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دفتر

تفاضل وتكامل 2

(فاينال)

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Infinite Series

$a_1, a_2, a_3, a_4, \dots$ seq

$a_1 + a_2 + a_3 + a_4 + \dots$ series $\sum_{n=1}^{+\infty} a_n$

Example

1. $1 + 2 + 3 + 4 + 5 + \dots \quad \sum_{n=1}^{+\infty} n$

2. $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \quad \sum_{n=1}^{+\infty} \frac{1}{n}$

Defn: let $\sum_{n=1}^{+\infty} a_n$, the seq of partal sum $\{S_n\}_{n=1}^{+\infty} : \{S_1, S_2, S_3, \dots\}$

$S_1 = a_1 \longrightarrow S_1 = a_1$

$S_2 = a_1 + a_2 \longrightarrow S_2 = S_1 + a_2$

$S_3 = a_1 + a_2 + a_3 \longrightarrow S_3 = S_2 + a_3$

$S_4 = a_1 + a_2 + a_3 + a_4 \quad S_4 = S_3 + a_4$

$\vdots \qquad\qquad \vdots$

$S_n = a_1 + a_2 + \dots + a_{n-1} + a_n \quad S_n = S_{n-1} + a_n$

note: $\{S_n\}_{n=1}^{+\infty}$ conv ($\lim_{n \to +\infty} S_n = L$) متقاربة

↳ seq of partal sum

$\rightarrow \sum_{n=1}^{+\infty} a_n$ conv $\rightarrow \sum_{n=1}^{+\infty} a_n = L$

note : $\{S_n\}_{n=1}^{+\infty}$ div (has no sum)

$\rightarrow \sum_{n=1}^{+\infty} a_n$ div

Example If the series $\sum_{n=1}^{+\infty} a_n$ has the nth partial sum $S_n = 2 + \frac{3n}{n+1}$, find the sum of $\sum_{n=1}^{\infty} a_n$?

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2 + \frac{3n}{n+1} = 2+3 = 5 \text{ conv}$$

$$\sum_{n=1}^{\infty} a_n \text{ conv} \longrightarrow \sum_{n=1}^{\infty} a_n = 5$$

Example Conv or div ?

① $\sum_{n=1}^{+\infty} (-1)^n = -1 + 1 + -1 + 1 + \cdots$

$S_1 = a_1 = -1$

$S_2 = 0$

$S_3 = -1$

$S_4 = 0$

$\{-1, 0, -1, 0, -1, 0, \ldots\}$

$\lim_{n\to\infty} S_n$ d.n.e div

$\sum_{n=1}^{\infty} (-1)^n$ div has no sum.

② $\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \dots$

$S_1 = 1$

$S_2 = 2$

$S_3 = 3$

$\vdots$

$S_n = n \qquad \lim_{n \to +\infty} S_n = \lim_{n \to \infty} n = +\infty$

div

$\sum_{n=1}^{\infty} 1$ div has no sum.

Example determine whether the series is conv or div, If it is conv find the sum :

① $\sum_{n=1}^{+\infty} \frac{1}{n} - \frac{1}{n+1}$

$S_1 = \frac{1}{1} - \frac{1}{2}$ telescoping sum

$S_2 = S_1 + a_2 = 1 - \cancel{1/2} + \cancel{1/2} - 1/3 = 1 - 1/3$

$S_3 = S_2 + a_3 = 1 - \cancel{1/3} + \cancel{1/3} - 1/4 = 1 - 1/4$

$S_4 = S_3 + a_4 = 1 - \cancel{1/4} + \cancel{1/4} - 1/5 = 1 - 1/5$

$S_n = 1 - 1/n+1$

$\lim_{n \to +\infty} S_n = \lim_{n \to +\infty} 1 - \cancel{\frac{1}{n+1}}^{\frac{1}{\infty}} = 1$ conv

$\therefore \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ conv $= 1$

② $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

لاستخدام ال telescoping sum لازم
يكون فيه اشارة - !

$$\sum_{n=1}^{\infty} \ln(n+1) - \ln n$$

$$S_1 = \ln 2 - \cancel{\ln 1} = \ln 2$$

$$S_2 = S_1 + a_1 = \cancel{\ln 2} + \ln 3 - \cancel{\ln 2} = \ln 3$$

$$S_3 = S_2 + a_3 = \cancel{\ln 3} + \ln 4 - \cancel{\ln 3} = \ln 4$$

$$S_n = \ln(n+1)$$

$$\lim_{n \to +\infty} \ln(n+1) = +\infty \quad \text{div}$$

$\therefore \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$ div has no sum.

③ $\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2}$

لاستخدام ال telescoping sum لازم
كسور جزئية :

$$\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2} = \frac{1}{(3n-1)(3n+2)} = \frac{A}{3n-1} + \frac{B}{3n+2}$$

$$= \frac{A(3n+2) + B(3n-1)}{(3n-1)(3n+2)}$$

$$1 = A(3n+2) + B(3n-1)$$

$$n = -2/3 \rightarrow 1 = -3B \rightarrow B = -1/3$$

$$n = 1/3 \rightarrow 1 = 3A \rightarrow A = 1/3$$

$$\sum_{n=1}^{+\infty} \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$$

No. lec 20

$S_1 = \frac{1}{6} - \frac{1}{15}$

$S_2 = S_1 + a_2 = 1/6 - \cancel{1/15} + \cancel{1/15} - 1/24$

$S_3 = S_2 + a_3 = 1/6 - \cancel{1/24} + \cancel{1/24} - 1/33$

$S_n = \frac{1}{6} - \frac{1}{3(3n+2)}$

$\lim_{n \to +\infty} \frac{1}{6} - \frac{1}{3(3n+2)} = \frac{1}{6}$ conv

$\sum_{n=1}^{\infty} \frac{1}{3(3n-1)} - \frac{1}{3(3n+2)}$ conv $= \frac{1}{6}$

$\sum_{n=1}^{\infty} \frac{1}{9n^2+3n-2} = \frac{1}{6}$

FIVE APPLE

Geometric Series

$$\sum_{n=m}^{+\infty} r^n \qquad r: \text{ratio}$$

الأساس ثابت / الأس متحرك

Ex $5^1 + 5^2 + 5^3 + 5^4 + 5^5 + \cdots$

$\frac{5^2}{5} = 5 \qquad \frac{5^3}{5^2} = 5 \qquad \frac{5^4}{5^3} = 5 \rightarrow$ ratio

Example which of the following is G.S?

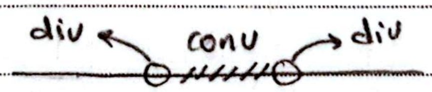
① $\sum_{n=5}^{\infty} 4^n = 4^5 + 4^6 + 4^7 + \cdots$ G.S

② $\sum_{k=1}^{\infty} 1/3^{k+1} = \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} \cdots$ G.S

③ $\sum_{k=1}^{\infty} k^3 = 1^3 + 2^3 + 3^3 + \cdots$ not G.S

④ $\sum_{k=1}^{\infty} (2k)^k = 2^1 + 4^2 + 6^3 + \cdots$ not G.S

Thrm: $\sum_{n=m}^{\infty} r^n = \begin{cases} \text{conv} & \text{if } |r| < 1 \ (-1 < r < 1) \\ \text{div} & \text{if } |r| \geq 1 \ (r \geq 1 \text{ or } r \leq -1) \end{cases}$



If $\sum_{n=m}^{+\infty} r^n$ conv the $\sum_{n=m}^{+\infty} r^n = \frac{r^m}{1-r} \rightarrow$ first term

Thrm: $\sum a_n$ conv then:

$\sum c a_n = c \sum a_n$

Example Determine whether the series conv or div, if conv find its sum?

