)=nx^{n-1}$

Ex: $f(x)=x^5 \longrightarrow f'(x)=5x^4$
$y=x^{-7} \longrightarrow y'=-7x^{-8}$

3) $\frac{d}{dx}(cf)=cf'$

Ex: $y=7x^2 \longrightarrow y'=7(2)x=14x$

4) $\frac{d}{dx}(f\pm g)=f'\pm g'$

Ex: $f(x)=\sqrt{x}+\sqrt[3]{x^2}+\frac{12}{x}+7$

$f(x)=x^{\frac{1}{2}}+x^{\frac{2}{3}}+12x^{-1}+7$

$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}+\frac{2}{3}x^{-\frac{1}{3}}-12x^{-2}$

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5) $\frac{d}{dx}(fg) = fg' + gf'$

Ex: $y = (8x+3)(x^2-3x)$

$\frac{dy}{dx} = (8x+3)(2x-3)+(x^2-3x)(8)$

6) $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{gf' - g'f}{g^2}$

Ex1: $f(x) = \frac{x^2}{x^3+5x}$

$f'(x) = \frac{(2x)(x^3+5x) - (x^2)(3x^2+5)}{(x^3+5x)^2}$

Ex2: $f(x) = \frac{x^4-1}{8x^3+2x^2}$

$f'(x) = \frac{(4x^3)(8x^3+2x^2) - (x^4-1)(24x^2+4x)}{(8x^3+2x^2)^2}$

7) $\frac{d}{dx}\left(\frac{c}{g}\right) = \frac{-cg'}{g^2}$

Ex: $f(x) = \frac{7}{x^3+1}$

$f'(x) = \frac{-7(3x^2)}{(x^3+1)^2} = \frac{-21x^2}{(x^3+1)^2}$

Differentiation Rules Chapter 3

Examples:

1) $f(x) = \frac{7x-1}{x^2+5}$, find $f'(2)$

$$f'(x) = \frac{(7)(x^2+5)-(7x-1)(2x)}{(x^2+5)^2}$$

$$f'(2) = \frac{(7)(4+5)-(14-1)(4)}{(4+5)^2} = \frac{63-52}{81} = \frac{11}{81}$$

2) $f(x) = (10x-1)(8x^2+5)$, find $f'(1)$

$$f'(x) = (10)(8x^2+5) + (10x-1)(16x)$$

$$f'(1) = (10)(13) + (9)(16)$$

$$= 130 + 144$$

$$= 274$$

* Chain Rule قاعدة السلسلة :

$$\frac{d}{dx} f(g(x)) = f'(g(x)g'(x)$$

Ex: $f(x) = x^2+x$, $g(x) = \frac{4}{x^2+1}$, Find $(f \circ g)'(1)$?

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

$$= f'(g(1))g'(1$$

$$= 5 * -2 = -10$$

$g(1) = 2$

$g'(1) = -2$

$f'(2) = 5$

Chain Rule

$$* \quad \frac{d}{dx}\left(g(x)\right)^{n} = n\left(g(x)\right)^{n-1} g'(x)$$

EX : Find y'

1) $y = (x^2+5)^{10}$

$y' = 10(x^2+5)^9(2x)$

2) $y = (x^7-2)^{-5}$

$y' = -5(x^7-2)^{-6}(7x^6)$

3) $y = \sqrt[3]{x^2+1} \longrightarrow y = (x^2+1)^{\frac{1}{3}}$

$y' = \frac{1}{3}(x^2+1)^{-\frac{2}{3}}(2x)$

4) $y = \sqrt{x^3+x^2} \longrightarrow y = (x^3+x^2)^{\frac{1}{2}}$

$y' = \frac{1}{2}(x^3+x^2)^{-\frac{1}{2}}(3x^2+2x)$

$= \frac{3x^2+2x}{2\sqrt{x^3+x^2}}$

* فيه قانون أسهل لاشتقاق الجذر:

$y = \sqrt{f(x)} \Rightarrow y' = \frac{f'(x)}{2\sqrt{f(x)}}$

Chain Rule

Chapter 3

$*$ If $f(2x+5) = x^3+2x^2+4$, find $f'(3)$

$f'(2x+5)(2) = 3x^2+4x$

$f'(3)(2) = -1$

$f'(3) = \frac{-1}{2}$

$2x+5 = 3$

$2x = -2$

$x = -1$

$*$ $\frac{d}{dx}\left(f(x^3)\right) = 12x^{17}$, find $f'(x)$

$f'(x^3)(3x^2) = 12x^{17}$

$f'(x^3) = \frac{12x^{17}}{3x^2}$

$f'(x^3) = 4x^{15}$

$f'(y) = 4(y^{\frac{1}{3}})^{15}$

$f'(y) = 4y^5$

$f'(x)\ 4x^5$

$x^3 = y$

$x = y^{\frac{1}{3}}$

Derivatives of trigonometric chapter 3

* $\frac{d}{dx}\left(\sin(f(x))\right) = \cos f(x)\ f'(x)$

Ex: $y = \sin(x^3) = 3x^2 \cos x^3$

* $\frac{d}{dx}\left(\cos(f(x))\right) = -\sin f(x)\ f'(x)$

Ex: $y = \cos(5x^2+10) \longrightarrow y' = -\sin(5x^2+10)\ (10x)$

* $\frac{d}{dx}\left(\tan(f(x))\right) = \sec^2(f(x))\ f'(x)$

Ex: $y = \sqrt{\tan x} \longrightarrow y = (\tan x)^{\frac{1}{2}} \longrightarrow y' = \frac{\sec^2 x}{2\sqrt{\tan x}}$

* $\frac{d}{dx}\left(\cot(f(x))\right) = -\csc^2(f(x))\ f'(x)$

Ex: $y = \cot^3(x^2) = \left(\cot(x^2)\right)^3 = 3\cot^2(x^2)\left(-\csc^2(x^2)\right)(2x)$

* $\frac{d}{dx}\left(\sec f(x)\right) = \sec f(x) \tan f(x)\ f'(x)$

Ex: $y = \sec\left(\frac{2}{x+1}\right) \longrightarrow y' = \sec \frac{2}{x+1} \tan \frac{2}{x+1} \left(\frac{-2}{(x+1)^2}\right)$

* $\frac{d}{dx}\left(\csc f(x)\right) = -\csc f(x) \cot f(x)\ f'(x)$

Ex: $y = \csc x^3 \Rightarrow y' = -\csc x^3 \cot x^3\ (3x^2)$

Derivative of exponential function

$$* \frac{d}{dx}\left(b^{f(x)}\right) = b^{f(x)} f'(x) \ln b$$

Ex: 1) $y = 5^{x^2} \longrightarrow y' = 5^{x^2}(2x)\ln 5$

2) $y = 2^{\cos x} \longrightarrow y' = 2^{\cos x}(-\sin x)\ln 2$

3) $y = 3^{x} \longrightarrow y' = 3^{x}\ln 3$

$$* \frac{d}{dx}\left(e^{f(x)}\right) = e^{f(x)} f'(x)$$

Ex: 1) $y = e^{x\tan x} \longrightarrow y' = e^{x\tan x}(x\sec^2 x + \tan x)$

2) $y = e^{\sqrt{1-5x^3}} \longrightarrow y' = e^{\sqrt{1-5x^3}}\left(\frac{-15x^2}{2\sqrt{1-5x^3}}\right)$

3) $y = e^{x} \longrightarrow y' = e^{x}$

4) $y = e^{-x} + \pi^{-x} \longrightarrow y' = e^{-x}(-1) + (\pi^{-x})(-1)\ln\pi$

Ex: If $f(x) = e^{x}g(x)$, $g(0) = 2$, $g'(0) = 5$, Find $f'(0)$?

$f'(x) = e^{x}g(x) + e^{x}g'(x)$

$f'(0) = 2 + 5$

$f'(0) = 7$

Derivatives of logarithmic

Chapter 3

$$* \frac{d}{dx}\left(\log_b f(x)\right) = \frac{f'(x)}{f(x)\ln b} \quad , f(x) > 0$$

Ex : Find y' :

1) $y = \log_2 (x^3+5) \longrightarrow y' = \frac{3x^2}{(x^3+5)\ln 2}$

2) $y = \log \sin x \longrightarrow y' = \frac{\cos x}{\sin x \ln 10} = \frac{\cot x}{\ln 10}$

$$* \frac{d}{dx} \ln(f(x)) = \frac{f'(x)}{f(x)} \quad , f(x) > 0$$

Ex : Find y' :

1) $y = \ln x \longrightarrow y' = \frac{1}{x}$

2) $y = \ln(x^2+1) \longrightarrow y' = \frac{2x}{x^2+1}$

3) $y = \ln(\cos e^x) \longrightarrow y' = \frac{-e^x \sin e^x}{\cos e^x} = -e^x \tan e^x$

4) $y = \ln(\ln x)$, $\left.\frac{dy}{dx}\right|_{x=e}$

$$y' = \frac{\frac{1}{x}}{\ln x} = \left.\frac{1}{x \ln x}\right|_{x=e} = \frac{1}{e}$$

5) $y = \ln|\sin x + x|$, $\frac{d}{dx}\ln|x| = \frac{1}{x}$ بنتجاهل ال | |

$y' = \frac{\cos x + 1}{\sin x + x}$

Ex: Find y'.

1) $y = \log_7 x$

$y' = \frac{1}{x\ln 7}$

2) $y = \log_x 7$

$y' = \frac{\ln 7}{\ln x}$

$y' = \frac{(-\ln 7)(\frac{1}{x})}{(\ln x)^2}$

$= \frac{-\ln 7}{x(\ln x)^2}$

Ex: Find y'.

