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ملخص

معادلات تفاضلية عادية 1

إعداد : مجدولين سمارة

New 2018

اللجنة الأكاديمية للهندسة المدنية

تلخيص

معادلات تفاضلية عادية

مجدولين سمارة

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$$* \quad \sec(x) = \frac{1}{\cos(x)}$$

$$\csc(x) = \frac{1}{\sin(x)}$$

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

- same for sinh(x), cosh(x) ...

$$* \quad \sin(-x) = -\sin(x)$$

$$\tan(-x) = -\tan(x)$$

$$\cos(-x) = \cos(x)$$

- same for sinh(-x), cosh(-x), ...

$$* \quad \sin^2(x) + \cos^2(x) = 1$$

$$\cosh^2(x) - \sinh^2(x) = 1$$

$$\sec^2(x) = 1 + \tan^2(x)$$

$$\csc^2(x) = 1 + \cot^2(x).$$

$$* \quad \sin(2x) = 2\sin(x)\cos(x)$$

$$\cos(2x) = \begin{cases} \cos^2(x) - \sin^2(x) \\ 2\cos^2(x) - 1 \\ 1 - 2\sin^2(x). \end{cases}$$

$$\sinh(2x) = 2\sinh(x)\cosh(x).$$

$$* \quad \cosh(x) = \frac{e^{x} + e^{-x}}{2}$$

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2}$$

نفس الاشارة

$$* \quad \sin(a \pm b) = \sin(a)\cos(b) \pm \cos(a)\sin(b).$$

عكس الاشارة :

$$\cos(a \pm b) = \cos(a)\cos(b) \mp \sin(a)\sin(b).$$

$$* \quad \sin(a)\cos(b) = \frac{\sin(a+b)}{2} + \frac{\sin(a-b)}{2}$$

$$\cos(a)\sin(b) = \frac{\sin(a+b)}{2} - \frac{\sin(a-b)}{2}$$

$$\cos(a)\cos(b) = \frac{\cos(a+b)}{2} + \frac{\cos(a-b)}{2}$$

$$\sin(a)\sin(b) = \frac{\cos(a+b)}{2} - \frac{\cos(a-b)}{2}$$

- same for sinh(x), cosh(x) ... -

* ملاحظة :

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \left(\frac{1}{x^2+1} \cdot \text{مشتقة الزاوية}\right)$$

- المتطابقات المطلوبة في المادة ...

- تلخيص دق 1 سكند + فاينل - مجدولين سماره.

انواع المعادلات :

* طريقة الحل :

نستبدل كل y ب r أس رتبة المشتقة .

1) Homogeneous $\rightarrow y'' + p(x) y' + q(x) y = \underline{\underline{0}}$ ← zero

عندما نكرنه بدلاله r

$= r^2 + p(x) r^1 + q(x) r^0 = 0$

تسمى auxillary eq

$= r^2 + p(x) r + q(x) = 0$

* طريقة الحل :

نفس ال homog. لكن نستخدم فرضا ل g(x) سيشرح في وقته .

2) non-homogeneous $\rightarrow y'' + p(x) y' + q(x) y = \underline{\underline{g(x)}}$ ← ثابت / متغير بدلالة x.

not-Zero

$= r^2 + p(x) r + q(x) = \cdots\cdots$

3) caushy $\rightarrow a x^2 y'' + b x y' + c y = \square$ ← $\overset{0}{g}(x)$.

كيف نميزها caushy. → $\underline{\underline{2}}$ أس x ، مشتقة $\underline{\underline{2}}$ ، أس x $\underline{\underline{1}}$ ، مشتقة $\underline{\underline{1}}$ ، لا يوجد مشتقة وأس ال x = 0.

الحل سيكون بدلالة t

* طريقة الحل $\xrightarrow[\boxed{t = \ln x}]{\text{تحول إلى homog}}$ $a y'' + (b-a) y' + c y = \square$

حفظ ، قانون التحويل من caushy الى homog.

وعند التكرار نقرب بـ t

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* Examples: (للتمييز بين انواع المعادلات)

1) $y'' + \sin(x) y' + y = 0 \longrightarrow$ homog.

2) $x^2 y'' + y' + x y = \dfrac{1}{2x} \longrightarrow$ non-homog. (الطرف الذي بعد المساوي ليس صفر)

3) $2 x^2 y'' + 5 x y' + 6 y = 0 \longrightarrow$ caushy.

4) $x(x^2 y'' + 5 x y' + y) = 0 \longrightarrow x^3 y'' + 5 x^2 y' + x y = 0$ (ليست caushy)

استراتيجية الحل

- r diff roots → $y = e^{rx}$ (* جذور مختلفة)
- r repeted roots → $y_1 = e^{rx}$, $y_2 = \underline{x} e^{rx}$ (* جذور مكررة)
- complex roots → $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ → $e^{\alpha x} \sin(\beta x)$, $e^{\alpha x} \cos(\beta x)$.

  المميز سالب $= b^2 - 4ac < 0$

  $r = \alpha \pm \beta i$

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Homog:

① **Solve**: $y'' - 5y' + 6y = 0$

sol → $r^2 - 5r + 6 = 0$

$(r-3)(r-2) = 0$

$\therefore r_1 = 3 \;,\; r_2 = 2$ (diffrent roots)

$\therefore y_1 = e^{3x} \;,\; y_2 = e^{2x}$

First sol. — second sol.

general sol. or complementry sol. → $y_c = c_1 e^{3x} + c_2 e^{2x}$

لاحظ
الحل العام للـ homog
ظهر فيه e
والـ r هي
أس الـ e.

② $y'' - 2y' + y = 0$.

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r_1 = 1 \;,\; r_2 = 1$ (repeted roots)

$\therefore y_1 = e^{1x} \;,\; y_2 = x e^{1x}$

لانه تكرر

$y_c = c_1 e^{x} + c_2 x e^{x}$.

③ Find the general sol. of:

$y'' + 2y' + 3y = 0$

sol $\underset{a}{\boxed{1}} r^2 + \underset{b}{\boxed{2}} r + \underset{c}{\boxed{3}} = 0$

المميز

$\Delta = b^2 - 4ac$
$= 4 - 4 \cdot 1 \cdot 3$
$= 4 - 12 = -8$

بما أنه المميز سالب
إذاً complex root

$r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{-2 \pm \sqrt{-8}}{2 \cdot 1}$

$\sqrt{-8} = \sqrt{-1} \cdot \sqrt{8} = i \cdot \sqrt{8}$

$= \dfrac{-2}{2} \pm \dfrac{\sqrt{8}\, i}{2} = \underbrace{-1}_{\alpha} \pm \underbrace{\dfrac{\sqrt{8}}{2}}_{\beta} i$

$\therefore y_1 = e^{\alpha x} \sin(\beta x) = e^{-1x} \sin(\frac{\sqrt{8}}{2} x)$

$y_2 = e^{\alpha x} \cos(\beta x) = e^{-1x} \cos(\frac{\sqrt{8}}{2} x)$.

$\therefore y_c = c_1 e^{-x} \sin(\frac{\sqrt{8}}{2} x) + c_2 e^{-x} \cos(\frac{\sqrt{8}}{2} x)$

* Find the general sol. or/How many sol. we have?