1. $\sum_{n=0}^{+\infty} (-1/5)^n$ $\quad -1 < -1/5 < 1$ conv

$$\sum_{n=0}^{\infty} (-1/5)^n = \frac{(-1/5)^0}{1 - -1/5} = \frac{1}{1+1/5} = 5/6$$

2. $\sum_{K=1}^{+\infty} (\pi/e)^K$ div $\quad \frac{\pi}{e} = \frac{3.14}{2.7} > 1$

3. $\sum_{n=1}^{+\infty} 5^n 7^{-n+2} = \sum_{n=1}^{\infty} 5^n 7^{-n} . 7^2$

$$= \sum_{n=1}^{\infty} 7^2 \left(\frac{5}{7}\right)^n = 7^2 \sum_{n=1}^{\infty} (5/7)^n$$

$\therefore$ conv $\quad -1 < 5/7 < 1$

$$7^2 \left(\frac{(5/7)^1}{1 - 5/7}\right) = \frac{245}{2}$$

Example If $\sum_{n=2}^{\infty} 2r^n = 1$ find r?

$\sum_{n=2}^{\infty} 2r^n$ conv $\therefore = 2\sum_{n=2}^{\infty} r^n = 1$

G.S

$$\sum_{n=2}^{\infty} r^n = \frac{1}{2} \rightarrow \frac{r^2}{1-r} = \frac{1}{2} \rightarrow 2r^2 = 1 - r$$

$$2r^2 + r - 1 = 0 \rightarrow (2r-1)(r+1) = 0$$

No. lec 21

$r = \frac{1}{2}$, $r = -1$

G.S conv $\longrightarrow$ $|r| < 1$ $\therefore$ $r = 1/2$

The Divergence Test

Σa_n, if $\lim_{n\to\infty} a_n \neq 0$, then Σa_n div

if $\lim_{n\to\infty} a_n = 0$ (Test fail), Σa_n mayether conv or div.

Example conv or div?

1. $\sum_{n=1}^{\infty} \frac{2n-1}{3n+4}$ $\quad \lim_{n\to\infty} \frac{2n-1}{3n+4} = \frac{2}{3} \neq 0$

$\therefore$ by div.T $\sum_{n=1}^{\infty} \frac{2n-1}{3n+4}$ div

2. $\sum_{n=1}^{\infty} (1-\frac{3}{n})^n$ $\quad \lim_{n\to\infty} (1-\frac{3}{n})^n = e^{-3} \neq 0$

$\therefore$ by div.T $\sum_{n=1}^{\infty} (1-3/n)^n$ div.

3. $\sum_{n=1}^{\infty} \cos n\pi$ $\quad \lim_{n\to\infty} \cos n\pi = \text{d.n.e} \neq 0$ $\hookrightarrow \{-1, 1, -1, 1, \dots\}$

$\therefore$ by div.T $\sum_{n=1}^{+\infty} \cos n\pi$ div

4. $\sum_{n=1}^{\infty} \ln n$ $\quad \lim_{n\to\infty} \ln n = +\infty \neq 0$

$\therefore$ by div.T $\Sigma \ln n$ div

5. $\sum_{n=1}^{\infty} \frac{1}{e^n}$ $\quad \lim_{n\to\infty} 1/e^n = \frac{1}{e^\infty} = 0$ Test fail

$$\sum_{n=1}^{\infty} (1/e)^n$$ Geometric series conv

$-1 < 1/e < 1$

6. $$\sum_{n=1}^{\infty} \frac{1}{n}$$

$\lim_{n \to \infty} \frac{1}{n} = 0$ Test fail

$$\sum_{n=1}^{\infty} \frac{1}{n}$$ (P-series) div

$\longrightarrow \lim_{n \to \infty} a_n = 0$ → conv / → div.

The Integral Test

$\sum_{n=m}^{+\infty} a_n \ , \ a_n = f(n) \quad [m, \infty)$

1. f positive
2. f continuous
3. f decreasing

$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\} [m, \infty)$

if $\int_m^{+\infty} f(x)\,dx$ conv (impro.) $\Rightarrow \sum_m^{+\infty} a_n$ conv

$\int_m^{+\infty} f(x)\,dx$ div $\Rightarrow \sum_m^{+\infty} a_n$ div

Example conv or div ?

① $\sum_{n=3}^{\infty} \ln n / n$

Div. test : $\lim_{n\to\infty} \frac{\ln n}{n} = 0$ test fail.

Int. test : $f(x) = \frac{\ln x}{x}$ (functions ser بنكتب ال)

1. $\frac{\ln x \to \oplus}{x \to x > 3 \ \oplus} > 0$

2. $f(x) = \frac{\ln x}{x}$, $x = 0 \notin [0, 3)$
$\therefore f(x)$ cont. $[3, \infty)$

3. $f(x) = \frac{\ln x}{x}$ $\quad f'(x) = \frac{1 - \ln x}{x^2}$

$1 - \ln x = 0$

$1 = \ln x$

$x = e$

$$\underset{e}{////}\ \underset{3}{- - -}\ \text{dec.}$$

∴ تحققت الـ 3 شروط

$$\int_3^{\infty} \frac{\ln x}{x}\, dx = \lim_{t \to +\infty} \int_3^{t} \frac{\ln x}{x}\, dx$$

$= \int \ln x / x \, dx$ | $u = \ln x$ بالتعويض

$= \int \frac{u}{\cancel{x}} \cdot \cancel{x}\, dx$ | $dx = x\, du$

$= \int u\, du = \frac{u^2}{2}$

$= \frac{(\ln x)^2}{2} + c$

$$\hookrightarrow \lim_{t \to +\infty} \left.\frac{(\ln x)^2}{2}\right|_3^t$$

$$= \lim_{t \to \infty} \frac{(\ln t)^2}{2} - \frac{(\ln 3)^2}{2} = +\infty$$

$\therefore \int_3^{\infty} \ln x / x\, dx$ div $\rightarrow \sum_3^{\infty} \frac{\ln n}{n}$ div (Int.T)

(2) $\sum_1^{\infty} n e^{-n^2}$ $\quad f(x) = x e^{-x^2}$

1. $x e^{-x^2}$ ⊕ > 0

2. $f(x) = \frac{x}{e^{x^2}}$, $e^{x^2} \neq 0$ cont $[1, \infty)$

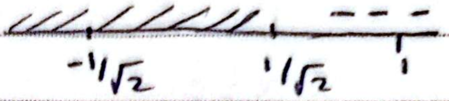
3. $f'(x) = -2x^2 e^{-x^2} + e^{-x^2}$

$f'(x) = e^{-x^2}(-2x^2 + 1) = 0$

$e^{-x^2} \neq 0 \qquad -2x^2 + 1 = 0$

$x^2 = 1/2$ ال ser شبامنها

$x = \pm 1/\sqrt{2}$



$\therefore$ dec $[1, \infty)$

$$\int_1^\infty x e^{-x^2}\,dx = \lim_{t\to\infty} \int_1^t x e^{-x^2}\,dx$$

$$\int x e^{-x^2}\,dx \qquad \Big|\; u = x^2, \quad dx = du/2x$$

$$\int \cancel{x} e^{-u} \frac{du}{2\cancel{x}}$$

$$\frac{1}{2}\int e^{-u}\,du = -\frac{1}{2}e^{-u} = -\frac{1}{2}e^{-x^2} + C$$

$$\hookrightarrow \lim_{t\to\infty} -\frac{1}{2}e^{-x^2}\Big|_1^t$$

$$= \lim_{t\to\infty} \cancel{-\frac{1}{2}e^{-t^2}}^{\,0} + \frac{1}{2}e^{-1} = \frac{1}{2e}$$