1) $y = \ln\left(\frac{x^2 \sin x}{\sqrt{1+x}}\right)$

$y = 2\ln x + \ln\sin x - \frac{1}{2}\ln(1+x)$

$y = \frac{2}{x} + \frac{\cos x}{\sin x} - \frac{1}{2(1+x)}$

2) $y = \log_4\left(\frac{\cos x}{(x-1)^3(x^2+1)^4}\right)$

$y = \log_4 \cos x - 3\log_4 (x-1) + 4\log_4 (x^2+1)$

$y' = \frac{-\sin x}{\cos x \ln 4} - \frac{3}{(x-1)\ln 4} + \frac{8x}{(x^2+1)\ln 4}$

$y' = \frac{1}{\ln 4}\left(-\tan x - \frac{3}{x-1} + \frac{8x}{x^2+1}\right)$

* Derivatives of Inverse Trigonometric functions:

$\frac{d}{dx}\left(\sin^{-1}(f(x))\right) = \frac{f'(x)}{\sqrt{1-(f(x))^2}}$

Find y':

$y = \sin^{-1} x \rightarrow y' = \frac{1}{\sqrt{1-x^2}}$

$y = \sin^{-1} \ln x \rightarrow y' = \frac{\frac{1}{x}}{\sqrt{1-(\ln x)^2}}$

$\frac{d}{dx}\left(\cos^{-1}(f(x))\right) = \frac{-f'(x)}{\sqrt{1+(f(x))^2}}$

Find y':

$y = \cos^{-1} x^2 \rightarrow y' = \frac{-2x}{\sqrt{1+x^4}}$

$\frac{d}{dx}\left(\tan^{-1}(f(x))\right) = \frac{f'(x)}{1+(f(x))^2}$

$y = \tan^{-1} x \rightarrow y' = \frac{1}{1+x^2}$

$y = \tan^{-1} e^x \rightarrow y' = \frac{e^x}{1+(e^x)^2}$

$\frac{d}{dx}\left(\cot^{-1}(f(x))\right) = -\frac{f'(x)}{1+(f(x))^2}$

$y = \cot^{-1}(x^2+1) = \frac{-2x}{1+(x^2+1)^2}$

Derivatives of inversetrignometric

Chapter 3

$$\frac{d}{dx}\left(\sec^{-1}(f(x))\right) = \frac{f'(x)}{|f(x)|\sqrt{(f(x))^2-1}}$$

Find y' :

$y = \sec^{-1}x \longrightarrow y' = \dfrac{1}{|x|\sqrt{x^2-1}}$

$y = \sec^{-1}x^3 \longrightarrow y' = \dfrac{3x^2}{|x^3|\sqrt{(x^3)^2-1}}$

$$\frac{d}{dx}\left(\csc^{-1}(f(x))\right) = \frac{-f'(x)}{|f(x)|\sqrt{f(x)^2-1}}$$

Find y' :

$y = \csc^{-1}(\ln x) \rightarrow y' = -\dfrac{\frac{1}{x}}{|\ln x|\sqrt{(\ln x)^2-1}}$

* Higher Derivatives المشتقات العليا :

$f, f', f'', f''', f^{(4)}, f^{(5)}, \ldots\ldots, f^{(n)}$

$y, \dfrac{d}{dx}, \dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}, \ldots\ldots, \dfrac{d^n}{dx^n}$

$\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \dfrac{d^3y}{dx^3}, \ldots\ldots, \dfrac{d^ny}{dx^n}$

Ex : $f(x) = x^5,\ f^{(4)}, f^{(5)}, f^{(6)}$

$f'(x) = 5x^4$

$f''(x) = 20x^3$

$f'''(x) = 60x^2$

$f^{(4)}(x) = 120x$

$f^{(5)}(x) = 120$

$f^{(6)}(x) = 0$

$f^{(7)}(x) = 0$

Higher derivatives

Chapter 3

$f(x) = a_0 + a_1 x + \text{------} + a_n x^n$

$f^{(n)}(x) = n!$, $f^{m}(x) = 0$, $m > n$

Ex: $f(x) = x^7$, Find $f^{(7)}$, $f^{(8)}$

$f^{(7)}(x) = 7!$; $f^{(8)}(x) = 0$

Ex: $y = 3^x$, Find $\left.\frac{d^{10}y}{dx^{10}}\right|_{x=0} = ?$

$y' = 3^x \ln 3$

$y'' = 3^x \ln 3 \ln 3 = 3^x (\ln 3)^2$

$y''' = 3^x (\ln 3)^3$

$\frac{d^{10}y}{dx^{10}} = 3^x (\ln 3)^{10}$

$\left.\frac{d^{10}y}{dx^{10}}\right|_{x=0} = (\ln 3)^{10}$

Ex: $f(4) = 3$, $f'(4) = 2$, $f''(4) = 5$, Find $\left(\frac{f'}{f}\right)'(4) = ?$

$$\left(\frac{f'}{f}\right)'(x) = \frac{f(x)f''(x) - f'(x)f'(x)}{(f(x))^2}$$

$$= \frac{f(4)f''(4) - (f'(4))^2}{(f(4))^2}$$

$$= \frac{15 - 4}{9} = \frac{11}{9}$$

Higher derivatives

Chapter 3

Ex: $y = \cos^3 x + 5\tan x$, find y'' ?

$y' = -3\cos^2 x \sin x + 5\sec^2 x$

$y'' = +6\cos x \sin x + -3\cos^3 x + 10\sec^2 x \tan x$

* Implict differentiation الاشتقاق الضمني :

1) $x^3 + y^3 = 1$

$3x^2 + 3y^2 y' = 0$

$3y^2 y' = -3x^2$

$y' = -\dfrac{3x^2}{3y^2} \longrightarrow y' = -\dfrac{x^2}{y^2}$

2) $e^y \sin x = x + x^3 y^2$

$y' e^y \sin x + e^y \cos x = 1 + 3x^2 y^2 + 2yy' x^3$

$y' e^y \sin x - 2yy' x^3 = 1 + 3x^2 y^2 - e^y \cos x$

$y' (e^y \sin x - 2y x^3) = 1 + 3x^2 y^2 - e^y \cos x$

$y' = \dfrac{1 + 3x^2 y^2 - e^y \cos x}{e^y \sin x - 2y x^3}$

Implict differentiation

Chapter 3

Ex: $f(1) = 3$

$f'(1) = 2$

and $f(y^2x) = 2y\,f(x) - 1$, find $\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)}$

$\downarrow$

$f(y^2x) = 2y\,f(x) - 1$

$f'(y^2x)(y^2 + 2yy'x) = 2y\,f'(x) + f(x)2y'$

$f'(1)(1+2y') = 2f'(1) + 2f(1)y'$

$2(1+2y') = 4+6y'$

$1+2y' = 2+3y'$

$\left.\frac{dy}{dx}\right|_{(x,y)=(1,1)} \; y' = -1$

1) $y = x^2 \longrightarrow y' = 2x$

2) $y = 2^x \longrightarrow y' = 2^x \ln 2$

3) $y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$\frac{y'}{y} = x * \frac{1}{x} + \ln x$

$\frac{y'}{y} = 1 + \ln x$

$y' = x^x(1+\ln x)$

Implict differentiation Chapter 3

Ex:

1) $y = (x^2+1)^{\sin x}$

$\ln y = \sin x \ln(x^2+1)$

$\frac{y'}{y} = \sin x \left(\frac{2x}{x^2+1}\right) + (\ln(x^2+1)) \cos x$

$y' = y\left(\frac{2x \sin x}{x^2+1} + \cos x \ln(x^2+1)\right)$

$y' = (x^2+1)^{\sin x}\left(\frac{2x\sin x}{x^2+1} + \ln(x^2+1)^{\cos x}\right)$

2) $x^y = y^x$, Find y'

$\ln x^y = \ln y^x$

$y \ln x = x \ln y$

$y\frac{1}{x} + \ln x y' = x\frac{y'}{y} + \ln y$

$\ln x y' - \left(\frac{x}{y}\right) y' = \ln y - \frac{y}{x}$

$y'\left(\ln x - \frac{x}{y}\right) = \ln y - \frac{y}{x}$

$y' = \frac{\ln y - \frac{y}{x}}{\ln x - \frac{x}{y}}$

Implict differentiation

3) $y = (x^3-2x)^{\ln x}$, find y'

$$\ln y = \ln x \ln(x^3-2x)$$

$$\frac{y'}{y} = (\ln x)\,\frac{(3x^2-2)}{x^3-2x} + \ln(x^3-2x) * \frac{1}{x}$$

$$y' = y\left((\ln x)\frac{3x^2-2}{x^3-2x} + \frac{\ln(x^3-2x)}{x}\right)$$

$$y' = (x^3-2x)^{\ln x}\left((\ln x)\frac{3x^2-2}{x^3-2x} + \frac{\ln(x^3-2x)}{x}\right)$$

* The relationship between differentiability and continuity :

العلاقة بين الاتصال وقابلية الاشتقاق :

Theorem: If a function f(x) is differentiable at C then f continuous at C

diff $\longrightarrow$ cts

not cts $\longrightarrow$ not diff

cts $\longrightarrow$??

