12.2

Vectors

A quantity such as force, displacement, or velocity is called a **vector** and is represented by a **directed line segment**.

DEFINITIONS The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

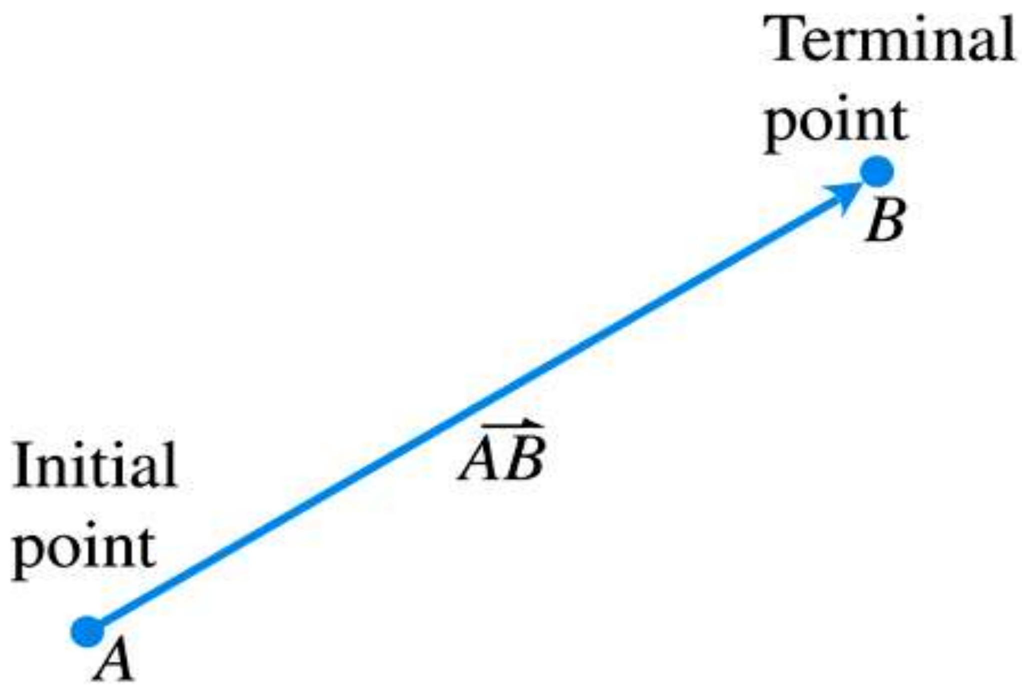


FIGURE 12.7 The directed line segment $\overrightarrow{AB}$ is called a vector.

The arrow points in the direction of the action and its length gives the magnitude of the action in terms of a suitably chosen unit.

For example, a velocity vector points in the direction of motion and its length is the speed of the moving object. (See Figure 12.8)

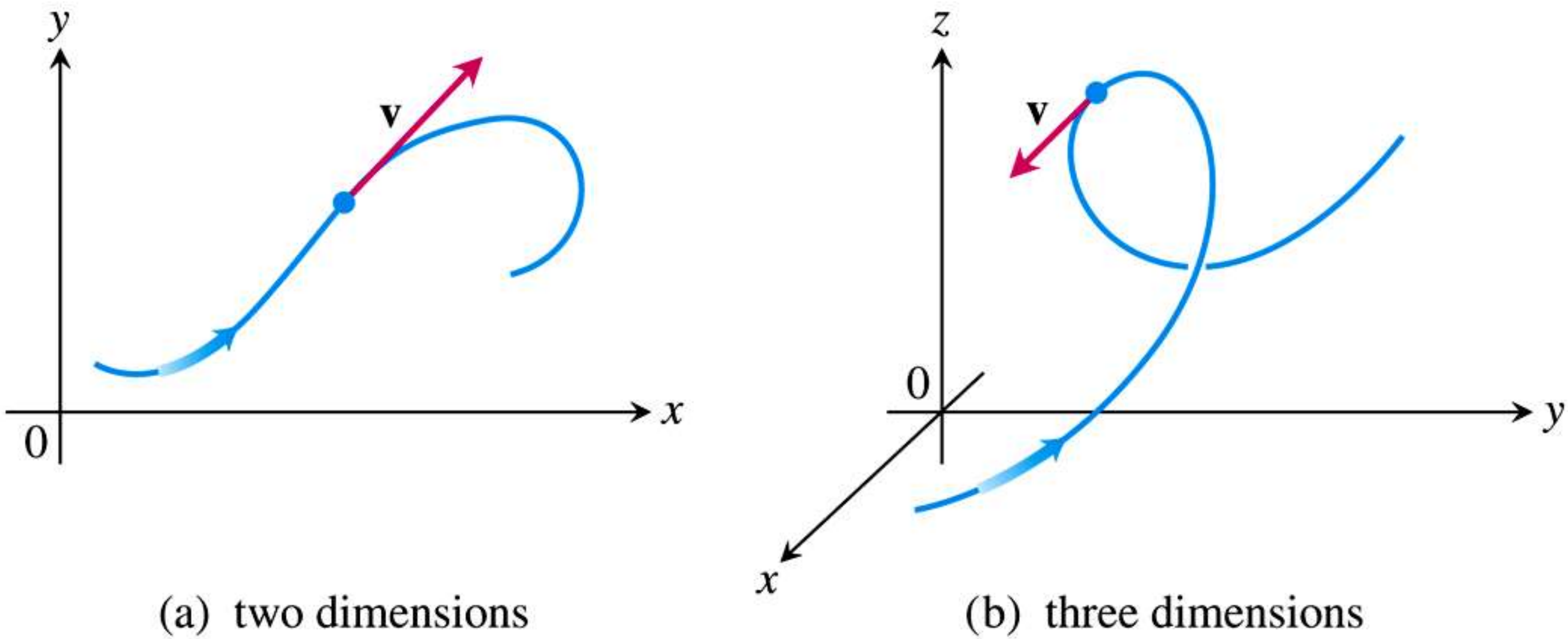


FIGURE 12.8 The velocity vector of a particle moving along a path (a) in the plane (b) in space. The arrowhead on the path indicates the direction of motion of the particle.