$\therefore \sum_{n=1} n e^{-n^2}$ conv by Inte. test

P - series

1. $\sum_{n=1}^{\infty} \frac{1}{n}$ conv or diverge ?   lim=0 X div.T
   : Int.T

1) $f(x) = \frac{1}{x} > 0 \quad [1, \infty)$
2) $f(x) = \frac{1}{x} \quad x=0 \notin [1, \infty)$ cont
3) $f'(x) = \frac{-1}{x^2} < 0$ decreasing

$$\int_1^{\infty} 1/x \, dx = \lim_{t \to \infty} \int_1^t 1/x \, dx = \lim_{t \to \infty} \ln|x| \Big|_1^t$$

$$= \lim_{t \to \infty} \ln t - \ln 1 = +\infty \text{ div}$$

$\therefore \sum_{n=1} 1/n$ div (harmonic series) ↓ or ≅

p-series

p>1 div

* $$\sum_{n=1}^{\infty} \frac{1}{n^p} = \begin{cases} \text{conv}, & p>1 \\ \text{div}, & p \le 1 \end{cases}$$

Example conv or div?

1. $\sum_{n=1}^{\infty} 1/n^2$ conv (p-series) $2 > 1$

2. $\sum_{n=1}^{\infty} 1/n^{\pi}$ conv (p-series) $\pi > 1$

3. $\sum_{n=3}^{\infty} 1/n^{2/5}$ div (p-series) $2/5 < 1$

4. $\sum_{n=2}^{\infty} 1/\sqrt{n+3} = \sum_{n=5}^{\infty} 1/\sqrt{n}$ div (p-series)

$1/\sqrt{5}$ $1/\sqrt{5}$

The limit Comparison Test $\rightarrow$ جذر | $\frac{\text{كثير حدود}}{\text{كثير حدود}}$

$\sum a_n$, $\sum b_n$ are series with positive terms and $c = \lim\limits_{n \to \infty} \frac{a_n}{b_n}$.

If $c > 0$ and $c \neq \infty$, then both series are conv or both are div.

Example conv or div?

① $\sum\limits_{1}^{\infty} \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2}$

lim = 0 x div.T

∫ صعب x Int.T

اكبر قوة ومعاملها

$\sum b_n = \sum\limits_{1}^{\infty} \frac{3n^3}{n^7} = \sum\limits_{1}^{\infty} 3/n^4$

$\sum\limits_{1}^{\infty} 3/n^4$ (p-series) conv $p > 1$

$$c = \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \underbrace{\frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2}}_{a_n} \cdot \underbrace{\frac{n^4}{3}}_{\div b_n}$$

اكبر قوة ومعاملها

$$= \lim_{n \to \infty} \frac{3n^7 - 2n^6 + 4n^4}{3n^7 - 3n^3 + 6} = \lim_{n \to \infty} \frac{3n^7}{3n^7} = 1 > 0, \neq \infty$$

$\therefore \underbrace{\sum a_n}_{\text{p-ser.}}$, $\sum b_n$ are conv by L.C.T

$\therefore \sum \frac{3n^3 - 2n^2 + 4}{n^7 - n^3 + 2}$ conv

No. lec 25

② $\sum_{1}^{\infty} \frac{2n^2+3n}{\sqrt{5+n^5}}$

اكبر قوة ومعاملها

$\sum b_n = \sum \frac{2n^2}{n^{5/2}} = \sum 2/n^{\frac{1}{2}}$

$\sum \frac{2}{n^{\frac{1}{2}}}$ (p-series) div $1/2 < 1$

$a_n \div b_n$

$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2n^2+3n}{\sqrt{5+n^5}} \cdot \frac{\sqrt{n}}{2}$

$= \frac{1}{2} \lim_{n\to\infty} \frac{\sqrt{n}\; n^2 (2+3/n)}{\sqrt{n^5 (5/n^5+1)}} = \frac{1}{2} \lim \frac{\cancel{n^{5/2}} (2+3/n)}{\cancel{n^{5/2}} \sqrt{5/n^5+1}}$

$= \frac{1}{2} \lim_{n\to\infty} \frac{2+\cancel{3/n}^{\,0}}{\sqrt{\cancel{5/n^5}^{\,0}+1}} = \frac{1}{2} \cdot \frac{2}{1} = 1 > 0 \neq \infty$

$\therefore \sum a_n , \sum b_n$ are div by L.C.T

$\therefore \sum \frac{2n^2+3n}{\sqrt{5+n^5}}$ div

③ $\sum_{n=1}^{\infty} \frac{3+9^n}{5+10^n}$ $\therefore$ conv by L.C.T

$\sum b_n = \sum \frac{9^n}{10^n} = \sum \left(\frac{9}{10}\right)^n$ conv (Geo.ser)

$-1 < \frac{9}{10} < 1$

$c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{3+9^n}{5+10^n} \cdot \frac{10^n}{9^n}$

$= \lim_{n\to\infty} \frac{\cancel{9^n}(3/9^n+1)\cancel{10^n}}{\cancel{10^n}(5/10^n+1)\cancel{9^n}} = \lim_{n\to\infty} \frac{\cancel{3/9^n}^{\,0}+1}{\cancel{5/10^n}_{\,0}+1}$

$= 1 > 0 \neq \infty$ $\therefore \sum a_n , \sum b_n$ conv (L.C.T)

$\therefore$

The Comparison Test

suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms, such that:

$$\sum a_n \leq \sum b_n$$

- If $\sum b_n$ is conv then $\sum a_n$ is also conv.
- If $\sum a_n$ is div then $\sum b_n$ is also div.

Example conv or div?