Ex : Is f(x) differentiable at $x=1$, $f(x) = \begin{cases} x^2+1, & x \leq 1 \\ 2x, & x > 1 \end{cases}$

$$\left.\begin{array}{l} \lim\limits_{x\to 1^+} 2x = 2 \\ \lim\limits_{x\to 1^-} x^2+1 = 2 \\ f(1) = 2 \end{array}\right\} \longrightarrow f \text{ cts at } x=1$$

$$f(x) = \begin{cases} 2x, & x<1 \\ x, & x>1 \end{cases}$$

$$f'_+(1) \overset{?}{=} f'_-(1)$$

$$2 = 2 \checkmark$$

f(x) diff at $x=1$

The relationship between differetiability   Chapter 3

* $f(x) = \begin{cases} x^2+2, & x \leq 1 \\ 2x, & x > 1 \end{cases}$ Is $f(x)$ diff at $x=1$?

$\lim_{x \to 1^+} 2x = 2$
$\neq$ $f(x)$ discts at $x=1$
$\lim_{x \to 1^-} = 3$ $f(x)$ not diff at $x=1$

* Is $f(x) = |x-5|$ diff at $x=5$?

$f(x) = \begin{cases} x-5, & x \geq 5 \\ 5-x, & x < 5 \end{cases}$

$|x-5| = x-5=0$
$x=5$

$$\begin{array}{ccc} - - - & & + + + \\ \hline 5-x & 5 & x-5 \end{array}$$

$$\left.\begin{array}{l} \lim_{x \to 5^+} x-5 = 0 \\ \lim_{x \to 5^-} 5-x = 0 \\ f(5) = 0 \end{array}\right\} \rightarrow f(x) \text{ cts at } x=5$$

$f'(x) = \begin{cases} 1, & x > 5 \rightarrow f'^{+}(5) = 1 \\ -1, & x < 5 \rightarrow f'^{-}(5) = -1 \end{cases}$ $\neq$ not diff at $x=5$

$f(x) = |x-5|$ not diff at $x=5$
diff $\mathbb{R} - \{5\}$

Equation of tangent and normal line

Chapter 3

$$*f(x)=\begin{cases} x^2-1 , & x\le 1 \\ kx-k , & x>1 \end{cases}$$

Find the Value of k

1) If f(x) continuous ? $k\in\mathbb{R}$

2) If f(x) differentiable ? $k=2$

3) If f(x) continuous but not diff ?

$\mathbb{R}-\{2\}$

1) $\lim\limits_{x\to1^+} kx-k = \lim\limits_{x\to1^-} x^2-1$

$0=0$?? $k\in\mathbb{R}=(-\infty,\infty)$

2) $$f'(x)=\begin{cases} 2x , & x\le 1 \\ k , & x>1 \end{cases}$$

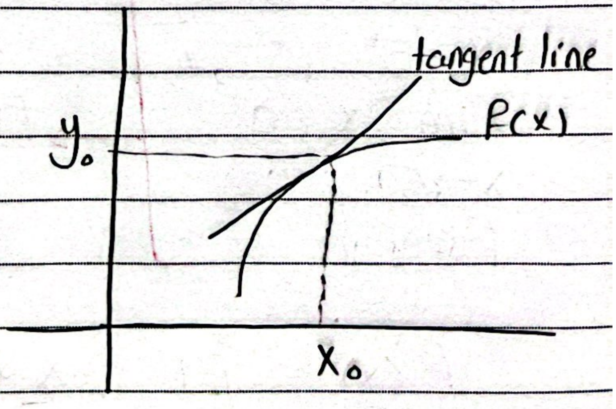
$\underset{+}{f'(1)} = \underset{-}{f'(1)}$

$\boxed{k=2}$

*Equation of tangent line : معادلة المماس

$$\boxed{y-y_0=m(x-x_0)}$$

$m = \text{slope} = f'(x_0)$



Equation of normal line
(Perpendicular to the tangent line)

معادلة العمودي على المماس

$$y-y_0=\frac{-1}{m}(x-x_0)$$

Equation of tangent and normal line

1) Find the slope of $f(x) = 2 + 3\tan^{-1} 2x$ at $x=1$

$$f'(x) = 3\left(\frac{2}{1+4x^2}\right) = \frac{6}{1+4x^2}$$

$$f'(1) = \frac{6}{5}$$ (slope of $f(x)$ at $x = 1$)

2) Find the slope of normal line of $f(x) = x^3$ at $x = -1$

$f'(x) = 3x^2$

$f'(-1) = 3$ (slope of tangent line)

Slope of normalline $= \frac{-1}{3}$

3) Find the equation of tangent line and normal line

$$f(x) = \frac{2}{1+e^{-x}} \quad \text{at } x=0$$

* (x_0, y_0) ?
$(0, 1)$

* $m = \text{slope} = f'(x) = \frac{-2(-e^{-x})}{(1+e^{-x})^2} = \frac{2e^{-x}}{(1+e^{-x})^2}$

$$f'(0) = \frac{2e^0}{(1+e^0)^2} = \frac{1}{2}$$

* equation of tangent line $\rightarrow y - 1 = \frac{1}{2}(x-0) \rightarrow y = \frac{1}{2}x + 1$
* equation of normal line $\rightarrow y - 1 = -2(x-0) \rightarrow y = -2x + 1$

4) For what value of X, the graph of $f(x) = 2x^3 - 6x$ have a horizontal tangent? $\longrightarrow$ slope $= m = 0$

$$f'(x) = 0$$

$$f'(x) = 6x^2 - 6 = 0$$

$$x^2 = 1$$

$$x = \pm 1 \in \mathbb{R}$$

at $x = \pm 1$ horizontal tangent

*Hyperbolic Functions :

1) Hyperbolic sine : $\sinh x = \frac{e^x - e^{-x}}{2}$

2) Hyperbolic cosine : $\cosh x = \frac{e^x + e^{-x}}{2}$

3) Hyperbolic tangent = $\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

4) Hyperbolic cotangent = $\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

5) Hyperbolic secant = $\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$

6) Hyperbolic cosecant = $\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$

Hyperbolic Functions

Chapter 3

Ex:

1) $\sinh 0 = \frac{e^0 - e^0}{2} = \frac{1-1}{2} = 0$

2) $\cosh(\ln 3) = \frac{e^{\ln 3} + e^{-\ln 3}}{2} = \frac{3 + \frac{1}{3}}{2} = \frac{10}{6}$

3) $\operatorname{Sech}(\ln 3) = \frac{1}{\cosh(\ln 3} = \frac{6}{10}$

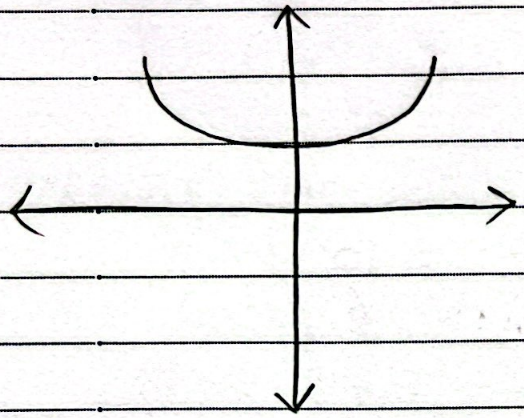
4) $\sinh(\ln 5) = \frac{e^{\ln 5} - e^{-\ln 5}}{2} = \frac{5 - \frac{1}{5}}{2} = \frac{24}{10}$

5) $\cosh(\ln 5) = \frac{e^{\ln 5} + e^{-\ln 5}}{2} = \frac{26}{10}$

6) $\tanh(\ln 5) = \frac{\sinh(\ln 5)}{\cosh(\ln 5)} = \frac{24}{26}$

7) $\operatorname{Csch}(\ln 5) = \frac{1}{\sinh(\ln 5)} = \frac{10}{24}$

*Graphs of hyperbolic functions :



$y = \cosh x$

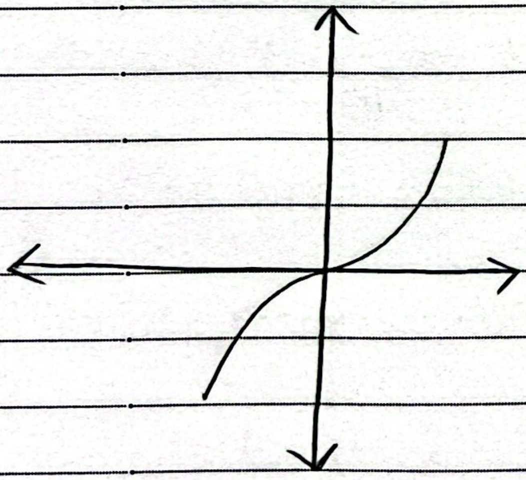
Domain: $\mathbb{R}$, Range : $[1, \infty)$

even $(\cosh(-x) = \cosh(x))$

$\lim\limits_{x \to +\infty} \cosh x = +\infty$

$\lim\limits_{x \to -\infty} \cosh x = +\infty$

No Horizontal and vertical asy



$y = \sinh x$

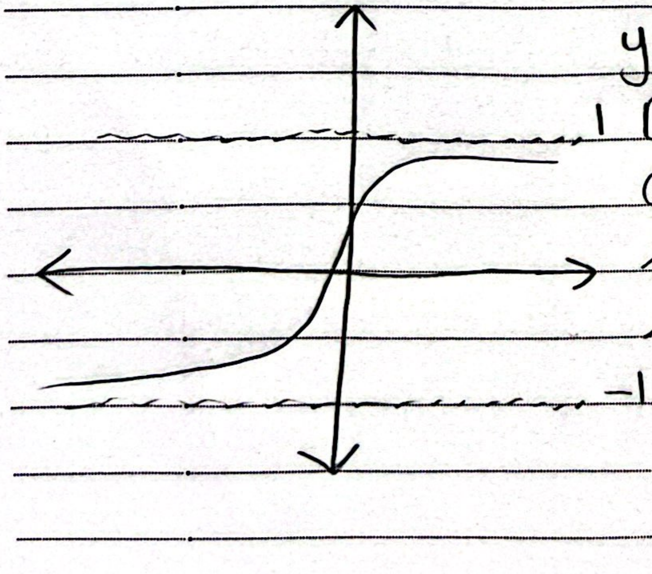
Domain : $\mathbb{R}$, Range: $\mathbb{R}$