$(r+5)^2 (r^2 - 9)^2 (r^2 + 3) = 0$

sol:

* $(r+5)^2 = 0 \to (r+5)(r+5) = 0 \;\therefore r_1 = -5 ,\; r_2 = -5 \to y_1 = e^{-5x} ,\; y_2 = x e^{-5x}$

* $(r^2 - 9)^2 = 0 \to (r^2 - 9)(r^2 - 9) = 0$

$\Rightarrow (r-3)(r+3) \; (r-3)(r+3) = 0$

$r_3 = 3 \quad r_4 = -3 \quad r_5 = 3 \quad r_6 = -3$

$y_3 = e^{3x} ,\; y_4 = e^{-3x}$

$y_5 = x e^{3x} \quad y_6 = x e^{-3x}$

* $(r^2 + 3) = 0 \to r^2 = -3 \to r = \pm\sqrt{-3} \to r = \pm\sqrt{3}\, i \;\therefore \alpha = 0 ,\; \beta = \sqrt{3}$

$y_7 = e^{0x} \cos(\sqrt{3} x) ,\; y_8 = e^{0x} \sin(\sqrt{3} x)$.

* general sol: $y_c = c_1 e^{-5x} + c_2 x e^{-5x} + \cdots\cdots + c_8 \sin(\sqrt{3} x)$.

we have 8 sol.s.

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يعني بده الي بدلالة r .

* Find the auxillary eq of the DE:

$$① \; y_c = c_1 e^{\overset{r_1}{\boxed{1}}x} + c_2 x e^{\overset{r_2}{\boxed{1}}x} + c_3 e^{\overset{r_3}{\boxed{2}}x}$$

اصلها homog لانه
ال general sol. ظهر فيه $\underline{e}$.
وال $\underline{r}$ هي أس ال $\underline{e}$.

$r = 1 \;,\; r = 1 \;,\; r = 2$

$(r-1)(r-1)(r-2) = 0 .$

$= r^3 - 3r + 2 = 0$ ← the auxillary eq.

$= y''' - 3y' + 2y = 0 .$

$$② \; y = e^{\overset{\alpha}{2}x} \cos(\overset{\beta}{3}x) .$$

فيها cos اذاً
الجذور complex
و r على صورة
$\alpha \pm \beta i$

$\alpha = 2 \;,\; \beta = 3 .$

$\therefore r = \alpha \pm \beta i$

$\therefore r = 2 \pm 3i$

$\Rightarrow r - 2 = \pm 3i$ ← نربع الطرفين
لكي نتخلص من i
وال $-1 = (i)^2$

$\therefore (r-2)^2 = (\pm 3i)^2$

$\therefore r^2 - 4r + 4 = -9$
$\quad +9 \qquad +9$

$\therefore r^2 - 4r + 13 = 0 .$

$y'' - 4y' + 13y = 0$ #

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* Caushy :-

تكون بدلالة $\underline{t}$
عند التكرار نضرب ب $\underline{t}$

solve: $① \; \underset{a}{\boxed{\;}} x^2 y'' \underset{b}{-2} x y' \overset{c}{-4} y = 0$

$\Downarrow \; \boxed{t = \ln x}$

عند ال caushy نحولها ل homog عن طريق القانون → $ay'' + (b-a)y' + cy = 0 .$

$\Rightarrow 1y'' + (-2-1)y' + -4y = 0 .$

$y'' - 3y' - 4y = 0 .$

$r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0$

$r_1 = 4 \;,\; r_2 = -1$ (diff rout) .

$y_1 = e^{4t} \;,\; y_2 = e^{-1t}$ (but $t = \ln x$)

$\therefore y_1 = e^{4\ln x} \;,\; y_2 = e^{-1\ln x}$

$y_1 = e^{\ln x^4} \;,\; y_2 = e^{\ln x^{-1}}$

$y_1 = x^4 \qquad y_2 = x^{-1} \longrightarrow y_c = c_1 \underline{x}^4 + c_2 \underline{x}^{-1}$

لاحظ
الحل العام لل
Caushy
ظهر فيه $\underline{x}$
وال r اس ال x .

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② $x^2y'' + xy' + 16y = 0$ (caushy)

$t = \ln x$

عالقانون → $ay'' + (b-a)y' + cy = 0$

$y'' + (1-1)y' + 16y = 0$

$y'' + 16y = 0$

$r^2 + 16 = 0$

$r^2 = -16$

$r = \pm\sqrt{-16}$

$r = \pm 4$

$\therefore \alpha = 0 , \beta = 4$

$\therefore y_1 = e^{0t}\cos(4t) \xrightarrow[t=\ln x]{\text{but}} = \cos(4\ln x)$

$y_2 = e^{0t}\sin(4t) = \sin(4\ln x)$

$\therefore y_c = c_1\cos(\ln x^4) + c_2\sin(\ln x^4)$

③ $y_c = c_1 x^{\boxed{-1}\,r_1} + c_2 x^{\boxed{-4}\,r_2}$ (caushy)

Sol

$r_1 = -1$, $r_2 = -4$

لانه فيها $x^{\square}$

$\therefore (r+1)(r+4) = 0$

$\therefore \boxed{\ }r^2 + \boxed{5}r + \boxed{4} = 0$

a b-a c

$y'' + 5y' + 4y = 0$

بما انها caushy فرضنا اصلاً على صورة $ay'' + (b-a)y' + cy$

شكل ال caushy $\boxed{ax^2y'' + bxy' + cy = 0}$

$\therefore b - a = 5$

$b - 1 = 5$

$\therefore b = 6$

$1x^2y'' + 6xy' + 4y = 0$. #

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* Wronskian:

قانون 1

$$w(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

det

** نستخدم هذا القانون اذا كانت معطيني y_1 و y_2

** اذا كان ال det ≠ 0 فهو linearly indep.

أمّا اذا كان ال det = 0 فهيي linearly dep.

* عندما يكون فوق القطر الرئيسي او تحته اصفار او كلاهما فإن ال det هو حاصل ضرب اعداد القطر الرئيسي :

مثلا

$$6\times4\times1 = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}$$ او $$4 = 2\times1\times2 = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix}$$

$24 =$

$$* \; w(y_1, y_2, y_3)(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

موقع y_1 الصف الاول والعمود الاول

الصف الاول ← العمود الاول

ما تبقى عندما نشطب الصف والعمود الذي يحوي y_1

$$= y_1(-1)^{1+1}\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} + y_2(-1)^{1+2}\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3(-1)^{1+3}\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

$$= (-1)^2 y_1\begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} + (-1)^3 y_2\begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + (-1)^4 y_3\begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

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*Ex: Find $w(\underbrace{\cos(2x)}_{y_1}, \underbrace{\sin(2x)}_{y_2})$ or show that $y_1 = \cos(2x)$ and $y_2 = \sin(2x)$ are Linearly independent ?

sol

$$w(\cos(2x), \sin(2x)) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix} = 2\cos^2(2x) - -2\sin^2(2x)$$

$$= 2(\cos^2(2x) + \sin^2(2x))$$

$$= 2 * 1 = \underline{2} \ \#$$

$\therefore w(y_1, y_2)(x) = 2 \neq 0 \ \therefore y_1 \ \& \ y_2$ Linearly indep.