1. $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$

ثابتة

عملنا ser ثانية للمقارنة

$$\frac{5}{2n^2+4n+3} < \frac{5}{2n^2}$$

(صغرنا المقام بشلت حدين موجبين)

$$\sum_{n=1} \frac{5}{2n^2} = \frac{5}{2}\sum_{n=1} \frac{1}{n^2} \text{ (p-series) conv } p=2$$

$$\underbrace{\sum \frac{5}{2n^2+4n+3}}_{\therefore \text{ conv by C.T}} < \underbrace{\sum \frac{5}{2n^2}}_{\text{conv}}$$

2. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin^2 n}{n^2}$

$$-1 \leq \sin n \leq 1$$

$$0 \leq \sin^2 n \leq 1$$

$$\frac{\sin^2 n}{n^2} \leq \frac{1}{n^2} \quad \div n^2$$

$\sum \frac{1}{n^2}$ (p-series) conv $p=2$

$$\sum \frac{\sin^2 n}{n^2} \leq \sum \frac{1}{n^2}$$

$\downarrow$ $\quad$ $\downarrow$

$\therefore$ conv by C.T $\quad$ conv

3. $\displaystyle\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{n^3}$

$$-\pi/2 < \tan^{-1} n < \pi/2$$

$\frac{\tan^{-1} n}{n^3} < \frac{\pi/2}{n^3} \quad \div n^3$ (انتبه اذا قسمت على - نقلب اشارة المتباينة)

$\displaystyle\sum \frac{\pi/2}{n^3} = \frac{\pi}{2} \sum_{n=1}^{\infty} 1/n^3$ (p-series) conv

$$\sum \frac{\tan^{-1} n}{n^3} < \sum \frac{\pi/2}{n^3}$$

$\downarrow$ $\quad$ $\downarrow$

$\therefore$ conv by C.T $\quad$ conv

4. $\displaystyle\sum_{n=2}^{\infty} 1/\ln n$

$\ln n < n$ اقلب

$$\frac{1}{\ln n} > \frac{1}{n}$$

$\displaystyle\sum_{n=1} \frac{1}{n}$ (p-series) div

$$\sum \frac{1}{n} < \sum \frac{1}{\ln n}$$

$\downarrow$ div $\qquad$ $\downarrow$ $\therefore$ div by C.T

$$5. \sum_{n=1}^{\infty} \frac{5^n+1}{2^n-1}$$

نجرب اقارن معها ser

$2^n - 1 < 2^n$ اقلب

$$\frac{1}{2^n-1} < \frac{1}{2^n} \quad * \; 5^n+1$$

$$\frac{5^n+1}{2^n-1} < \frac{5^n+1}{2^n}$$

$$\frac{5^n+1}{2^n} > \frac{5^n}{2^n}$$

$$\frac{5^n+1}{2^n-1} > \cancel{\frac{5^n+1}{2^n}} > \frac{5^n}{2^n}$$

$$\frac{5^n+1}{2^n-1} > \frac{5^n}{2^n}$$

$\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$ div Geo. series $r = 5/2$

$$\sum \frac{5^n}{2^n} < \sum \frac{5^n+1}{2^n-1}$$

$\downarrow$ div $\qquad$ $\downarrow$ $\therefore$ div by C.T

Alternating Series Test

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad , \quad a_n > 0$$

1. $\lim\limits_{n \to +\infty} a_n = 0$
2. a_n decreasing

$\Big\}$ $\sum (-1)^n a_n$ is conv

Example conv or div? (Alternating harmonic S)

① $$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{\infty} (-1)^n (1/n)$$

1. $\lim\limits_{n \to +\infty} 1/n = 0$

2. $f(x) = 1/x \longrightarrow f'(x) = -1/x^2 < 0$ decreasing

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \text{ conv by A.S.T}$$

② * $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{\ln(n+4)}$$

1. $\lim\limits_{n \to \infty} \frac{1}{\ln(n+4)} = \frac{1}{\ln \infty} = \frac{1}{\infty} = 0$

2. $f(x) = \frac{1}{\ln(x+4)} \longrightarrow f'(x) = \frac{-\left(1/(x+4)\right)}{\left(\ln(x+4)\right)^2}$ dec

$$\therefore \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\ln(n+4)} \text{ conv by A.S.T}$$

③ $\sum_{n=1} \frac{(-1)^n}{e^n} = \sum_{n=1}^{\infty} (-1)^n \frac{1}{e^n}$

1. $\lim_{n\to\infty} \frac{1}{e^n} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$

2. $f(x) = \frac{1}{e^x} = e^{-x} \longrightarrow f'(x) = -e^{-x}$ dec

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{e^n}$ conv by A.S.T

④ $\sum_{n=1}^{\infty} (-1)^n \frac{n+1}{2n+1}$

اكبر قوة ومعاملها

1. $\lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{\cancel{n}}{2\cancel{n}} = \frac{1}{2} \neq 0$ (T. fail)

$\therefore$ (A.S.T) fail $\longrightarrow$ Div. Test

$\hookrightarrow \lim_{n\to\infty} (-1)^n \frac{n+1}{2n+1} \xrightarrow{\text{even}} \lim_{n\to\infty} \frac{n+1}{2n+1} = 1/2$

$\xrightarrow{\text{odd}} \lim -\frac{n+1}{2n+1} = -1/2$

$1/2 \neq -1/2 \therefore$ d.n.e

$\therefore \sum_{n=1}^{\infty} (-1)^n \frac{n+1}{2n+1}$ div by Div.T

No. lec 28

Ratio Test $n!$

$$\lim_{n\to+\infty}\left|\frac{a_{n+1}}{a_n}\right| = c$$

1. $c<1 \rightarrow \sum a_n$ convergent (Absolutly conv)
2. $c>1$ or $c=\infty \rightarrow \sum a_n$ div
3. $c=1$ Test fail (no conclusion)

Example conv or div?

① $\sum_{n=1}^{\infty}\frac{3^n}{n!}$

لاني لازم كل حدود ال ser +

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{3^{n+1}}{(n+1)!} * \frac{n!}{3^n}\right|$$

$$= \lim_{n\to\infty}\frac{\cancel{3^n}\cdot 3}{(n+1)\cancel{n!}}\cdot\frac{\cancel{n!}}{\cancel{3^n}} = \lim_{n\to\infty}\frac{3}{n+1} = 0$$

$\therefore$ conv by Ratio Test

② $\sum_{n=1}^{\infty}\frac{n!}{n^3}$

"

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{(n+1)^3}\cdot\frac{n^3}{n!}\right|$$

$$= \lim_{n\to\infty}\frac{\cancel{(n+1)}\cancel{n!}\,n^3}{(n+1)^{\cancel{3}\,2}\cancel{n!}} = \lim_{n\to\infty}\frac{n^3}{(n+1)^2} = \lim_{n\to\infty}\frac{n^3}{n^2+2n+1}$$

اكبر قوة (ومعاملها)

$$= \lim_{n\to\infty}\frac{n^3}{n^2} = \lim_{n\to\infty} n = +\infty$$

$\therefore$ div by ratio test.

FIVE APPLE

③ $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^n}{n!}$

ضروري لأنه مش كل الحدود +

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} (n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{(-1)^n n^n} \right|$$

راحوا المطلق

$$= \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{(n+1)}(n+1)^n}{\cancel{(n+1)}\,\cancel{n!}} \cdot \frac{\cancel{n!}}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \overset{L.R}{=} e > 1$$

$\therefore$ div by ratio test.

No. lec 29

Root test $(\ )^n$

$$\lim_{n\to\infty} (|a_n|)^{1/n} = c \qquad \sqrt[n]{|a_n|}$$

1. $c<1 \rightarrow \sum a_n$ conv (Absolute conv)
2. $c>1$ or $c=\infty \longrightarrow \sum a_n$ div
3. $c=1$ Test fail (no conclusion)

Example conv or div?

1. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{2n+1}{4n-5}\right)^n$ ⊕ البسط ⊕ المقام

$$\lim_{n\to\infty} \left(\left|\left(\frac{2n+1}{4n-5}\right)^n\right|\right)^{1/n} = \lim_{n\to\infty} \left(\left(\frac{2n+1}{4n-5}\right)^{\cancel{n}}\right)^{\cancel{\frac{1}{n}}}$$

$$\lim_{n\to\infty} \frac{2n+1}{4n-5} = \lim_{n\to\infty} \frac{2\cancel{n}}{4\cancel{n}} = \frac{1}{2} < 1 \text{ conv}$$

by root . T

2. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{100}\right)^n \rightarrow ⊕$

$$\lim_{n\to\infty} \left(\left(\frac{n}{100}\right)^{\cancel{n}}\right)^{\cancel{\frac{1}{n}}} = \lim_{n\to\infty} n/100 = \infty \text{ div by R.T}$$

3. $\displaystyle\sum_{n=2}^{\infty} \left(\frac{-1}{\ln n}\right)^n$

$$\sum_{n=2}^{\infty} (-1)^n (1/\ln n)^n = \lim_{n\to\infty} \left((1/\ln n)^{\cancel{n}}\right)^{\cancel{\frac{1}{n}}}$$

$$\lim_{n\to\infty} 1/\ln n = 1/\infty = 0 < 1 \text{ conv by root .T}$$

Algebraic properties of infinite series

- if $\sum a_n$ and $\sum b_n$ are conv, then:
  $\sum a_n + b_n$ and $\sum a_n - b_n$ are conv
  $\sum a_n + b_n = \sum a_n + \sum b_n$
  $\sum a_n - b_n = \sum a_n - \sum b_n$

- if c is a nonzero constant, then:
  if $\sum a_n$ conv $\longrightarrow$ $\sum c a_n$ conv, $\sum c a_n = c \sum a_n$
  if $\sum a_n$ div $\longrightarrow$ $\sum c a_n$ div

Example $\sum_{n=1}^{\infty} \frac{3}{4^n} - \frac{2}{5^{n-1}}$ Geo.

$= \sum_{n=1}^{\infty} 3(1/4)^n - 2(1/5)^{n-1}$ Geo

$= 3\sum_{n=1}^{\infty} (1/4)^n - 2\sum_{n=1}^{\infty} (1/5)^{n-1}$

$\downarrow$ conv $-1 < 1/4 < 1$ $\qquad$ $\downarrow$ conv $-1 < 1/5 < 1$

$\sum_{n=1}^{\infty} (\frac{1}{4})^n = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = 1/3$

$\sum_{n=1}^{\infty} (\frac{1}{5})^{n-1} = \frac{(1/5)^0}{1-1/5} = \frac{1}{4/5} = 5/4$

$\hookrightarrow 3(1/3) - 2(5/4)$

$= 1 - 5/2$

$= -3/2$ conv

No. lec30

(2) $\sum_{n=1}^{\infty} 1/e^n + \frac{1}{n(n+1)}$

G.S

$(\frac{1}{e})^n \quad -1 < \frac{1}{e} < 1$ conv

$$\sum_{n=1}^{\infty} (1/e)^n = \frac{1/e}{1-1/e} = \frac{1}{e-1}$$

(telescoping sum) conv

$$\sum_{n=1} \frac{1}{n(n+1)} = \sum \frac{A}{n} + \frac{B}{n+1}$$

$1 = A(n+1) + nB$

$n=0 \longrightarrow A=1$

$n=-1 \longrightarrow B=-1$

$$\sum \frac{1}{n(n+1)} = \sum \frac{1}{n} - \frac{1}{n+1}$$

$S_1 = 1 - \frac{1}{2}$

$S_2 = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \frac{1}{3}$

$S_n = 1 - \frac{1}{n+1}$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 1 - \cancel{\frac{1}{n+1}} = 1$$

$$\hookrightarrow \sum_{n=1}^{\infty} 1/e^n + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$= \frac{1}{e-1} + 1 = \frac{e}{e-1}$$

- if $\sum a_n$ conv and $\sum b_n$ div, then:
  $\sum a_n \mp b_n$ div

Example conv or div ?

$$\sum_{n=1}^{\infty} (1/2)^n + 1/n$$

(G.S) conv $-1 < 1/2 < 1$ (Harmonic.S) div

$$\therefore \sum_{n=1}^{\infty} (1/2)^n + 1/2 \quad \text{div}$$

- If $\sum a_n$ and $\sum b_n$ are div then: $\sum a_n + b_n$ may either conv or div

Example

① $\sum_{n=1}^{\infty} -1$ div, $\sum_{n=1}^{\infty} 1$ div $\Big\}$ $\sum_{n=1}^{\infty} -1+1 = \sum_{n=1}^{\infty} 0 = 0+0+0\ldots$ conv

② $\sum_{n=1}^{\infty} 1/n$ (p-ser) div or harm., $\sum_{n=1}^{\infty} 5/n$ div (harmonic / p-ser) $\Big\}$ $\sum_{n=1}^{\infty} \frac{1}{n} + \frac{5}{n}$

$$= \sum_{n=1}^{\infty} 6/n \quad \text{div (harmonic ser)}$$

Strategy for test series

$= \Sigma$

1. Geometric series $\rightarrow \Sigma 5^n / \Sigma (1/4)^n / \Sigma (2/3)^n$

$$\sum_{n=m} r^n = \begin{cases} \text{conv if } |r|<1 \\ \text{div if } |r| \geq 1 \end{cases}$$

2. P- series $\rightarrow \Sigma 1/\sqrt{n} / \Sigma 1/n^3$

$$\sum_{n=1} \frac{1}{n^p} = \begin{cases} \text{conv}, & p>1 \\ \text{div}, & p \leq 1 \end{cases}$$

3. Ratio test $\rightarrow n! / 5! / 5^n / 3^n$

$c = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$ if

1. $c<1 \rightarrow \Sigma a_n$ conv
2. $c>1$ or $c=\infty \rightarrow \Sigma a_n$ div
3. $c=1$ test fail

4. Root test $\rightarrow (b_n)^n$

$c = \lim_{n\to\infty} \sqrt[n]{|a_n|}$ if

1. $c<1 \rightarrow \Sigma a_n$ conv
2. $c>1$ or $c=\infty \rightarrow \Sigma a_n$ div
3. $c=1$ test fail

5. Alternating Series test $\rightarrow (-1)^n$

$\sum_{n=1} (-1)^n a_n$, $a_n > 0$ if:

① $\lim a_n = 0$

② a_n decreasing

then $\Sigma (-1)^n a_n$ conv.

6. Divergence test → lim بقدر احسب

If $\lim a_n \neq 0$ then $\sum a_n$ div

7. limite Comparsion test → كثير حدود | $\sqrt{n}$ / كثير

$c = \lim a_n / b_n$, if $c > 0$ and $c \neq \infty$,

then both series are conv or both series are div.

8. Integral test

$\sum\limits_{n=m} a_n$, $f(n) = a_n$ if:

① $f(n)$ positive

② $f(n)$ cont.

③ $f(n)$ decreasing

then $\int_m^\infty f(x)\,dx$ div → $\sum\limits_{n=m} a_n$ div

$\int_m^\infty f(x)\,dx$ conv → $\sum\limits_{n=m} a_n$ conv

= Σ

9. Telescoping sum

if $\lim S_n = L$

then $\sum\limits_{n=1} a_n = L$

No. lec 31.