Odd $(\sinh(-x) = -\sinh(x))$

$\lim\limits_{x \to +\infty} \sinh x = +\infty$

$\lim\limits_{x \to -\infty} \sinh x = -\infty$

No Horizontal and vertical asy



$y = \tanh x$

Domain: $\mathbb{R}$, Range: $(-1, 1)$

Odd $(\tanh(-x) = -\tanh(x))$

$\lim\limits_{x \to +\infty} \tanh x = 1$

$\lim\limits_{x \to -\infty} \tanh x = -1$

Horizontal asy ✓

No Vertical asy

*Hyperbolic identities المتطابقات :

1) $\cosh^2 x - \sinh^2 x = 1$
2) $1 - \tanh^2 x = \operatorname{sech}^2 x$
3) $\sinh x + \cosh x = e^{x}$
4) $\cosh x - \sinh x = e^{-x}$
5) $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$
6) $\cosh(x+y) = \cosh x \cosh y + \sinh x$

Ex: Find the value of x :

1) $\cosh x + \sinh x = 4$

بناءً على ← $e^{x} = 4$

المتطابقة 3 $x = \ln 4 = 2\ln 2$

2) $\cosh x - \sinh x = \frac{1}{3}$

بناءً على المتطابقة ← $e^{-x} = \frac{1}{3}$

4

$-x = \ln\frac{1}{3}$

$x = \ln 3$

3) $\sinh x = \frac{e^{x}}{2} - 3$

$$\frac{e^{x} - e^{-x}}{2} = \frac{e^{x}}{2} - 3 \longrightarrow \frac{e^{x} - e^{-x}}{2} - \frac{e^{x}}{2} = -3$$

$$\frac{-e^{-x}}{2} = -3 \rightarrow e^{-x} = 6 \rightarrow -x = \ln 6 \rightarrow x = \ln\frac{1}{6}$$

Ex: If $\sinh x = \frac{1}{2}$, find $\cosh x$?

$$\cosh^2 x - \sinh^2 x = 1$$

$$\cosh^2 x - \frac{1}{4} = 1$$

$$\cosh^2 x = \frac{5}{4} \longrightarrow \cosh x = \pm\sqrt{\frac{5}{4}} = +\frac{\sqrt{5}}{2}$$

معلومة لأنه دائماً قيمة +

* Derivative of hyperbolic functions :

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^{x}-e^{-x}}{2}\right) = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

1) $\frac{d}{dx}(\sinh x) = \cosh x$

2) $\frac{d}{dx}(\cosh x) = \sinh x$

3) $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$

4) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$

5) $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$

6) $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$

* Find y' :

1) $y = x \sinh x$

$y' = x \cosh x + \sinh x$

2) $y = \cosh(\ln x)$

$y' = \frac{1}{x} \sinh(\ln x)$

$= \frac{1}{x} * \frac{e^{\ln x} - e^{-\ln x}}{2} = \frac{1}{x} * \frac{x - \frac{1}{x}}{2} = \frac{\frac{x^2-1}{x}}{2x} = \frac{x^2-1}{2x^2}$

3) $y = \ln(\tanh x)$

$y' = \frac{\operatorname{sech}^2 x}{\tanh x} = \frac{1}{\cosh^{\cancel{2}} x} * \frac{\cancel{\cosh x}}{\sinh x} = \frac{1}{\cosh x \sinh x} = \frac{1}{\frac{1}{2}\sinh 2x}$

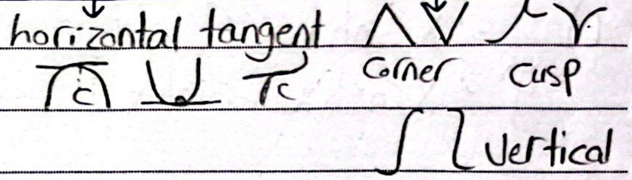
$= 2\operatorname{csch} 2x$

4) $y = 3^{\coth x}$

$y' = (3^{\coth x})(-\operatorname{csch}^2 x)\ln 3$

Def: A critical number of a function f is a number c in the domain of f, such that either $f'(c)=0$ OR $f'(c)$ d.n.e

$c \in D_f$



Ex: Find the critical numbers :

1) $f(x) = 2x^3 - 3x^2 - 36x$

$D_f = \mathbb{R}$ (لأنه كثير حدود)

$f'(x) = 6x^2 - 6x - 36$

$f'(x) = 0$

$6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

$(x+2)(x-3) = 0$

$x = -2 \; , \; x = 3 \in D_f \longrightarrow$ critical number إذاً هم ال

$\{-2, 3\}$

هون هو كثير حدود عشان هيك مستحيل يكون غير موجود

Critical Points

$(-2, f(-2)) \; , \; (3, f(3))$

١) بوجد المجال

٢) بشتق

٣) بساوي المشتقة بالصفر وبحلها

٤) بنشوف بعدها إذا الي طلع من ضمن المجال أو لا

٥) بشوف إذا الاقتران غير موجود d.n.e

2) $f(x) = \dfrac{x^2}{x^2 - 1}$

$D_f = R - \{1, -1\}$

$f'(x) = \dfrac{(2x)(x^2-1) - (x^2)(2x)}{(x^2-1)^2} = \dfrac{2x^3 - 2x - 2x^3}{(x^2-1)^2} = \dfrac{-2x}{(x^2-1)^2}$

$f'(x) = 0$

$-2x = 0$

$x = 0 \in D_f$

$f'(x)$ d.n.e

$(x^2-1)^2 = 0$

$x = \pm 1 \notin D_f$

Critical number $\rightarrow x = 0$

Critical Points

$(0, f(0))$

Critical numbers القيم الحرجة — Chapter 4.

3) $f(x) = 4x^{\frac{3}{5}} - x^{\frac{8}{5}}$

$D_f = \mathbb{R}$

$f'(x) = \frac{12}{5} x^{\frac{-2}{5}} - \frac{8}{5} x^{\frac{3}{5}}$

$f'(x) = \frac{12}{5x^{\frac{2}{5}}} - \frac{8}{5} x^{\frac{3}{5}} \longrightarrow \frac{12 - 8x}{5x^{\frac{2}{5}}}$

$f'(x) = 0$	$f'(x)$ d.n.e
$12 - 8x = 0$	$5x^{\frac{2}{5}} = 0$
$x = \frac{12}{8} \in D_f$	$x = 0 \in D_f$

Critical numbers : $x = \frac{12}{8}$, 0

Critical Points : $(0, f(0))$, $(\frac{12}{8}, f(\frac{12}{8}))$

4) $f(x) = |3x - 6|$

$3x - 6 = 0$

$x = 2$

$$\begin{array}{ccc} - - - & & + + + \\ \hline 6-3x & 2 & 3x-6 \end{array}$$

$D_f = \mathbb{R}$

$f(x) = \begin{cases} 3x - 6, & x \geq 2 \\ 6 - 3x, & x < 2 \end{cases}$

$f'(x) = \begin{cases} 3, & x > 2 \\ -3, & x < 2 \end{cases}$

$f'(x) = 0$	$f'(x)$ d.n.e
$3 \neq 0$ ✗	$f'_+(2) = 3$
$-3 \neq 0$ ✗	$f'_-(2) = -3$

$\neq$ إذاً غير موجودة عند $x = 2$

Critical number: $x = 2$

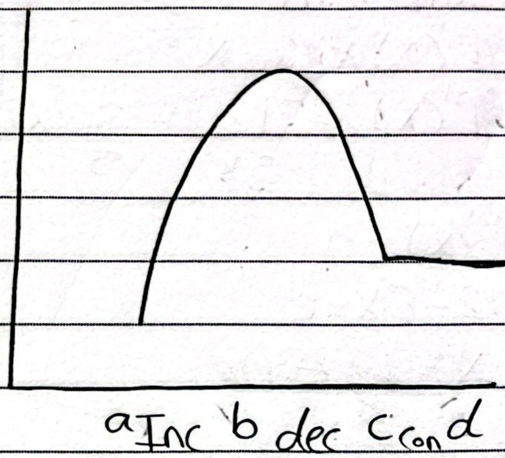
Critical Point: $(2, f(2))$

Increasind and decreasing function

Chapter 4

* Inc & dec Functions التزايد والتناقص

The shape of the function on the Interval		Test
Increasing ↑	تزايد	$f'(x) > 0$
decreasing ↓	تناقص	$f'(x) < 0$
Constant —	ثبات	$f'(x) = 0$