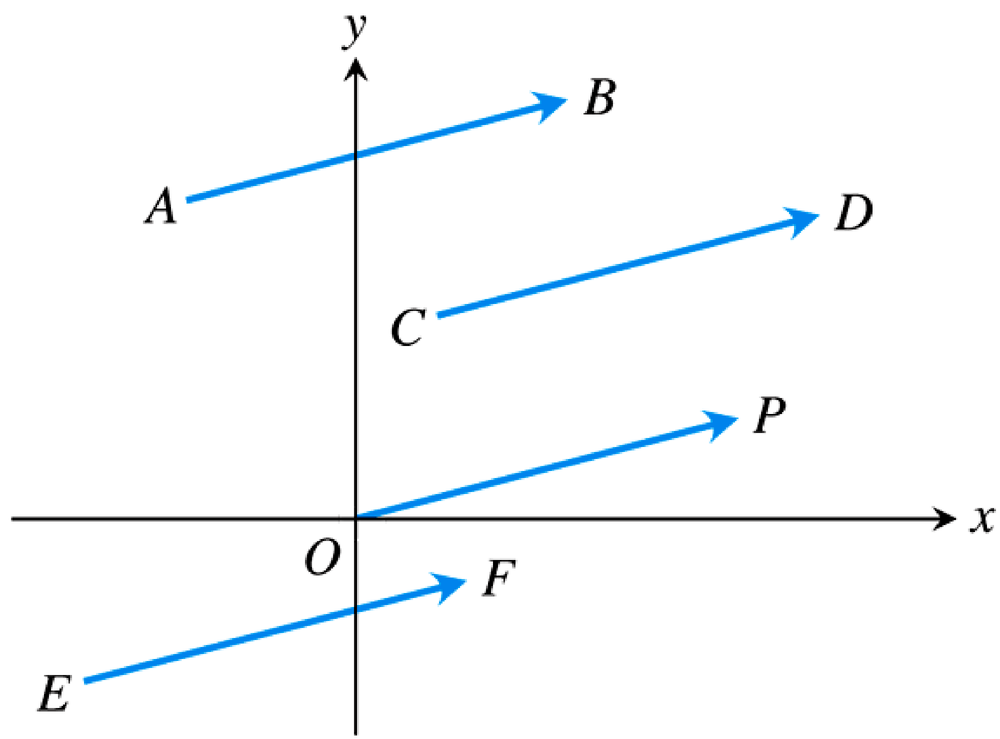


FIGURE 12.9 The four arrows in the plane (directed line segments) shown here have the same length and direction. They therefore represent the same vector, and we write $\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{OP} = \overrightarrow{EF}$.

Let $\boldsymbol{v} = \overrightarrow{PQ}$. There is one directed line segment equal to $\overrightarrow{PQ}$ whose initial point is the origin (Figure 12.10). It is the representative of $\boldsymbol{v}$ in **standard position** and is the vector we normally use to represent $\boldsymbol{v}$.

If $\boldsymbol{v}$ is a vector in standard position with terminal point (v_1, v_2, v_3), then the component form of $\boldsymbol{v}$ is

$$\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$$

Given the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$, the standard position vector $\boldsymbol{v} = \overrightarrow{PQ}$ is