Exercises

1. $\sum_{n=2}^{\infty} 1/n\sqrt{\ln n}$ Div.T ✗ $\lim = 0$

   Inte.T ✓

2. $\sum_{n=1}^{\infty} (2n+1)^{n}/n^{2n}$ Root.T $\lim \frac{2n+1}{n^2}$

3. $\sum_{n=1}^{\infty} 1/n^3$ P-series

4. $\sum_{n=1} 1/3^{n-1}$ Geo-series

5. $\sum_{n=1} 2^n n!/(n+2)!$ Ratio.T

   كثير حدود

6. $\sum_{n=1} \sqrt{n^2-1}/n^3+2n^2+5$ lim.comparasion.T

7. $\sum_{n=1} n^2+1/n^3+1$ lim.comparasion.T

8. $\sum_{n=1} n\sin(1/n)$ Div.T $\lim \frac{\sin 1/n}{1/n}$ L.R

9. $\sum_{n=1} (\sqrt[n]{2}-1)^{n}$ Root.T

10. $\sum_{n=1} (-1)^n/\cosh n$ Alt.S.T

11. $\sum_{n=1} e^{1/n} - e^{1/n+1}$ $S_1 = e - e^{1/2}$

    $S_2 = e - \cancel{e^{1/2}} + \cancel{e^{1/2}} - e^{1/3}$

    ↳ telescoping sum $S_n = e - e^{1/n+1}$

12. $\sum_{n=1}^{\infty} (n/n+1)^{n^2}$ Root test

13. $\sum_{n=1} (-1)^n \dfrac{\sqrt{n}}{n+5}$ Alt. S. T

14. $\sum_{n=1} \dfrac{(-5)^n}{n!} = \sum \dfrac{(-1)^n (5)^n}{n!}$ ← السهل Ratio test

15. $\sum_{n=1} n+1/2n+1$ $\lim = \frac{1}{2}$ Div. T

16. $\sum_{n=1} \dfrac{2^{n-1}\, 3^{n+1}}{n^n} = \sum_{n=1} \dfrac{2^n\, 2^{-1}\, 3^n\, 3}{n^n}$

$= \sum \dfrac{3}{2} \dfrac{6^n}{n^n} = \dfrac{3}{2} \sum \left(\dfrac{6}{n}\right)^n$ Root. T

17. $\sum_{n=1} (-3)^{n-1}/4^n = \sum \dfrac{(-3)^{-1} (-3)^n}{4^n}$

$= \sum (-3)^{-1} (-3/4)^n$ Geo. S

18. $\sum_{n=1} \dfrac{n^2\, 2^{n-1}}{(-5)^n} = \sum \dfrac{n^2\, 2^n . 2^{-1}}{(-1)^n . 5^n}$

$= 2^{-1} \sum \dfrac{n^2 (-1)^n 2^n}{5^n} = 2^{-1} \sum (-1)^n n^2 (2/5)^n$

Root. T or Ratio T or Alt. S. T

Absolute convergence , Conditional convergence

$\sum a_n = a_1 + a_2 + a_3 + \cdots$

$\sum |a_n| = |a_1| + |a_2| + |a_3| + \cdots$ } conv or div ?

$\sum_{n=1}^{\infty} (-1)^n/n = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} \cdots$ conv

$\sum_{n=1}^{\infty} |(-1)^n/n| = |-1| + |\frac{1}{2}| + |-\frac{1}{3}| + |\frac{1}{4}| \cdots$ → $\frac{1}{n}$ (harmo.)

$= 1 + 1/2 + 1/3 + 1/4 \cdots$ div

① If $\sum |a_n|$ is conv → $\sum a_n$ Absolutly conv

② If $\sum |a_n|$ is div and $\sum a_n$ conv → $\sum a_n$ conditionally conv.

③ If $\sum |a_n|$ is div and $\sum a_n$ div → $\sum a_n$ div.

Thrm: $\sum a_n$ Abs. conv → $\sum a_n$ conv

Example Determine whether the series is abs. conv , conditionally conv or div?

1. $\sum_{n=1}^{\infty} (-1)^n (1/2)^n$

$\sum_{n=1}^{\infty} |(-1)^n (1/2)^n| = \sum_{n=1} (1/2)^n$ conv (G.S)

$\therefore \sum_{n=1} (-1)^n (1/2)^n$ conv (Abs. conv)

2. $\sum_{n=1}^{\infty} (-1)^n n^3 / 3^n$

$\sum_{n=1}^{\infty} \left| (-1)^n n^3 / 3^n \right| = \sum_{n=1}^{\infty} n^3 / 3^n$ conv or div?

نختار test مناسب

ratio

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3}$

$= \lim_{n \to \infty} \frac{(n+1)^3 \cancel{3^n}}{\cancel{3^n} \cdot 3 \, n^3} = \lim \frac{1}{3} (n+1/n)^3$

$= \lim_{n \to \infty} \frac{1}{3} (1 + \overset{0}{\cancel{1/n}})^3 = \frac{1}{3} < 1$ conv

$\therefore \sum_{n=1} (-1)^n n^3 / 3^n$ Abs. conv $\longrightarrow$ conv

3. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$\sum_{n=1} \left| \frac{(-1)^n}{n} \right| = \sum 1/n$ div (p-series)

$\sum_{n=1} \frac{(-1)^n}{n}$ A.S.T 1. lim = 0 2. decreasing

$\therefore \sum_{n=1} \frac{(-1)^n}{n}$ conv by A.S.T

$\sum_{n=1} \frac{(-1)^n}{n}$ conv $\rightarrow \sum \left| \frac{(-1)^n}{n} \right|$ div

$\therefore$ conditionally conv.

4. $\sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)!}{3^n}$

ratio

$\sum_{n=1} \left| \frac{(-1)^n (2n-1)!}{3^n} \right| = \sum \frac{(2n-1)!}{3^n}$ conv or div? test

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(2(n+1)-1)!}{3^{n+1}} \cdot \frac{3^n}{(2n-1)!}$

$= \lim_{n\to\infty} \frac{(2n+1)!\,\cancel{3^n}}{\cancel{3^n}\cdot 3\,(2n-1)!} = \lim_{n\to\infty} \frac{(2n+1)(2n)\cancel{(2n-1)!}}{3\,\cancel{(2n-1)!}}$

$= \lim_{n\to\infty} \frac{1}{3}(2n+1)(2n) = \infty$ div

$\sum \frac{(2n-1)!}{3^n}$ div by ratio test

$\sum \frac{(-1)(2n-1)!}{3^n}$ div by ratio test (نفس الخطوات)

5. $a_1 = 2$, $a_{n+1} = \frac{5n+1}{4n+3} a_n$ ratio

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{5n+1}{4n+3} \cancel{a_n} * \frac{1}{\cancel{a_n}} \right|$

اكبر قوة وعاملها

$\lim_{n\to\infty} \frac{5n+1}{4n+3} = \frac{5}{4} > 1$ div by ratio. T

$\sum a_n$	$\sum \lvert a_n \rvert$	
conv	conv	$\sum a_n$ Abs. conv
conv	div	$\sum a_n$ c. conv
div	div	$\sum a_n$ div

Power Series

- $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

  x: variable

  c_n: constant (cofficients)

- $\sum_{n=0}^{\infty} c_n (x-a)^n$ Power series centered at a.