Ex: Finel the Intervals of Increase or decrease

1) $f(x) = xe^{-x}$

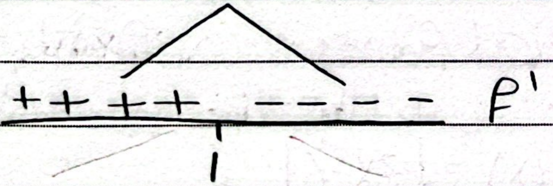
$D_f = \mathbb{R}$

$f'(x) = e^{-x} + -xe^{-x}$

$f'(x) = e^{-x}(1-x)$

$e^{-x}(1-x) = 0$

$\downarrow x \neq 0$ $\quad$ $\downarrow x = 1$ critical



Inc : $(-\infty, 1)$

Dec : $(1, \infty)$

2) $f(x) = x + 2\sin x \quad , x \in [0, 2\pi]$

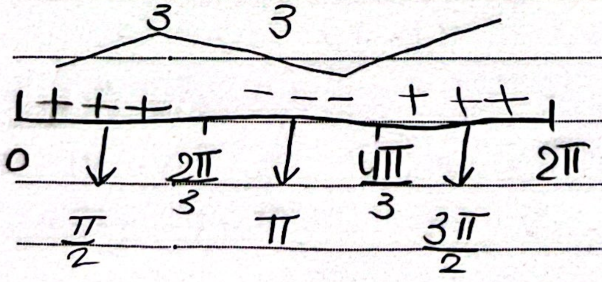
$f'(x) = 1 + 2\cos x$

$\cos x = \frac{-1}{2}$

Inc : $(0, \frac{2\pi}{3}), (\frac{4\pi}{3}, 2\pi)$

$x = \frac{2\pi}{3}, \frac{4\pi}{3} \leftarrow [0, 2\pi]$

Dec : $(\frac{2\pi}{3}, \frac{4\pi}{3})$



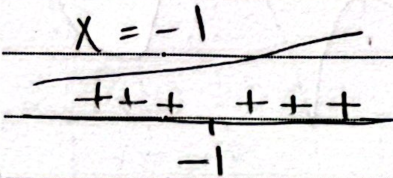
3) $f(x) = (x+1)^5$

$D_f = \mathbb{R}$ Inc : $(-\infty, +\infty)$

$f'(x) = 5(x+1)^4$

$5(x+1)^4 = 0$

$x = -1$



4) $f(x) = x^{\frac{1}{3}}(x+4)$

$D_f = \mathbb{R}$

$f'(x) = x^{\frac{1}{3}} + \frac{1}{3} x^{\frac{-2}{3}}(x+4)$

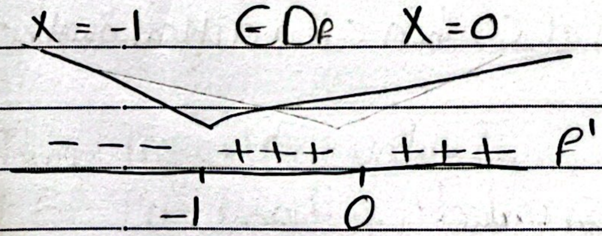
$f'(x) = x^{\frac{1}{3}} + \frac{x \overset{3}{+} 4}{3x^{\frac{2}{3}}}$

$f'(x) = \frac{4x+4}{3x^{\frac{2}{3}}}$

$f'(x) = 0$ $f'(x)$ d.n.e

$4x+4 = 0$ $3x^{\frac{2}{3}} = 0$

$x = -1 \in D_f$ $x = 0$



Inc : $(-1, \infty)$

Dec : $(-\infty, -1)$

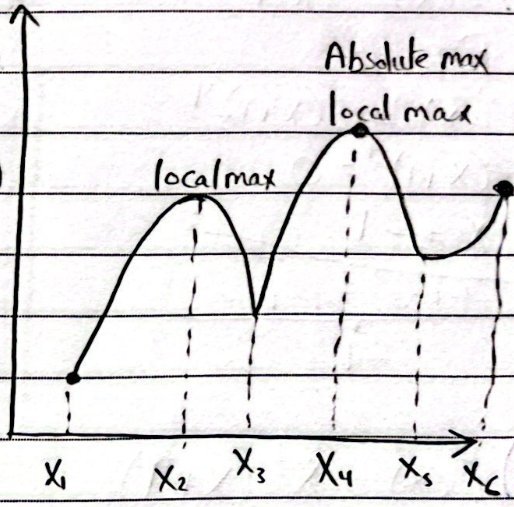
Extreme Values القيم القصوى

Chapter 4

* Extreme Values

max
- local محلي relative
- absolute مطلق (global)

min
- local محلي (relative)
- absolute مطلق (global)



* local max : $f(c) \geq f(x)$ لكل x قريبة من c

$$\underset{c}{++ \quad --} \quad f'$$

* Absolute max : $f(c) > f(x)$ لكل x تنتمي للـ D

* local min : $f(c) \leq f(x)$ لكل x قريبة من c

$$\underset{c}{-- \quad ++} \quad f'$$

* Absolute min : $f(c) < f(x)$ لكل x تنتمي للـ D

* Then : If f has a local max or min at c, then c is a critical number of $f(x)$

Ex : a) find the local max and min values
b) find the absolute max and min values

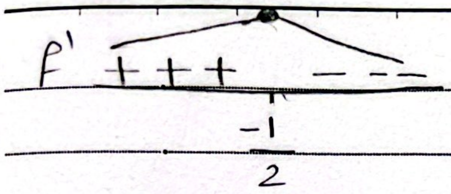
1) $f(x) = -x^2 - x - 2$

$D_f = \mathbb{R}$

$f'(x) = -2x - 1$

$-2x - 1 = 0 \longrightarrow x = -\frac{1}{2}$

Extreme Values القيم القصوى — Chapter 4



localmax at $x = -\frac{1}{2}$

absolute max at $x = -\frac{1}{2}$

local & Absolute max value $f(-\frac{1}{2})$

2) $f(x) = 2x^3 - 3x^2 - 12x$

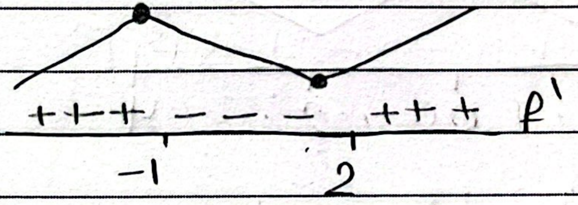
$D_f = \mathbb{R}$

$f'(x) = 6x^2 - 6x - 12$

$x^2 - x - 2 = 0$

$(x+1)(x-2) = 0$

$x = -1, 2$



at $x = -1$ local max value $f(-1)$

at $x = 2$ local min value $f(2)$

No absolute value

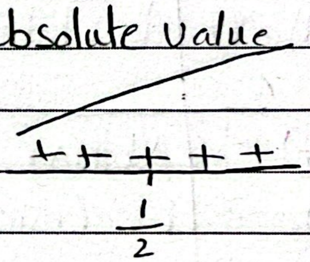
3) $f(x) = (2x-1)^3$

$D_f = \mathbb{R}$

$f'(x) = 6(2x-1)^2$

$6(2x-1)^2 = 0$

$x = \frac{1}{2}$

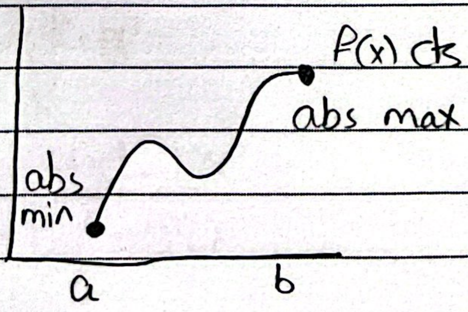


Increasing : $(-\infty, +\infty)$

No extreme values

Thm : The extreme value Thm :

If a function f is cts on a closed interval $[a,b]$ Then f has both an absolute max & absolute min on $[a,b]$



Extreme Values القيم القصوى Chapter 4

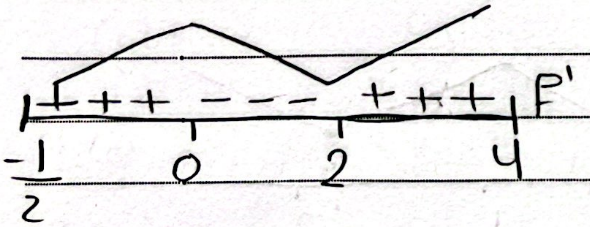
Ex: $f(x) = x^3 - 3x^2 + 1$ $[-\frac{1}{2}, 4]$

$f'(x) = 3x^2 - 6x$

$3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0 , x = 2$

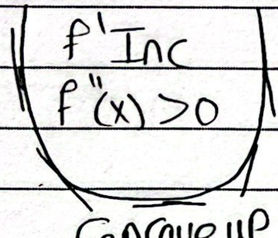


	X	f(x)	
max	0	$f(0) = 1$	local max
	4	$f(4) = 17$	Absolute max Value is $f(4) = 17$
min	$-\frac{1}{2}$	$f(\frac{-1}{2}) = \frac{1}{8}$	Nothing
	2	$f(2) = -3$	local min , Absolute min Value is $f(2) = -3$

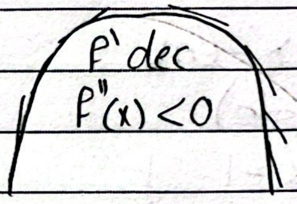
Concavity التقعر Chapter 4

* Concave up : The graph of f lies above all of its tangents on the interval

* Concave down : The graph of f lies below all of its tangents on the interval

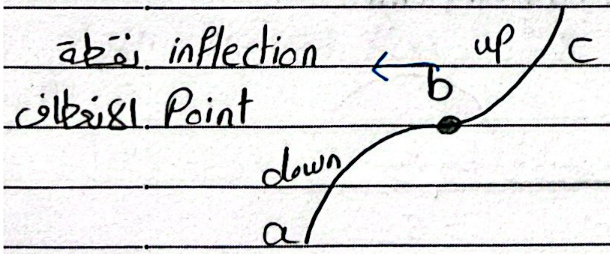


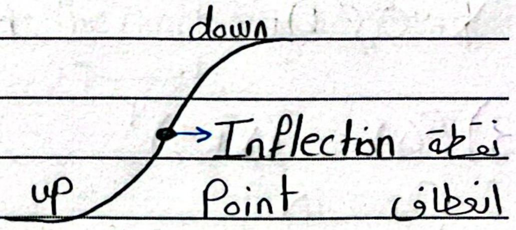
Concave up



Concave down

* Def: A point P on the curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes the direction of its concavity at P.