قانون 2

* $w(y_1, y_2)(x) = c \cdot e^{-\int p(x)\,dx}$ معامل y'

** لازم لما اطلق عالقانون يكون معامل $y'' = 1$.

** نستخدم هذا القانون اذا كان معطيني معادلة وبدو ال w .

* Ex: IF y_1 and y_2 sols For the D.E $[xy'' + (x-1)y' + 3y = 0]$, $x > 0$ Find $w(y_1, y_2)(x)$?

sol

$$\frac{\cancel{x}\,y''}{\cancel{x}} + \frac{(x-1)}{x}y' + \frac{3y}{x} = \frac{0}{x} \longleftarrow$$ عشان اطلق عالقانون لازم معامل $y'' = 1$

$$\rightarrow y'' + \underbrace{\left(\frac{x-1}{x}\right)}_{P(x)} y' + \frac{3}{x}y = 0$$

$$\therefore w(y_1, y_2)(x) = c \cdot e^{-\int \frac{x-1}{x}dx} \longrightarrow c \cdot e^{-\int 1 - \frac{1}{x}dx} \longrightarrow c \cdot e^{\ln x - x} \longrightarrow c\,x \cdot e^{-x}$$

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*Ex: $x^2y'' - 2y' + (3+x)y = 0$, and $w(y_1, y_2)(2) = 3$, Find $w(y_1, y_2)(5)$?

sol

$$x^2y'' - 2y' + (3+x)y = 0 \quad (\div x^2)$$

$$\therefore y'' - \underbrace{\frac{2}{x^2}}_{P(x)} y' + \left(\frac{3+x}{x^2}\right)y = 0 .$$

now: $w(y_1, y_2)(x) = c \cdot e^{-\int \frac{-2}{x^2}dx}$

$$= c \cdot e^{-\int -2x^{-2}dx}$$

$$w(y_1, y_2)(x) = c \cdot e^{\frac{-2}{x}}$$

$$w(y_1, y_2)(x) = 3e \cdot e^{\frac{-2}{x}}$$

$$\therefore w(y_1, y_2)(5) = 3 \cdot e^{\frac{5 \times 1}{5 \times 1}} \cdot e^{\frac{-2}{5}}$$

$$= 3e^{\frac{3}{5}} \ \#$$

To find the value of $\underline{(c)}$.

$$w(y_1, y_2)(2) = c \cdot e^{\frac{-2}{2}}$$

$$\frac{3}{e^{-1}} = c \cdot \frac{\cancel{e^{-1}}}{e^{-1}}$$

$$\therefore \boxed{c = 3e^1}$$

* extra ex: IF $w(y_1, y_2)(x) = xe^{-x}$, For $y'' + p(x)y' + \frac{3}{x}y = 0$.
Find $p(x)$.

Sol:

$$w(y_1, y_2)(x) = c \cdot e^{-\int p(x)\,dx}$$

$$xe^{-x} = c \cdot e^{-\int p(x)\,dx}$$

ناخذ ln للطرفين حتى نخلص من e.

$$\ln(xe^{-x}) = \ln(c \cdot e^{-\int p(x)\,dx})$$

من خواص ال ln انه في حالة الضرب بتحول الى جمع

$$\Rightarrow \ln(x) + \ln(e^{-x}) = \ln(c) + \ln e^{-\int p(x)\,dx}$$

$$\ln(x) + (-x) = \ln(c) + -\int p(x)\,dx$$

$$\ln(x) - x = \ln(c) - \int p(x)\,dx$$

$$\therefore \left(\int p(x)\,dx\right)' = \left(\underbrace{\ln(c)}_{\text{ثابت}} - \ln(x) + x\right)'$$

نشتق حتى نخلص من التكامل ونجد قيمة p(x)

$$\therefore p(x) = 0 - \frac{1}{x} + 1$$

$$\therefore \boxed{p(x) = 1 - \frac{1}{x}}$$

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* قانون 3

$$y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

* نستخدم هذا القانون اذا كان معطيني y_1 وبدو y_2 او الحل العام.

* لازم معامل $y'' = 1$.

* Example: if $x^2y'' + 2xy' - 2y = 0$, $x>0$, and the first sol. $y_1 = x$,
Find the general sol. or Find the second sol. (y_2)?

Sol:

بعد القسمة على x^2

$$y'' + \underbrace{\frac{2}{x}}_{p(x)}y' - \frac{2}{x^2}y = 0$$

$$\therefore y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$= x \cdot \int \frac{e^{-\int \frac{2}{x}dx}}{x^2}\,dx$$

$$= x \cdot \int \frac{e^{-2\ln x}}{x^2}\,dx$$

$$= x \int \frac{\cancel{e}^{\cancel{\ln} x^{-2}}}{x^2}\,dx$$

$$= x \int \frac{x^{-2}}{x^2}\,dx$$

$$\rightarrow y_2 = x \int x^{-4}\,dx$$

$$y_2 = x \cdot \frac{x^{-3}}{-3}$$

$$y_2 = \frac{-1}{3} \cdot x^{-2}$$

اذا طلب y_2 هاي الجواب #

$$\therefore y_c = c_1 \cdot x + c_2 \cdot \frac{-1}{3} x^{-2}$$

ممكن يدمج الثابت مع $\frac{-1}{3}$ فيصبح الجواب

$$y_c = c_1 x + c_2 x^{-2}$$

#

* non-homog.

حفظ
جدول الفرضيات لـ $g(x)$:

$g(x)$	y_p ← particular sol.
ثابت → 3	A
عبارة خطية → x	$Ax+B$
عبارة تربيعية → x^2+2	Ax^2+Bx+C
$3e^{\alpha x}$	$Ae^{\alpha x}$
$xe^{\alpha x}$	$(Ax+B)e^{\alpha x}$
$\sin(\beta x)$	$A\sin(\beta x)+B\cos(\beta x)$
$e^{\alpha x}\cos(\beta x)$	$Ae^{\alpha x}\cos(\beta x)+Be^{\alpha x}\sin(\beta x)$
$xe^{\alpha x}\sin(\beta x)$	$(Ax+B)e^{\alpha x}\sin(\beta x)+(Cx+D)e^{\alpha x}\cos(\beta x)$

عندما تكون المعادلة خطية ننتبه اذا وجد تكرار للصفر في الجزء ال homog

** non-homog : $\underbrace{y''+p(x)y'+q(x)y}_{\text{general sol, or Complementary sol. } (y_c)} = \underbrace{g(x)}_{\text{particular sol. } (y_p)}$

* $y = y_c + y_p$ ← general sol

---

* Example: Find the general sol. $[y''-9y=2x-1]$. ← non-homog

sol:

$y''-9y = 2x-1$ ← خطية.

$y''-9y=0$

$r^2-9=0$

$(r-3)(r+3)=0$

$r_1=3$, $r_2=-3$ (diff roots).