$$\boldsymbol{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

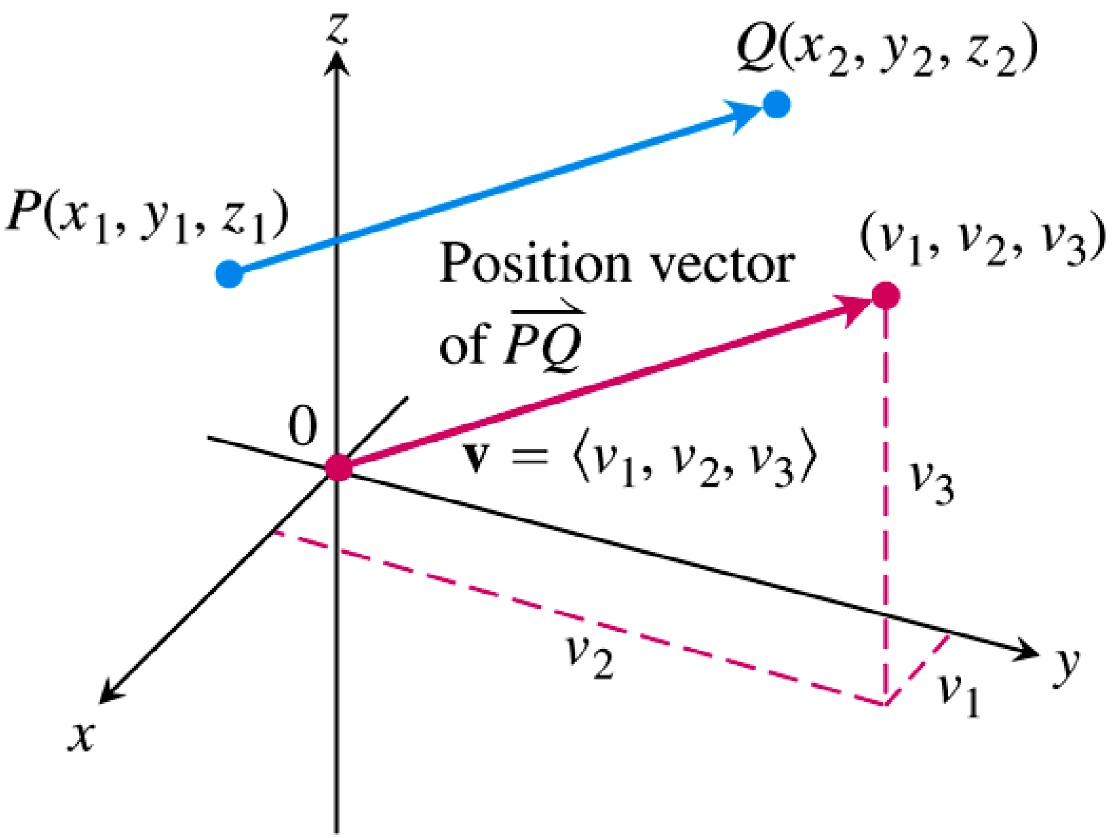


FIGURE 12.10 A vector $\overrightarrow{PQ}$ in standard position has its initial point at the origin. The directed line segments $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel and have the same length.

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(See Figure 12.10.)

EXAMPLE 1 Find the (a) component form and (b) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Solution

(a) The standard position vector v representing $\overrightarrow{PQ}$ has components

$$\boldsymbol{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle = \langle -2, -2, 1\rangle.$$

(b) The length or magnitude of $\boldsymbol{v} = \overrightarrow{PQ}$ is

$$|\boldsymbol{v}| = \left|\overrightarrow{PQ}\right| = \sqrt{(-\mathbf{2})^2+(-\mathbf{2})^2+(\mathbf{1})^2} = \mathbf{3}$$

DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

Addition: $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

Scalar multiplication: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

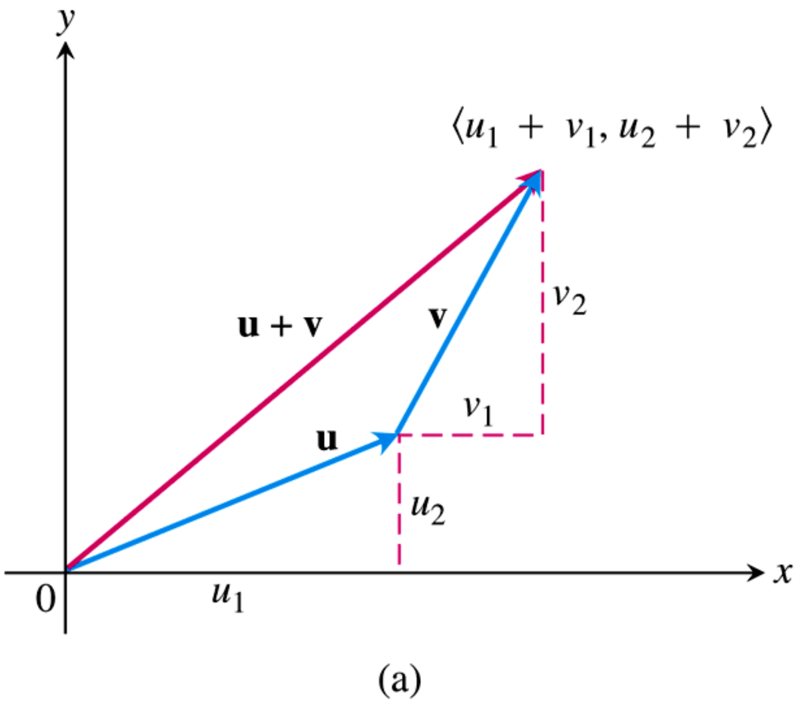


FIGURE 12.12 (a) Geometric interpretation of the vector sum. (b) The parallelogram law of vector addition.

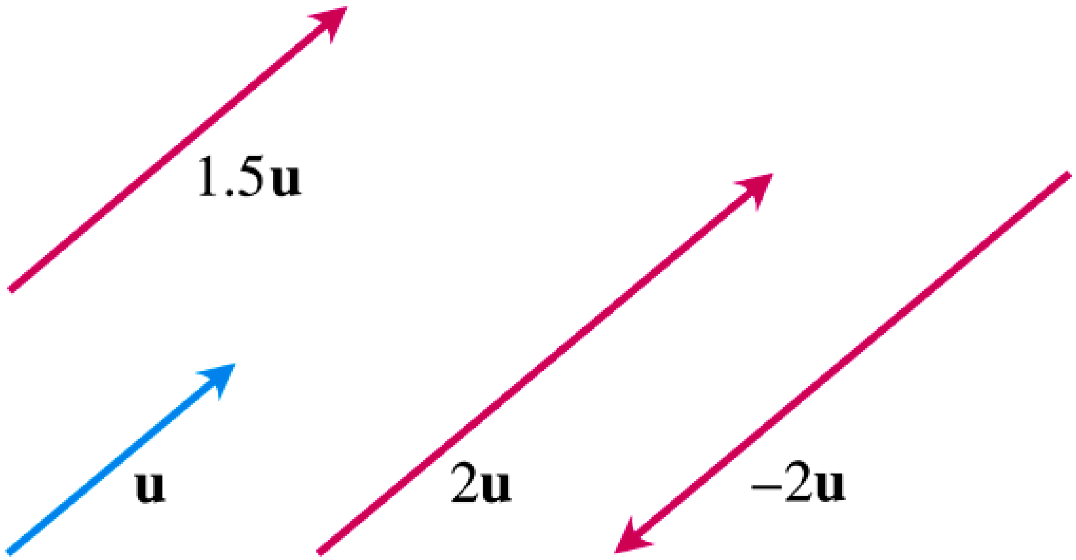


FIGURE 12.13 Scalar multiples of **u**.