Ex: a) Find the Intervals of concavity
b) Find the inflection Points

1) $f(x) = x^4 - 4x^3$

$D_f = \mathbb{R}$

$f'(x) = 4x^3 - 12x^2$

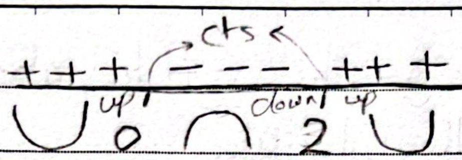
$f''(x) = 12x^2 - 24x$

Concavity التقعر   Chapter 4

$12x^2 - 24x = 0$

$12x(x-2) = 0$

$x = 0\ ,\ x = 2$



Concave up : $(-\infty, 0)\ ,\ (2, \infty)$

Concave down : $(0, 2)$

The points $(0, f(0)), (2, f(2))$ are inflection points

2) $f(x) = x^2 - x - \ln x$

$D_f = (0, \infty)$

$f'(x) = 2x - 1 - \frac{1}{x}$

$f''(x) = 2 + \frac{1}{x^2} = \frac{2x^2+1}{x^2}$

$2x^2 + 1 = 0$

$2x^2 = -1$ X

$x^2 = 0$

$x = 0 \notin D_f$

(+ + + + + +

0

Concave up $(0, \infty)$

No inflection point

3) $f(x) = -(x+5)^4$

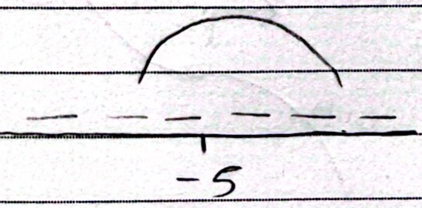
$D_f = \mathbb{R}$

$f'(x) = -4(x+5)^3$

$f''(x) = -12(x+5)^2$

$-12(x+5)^2 = 0$

$x = -5$



Concave down

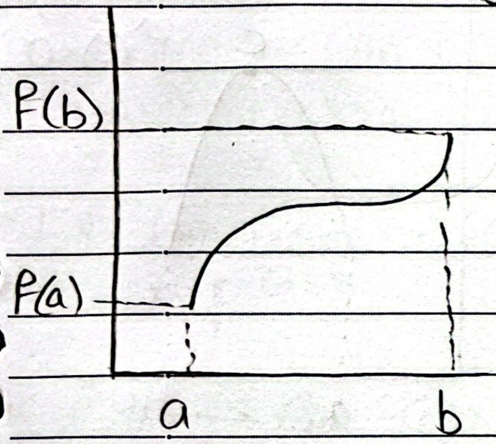
No inflection points

The mean value theorem

Chapter

* The mean value theorem :

If f is continuous on [a,b] &
f is defferentiable on (a,b), then
there is at least one Point c in (a,b)
Such that $f'(c) = \frac{f(b)-f(a)}{b-a}$



Ex: Find the number c that satisfies the conclusion of the mean Value theorem $f(x) = x^3 - x$, $[0,2] \rightarrow$ cts ✓

$f'(x) = 3x^2 - 1 \rightarrow$ diff ✓

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{6-0}{2-0} = 3$$

$$3c^2 - 1 = 3$$

$$3c^2 = 4$$

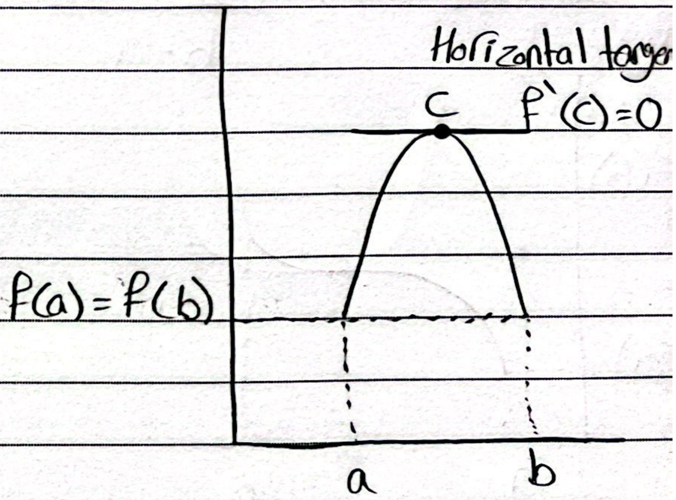
$$c^2 = \frac{4}{3}$$

$$c = \pm \frac{2}{\sqrt{3}}$$

مهمل لاتنتمي

Rolle's Theorem

*If f is continuous on [a,b] and
f is differentiable on (a,b) and
f(a) = f(b), then there is at least one Point c in (a,b) such tha
f'(c) = 0 (Horizontal tangent)



*Find the number c, that satisfies the conclusion of the Rolle's theorem $f(x) = 5-12x+3x^2$. [1,3]

$f(x) = 5-12x+3x^2$ [1,3]

* f(x) cts [1,3] ✓
* f(x) diff (1,3) ✓
* f(a) = f(b) ✓ $\overset{a}{[1},\overset{b}{3]}$

at x=2 Horizontal tangent, f'(2) = 0

$f(1) = -4$
$f(3) = -4$

$f'(x) = -12+6x$
$-12+6x = 0$
$6x = 12 \longrightarrow x = 2 \in (1,3)$

*l'Hopital Rule:

If f, g differentiable on an open Interval contains a and $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Ex: $\lim_{x\to \frac{\pi}{2}} \frac{1-\sin x}{\cos x} = \frac{1-\sin \pi/2}{\cos \pi/2} = \frac{0}{0}$

$l'H \longrightarrow \lim_{x\to \frac{\pi}{2}} \frac{-\cos x}{-\sin x} = \frac{0}{1} = 0$

Ex: $\lim_{x\to 0} \frac{e^x-1}{x^3} = \frac{0}{0}$

$l'H \to \lim_{x\to 0} \frac{e^x}{3x^2} = \frac{1}{0} = +\infty$

Ex: $\lim_{x\to +\infty} \frac{x^2}{e^x} = \frac{\infty}{\infty}$

$l'H \to \lim_{x\to +\infty} \frac{2x}{e^x} = \frac{\infty}{\infty}$

$l'H \to \lim_{x\to \infty} \frac{2}{e^x} = \frac{2}{\infty} = 0$

l'Hopital Rule

Ex: $\lim\limits_{x\to\infty} \dfrac{x^{-\frac{4}{3}}}{\sin(\frac{1}{x})} = \dfrac{1/x^{\frac{4}{3}}}{\sin 0} = \dfrac{0}{0}$

l'H $\longrightarrow \lim\limits_{x\to\infty} \dfrac{-\frac{4}{3}x^{-\frac{7}{3}}}{\frac{-1}{x^2}\cos\frac{1}{x}} = \lim\limits_{x\to\infty} \dfrac{4x^{-\frac{1}{3}}}{3\cos\frac{1}{x}} = \dfrac{4}{\infty} = 0$

Ex: $\lim\limits_{x\to 0^+} \dfrac{\ln x}{\csc x} = \dfrac{-\infty}{+\infty}$

l'H $\longrightarrow \lim\limits_{x\to 0^+} \dfrac{1/x}{-\csc x \cot x} = \lim\limits_{x\to 0^+} \dfrac{-1}{x\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x}}$

$= \lim\limits_{x\to 0^+} \dfrac{-\sin^2 x}{x\cos x} = \dfrac{0}{0}$

l'H $\longrightarrow \lim\limits_{x\to 0^+} \dfrac{-2\sin x\cos x}{-x\sin x+\cos x} = \dfrac{-2(0)(1)}{0+1} = \dfrac{0}{1} = 0$

*Indeterminate Forms: $0\cdot\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$