$y_1=e^{3x}$

$y_2=e^{-3x}$

$y_c=c_1e^{3x}+c_2e^{-3x}$

$y_p = Ax+B$ ← the Form

$y'=A$

$y''=0$

نعوض

$\Rightarrow 0-9(Ax+B)=2x-1$

$-9Ax-9B=2x-1$

$\therefore -9A=2 \rightarrow \boxed{A=\frac{-2}{9}}$

$-9B=-1 \rightarrow \boxed{B=\frac{1}{9}}$

بما انها خطية ننتبه اذا وجد تكرار للصفر في الحل ال homog. لكن الجذور هنا هي 3 و -3 لا يوجد صفر فلا نضرب بـ x

$\therefore y_p = \frac{-2}{9}x+\frac{1}{9}$ ← particular sol.

$\therefore y = y_c+y_p$ ← general sol

$\therefore y = \underbrace{c_1e^{3x}+c_2e^{-3x}}_{\text{يحوي c}} + \underbrace{\frac{-2}{9}x+\frac{1}{9}}_{\text{لا يحوي c}}$

لاحظ عندما تكون المعادلة non-homog فإن الحل العام لا يظهر في كل الحدود c والذي لم يظهر فيه c هو yp

*Ex: Find the Form of particular sol.

$$y'' + y = \sin(x)\cos(2x)$$

متى يصبح بإمكاني استخدام الفرض. نستخدم متطابقة → $\sin a \cdot \cos b = \frac{\sin(a+b)}{2} + \frac{\sin(a-b)}{2}$

Sol:

$y'' + y = 0$

$r^2 + 1 = 0$

$r = \pm i$

$\alpha = 0$, $\beta = 1$

$\therefore y_1 = \cos(x)$

$y_2 = \sin(x)$

$\therefore y_c = C_1\cos(x) + C_2\sin(x)$.

$= \frac{\sin(3x)}{2} + \frac{\sin(-x)}{2} = \frac{1}{2}\sin(3x) - \frac{1}{2}\sin(x)$.

$y_p = A\sin(3x) + B\cos(3) + Cx\sin(x) + Dx\cos(x)$

Form

ضربت ب x بسبب تكرار y_c ب $\cos(x)$ و $\sin(x)$

---

*Ex: Find the general sol. $[x^2y'' - xy' + y = \ln x]$

حولناها homog حسب القانون $ay'' + (b-a)y' + cy =$ —

sol: $x^2y'' - xy' + y = \ln(x) \xrightarrow[t=\ln x]{\text{(Caushy)}} \boxed{y'' - 2y' + y = t}$ ← خطية

$\therefore y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$

$r = 1$, $r = 1$

$y_1 = e^t \to e^{\ln x} \to x$

$y_2 = te^t \to (\ln x)e^{\ln x} \to x\ln(x)$

$y_c = C_1x + C_2x\ln(x)$

$y_p = At + B$ ← خطية ولا يوجد تكرار للصفر لا نضرب ب t.

$y' = A$

$y'' = 0$

$\therefore 0 - 2A + At + B = t$

$\therefore \boxed{A = 1}$

$-2A + B = 0 \to -2 + B = 0$ $\therefore \boxed{B = 2}$

$y_p = t + 2$ but $t = \ln x$

$\therefore y_p = \ln x + 2$

$\therefore y = y_c + y_p = \dots$

---

*Ex: $y''' - 2y'' + y' = x + 2e^{2x} - \cos(2x)$, Find the Form of y_p.

sol: $y''' - 2y'' + y' = 0$

$r^3 - 2r^2 + r = 0$

$r(r^2 - 2r + 1) = 0$

$r = 0$, $r = 1$, $r = 1$

$y_c = C_1e^{0x} + C_2e^x + C_2xe^x$

$y_p = (Ax + B)x + Ce^{2x} + (D\cos(2x) + E\sin(2x))$

نضرب المعادلة الخطية ب x بسبب تكرار الصفر مرة واحدة ولو تكرر مرتين نضرب ب x^2 وهكذا..

$\therefore y_p = Ax^2 + Bx + \dots$

Ex: $y^{(6)} - 81y'' = 7x + \sin(3x) + e^{-3x}$, Find the form of y_p.

sol:

$y^{(6)} - 81y'' = 0$

$r^6 - 81r^2 = 0$

$r^2(r^4 - 81) = 0$

$r^2(r^2 - 9)(r^2 + 9) = 0$

$r^2(r-3)(r+3)(r^2+9) = 0$

$r = 0, 0, 3, -3, \pm 3i$

$y_1 = c_1 \;/\; y_2 = c_2 x \;/\; y_3 = e^{3x}$

$y_4 = e^{-3x} \;/\; y_5 = \sin(3x)$

$y_6 = \cos(3x)$.

$y_p = (Ax+B)x^2 + C\sin(3x)x + D\cos(3x)x + Exe^{-3x}$

نضرب المعادلة كلها بـ x^2 لتكرار الصفر مرتين

نضرب بـ x لتكرار $\sin(3x)$ مرة واحدة.

نضرب بـ x لتكرار $\cos(3x)$ مرة واحدة

نضرب بـ x لتكرار e^{-3x} مرة واحدة

---

* if $y = \underbrace{c_1 x + c_2 x \ln x}_{} + \underbrace{\ln x + 2}_{}$, Find the D.E.

بما انها تحوي c اذاً هاي y_c وبما انه على صورة $x^{\square}$ فهي Caushy.

لا تحوي c اذاً هاي y_p. ← non-homog.

sol:

$r = 1, r = 1$

$(r-1)(r-1) = 0$

$r^2 - 2r + 1 = 0$

$\underset{a}{1}y'' \underset{b-a}{-2}y' + \underset{c}{1}y = 0$

$\therefore 1x^2y'' - xy' + y = 0$

عوض

$y_p = \ln x + 2$

$y' = \frac{1}{x}$

$y'' = \frac{-1}{x^2}$

$\Rightarrow \cancel{x^2} \cdot \frac{-1}{\cancel{x^2}} - \cancel{x} \cdot \frac{1}{\cancel{x}} + \ln x + 2 = g(x)$

$-\cancel{1} - \cancel{1} + \ln x + \cancel{2} = g(x)$

$\therefore g(x) = \ln(x)$. #

$\therefore$ the eq $\rightarrow x^2y'' - xy' + y = \ln x$. #

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اذا كانت $g(x)$ ليست من الجدول نستخدم القانون التالي :

$$y_p = v_1 y_1 + v_2 y_2$$

sol: $v_1 = \int \frac{w_1}{w} dx$, $v_2 = \int \frac{w_2}{w} dx$.

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} , \quad w_1 = \begin{vmatrix} 0 & y_2 \\ g(x) & y_2' \end{vmatrix} , \quad w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g(x) \end{vmatrix}$$

Ex: Find the general sol. of $y'' + y = \tan(x)$.

$g(x)$ ليست من الجدول ولا يمكن تعديلها $g(x)$ من الجدول فنحل على القانون السابق

sol: $y'' + y = \tan(x)$.

$y'' + y = 0$

$r^2 + 1 = 0 \rightarrow r = \pm i \rightarrow y_1 = \cos(x)$

$\alpha = 0 , \beta = 1$ $\quad y_2 = \sin(x)$.