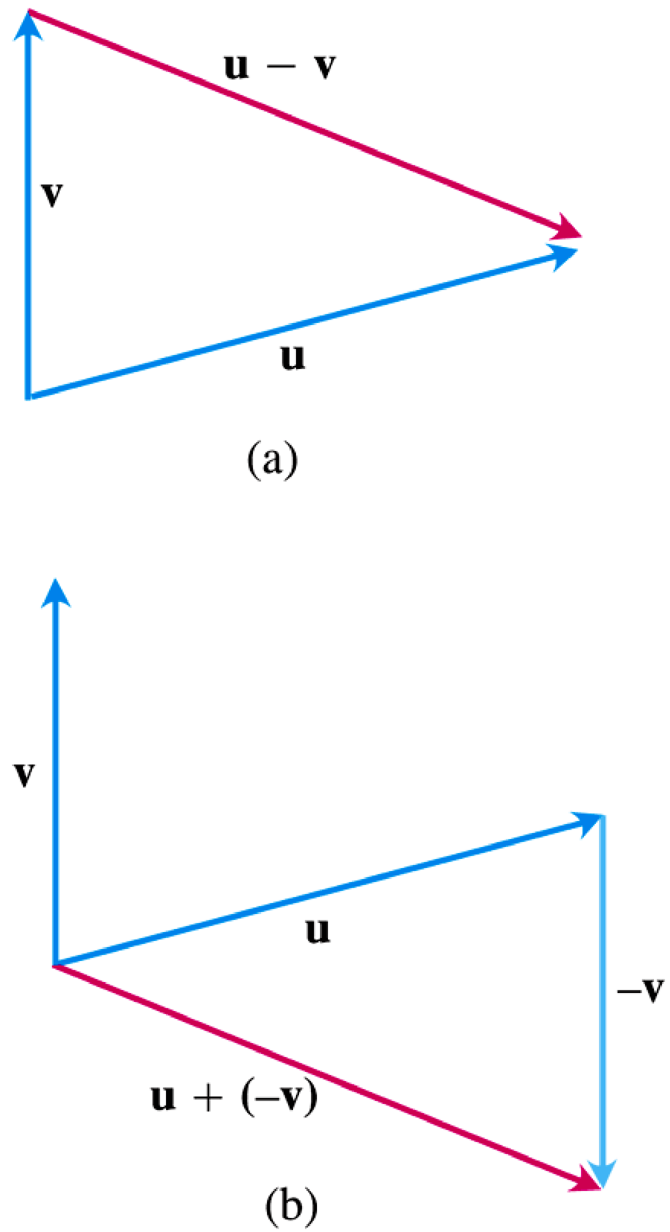


FIGURE 12.14 (a) The vector $\mathbf{u} - \mathbf{v}$, when added to $\mathbf{v}$, gives $\mathbf{u}$. (b) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v})$.

EXAMPLE 3 Let $\mathbf{u} = \langle -1, 3, 1 \rangle$ and $\mathbf{v} = \langle 4, 7, 0 \rangle$. Find the components of

(a) $2\mathbf{u} + 3\mathbf{v}$ **(b)** $\mathbf{u} - \mathbf{v}$ **(c)** $\left| \frac{1}{2}\mathbf{u} \right|$.

Solution

(a) $2\mathbf{u} + 3\mathbf{v} = 2\langle -1, 3, 1 \rangle + 3\langle 4, 7, 0 \rangle = \langle -2, 6, 2 \rangle + \langle 12, 21, 0 \rangle = \langle 10, 27, 2 \rangle$

(b) $\mathbf{u} - \mathbf{v} = \langle -1, 3, 1 \rangle - \langle 4, 7, 0 \rangle = \langle -1 - 4, 3 - 7, 1 - 0 \rangle = \langle -5, -4, 1 \rangle$

(c) $\left| \frac{1}{2}\mathbf{u} \right| = \left| \left\langle -\frac{1}{2}, \frac{3}{2}, \frac{1}{2} \right\rangle \right| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{2}\sqrt{11}.$

Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors and a, b be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$
5. $0\mathbf{u} = \mathbf{0}$
6. $1\mathbf{u} = \mathbf{u}$
7. $a(b\mathbf{u}) = (ab)\mathbf{u}$
8. $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
9. $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

Unit Vectors

A vector $\boldsymbol{v}$ of length 1 is called a unit vector.

The standard unit vectors are

$$\boldsymbol{i} = \langle 1,0,0 \rangle, \qquad \boldsymbol{j} = \langle 0,1,0 \rangle, \qquad \boldsymbol{k} = \langle 0,0,1 \rangle.$$

Any vector $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors as follows:

$$\begin{aligned}
\boldsymbol{v} &= \langle v_1, v_2, v_3 \rangle \\
&= \langle v_1, 0,0 \rangle + \langle 0, v_2, 0 \rangle + \langle 0,0, v_3 \rangle \\
&= v_1 \langle 1,0,0 \rangle + v_2 \langle 0,1,0 \rangle + v_3 \langle 0,0,1 \rangle \\
&= v_1 \boldsymbol{i} + v_2 \boldsymbol{j} + v_3 \boldsymbol{k}
\end{aligned}$$

We call the number v_1 the i-component of the vector $\boldsymbol{v}$, v_2 the j-component, and v_3 the k-component.

In component form, the vector from $P(x_1, y_1, z_1)$ to $Q(x_2, y_2, z_2)$ is

$$\overrightarrow{PQ} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

See Figure 12.15.