1) $0\cdot\infty$: * $\lim\limits_{x\to 0^+} x\ln x = 0, -\infty$

بنخلّي الأسهل والأبسط في المقام $f/\frac{1}{g}$

$\lim\limits_{x\to 0^+} \dfrac{\ln x}{1/x} = \dfrac{-\infty}{+\infty}$

l'H $\longrightarrow \dfrac{1/x}{-1/x^2} = \lim\limits_{x\to 0} -x = 0$

l'Hopital Rules

$$*\lim_{x\to\frac{\pi}{4}} (1-\tan x)\sec 2x = (1-1)\sec\frac{\pi}{2} = 0\cdot\infty$$

$$\lim_{x\to\frac{\pi}{4}} \frac{1-\tan x}{\cos 2x} = \frac{0}{0}$$

$$l'H \rightarrow \frac{-\sec^2 x}{-2\sin 2x} = \frac{(\sqrt{2})^2}{2} = \frac{2}{2} = 1$$

$$*\lim_{x\to\infty} x\sin\frac{\pi}{x} = \infty\cdot 0$$

$$\lim_{x\to\infty} \frac{\sin\pi/x}{1/x} = \frac{0}{0}$$

$$l'H \rightarrow \frac{\cos(\pi/x)(-\pi/x^2)}{-1/x^2}$$

$$\lim_{x\to\infty} \pi\cos\frac{\pi}{x} = \pi$$

2) $\infty-\infty$: $*\lim_{x\to 0} \frac{1}{x} - \frac{1}{\sin x} = \infty - \infty$ توحيد مقامات

$$\lim_{x\to 0} = \frac{\sin x - x}{x\sin x} = \frac{0}{0} \longrightarrow l'H \rightarrow \lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x} = \frac{0}{0}$$

$$l'H \rightarrow \lim_{x\to 0} \frac{-\sin x}{-x\sin x + \sin x + \cos x} = \frac{0}{2} = 0$$

$$* \lim_{x\to\infty} x - \ln(x^2+1) = \infty - \infty$$

$\uparrow$ اكتبها

$$\lim_{x\to\infty} \ln e^x - \ln(x^2+1) = \lim_{x\to\infty} \ln \frac{e^x}{x^2+1}$$

$$= \ln\left(\lim_{x\to\infty} \frac{e^x}{x^2+1}\right) = \frac{\infty}{\infty}$$

$$l'H \longrightarrow \ln\left(\lim_{x\to\infty} \frac{e^x}{2x}\right) \quad \frac{\infty}{\infty}$$

$$l'H \longrightarrow \ln\left(\lim_{x\to\infty} \frac{e^x}{2}\right) = \ln(+\infty) = +\infty$$

3) $0^0, \infty^0, 1^\infty$;

$$* \lim_{x\to\infty} x^{\frac{1}{x}} = \infty^0$$

$$y = x^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x} \ln x$$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \quad \frac{\infty}{\infty}$$

$$\ln y = \lim_{x\to\infty} \frac{1/x}{1}$$

$$\ln y = 0 \longrightarrow y = e^0 \Rightarrow y = 1$$

l'Hopital Rules

Chapter

$$* \lim_{x\to\infty} (e^x+x)^{\frac{1}{x}} = \infty^0$$

$$y = (e^x+x)^{\frac{1}{x}}$$

$$\ln y = \frac{1}{x}\ln(e^x+x)$$

$$\lim \ln y = \lim_{x\to\infty} \frac{\ln(e^x+x)}{x} \quad \frac{\infty}{\infty}$$

$$l'H \longrightarrow \ln y = \lim \frac{\frac{e^x+1}{e^x+x}}{1}$$

$$\ln y = \lim \frac{e^x+1}{e^x \mp x} \quad \frac{\infty}{\infty}$$

$$\ln y = \lim_{x\to\infty} \frac{e^x}{e^x+1} = \frac{\infty}{\infty}$$

$$\ln y = \lim \frac{e^x}{e^x} = 1$$

$$\ln y = 1 \longrightarrow y = e^1 \Rightarrow y = e$$

$$* \lim_{x\to 0} (1+ax)^{\frac{b}{x}} = e^{ab} \quad , \lim (1+\frac{a}{x})^{bx} = e^{ab}$$

$$1) \lim_{x\to+\infty} (1-\frac{3}{x})^{2x} = 1^\infty$$

يا إمّا بنحل على طريقة لوبيتال
يا إمّا على القاعدة السريعة هاي

$$e^{(-3)(2)} = e^{-6}$$

2) $\lim_{X\to 0}(1-8X)^{\frac{2}{x}} = e^{-16}$

3) $\lim_{X\to\infty}\left(\frac{X}{X+2}\right)^{-3X}$

$= \lim_{X\to\infty}\left(\frac{X+2}{X}\right)^{3X} = \lim_{X\to\infty}\left(1+\frac{2}{X}\right)^{3X} = e^{6}$

4) $\lim_{X\to\infty}\left(\frac{X-3}{X-2}\right)^{X}$

$\lim_{X\to\infty}\left(\frac{x\left(1-\frac{3}{x}\right)}{x\left(1-\frac{2}{x}\right)}\right)^{x}$

$= \lim_{X\to\infty}\frac{\left(1-\frac{3}{x}\right)^{x}}{\left(1-\frac{2}{x}\right)^{x}} = \frac{e^{-3}}{e^{-2}} = e^{-3+2} = e^{-1}$

* Indefinite integral التكامل غير المحدود :

$$\int f'(x)\,dx = f(x) + C \longrightarrow \text{Constant}$$

1) $\int k\,dx = kx + C$

2) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$

3) $\int x^{-1}\,dx = \ln|x| + C$

4) $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

5) $\int k f(x)\,dx = k\int f(x)\,dx$

Ex: $\int x^2 - 5x^3 + x^{\frac{1}{2}}\,dx = \frac{x^3}{3} - \frac{5x^4}{4} + \frac{2}{3}x^{\frac{3}{2}} + C$

6) $\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$

Ex: $\int e^{7x}\,dx = \frac{e^{7x}}{7} + C$

Indefinite integral

Chapter

7) $\int b^{ax+k}\,dx = \frac{b^{ax+k}}{a\ln b} + c$

8) $\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$ لازم يكون الي داخل القوس خطي

Ex: $\int (4+x^2)^2\,dx = \int 16+8x^2+x^4\,dx = 16x + \frac{8}{3}x^3 + \frac{x^5}{5} + c$

Ex: $\int \frac{x^3-2\sqrt{x}}{x} = \int \frac{x^3}{x} - \frac{2\sqrt{x}}{x}\,dx = \int x^2 - 2x^{-\frac{1}{2}} = \frac{x^3}{3} - \frac{2x^{\frac{1}{2}}}{\frac{1}{2}} + c$

9) $\int \frac{g'(x)}{g(x)}\,dx = \ln|g(x)| + c$

Ex: $\int \frac{x}{x^2+5}\,dx = \frac{1}{2}\ln|x^2+5| + c$

Indefinite integral

Chapter

* Trigonometric functions :

1) $\int \cos(ax+b)\,dx = \frac{\sin(ax+b)}{a} + C$

2) $\int \sin(ax+b)\,dx = -\frac{\cos(ax+b)}{a} + C$

3) $\int \sec^2(ax+b)\,dx = \frac{\tan(ax+b)}{a} + C$

4) $\int \csc^2(ax+b)\,dx = -\frac{\cot(ax+b)}{a} + C$

5) $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{\sec(ax+b)}{a} + C$

6) $\int \csc(ax+b)\cot(ax+b)\,dx = -\frac{\csc(ax+b)}{a} + C$

Ex:

1) $\int \frac{\sin x}{\cos^2 x}\,dx = \int \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\,dx = \int \sec x \tan x\,dx = \sec x + C$

2) $\int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + C$

Indefinite integral

Chapter

3) $\int \tan x\, dx = -\ln|\cos x| + c$

* * *

* $\int \frac{f'(x)}{\sqrt{a^2-(f(x))^2}}\, dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + C$

1) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1}\left(\frac{x}{1}\right) + c$

2) $\int \frac{2x}{\sqrt{5-x^4}}\, dx = \sin^{-1}\left(\frac{x^2}{\sqrt{5}}\right) + c$

3) $\int \frac{e^x}{\sqrt{9-e^{2x}}}\, dx = \sin^{-1}\frac{e^x}{3} + c$

* $\int \frac{f'(x)}{a^2+(f(x))^2}\, dx = \frac{1}{a} \tan^{-1}\left(\frac{f(x)}{a}\right) + C$

1) $\int \frac{1}{1+x^2}\, dx = \tan^{-1}(x) + c$

2) $\int \frac{1}{7+x^2}\, dx = \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + c$

3) $\int \frac{e^{2x}}{3+e^{4x}}\, dx = \frac{1}{2}\left(\frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{e^{2x}}{\sqrt{3}}\right)\right) + c$

* Hyperbolic functions :

1) $\int \sinh(ax+b)\,dx = \frac{\cosh(ax+b)}{a} + C$

2) $\int \cosh(ax+b)\,dx = \frac{\sinh(ax+b)}{a} + C$

3) $\int \text{sech}^2(ax+b)\,dx = \frac{\tanh(ax+b)}{a} + C$

4) $\int \text{sech}(ax+b)\tanh(ax+b)\,dx = \frac{-\text{sech}(ax+b)}{a} + C$

5) $\int \text{csch}(ax+b)\coth(ax+b)\,dx = -\frac{\text{csch}(ax+b)}{a} + C$

6) $\int \text{csch}^2(ax+b)\,dx = -\frac{\coth(ax+b)}{a} + C$

Ex :

1) $\int \sinh(2x)\,dx = \frac{\cosh 2x}{2} + C$

2) $\int \cosh(3x+5)\,dx = \frac{\sinh(3x+5)}{3} + C$

Indefinite Integral

$$3)\ \int \operatorname{sech}(3x)\tanh(3x)\,dx = \frac{-\operatorname{sech}(3x)}{3} + C$$

$$4)\ \int \frac{\sinh x}{e^x}\,dx = \int \frac{e^x - e^{-x}}{2} * \frac{1}{e^x}\,dx$$

$$= \frac{1}{2}\int \frac{e^x}{e^x} - \frac{e^{-x}}{e^x}\,dx = \frac{1}{2}\int 1 - e^{-2x}\,dx = \frac{1}{2}\left(x - \frac{e^{-2x}}{-2}\right) + C$$

* Definite Integral التكامل المحدود :

$$\int_a^b f'(x)\,dx = f(x)\Big|_a^b = f(b) - f(a)$$

$$\int_a^b k f(x)\,dx = k\int_a^b f(x)\,dx$$

$$\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\text{Ex:}\ \int_1^2 x^2 - 2x + 1\,dx = \left(\frac{x^3}{3} - x^2 + x\right)\Big|_1^2$$

$$\left(\frac{8}{3} - 4 + 2\right) - \left(\frac{1}{3} - 1 + 1\right)$$

$$\frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Definite Integral

Chapter

Ex: $\int_{0}^{\frac{\pi}{4}} \sec^2 x \, dx = \tan x \Big|_{0}^{\frac{\pi}{4}} = 1 - 0 = 1$

Ex: $\int_{e}^{e^2} x^{-1} dx = \ln|x| \Big|_{e}^{e^2} = \ln e^2 - \ln e = 2 - 1 = 1$

* خصائص بتميّز التكامل المحدود :

1) $\int_{a}^{b} k \, dx = k(b-a)$

Ex: $\int_{3}^{6} 7 dx = 7(6-3) = 21$

2) $\int_{a}^{a} f(x) \, dx = 0$

Ex: $\int_{3}^{3} x^2 + 1 \, dx = 0$

3) $\int_{a}^{b} f(x) dx = -\int_{b}^{a} f(x) dx$

Ex: If $\int_{1}^{3} f(x) dx = 4$, find $\int_{3}^{1} f(x) dx$?