$\therefore y_c = C_1 \cos(x) + C_2 \sin(x)$

* now to find y_p :

$$w = \begin{vmatrix} \cos(x) & \sin(x) \\ -\sin(x) & \cos(x) \end{vmatrix} = \cos^2(x) + \sin^2(x) = \boxed{1}$$

$$w_1 = \begin{vmatrix} 0 & \sin(x) \\ \tan(x) & \cos(x) \end{vmatrix} = -\sin(x)\tan(x) .$$

$$w_2 = \begin{vmatrix} \cos(x) & 0 \\ -\sin(x) & \tan(x) \end{vmatrix} = \sin(x) .$$

$$\therefore v_1 = \int \frac{-\sin(x)\tan(x)}{1} dx = -\int \frac{\sin^2(x)}{\cos(x)} dx = -\int \frac{1-\cos^2(x)}{\cos(x)} = -\int \sec(x) - \cos(x) dx$$

$$= -\ln|\sec(x) + \tan(x)| + \sin(x)$$

$$v_2 = \int \frac{\sin(x)}{1} dx = -\cos(x) .$$

$\therefore y_p = v_1 y_1 + v_2 y_2$

$y_p = (\sin(x) - \ln|\sec(x) + \tan(x)|) \cdot \cos(x) + (-\cos(x)) \cdot \sin(x)$

$y = y_c + y_p$.

لاتنسونا من صالح دعائكم ♡

majdoleen samara .

* Laplace Transformation:

$$L(f(t)) = \int_0^\infty e^{-st} \cdot f(t)\, dt$$

القانون الذي أوجد منه القواعد التالية

$f(t) = 1$ const.

Ex: Find $L(1)$

$$L(1) = \int_0^\infty e^{-st} \cdot 1\, dt$$

$$= \lim_{k \to \infty} \int_0^k e^{-st}\, dt$$

$$= \lim_{k \to \infty} \left. \frac{-e^{-st}}{s} \right|_0^k$$

$$= \lim_{k \to \infty} \left(\frac{-e^{-ks}}{s} + \frac{1}{s} \right)$$

$$= \lim_{k \to \infty} \frac{-e^{-ks}}{s} + \lim_{k \to \infty} \frac{1}{s}$$

$$= 0 + \frac{1}{s}$$

$$= \frac{1}{s}$$

$$\therefore L(1) = \frac{1}{s}$$

$$L(a) = \frac{a}{s} \quad \#$$

$L^{-1}(F(s)) = f(t)$ $L(f(t)) = F(s)$

const. $a \longrightarrow \dfrac{a}{s}$

$t^{a=n} \longrightarrow \dfrac{a!}{s^{a+1}}$ ($\dfrac{n!}{s^{n+1}}$)

$e^{at} \longrightarrow \dfrac{1}{s-a}$

$\sin(at) \longrightarrow \dfrac{a}{s^2+a^2}$

$\cos(at) \longrightarrow \dfrac{s}{s^2+a^2}$

$\sinh(at) \longrightarrow \dfrac{a}{s^2-a^2}$

$\cosh(at) \longrightarrow \dfrac{s}{s^2-a^2}$

---

* Examples: , Find:

① $L(2) = \dfrac{2}{s}$ (a)

② $L(t) = \dfrac{1!}{s^{1+1}} = \dfrac{1}{s^2}$ ($a=1$)

③ $L(t^3) = \dfrac{3!}{s^{3+1}} = \dfrac{3!}{s^4}$

④ $L(e^{7t}) = \dfrac{1}{s-7}$

⑤ $L(e^{-2t}) = \dfrac{1}{s+2}$

⑥ $L(\cos\sqrt{7}t) = \dfrac{s}{s^2+(\sqrt{7})^2} = \dfrac{s}{s^2+7}$

⑦ $L(\sinh(-8t)) = \dfrac{-8}{s^2-64}$

* properties of Laplace:

① $L(\alpha f(t) \mp \beta g(t)) = \alpha L(f(t)) \mp \beta L(g(t))$

يعني: الـ Laplace بتوزع على الجمع والطرح والثابت يخرج خارج الـ "L"

② $L(e^{at} \cdot f(t)) = L(f(t))\Big|_{s \to s-a}$ → "shifting property"

③ $L(t^n \cdot f(t)) = (-1)^n \dfrac{d^n}{ds^n} [\underbrace{L(f(t))}_{F(s)}]$

---

* Examples:

① $L(3 + 2e^{-7t}) = L(3) + 2L(e^{-7t})$

$= \dfrac{3}{s} + 2 \cdot \dfrac{1}{s-7} = \dfrac{3}{s} + \dfrac{2}{s+7}$.

② $L(e^{2t+3}) = L(e^{2t} \cdot \underbrace{e^3}_{\text{const.}}) = e^3 L(e^{2t})$

$= e^3 \cdot \dfrac{1}{s-2}$.

③ $L(\sin(2t)\cos(3t)) = L\left(\dfrac{\sin(-t)}{2} + \dfrac{\sin(5t)}{2}\right)$

$= \dfrac{-1}{2} L(\sin(t)) + \dfrac{1}{2} L(\sin(5t))$

$= \dfrac{1}{2} \cdot \dfrac{1}{s^2+1} + \dfrac{1}{2} \cdot \dfrac{5}{s^2+25}$

④ $L(\cos(2t + \frac{\pi}{6})) = L(\cos(2t) \cdot \cos(\frac{\pi}{6}) - \sin(2t) \cdot \sin(\frac{\pi}{6}))$

$= L(\cos(2t) \cdot \frac{\sqrt{3}}{2} - \sin(2t) \cdot \frac{1}{2})$

$= \dfrac{\sqrt{3}}{2} L(\cos(2t)) - \dfrac{1}{2} L(\sin(2t))$

$= \dfrac{\sqrt{3}}{2} \cdot \dfrac{s}{s^2+4} - \dfrac{1}{2} \cdot \dfrac{2}{s^2+4}$ #

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(5) $L(\sinh^2(3t)) = L\left(\left(\frac{e^{3t} - e^{-3t}}{2}\right)^2\right)$

(7)

$$= \frac{1}{4} L\left(e^{6t} - 2e^{3t-3t} + e^{-6t}\right)$$

$$= \frac{1}{4} L\left(e^{6t} - 2 + e^{-6t}\right)$$

$$= \frac{1}{4}\left(\frac{1}{s-6} - \frac{2}{s} + \frac{1}{s+6}\right) \quad \#$$

shifting

(6) $L(e^{5t} \cdot \cosh(4t)) = L(\cosh(4t))\Big|_{s \to s-5}$

(3)

$$= \frac{s}{s^2 - 16}\Big|_{s \to s-5} = \frac{(s-5)}{(s-5)^2 - 16}$$

(7) $L(\cosh(2t) \cdot \sin(3t)) = L\left(\left(\frac{e^{2t} + e^{-2t}}{2}\right) \cdot \sin(3t)\right)$

(8)

$$= \frac{1}{2} L\left(e^{2t} \cdot \sin(3t)\right) + \frac{1}{2} L\left(e^{-2t} \cdot \sin(3t)\right)$$

$$= \frac{1}{2} \cdot L(\sin(3t))\Big|_{s \to s-2} + \frac{1}{2} L(\sin(3t))\Big|_{s \to s+2}$$

$$= \frac{1}{2} \cdot \frac{3}{s^2+9}\Big|_{s \to s-2} + \frac{1}{2} \cdot \frac{3}{s^2+9}\Big|_{s \to s+2}$$