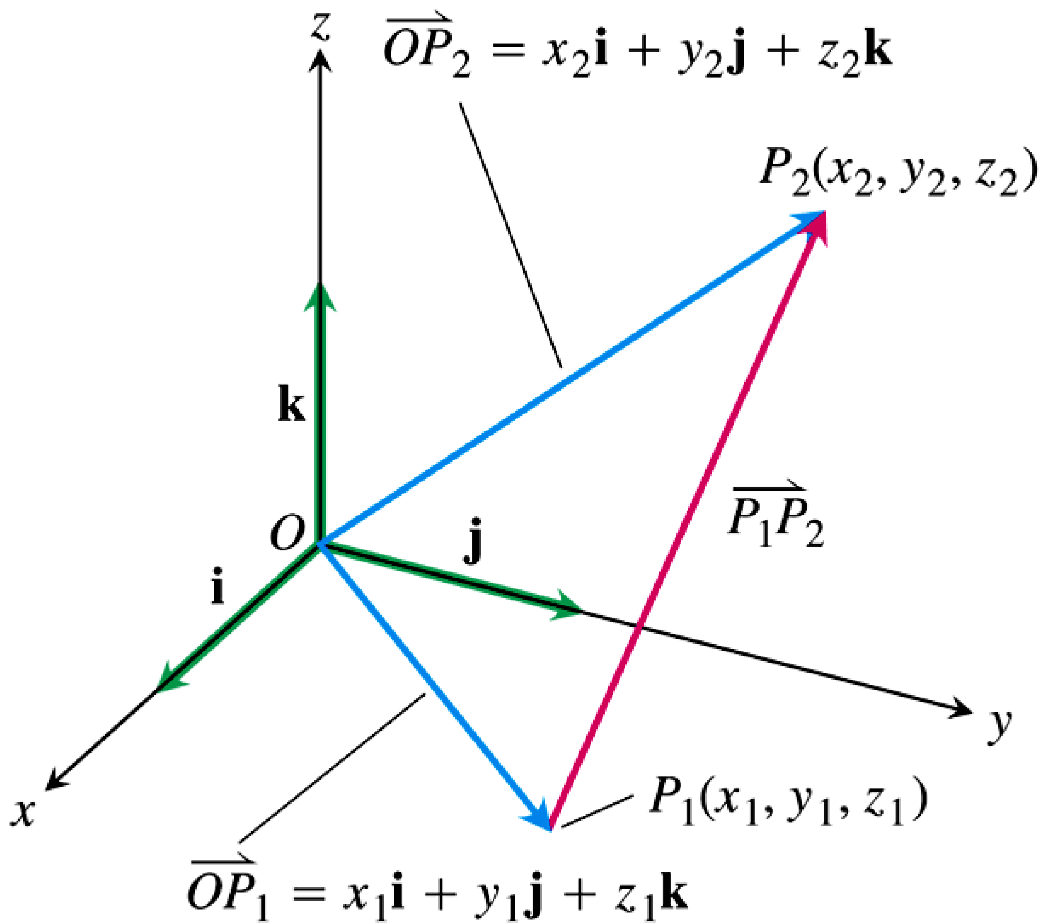


FIGURE 12.15 The vector from P_1 to P_2 is $\overrightarrow{P_1P_2} = (x_2 - x_1)\mathbf{i} + (y_2 - y_1)\mathbf{j} + (z_2 - z_1)\mathbf{k}$.

If $\mathbf{v} \neq \mathbf{0}$, then

1. $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ is a unit vector in the direction of $\mathbf{v}$;
2. the equation $\mathbf{v} = |\mathbf{v}|\dfrac{\mathbf{v}}{|\mathbf{v}|}$ expresses $\mathbf{v}$ as its length times its direction.

EXAMPLE 4 Find a unit vector $\boldsymbol{u}$ in the direction of the vector from $P_1(1,0,1)$ to $P_2(3,2,0)$.

Solution

First we find the coordinates of the vector

$$\overrightarrow{P_1P_2} = \langle 3-1, 2-0, 0-1\rangle = \langle 2,2,-1\rangle.$$

Next we find the length of the vector

$$\left|\overrightarrow{P_1P_2}\right| = \sqrt{(2)^2+(2)^2+(-1)^2} = 3.$$

The unit vector $\boldsymbol{u} = \dfrac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|}$ has the same direction as $\overrightarrow{P_1P_2}$:

$$\boldsymbol{u} = \frac{\overrightarrow{P_1P_2}}{\left|\overrightarrow{P_1P_2}\right|} = \left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle$$