  $= c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$

Example

① $\sum_{n=0}^{\infty} x^n/5^n$ centered at 0 (x - عدد (ماعندي))

n=0 عوض   عوض 1

$= \underset{\downarrow}{1} + \underset{\downarrow}{x/5} + x^2/5^2 + x^3/5^3 + \cdots$

$c_1 = 1 \quad c_2 = 1/5 \quad c_3 = 1/25 \ \cdots$

② $\sum_{n=n}^{\infty} n x^n$ centered at 0

$= x + 2x^2 + 3x^3 + 4x^3 + \cdots$

$c_1 = 1 \quad c_2 = 2 \quad c_3 = 3 \quad \cdots$

③ $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n+1}$ centered at 3 $(x-3=0)$ $x=3$

$= \frac{x-3}{2} + \frac{(x-3)^2}{3} + \frac{(x-3)^3}{4} + \cdots$

④ $\sum_{n=1}^{\infty} \frac{(2x-1)^n}{\sqrt{n}}$ centered at $1/2$ $\quad 2x-1=0$

$$= \frac{2x-1}{1} + \frac{(2x-1)^2}{\sqrt{2}} + \frac{(2x-1)^3}{\sqrt{3}} + \dots$$

- $f(x) = \sum_{n=1}^{\infty} C_n x^n$, Domain : the set of all x for which the series conv.

Example $\sum_{n=1}^{\infty} x^n / 2^n$

$x = 1 \rightarrow \sum \frac{1}{2^n}$ conv by G.S $-1 < 1/2 < 1$

$x = 6 \rightarrow \sum \frac{6^n}{2^n} = \sum 3^n$ div by G.S

$\hookrightarrow \sum_{n=1}^{\infty} (x/2)^n$ G.S $\quad -1 < x/2 < 1$ غيرها div

$-2 < x < 2$ Domain $\leftarrow$ conv

Thrm: $\sum_{n=0}^{\infty} C_n (x-a)^n$ Ratio / Root test

	Radius of conv	Interval of conv
1. The series conv only when $x = a$	$R = 0$	$\{a\}$
2. The series conv for all x	$R = +\infty$	$(-\infty, \infty)$
3. There's a positive no. R such that series conv if $\lvert x-a \rvert < R$, div if $\lvert x-a \rvert > R$	R	$[(a-R, a+R)]$

No. lec 33

Example Find the radius of conv and the internal of conv?

لـ قيم x اللي تخلي pow.ser conv ؟

:. power series'

1. $\sum_{n=1}^{\infty} (-1)^n x^n / 3^n(n+1)$

بسبب المطلق

$* \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\cancel{(-1)}^{n+1} x^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{\cancel{(-1)^n} x^n}\right|$

$\lim_{x\to\infty} \left|\frac{\cancel{x^n} x \cdot \cancel{3^n}(n+1)}{\cancel{3^n}\, 3\,(n+2)\,\cancel{x^n}}\right| = \lim_{n\to\infty} \left|\frac{x(n+1)}{3(n+2)}\right|$

$\lim_{n\to\infty} \frac{1}{3} \frac{(n+1)}{(n+2)} |x| = \frac{1}{3}|x|$

conv (Ratio test) $c < 1$

$\frac{1}{3}|x| < 1$

$|x| < 3$ Radius of conv $R = 3$

نفكها

$*$ the internal of conv $-3 < x < 3$ نفحص

$\boxed{x=-3} \to \sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n(n+1)} = \sum_{n=0}^{\infty} \cancel{3^n}/\cancel{3^n}(n+1)$

$= \sum_{n=0}^{\infty} 1/n+1$ (P-series) div مفتوحة :.

$\boxed{x=3} \to \sum_{n=0}^{\infty} \frac{(-1)^n \cancel{(3)^n}}{\cancel{3^n}(n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$

(A.S.T) conv مغلقة :.

:. $(-3, 3]$ internal of conv.

2. $\sum_{n=1}^{\infty} (x-2)^n / n^2+1$ Ratio test

* $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{(n+1)^2+1} \cdot \frac{n^2+1}{(x-2)^n} \right|$

$\lim_{n\to\infty} \left| \frac{\cancel{(x-2)^n}(x-2)(n^2+1)}{(n^2+2n+2)\cancel{(x-2)^n}} \right|$

اكبر قوة وحاملها $\lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2} |x-2| = |x-2| < 1$ conv

$R = 1$ Raduis of conv

* $|x-2| < 1$ فكها

$-1 < x-2 < 1$

$1 < x < 3$ نفحص ← بتعويضها بالسؤال

$\boxed{x=1} \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1}$ 1. $\lim \frac{1}{n^2+1} = 0$

2. $\frac{1}{n^2+1}$ decreasing

$\therefore$ conv by A.S.T

$\boxed{x=3} \rightarrow \sum_{n=1}^{\infty} 1/n^2+1$

$\sum b_n = \sum 1/n^2$ $\lim \frac{a_n}{b_n} = \lim \frac{1}{n^2+1} \cdot n^2$

$= \lim n^2/n^2+1 = 1 > 0$ $\therefore$ conv by L.C.T

$\therefore$ $[1,3]$ interval of conv.

No. lec 33

3. $\sum_{n=1}^{\infty} x^n / n!$ ratio test او root

* $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right|$

$\lim_{n\to\infty} \left| \frac{x^n x \, n!}{(n+1)\cancel{n!}\,\cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right|$

$\lim_{n\to\infty} \frac{1}{n+1} |x| = 0 < 1$

↳ conv for all values of x حسب كل ال

↳ interval of conv $(-\infty, \infty)$

Radius " " $R = +\infty$

4. $\sum_{n=1}^{\infty} n^n (x-3)^n$ root test او ratio

* $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left| \cancel{n^n} (x-3)^{\cancel{n}} \right|^{1/n}$

$\lim_{n\to\infty} |n(x-3)| = \lim_{n\to\infty} n\,|x-3| = \infty$

div by root test

$\lim_{n\to\infty} n\,|x-3| \xrightarrow{x=3} \lim_{n\to\infty} n \cdot 0 = 0$ conv

حالة 1

↳ Radius of conv $R=0$

↳ interval of " $\{3\}$

FIVE APPLE

Taylor and Maclaurin Series

- Taylor series: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$

- $f(x) = f(a) + \underbrace{\frac{f'(a)}{1!}}_{c_1} (x-a)^1 + \underbrace{\frac{f''(a)}{2!}}_{c_2} (x-a)^2 + \dots$

$c_n = \frac{f^n(a)}{n!}$

$\sum_{n=0}^{\infty} c_n (x-a)^n$

- If $a=0$, we call it Maclaurin series:

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n$

$f(x) = f(0) + \frac{f'(0)}{1!} x^1 + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \dots$

Example find taylar series for the function $f(x) = 2^x$ centered at $x=1$

$f(x) = \sum_{n=0}^{\infty} \frac{f^n(1)}{n!} (x-1)^n$ (taylor)