1 = t أس 3

$$= \frac{1}{2} \frac{3}{(s-2)^2+9} + \frac{1}{2} \cdot \frac{3}{(s+2)^2+9}$$

(8) $L(t \cdot \sin(3t)) = (-1)^1 \cdot \frac{d^1}{ds^1} L[\sin(3t)]$

(4)

$$= -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$$

$$= -\frac{-3(2s)}{(s^2+9)^2} = \frac{6s}{(s^2+9)^2}$$

(9) $L(t^2 \cdot e^{3t}) = (-1)^2 \cdot \frac{d^2}{ds^2} L(e^{3t})$

المشتقة الثانية ←

$$= \frac{d^2}{ds^2}\left(\frac{1}{s-3}\right)$$

$$= \frac{2s-6}{(s-3)^4}$$

or $= \frac{2}{(s-3)^3}$ #

$= \frac{1}{s-3}$

المشتقة الاولى

$= \frac{-1}{(s-3)^2}$

المشتقة الثانية

$= \frac{--1 * 2(s-3)}{(s-3)^4}$

$= \frac{2s-6}{(s-3)^4}$

or $= \frac{2\cancel{(s-3)}}{(s-3)^{\cancel{4}}}$

$\boxed{= \frac{2}{(s-3)^3}}$

(10) $L(t \cdot e^{-2t} \cdot \sin(2t)) = L(t\sin(2t))\Big|_{s \to s+2}$

اشتق بعدين ← اعمل shifting .

$$= (-1)^1 \frac{d}{ds} L(\sin(2t))\Big|_{s \to s+2}$$

$$= -1 \frac{d}{ds}\left(\frac{2}{s^2+4}\right)\Big|_{s \to s+2}$$

$$= \cancel{-}\frac{\cancel{-}2 \cdot 2s}{(s^2+4)^2}\Big|_{s \to s+2} = \frac{4s}{(s^2+4)^2}\Big|_{s \to s+2} = \frac{4(s+2)}{((s+2)^2+4)^2}$$

or

اعمل shifting بعدين اشتق ←

$$= (-1)^1 \frac{d}{ds} L(e^{-2t}\sin(2t))$$

$$= -\frac{d}{ds}\left(\frac{2}{s^2+4}\Big|_{s \to s+2}\right)$$

$$= -\frac{d}{ds}\left(\frac{2}{(s+2)^2+4}\right)$$

$$= \cancel{-}\frac{\cancel{-}2 \cdot 2(s+2)}{((s+2)^2+4)^2}$$

$$= \frac{4(s+2)}{((s+2)^2+4)^2}$$ #

* Inverse of Laplace:

$$\mathcal{L}(f(t)) = F(s)$$

$$\to \cancel{\mathcal{L}^{-1}}\cancel{\mathcal{L}}(f(t)) = \mathcal{L}^{-1}(F(s))$$

$$\to \boxed{f(t) = \mathcal{L}^{-1}(F(s))}.$$

* Examples:

① $\mathcal{L}^{-1}\left(\frac{2}{s}\right) = 2$

② $\mathcal{L}^{-1}\left(\frac{\overset{2! = a!}{2}}{\underset{2+1=a+1}{s^3}}\right) = t^2 \sim \boxed{\mathcal{L}^{-1}\left(\frac{a!}{s^{a+1}}\right)}$

③ $\mathcal{L}^{-1}\left(\frac{3}{s^5}\right) = \frac{3}{4!}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) = \frac{3}{4!} \cdot t^4$

④ $\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) = e^{-4t} \sim \boxed{\frac{1}{s--a}}$

⑤ $\mathcal{L}^{-1}\left(\frac{s}{s^2+9}\right) = \cos(3t)$

$(3)^2$

⑥ $\mathcal{L}^{-1}\left(\frac{1}{s^2+7}\right) = \frac{1}{\sqrt{7}}\mathcal{L}^{-1}\left(\frac{\sqrt{7}}{s^2+7}\right) = \frac{1}{\sqrt{7}}\sin(\sqrt{7}t).$

a

a^2

$a^2 = 7$

$\underline{a = \sqrt{7}}$

⑦ $\mathcal{L}^{-1}\left(\frac{1}{4s^2-3}\right) = \mathcal{L}^{-1}\left(\frac{1}{4(s^2-\frac{3}{4})}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s^2-\frac{3}{4}}\right)$

$a^2 = \frac{3}{4}$

$\underline{a = \frac{\sqrt{3}}{2}}$

$$= \frac{\frac{1}{4}}{\frac{\sqrt{3}}{2}}\mathcal{L}^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{s^2-\frac{3}{4}}\right)$$

$= \frac{1}{4} \div \frac{\sqrt{3}}{2}$

$= \frac{1}{\cancel{4}_2} \times \frac{\cancel{2}}{\sqrt{3}}$

$= \frac{1}{2\sqrt{3}}.$

$$= \frac{1}{2\sqrt{3}}\sinh\left(\frac{\sqrt{3}}{2}t\right)$$

⑥

2 ⑨ $L^{-1}\left(\frac{s}{(s-2)^2+16}\right) = L^{-1}\left(\frac{s-2+2}{(s-2)^2+16}\right) = L^{-1}\left(\frac{s-2}{(s-2)^2+16}\right) + L^{-1}\left(\frac{2}{(s-2)^2+16}\right)$

لازم تكون s-2

$= e^{2t}\cos(4t) + \frac{1}{2}e^{2t}\sin(4t)$

1 ⑧ $L^{-1}\left(\frac{s}{s^2-\sqrt{e}}\right) = \cosh(\sqrt[4]{e}\,t)$

$a^2 = e^{\frac{1}{2}}$

$a = e^{\frac{1}{4}} \rightarrow a = \sqrt[4]{e}$

⑩ $L^{-1}\left(\frac{1}{s^2-2s-3}\right) = L^{-1}\left(\frac{1}{\underbrace{s^2-2s+1}_{(s-1)(s-1)}\underbrace{-1-3}}\right) = L^{-1}\left(\frac{1}{(s-1)(s-1)-4}\right)$

وإكمال المربع يكون ب ± $\left(\frac{\text{معامل } s}{2}\right)^2$

طريقة ① لإكمال المربع

$= \frac{1}{2}L^{-1}\left(\frac{1\times 2}{(s-1)^2-4}\right) = \frac{1}{2}e^{1t}\sinh(2t) = \frac{e^t}{2}\cdot\frac{e^{2t}-e^{-2t}}{2}$

$= \frac{e^{3t}-e^{-t}}{4}$

توحيد مقامات

طريقة ② كسور جزئية (partial Fraction)

$\rightarrow L^{-1}\left(\frac{1}{s^2-2s-3}\right) \Rightarrow \frac{1}{(s-3)(s+1)} = \frac{A}{s-3}\,(s+1) + \frac{B}{s+1}\,(s-3)$

$\rightarrow \quad 1 = A(s+1) + B(s-3)$

$s=-1 \rightarrow 1 = A(0) + B(-4) \Rightarrow \boxed{B = \frac{-1}{4}}$

$s=3 \rightarrow 1 = A(4) + B(0) \rightarrow \boxed{A = \frac{1}{4}}$

$\rightarrow L^{-1}\left(\frac{\frac{1}{4}}{s-3} - \frac{\frac{1}{4}}{s+1}\right) = \frac{1}{4}L^{-1}\left(\frac{1}{s-3}\right) + \frac{-1}{4}L^{-1}\left(\frac{1}{s+1}\right)$

$= \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}$ #

⑪ $L^{-1}\left(\frac{3}{(s^2+1)(s^2+4)}\right)$ sol 1 → by partial Fraction $\frac{3}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{s^2+4}$ ....