The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

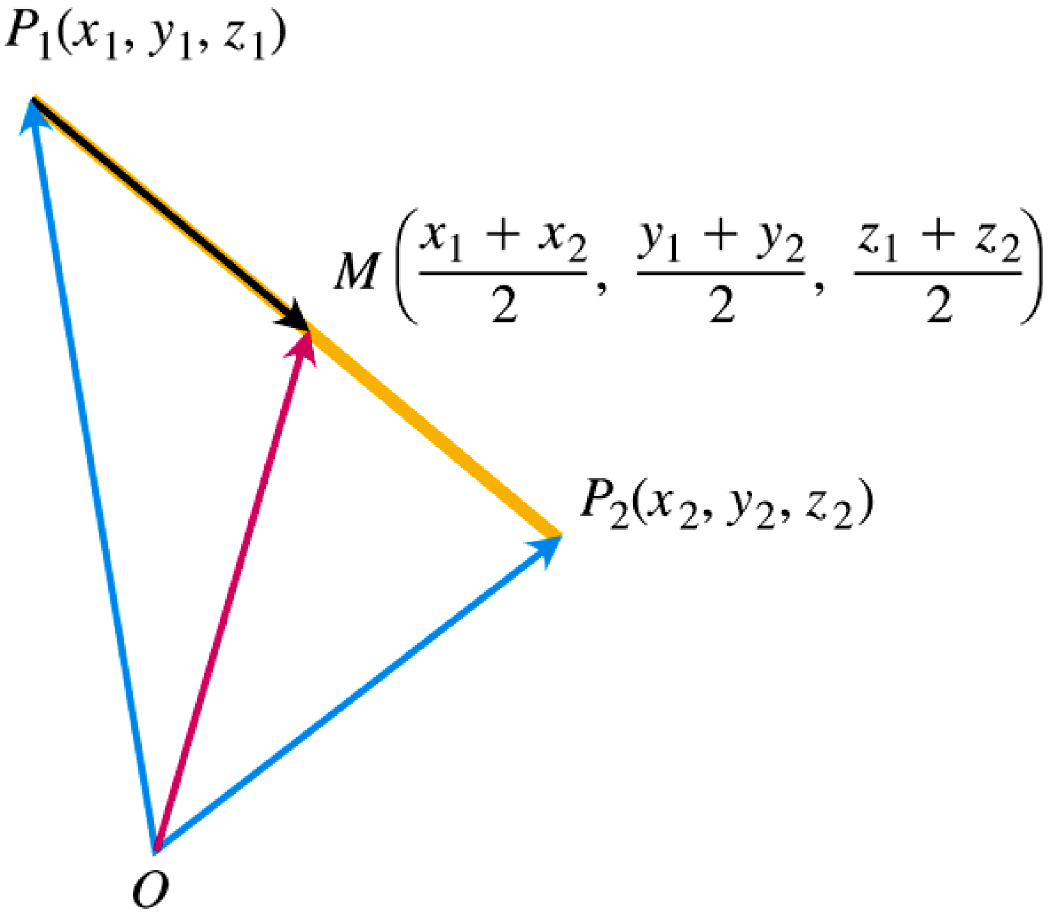


FIGURE 12.16 The coordinates of the midpoint are the averages of the coordinates of P_1 and P_2.

12.3

The Dot Product

When two nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are placed so their initial points coincide, they form an angle θ of measure $0 \leq \theta \leq \pi$.

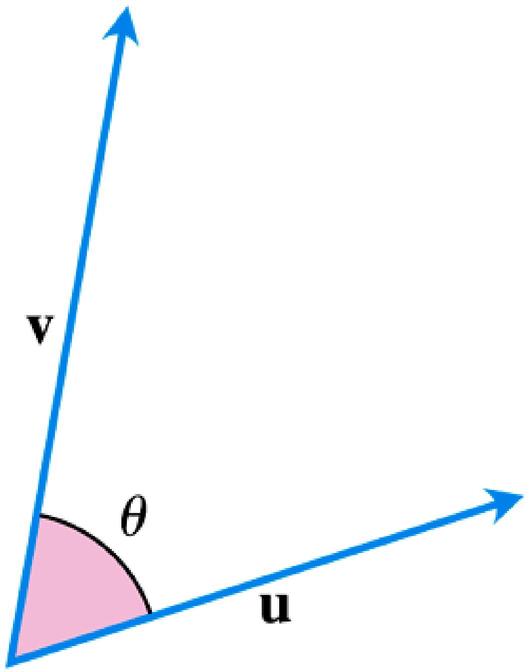


FIGURE 12.20 The angle between **u** and **v**.

THEOREM 1—Angle Between Two Vectors The angle θ between two nonzero vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{u_1 v_1 + u_2 v_2 + u_3 v_3}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

DEFINITION The **dot product** $\mathbf{u} \cdot \mathbf{v}$ ("$\mathbf{u}$ dot $\mathbf{v}$") of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

EXAMPLE 1

(a) $$\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle = (1)(-6) + (-2)(2) + (-1)(-3) = -6 - 4 + 3 = -7$$

(b) $$\left(\frac{1}{2}\mathbf{i} + 3\mathbf{j} + \mathbf{k}\right) \cdot (4\mathbf{i} - \mathbf{j} + 2\mathbf{k}) = \left(\frac{1}{2}\right)(4) + (3)(-1) + (1)(2) = 1$$

In the notation of the dot product, the angle between two vectors **u** and **v** is

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right).$$

This leads to the formula:

$$\boldsymbol{u} \cdot \boldsymbol{v} = |\boldsymbol{u}||\boldsymbol{v}| cos\theta$$

EXAMPLE 2 Find the angle between $\mathbf{u} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and $\mathbf{v} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$.

Solution We use the formula above:

$$\mathbf{u} \cdot \mathbf{v} = (1)(6) + (-2)(3) + (-2)(2) = 6 - 6 - 4 = -4$$

$$|\mathbf{u}| = \sqrt{(1)^2 + (-2)^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\mathbf{v}| = \sqrt{(6)^2 + (3)^2 + (2)^2} = \sqrt{49} = 7$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}\right) = \cos^{-1}\left(\frac{-4}{(3)(7)}\right) \approx 1.76 \text{ radians}.$$

DEFINITION Vectors **u** and **v** are **orthogonal** (or **perpendicular**) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

EXAMPLE 4 To determine if two vectors are orthogonal, calculate their dot product.

(a) $\mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 2\mathbf{j} + 4\mathbf{k}$ are orthogonal because

$$u \cdot v = (3 \times 0) + (-2 \times 2) + (1 \times 4) = 0.$$

(b) $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 4\mathbf{i} + \mathbf{j} - \mathbf{k}$ are **not** orthogonal because

$$u \cdot v = (1 \times 4) + (-2 \times 1) + (3 \times -1) = -1 \neq 0.$$

Properties of the Dot Product

If **u**, **v**, and **w** are any vectors and c is a scalar, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$
2. $(c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$
3. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$
4. $\mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2$
5. $\mathbf{0} \cdot \mathbf{u} = 0$.