$f(x) = 2^x \longrightarrow f(1) = 2$

$f'(x) = 2^x \ln 2 \longrightarrow f'(1) = 2 \ln 2$

$f''(x) = 2^x (\ln 2)^2 \longrightarrow f''(1) = 2(\ln 2)^2$

$f'''(x) = 2^x (\ln 2)^2 \longrightarrow f'''(1) = 2(\ln 2)^3$

$f^n(x) = 2^x (\ln 2)^n \longrightarrow f^n(1) = 2(\ln 2)^n$

$\hookrightarrow f(1) + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \dots + \frac{f^n(1)}{n!}(x-1)^n$

$$= 2 + 2\ln 2(x-1) + \frac{2(\ln 2)^2}{2!}(x-1)^2 + \dots + \frac{2(\ln 2)^n}{n!}(x-1)^n$$

$$2^x = \sum_{n=0}^{\infty} \frac{2(\ln 2)^n}{n!}(x-1)^n$$

Example find the Maclaurin series for the $f(x) = e^x$? $x=0$

$$\sum_{n=0}^{\infty} \frac{f^n(0)}{n!} x^n \quad \text{(Maclaurin)}$$

$f(x) = e^x \longrightarrow f(0) = e^0 = 1$

$f'(x) = e^x \longrightarrow f'(0) = 1$

$f''(x) = e^x \longrightarrow f''(0) = 1$

$f^n(x) = e^x \longrightarrow f^n(0) = 1$

$$\hookrightarrow f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n$$

$$1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Example Find the Maclaurin series for the function $f(x) = \sin x$? $\hookrightarrow x=0$

$f(x) = \sin x \longrightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \longrightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \longrightarrow f''(0) = 0$

$f'''(x) = -\cos x \longrightarrow f'''(0) = -1$

$f^4(x) = \sin x \longrightarrow f^4(0) = 0$

$f^5(x) = \cos x \longrightarrow f^5(0) = 1$

زوجي $f^{(2n)}(x) = (-1)^n \sin x \xrightarrow{\text{نعوض}} f^{2n}(0) = (-1)^n \sin 0 = 0$

فردي $f^{(2n+1)}(x) = (-1)^n \cos x \xrightarrow{\text{"}} f^{2n+1}(0) = (-1)^n \cos 0 = (-1)^n$

$\hookrightarrow f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{2n}(0)}{(2n)!}x^{2n} + \frac{f^{2n+1}(0)\, x^{2n+1}}{(2n+1)!}$

$= 0 + x + 0 + \frac{-x^3}{3!} + 0 + \frac{x^5}{5!} + \cdots + 0 + \frac{(-1)^n}{(2n+1)!} x^{2n+1}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

- $\dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n =$
- $e^x = \sum_{n=0}^{\infty} \dfrac{x^n}{n!}$
- $\sin x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$
- $\cos x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$
- $\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$
- $\ln(1+x) = \sum_{n=0}^{\infty} (-1)^{n-1} \dfrac{x^n}{n}$
- $(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n =$

Example. Find the Maclaurin Series?

1. $f(x) = \sin 3x \xrightarrow{\text{نشبه}} \sin x = \sum_{0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$= \sum_{0}^{\infty} \frac{(-1)^n (3x)^{2n+1}}{(2n+1)!}$$

2. $f(x) = \dfrac{1}{1+x^2} \xrightarrow{\text{"}} \dfrac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

f(x) نعدل $\quad f(x) = \dfrac{1}{1-(-x^2)} = \sum_{0}^{\infty} (-x^2)^n$

$$= \sum_{0}^{\infty} (-1)^n \cdot x^{2n}$$

3. $f(x) = \dfrac{1}{x+2}$ $\xrightarrow{\text{نستخدم}}$ $\dfrac{1}{1-x} = \sum_{0}^{\infty} x^n$

نعدل $f(x)$

$$f(x) = \frac{1}{2(x/2+1)} = \frac{1}{2(1-(-x/2))} = \frac{1}{2}\left[\frac{1}{1-(-\frac{x}{2})}\right]$$

$$\therefore = \frac{1}{2}\sum_{0}^{\infty}(-x/2)^n = \sum_{0}^{\infty}\frac{1}{2}\,\frac{(-1)^n x^n}{2^n}$$

$$\frac{1}{x+2} = \sum_{0}^{\infty}\frac{(-1)^n}{2n+1}x^n$$

4. $f(x) = xe^{-x}$ $\xrightarrow{\text{نستخدم}}$ $e^x = \sum_{0}^{\infty}\frac{x^n}{n!}$

نعدل f

$$e^{-x} = \sum_{0}^{\infty}\frac{(-x)^n}{n!}$$

$$xe^{-x} = x\sum_{0}^{\infty}\frac{(-1)^n x^n}{n!} = \sum_{0}^{\infty}\frac{x(-1)^n x^n}{n!}$$

$$\therefore\ xe^{-x} = \sum_{0}^{\infty}\frac{(-1)^n x^{n+1}}{n!}$$

5. $f(x) = \sin^2 x$

نعدل f

$$\sin^2 x = \frac{1}{2}(1-\cos 2x) \rightsquigarrow \cos x = \sum_{0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}$$

$$= \frac{1}{2}\left(1-\sum_{0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right)$$

موجوده

$$= \frac{1}{2}\left(\cancel{1}-\cancel{1}-\sum_{n=1}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right)$$

$$= -\frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)(-1)^n (2)^{2n} (x)^{2n}}{2(2n)!}$$

$$\sin^2 x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2^{2n-1}\, x^{2n}}{(2n)!}$$

6. $f(x) = \cosh x$

$$\cosh x = \frac{e^x + e^{-x}}{2} \rightsquigarrow e^x = \sum_{0}^{\infty} \frac{x^n}{n!}$$

$$e^{-x} = \sum_{0}^{\infty} \frac{(-x)^n}{n!}$$

$$e^x = 1 + \cancel{x} + \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} + \cancel{\frac{x^5}{5!}} + \cdots$$

$$e^{-x} = 1 - \cancel{x} + \frac{x^2}{2!} - \cancel{\frac{x^3}{3!}} + \frac{x^4}{4!} - \cancel{\frac{x^5}{5!}} \cdots$$

$$e^x + e^{-x} = 2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} \cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

No. lec 34

Example Find the sum of the series ?

conv إذا كان

1. $\sum_{n=0}^{\infty} \frac{2^n}{n!} \rightsquigarrow e^x = \sum_{0}^{\infty} \frac{x^n}{n!}$

$= e^2$

2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{n!} \rightsquigarrow e^x = \sum_{0}^{\infty} \frac{x^n}{n!}$

$= \sum_{0}^{\infty} \frac{(-1)^n (x^4)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-x^4)^n}{n!} = e^{-x^4}$

3. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n5^n} \rightsquigarrow \ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (3/5)^n}{n} = \ln(1+3/5) = \ln \frac{8}{5}$ conv

4. $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+2}}{4^{2n+2} (2n+1)!} \rightsquigarrow \sin x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\cos x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$= \sum_{0}^{\infty} \frac{(-1)^n (\pi/4)^{2n+2}}{(2n+1)!} = \sum_{0}^{\infty} \frac{(-1)^n (\pi/4)(\pi/4)^{2n+1}}{(2n+1)!}$

$= \frac{\pi}{4} \sum_{0}^{\infty} \frac{(-1)^n (\pi/4)^{2n+1}}{(2n+1)!} = \frac{\pi}{4} \sin \frac{\pi}{4} = \frac{\pi}{4\sqrt{2}}$

5. $-\frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots$

$\rightsquigarrow \sin x = \sum_{0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$\sin \pi = \sum_{0}^{\infty} \frac{(-1)^n (\pi)^{2n+1}}{(2n+1)!} = \pi + \sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$

$\hookrightarrow = \sum_{n=1}^{\infty} \frac{(-1)^n \pi^{2n+1}}{(2n+1)!}$

$= \sin \pi - \pi = -\pi$ conv