كسور جزئية

sol 2 or

$\Rightarrow 3\cdot\frac{1}{-3}L^{-1}\left(\frac{(s^2+1)-(s^2+4)}{(s^2+1)(s^2+4)}\right)$

$s^2+1-s^2-4 = -3$

$= \frac{1}{2}\cdot -L^{-1}\left(\frac{\cancel{s^2+1}\cdot 2}{\cancel{(s^2+1)}(s^2+4)}\right) + L^{-1}\left(\frac{\cancel{s^2+4}}{(s^2+1)\cancel{(s^2+4)}}\right)$

$= \frac{-1}{2}\sin(2t) + \sin(t)$.

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* we know: $L(t^n f(t)) = (-1)^n \dfrac{d^n}{ds^n}[\underbrace{L(f(t))}_{F(s)}]$

when $n=1$ $\rightarrow$ $L(t\, f(t)) = (-1)\dfrac{d}{ds}[F(s)]$

نشغل L^{-1} للطرفين

نقسم ع t $\rightarrow$ $t\, f(t) = (-1) L^{-1}\dfrac{d}{ds}[F(s)]$

$$f(t) = \frac{-1}{t} L^{-1} \frac{d}{ds}[F(s)]$$

but $f(t) = L^{-1}[F(s)]$ $\Rightarrow$ $\boxed{L^{-1}[F(s)] = \dfrac{-1}{t} L^{-1}\dfrac{d}{ds}[F(s)]}$

لما يطلب مني L^{-1} لاقترانه

مش من الجدول

Ex 1 Find $L^{-1}\left\{\ln\left(\dfrac{s^2-9}{s+1}\right)\right\}$

sol: $= L^{-1}\left(\ln(s^2-9) - \ln(s+1)\right)$

$= L^{-1}(\ln(s^2-9)) - L^{-1}(\ln(s+1))$

$= \dfrac{-1}{t} L^{-1}\left(\dfrac{2s}{s^2-9}\right) - \dfrac{-1}{t} L^{-1}\left(\dfrac{1}{s+1}\right)$

$= \dfrac{-2}{t}\cosh(3t) + \dfrac{1}{t} e^{-t}$ #

2 $L^{-1}\left(\tan^{-1}\left(\dfrac{s}{3}\right) + \dfrac{\pi}{2}\right)$ $\longrightarrow$ مشتقة $\tan^{-1}x = \dfrac{1}{1+x^2} \cdot$ (مشتقة الزاوية)

$= \dfrac{-1}{t} L^{-1}\left(\dfrac{1}{3} \cdot \dfrac{1}{\frac{s^2}{3^2} + \frac{1 \times 9}{9}} + 0\right)$

$= \dfrac{-1}{3t} L^{-1}\left(\dfrac{1 \times 9}{s^2+9}\right)$

$= \dfrac{-3}{3 \times t} L^{-1}\left(\dfrac{1 \times 3}{s^2+9}\right)$

$= \dfrac{-1}{t}\sin(3t).$

* IF $L\left(\frac{1}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}}$, Find: $L\left(t \cdot \sqrt{\frac{3}{\pi t}}\right)$?

sol

$$L\left(t \cdot \sqrt{\frac{3}{\pi t}}\right) = (-1)^1 \frac{d}{ds} L\Big(\underbrace{\sqrt{\frac{3}{\pi}}}_{\text{const.}} \cdot \frac{1}{\sqrt{t}}\Big)$$

$$= -\frac{\sqrt{3}}{\sqrt{\pi}} \frac{d}{ds} L\left(\frac{1}{\sqrt{t}}\right).$$

$$= \frac{-\sqrt{3}}{\cancel{\sqrt{\pi}}} \frac{d}{ds}\Big(\underbrace{\cancel{\sqrt{\pi}}}_{\text{Const}} \cdot \frac{1}{\sqrt{s}}\Big)$$

$$= -\sqrt{3} \frac{d}{ds}\left(\frac{1}{\sqrt{s}}\right)$$

$$= \cancel{-}\sqrt{3}\left[\frac{\cancel{-}1 \cdot \frac{1}{2\sqrt{s}}}{(\cancel{\sqrt{s}})^{\cancel{2}}}\right]$$

$$= \sqrt{3} \times \frac{1}{2\sqrt{s}} \div s$$

$$= \sqrt{3} \cdot \frac{1}{2\sqrt{s}} \times \frac{1}{s}$$

$$= \frac{\sqrt{3}}{2s\sqrt{s}} \quad \#$$

eq: IF $L^{-1}\left(\ln\left(\frac{s-5}{s+2}\right)\right) = F(t)$, Find $\underbrace{\int_0^\infty F(t)\cdot e^{-6t}\,dt}$. (8)

يعني مطلوب إيجاد $L(F(t))$

عندما $\boxed{t+6=s}$

sol take "L" For both side

$$\cancel{L}\{\cancel{L^{-1}}\left(\ln\left(\frac{s-5}{s+2}\right)\right)\} = L(F(t))$$

$$\ln\left(\frac{s-5}{s+2}\right) = L(F(t))$$

$\boxed{s=6} \rightarrow \ln\left(\frac{6-5}{6+2}\right) = L(F(t))$

$\rightarrow \ln\left(\frac{1}{8}\right) = L(F(t))$ #

$\ln(8^{-1}) = L(F(t))$

** Solving I.V.P by laplace transformation.

$$L(y^{n}(t)) = s^{n} L(y(t)) - s^{n-1} y(0) - s^{n-2} y(0) - \dots - y^{n-2}(0) - y^{n-1}(0).$$

$n=1$

$\Rightarrow L(y^{(1)}(t)) = s^{1} L(y(t)) - y(0).$

$n=2$

$\Rightarrow L(y''(t)) = s^{2} L(y(t)) - s\,y(0) - y'(0)$

حفظ

$n=3$

$\Rightarrow L(y'''(t)) = s^{3} L(y(t)) - s^{2} y(0) - s\,y'(0) - y''(0).$

* Example: use laplace transformation to solve the I.V.P.

$$y'' - 4y' + 4y = t^{3} e^{2t}, \quad y(0) = 0, \quad y'(0) = 0$$

$$\underset{n=2}{L(y'')} - 4\underset{n=1}{L(y')} + 4L(y) = L(t^{3} e^{2t})$$

$$s^{2} L(y) - s\,y(0) - y'(0) - 4[s\,L(y) - y(0)] + 4L(y) = \frac{3!}{(s-2)^{4}}$$

$$s^{2} L(y) - \cancel{s\,y(0)} - \cancel{y'(0)} - 4s\,L(y) + \cancel{4\,y(0)} + 4L(y) = \frac{3!}{(s-2)^{4}}$$