The vector projection of $\boldsymbol{u} = \overrightarrow{PQ}$ onto a nonzero vector $\boldsymbol{v} = \overrightarrow{PS}$ (Figure 12.23) is the vector $\overrightarrow{PR}$ determined by dropping a perpendicular from Q to the line PS. The notation for this vector is $proj_{\boldsymbol{v}}\boldsymbol{u}$.

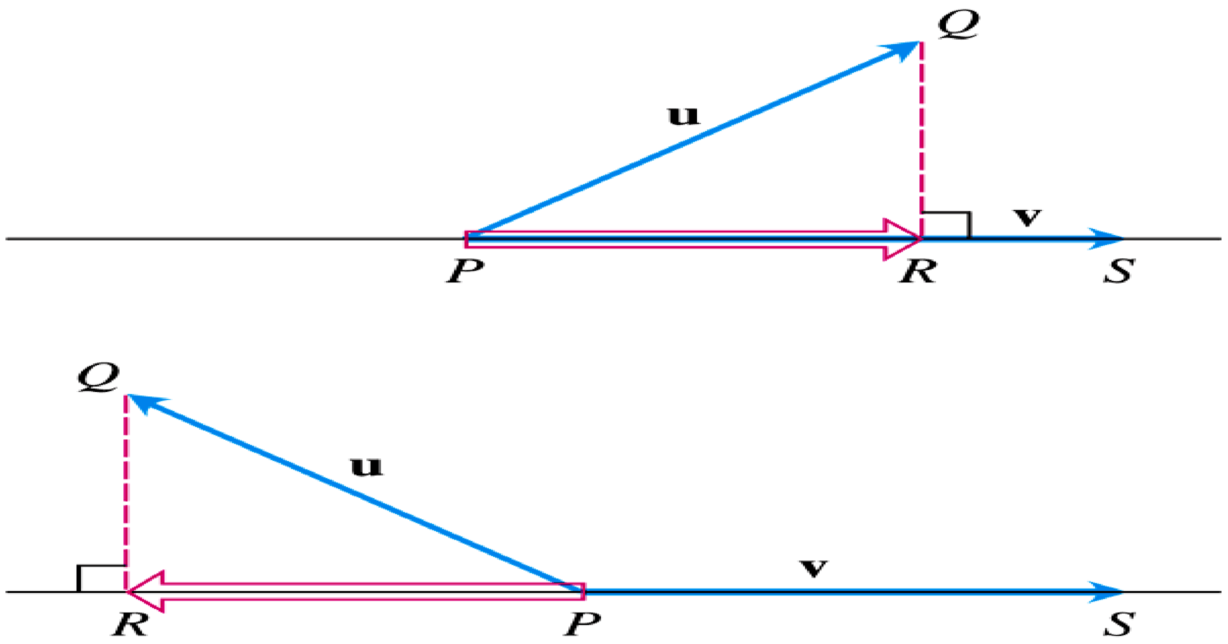


FIGURE 12.23 The vector projection of **u** onto **v**.

The vector projection of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is the vector

$$proj_{v}\boldsymbol{u} = \left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{v}|^{2}}\right)\boldsymbol{v}$$

The scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$ is the scalar $|\boldsymbol{u}|cos\theta$ which can be computed also using that