Answer: -4

4) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

5) $\int_{-a}^{a} f(x)\,dx = 0$, If $f(x)$ odd function

Ex: $\int_{-3}^{3} \sin x\,dx = 0$, $\sin x$ odd function

6) $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$, $f(x)$ even function

Examples:

1) $\int_0^3 7f(x)\,dx = 14$, $\int_5^3 f(x)\,dx = 9$, Find:

* $\int_0^3 f(x)\,dx = 7\int_0^3 f(x)\,dx = 14 = \int_0^3 f(x)\,dx = 2$

* $\int_3^5 f(x)\,dx = -9$

* $\int_0^5 f(x)\,dx = \int_0^3 f(x)\,dx + \int_3^5 f(x)\,dx = 2 + -9 = -7$

Definite Integral — Chapter

2) $\int_0^2 |x-1|\,dx$

$x-1=0$

$x=1$

$$\begin{array}{ccccc} & - & & + & \\ 0 & 1-x & 1 & x-1 & 2 \end{array}$$

$$\int_0^1 1-x\,dx + \int_1^2 x-1\,dx$$

$$\left(x-\frac{x^2}{2}\right)\Big|_0^1 + \left(\frac{x^2}{2}-x\right)\Big|_1^2$$

$$\left(\frac{1}{2}\right) + \left(0+\frac{1}{2}\right) = 1$$

* Integration by substitution التكامل بالتعويض :

1) $\int x\sqrt{3+x^2}\,dx$

$y = 3+x^2$

$dy = 2x\,dx$

$\frac{dy}{2x} = dx$

$$\int \cancel{x}\, y^{\frac{1}{2}} \frac{dy}{2\cancel{x}} = \frac{1}{2}\int y^{\frac{1}{2}} dy$$

$$= \frac{1}{2}\left(\frac{2y^{\frac{3}{2}}}{3}\right) + C$$

$$= \frac{(3+x^2)^{\frac{3}{2}}}{3} + C$$

2) $\int_0^{\frac{\pi}{2}} e^{\sin x} \cos x \, dx$

$y = \sin x$

$dy = \cos x \, dx$

$\frac{dy}{\cos x} = dx$

$\int_0^1 e^y \cancel{\cos x} \frac{dy}{\cancel{\cos x}}$

$x = 0 \rightarrow y = 0$

$x = \frac{\pi}{2} \rightarrow y = 1$

$\int_0^1 e^y dy = e^y \Big|_0^1 = e^1 - e^0 = e - 1$

3) $\int \frac{\ln x}{x} dx$

$y = \ln x$

$dy = \frac{1}{x} dx$

$x dy = dx$

$\int \frac{y}{\cancel{x}} * \cancel{x} dy = \int y dy = \frac{y^2}{2} + c$

$= \frac{(\ln x)^2}{2} + c$

4) $\int x (2x+5)^8 dx$

$y = 2x+5$

$dy = 2 dx$

$\frac{dy}{2} = dx$

$\int x y^8 \frac{dy}{2}$

$\frac{y-5}{2} = x$

$\int \left(\frac{y-5}{2}\right) y^8 \frac{dy}{2} = \frac{1}{4} \int y^9 - 5y^8 dy$

$= \frac{1}{4} \left(\frac{y^{10}}{10} - \frac{5}{9} y^9 \right) + C \rightarrow \frac{1}{4} \left(\frac{(2x+5)^{10}}{10} - \frac{5}{9} (2x+5)^9 \right) + c$

The Fundamental Theorem

* The Fundamental Theorem of calculus النظرية الأساسية بالتفاضل والتكامل:

⇒ If f is continuous and g&h are differentiable functions Then

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$$

EX: Find the derivative of the functions:

1) $y = \int_{3}^{\tan x} \sqrt{t+\sqrt{t}}\,dt$

$$y' = \frac{d}{dx}\int_{3}^{\tan x} \sqrt{t+\sqrt{t}}\,dt = \sqrt{\tan x+\sqrt{\tan x}}\,(\sec^2 x) - \cancel{\left(\sqrt{3+\sqrt{3}}\right)(0)}$$

2) $y = \int_{1-3x}^{1} \frac{t^3}{1+t^3}\,dt = \cancel{\left(\frac{1}{2}\right)(0)} - \left(\frac{(1-3x)^3}{1+(1-3x)^3}\right)\left(-3\right)$

3) $y = \int_{0}^{x^4} \cos^2\theta\,d\theta$

$$y' = \frac{d}{dx}\int_{0}^{x^4} \cos^2\theta\,d\theta = (\cos^2(x^4))(4x^3) - (\cos^2 0)(0)$$

4) $y = \int_{e^x}^{\sqrt{x}} \ln t\,dt$

$$y' = \frac{d}{dx}\int_{e^x}^{\sqrt{x}} \ln t\,dt = (\ln\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right) - (\ln e^x)(e^x)$$

The Fundamental Theorem

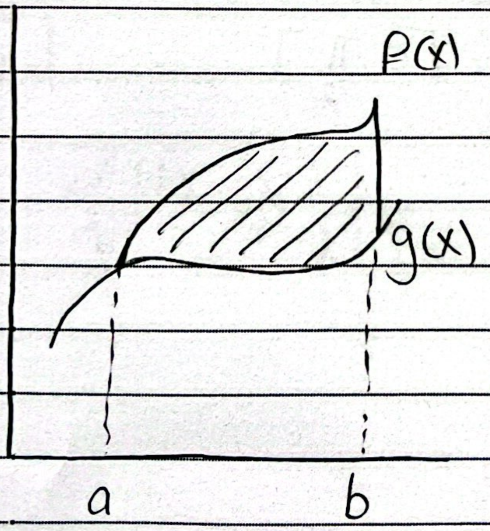
Chapter

5) $y = \int_{\sin x}^{\cos x} \sqrt{1+t^2}\, dt$

$$y' = \frac{d}{dx}\int_{\sin x}^{\cos x} \sqrt{1+t^2}\, dt = \left(\sqrt{1+(\cos x)^2}\right)(-\sin x) - \left(\sqrt{1+(\sin x)^2}\right)(\cos x)$$

* Areas between Curves المساحة بين المنحنيات :

⇒ If f and g are continuous functions on the Interval $[a,b]$ such that $f(x) \geq g(x)$ for all $x \in [a,b]$, then the area of the region bounded above by $f(x)$, below by $g(x)$ on the left $x=a$ on the right $x=b$ is



$f(x) > g(x)$

$$A = \int_a^b (f(x) - g(x))\, dx$$

Areas between Curves

1) Find the area of the region that enclosed by $y = x^2$ & $y = x + 6$?

$x^2 = x + 6$

$x^2 - x - 6 = 0$

$(x + 2)(x - 3) = 0$

$x = -2 , x = 3$

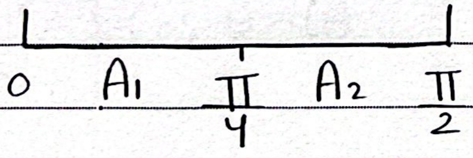
$$A = \int_{-2}^{3} (x+6) - (x^2)\, dx$$

$$= \left(\frac{x^2}{2} + 6x - \frac{x^3}{3}\right)\Bigg|_{-2}^{3} = \frac{125}{6}$$

2) Area $y = \cos x$, $y = \sin x$, $[0, \frac{\pi}{2}]$

$\cos x = \sin x$

$x = \frac{\pi}{4}$



$A = A_1 + A_2$

$$A = \int_{0}^{\frac{\pi}{4}} ((\cos x - \sin x) dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x) dx$$

Area between Curves Chapter

3) Area $f(x) = e^x$ & x-axis $[0, \ln 4]$

$e^x = 0 \quad x$

$$A = \int_0^{\ln 4} e^x \, dx = e^x \Big|_0^{\ln 4} = (4) - (1) = 3$$

4) Find the area of the region that enclosed by $y = x^2 - 4$ & x-axis

$x^2 - 4 = 0$

$x = \pm 2$

$$A = \int_{-2}^{2} (0 - (x^2 - 4) \, dx = \int_{-2}^{2} 4 - x^2 \, dx$$

$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right)$$