عامل مشترك

$$L(y)(s^{2} - 4s + 4) = \frac{6}{(s-2)^{4}}$$

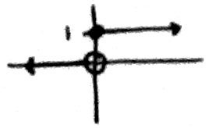
$$\therefore L(y) = \frac{6}{(s-2)^{4}\,(s^{2}-4s+4)} \Rightarrow L(y) = \frac{6}{(s-2)^{6}} \Rightarrow y = L^{-1}\left(\frac{6}{(s-2)^{6}}\right)$$

$(s-2)(s-2)$

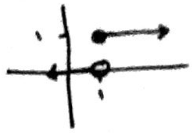
$$\therefore y = 6\cdot e^{2t}\cdot\frac{t^{5}}{5!}$$ #

The unit step funct.

$$u(t)=\begin{cases}0 & t<0\\ 1 & t\geq 0\end{cases}$$ رسمة →



$$u(t-1)=\begin{cases}0 & t<1\\ 1 & 0>t\geq 1\end{cases}$$ →



بصورة عامة
$$u(t-a)=\begin{cases}0 & t<a\\ 1 & 0>t\geq a\end{cases}$$

* **Rules:**

① $L(u(t-a))=\dfrac{e^{-as}}{s}$.

② $L(f(t-a)\,u(t-a))=e^{-as}L(f(t))$.

③ $L(f(t)\,u(t-a))=e^{-as}L(f(t+a))$.

Ex:

قانون 1

1) $L(u(t-\overset{a}{3}))=\dfrac{e^{-3s}}{s}$

2) $L(u(t-\overset{a}{\tfrac{\pi}{2}}))=\dfrac{e^{-\frac{\pi}{2}s}}{s}$

قانون 2

3) $L(\underbrace{e^{t-3}}_{f(t-3)}\cdot\underbrace{u(t-\overset{a}{3})}_{u(t-3)})=e^{-3t}\cdot L(e^{t})=e^{-3t}\cdot\dfrac{1}{s-1}$

4) $L(\sin(t-\tfrac{\pi}{2})\cdot u(t-\overset{a}{\tfrac{\pi}{2}}))=e^{-\frac{\pi}{2}s}\cdot L(\sin(t))=e^{-\frac{\pi}{2}s}\cdot\dfrac{1}{s^2+1}$.

قانون 3

متطابقة $\cos(t)$

5) $L(\sin(t)\cdot u(t-\overset{a}{\tfrac{\pi}{2}}))=e^{-\frac{\pi}{2}s}\cdot L(\sin(t+\tfrac{\pi}{2}))=e^{-\frac{\pi}{2}s}\cdot L(\cos(t))=e^{-\frac{\pi}{2}s}\cdot\dfrac{s}{s^2+1}$

6) $L(t^2\cdot u(t-\overset{a}{5}))=e^{-5s}L((t+5)^2)$

$=e^{-5s}L(t^2+10t+25)$

$=e^{-5s}(L(t^2)+10L(t)+L(25))$

$=e^{-5s}\left(\dfrac{2}{s^3}+10\cdot\dfrac{1}{s^2}+\dfrac{25}{s}\right)$ #

7) $L(\underbrace{e^{3t}}_{\text{shift}} \cdot t \cdot u(t-2)) = L(t \cdot u(t-2))\Big|_{s \to s-3}$

$$= e^{-2s} \cdot L(t+2)\Big|_{s\to s-3} = e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)\Big|_{s\to s-3}$$

$$= e^{-2(s-3)}\left(\frac{1}{(s-3)^2} + \frac{2}{(s-3)}\right)$$

* $f(t) = \begin{cases} \boxed{2} & t<1 \\ 0 & 1<t<2 \\ e^t & t>2 \end{cases}$, Find $L(f(t))$??

$$f(t) = 2 + (0-2)\cdot u(t-1) + (e^t - 0)\cdot u(t-2)$$

$$f(t) = 2 - 2\,u(t-1) + e^t\,u(t-2)$$

$$L(f(t)) = L(2) - 2\,L(u(t-1)) + L(e^t \cdot u(t-2))$$

$$= \frac{2}{s} - 2\,\frac{e^{-1s}}{s} + e^{-2s}\cdot L(e^{t+2}) \qquad L(e^t \cdot \underbrace{e^2}_{\text{ثابت}})$$

$$= \frac{2}{s} - \frac{2e^{-s}}{s} + e^{-2s}\cdot e^{2} \cdot L(e^t)$$

$$= \frac{2}{s} - \frac{2e^{-s}}{s} + e^{-2s+2}\cdot\frac{1}{s-1}$$

* $f(t) = \begin{cases} t & t<3 \\ 2 & 3<t<5 \\ e^t & t>5 \end{cases}$, Find $L(f(t))$?

$$f(t) = t + (2-t)\,u(t-3) + (e^t - 2)\cdot u(t-5)$$

$$L(f(t)) = L(t) + 2\,L(u(t-3)) - L(t\cdot u(t-3)) + L(e^t \cdot u(t-5)) - 2\,L(u(t-5))$$

⋮

أكمل :

* $L^{-1}\left(\frac{e^{-as}}{s}\right) = u(t-a)$

* $L^{-1}\left(e^{-as} F(s)\right) = F(t-a) \cdot L^{-1}(F(s))\Big|_{t \to t-a}$

* Ex:

① $L^{-1}\left(\frac{e^{-2s}}{s}\right) = u(t-2)$

② $L^{-1}\left(\frac{e^{-2s}}{s^2+9}\right) = L^{-1}\left(e^{-2s} \cdot \frac{1}{s^2+9}\right)$

$= u(t-2) \cdot \frac{1}{3} L^{-1}\left(\frac{1*3}{s^2+9}\right)\Big|_{t \to t-2}$

$= \frac{1}{3} u(t-2) \cdot \sin(3t)\Big|_{t \to t-2}$

$= \frac{1}{3} u(t-2) \cdot \sin(3(t-2))$.

③ $L^{-1}\left(\frac{e^{-3s}}{(s-4)}\right) = L^{-1}\left(e^{-3s} \cdot \frac{1}{s-4}\right)$

$= u(t-3) \; L^{-1}\left(\frac{1}{s-4}\right)\Big|_{t \to t-3}$

$= u(t-3) \cdot e^{4t}\Big|_{t \to t-3} = u(t-3) \cdot e^{4(t-3)}$.

④ $L^{-1}\left(e^{-4s} \frac{s-2}{(s-2)^2+81}\right)$ e^{2t} shift

$= u(t-4) \cdot L^{-1}\left(\frac{s-2}{(s-2)^2+81}\right)\Big|_{t \to t-4}$ $a=9$

$= u(t-4) \cdot \left[e^{2t} \cdot \cos(9t)\right]_{t \to t-4}$

$= u(t-4) \cdot e^{2(t-4)} \cdot \cos(9(t-4))$. #