$$|\boldsymbol{u}|cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{v}|}$$

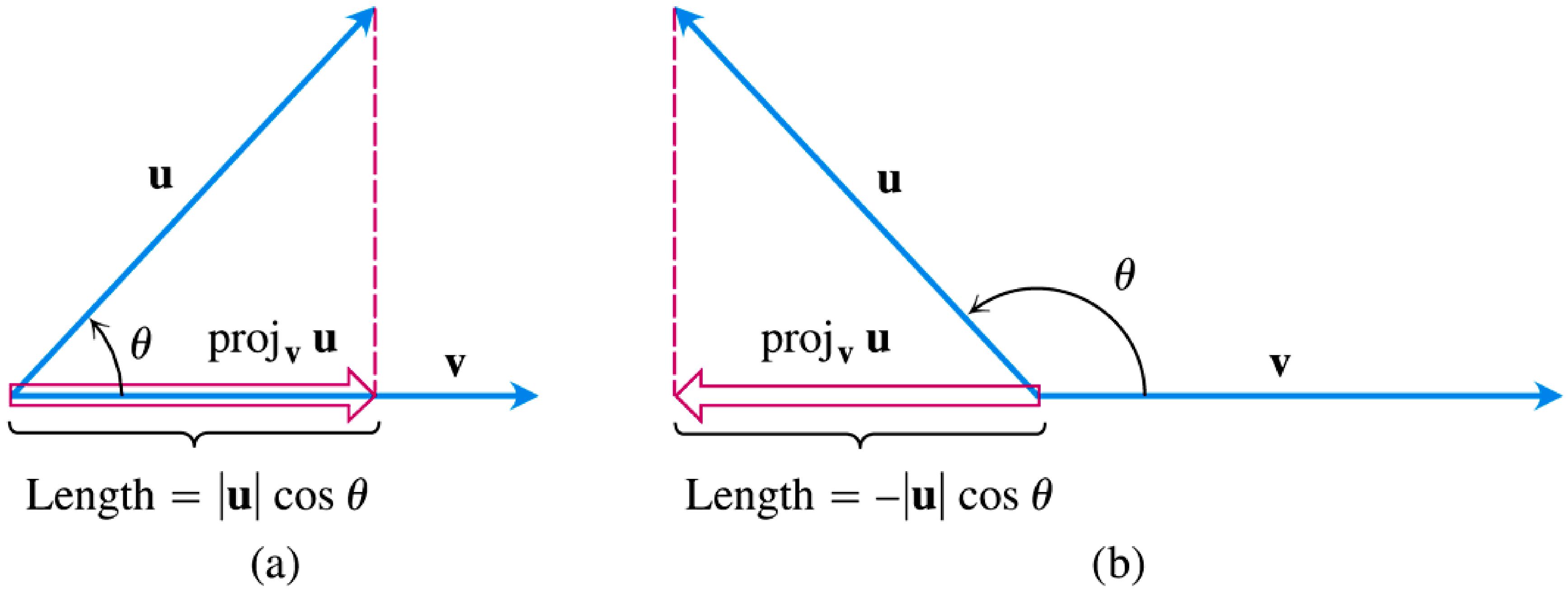


FIGURE 12.25 The length of $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ is (a) $|\mathbf{u}|\cos\theta$ if $\cos\theta \geq 0$ and (b) $-|\mathbf{u}|\cos\theta$ if $\cos\theta < 0$.

EXAMPLE 5 Find the vector projection of $\mathbf{u} = 6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ onto $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 2\mathbf{k}$ and the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$.

Solution We find $\text{proj}_{\mathbf{v}}\,\mathbf{u}$ from Equation (1):

$$\begin{aligned}\text{proj}_{\mathbf{v}}\,\mathbf{u} &= \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{6 - 6 - 4}{1 + 4 + 4}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) \\ &= -\frac{4}{9}(\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}) = -\frac{4}{9}\mathbf{i} + \frac{8}{9}\mathbf{j} + \frac{8}{9}\mathbf{k}.\end{aligned}$$

We find the scalar component of $\mathbf{u}$ in the direction of $\mathbf{v}$ from Equation (2):

$$\begin{aligned}|\mathbf{u}|\cos\theta &= \mathbf{u}\cdot\frac{\mathbf{v}}{|\mathbf{v}|} = (6\mathbf{i} + 3\mathbf{j} + 2\mathbf{k})\cdot\left(\frac{1}{3}\mathbf{i} - \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}\right) \\ &= 2 - 2 - \frac{4}{3} = -\frac{4}{3}.\end{aligned}$$

12.4

The Cross Product

Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two nonzero vectors in space. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are not parallel, they determine a plane. We select a unit vector $\boldsymbol{n}$ perpendicular to the plane by the right-hand rule. This means that we choose $\boldsymbol{n}$ to be the unit (normal) vector that points the way your right thumb points when your fingers curl through the angle θ from $\boldsymbol{u}$ to $\boldsymbol{v}$ (Figure 12.27).

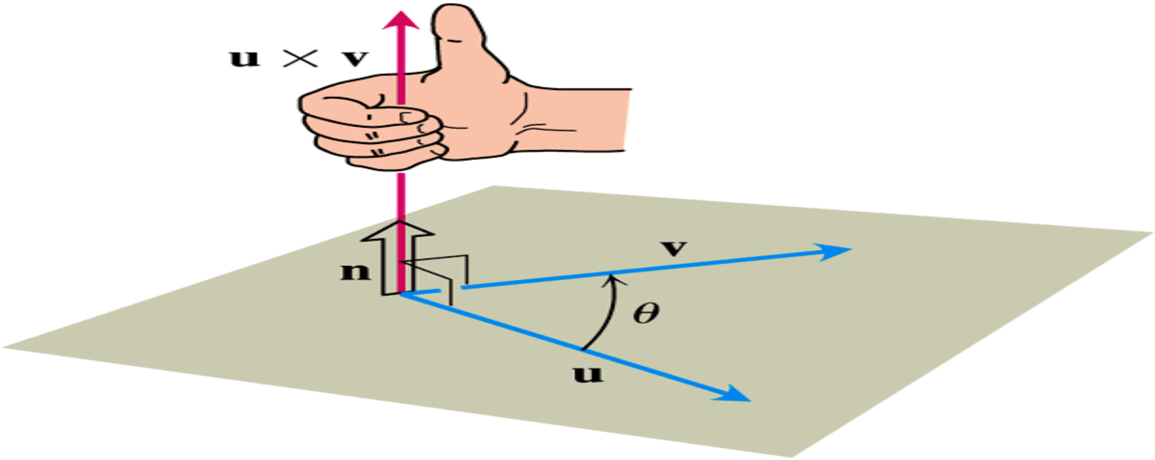


FIGURE 12.27 The construction of $\mathbf{u} \times \mathbf{v}$.

DEFINITION

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin\theta)\, \mathbf{n}$$

Because $\boldsymbol{n}$ *is a* unit vector, the magnitude of $\boldsymbol{u} \times \boldsymbol{v}$ is

$$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}||\mathbf{v}|\, |\sin\theta||\mathbf{n}| = |\mathbf{u}||\mathbf{v}| \sin\theta.$$

Parallel Vectors

Nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if and only if $\mathbf{u} \times \mathbf{v} = \mathbf{0}$.

Properties of the Cross Product

If $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are any vectors and r, s are scalars, then

1. $(r\mathbf{u}) \times (s\mathbf{v}) = (rs)(\mathbf{u} \times \mathbf{v})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
3. $\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v})$
4. $(\mathbf{v} + \mathbf{w}) \times \mathbf{u} = \mathbf{v} \times \mathbf{u} + \mathbf{w} \times \mathbf{u}$
5. $\mathbf{0} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$

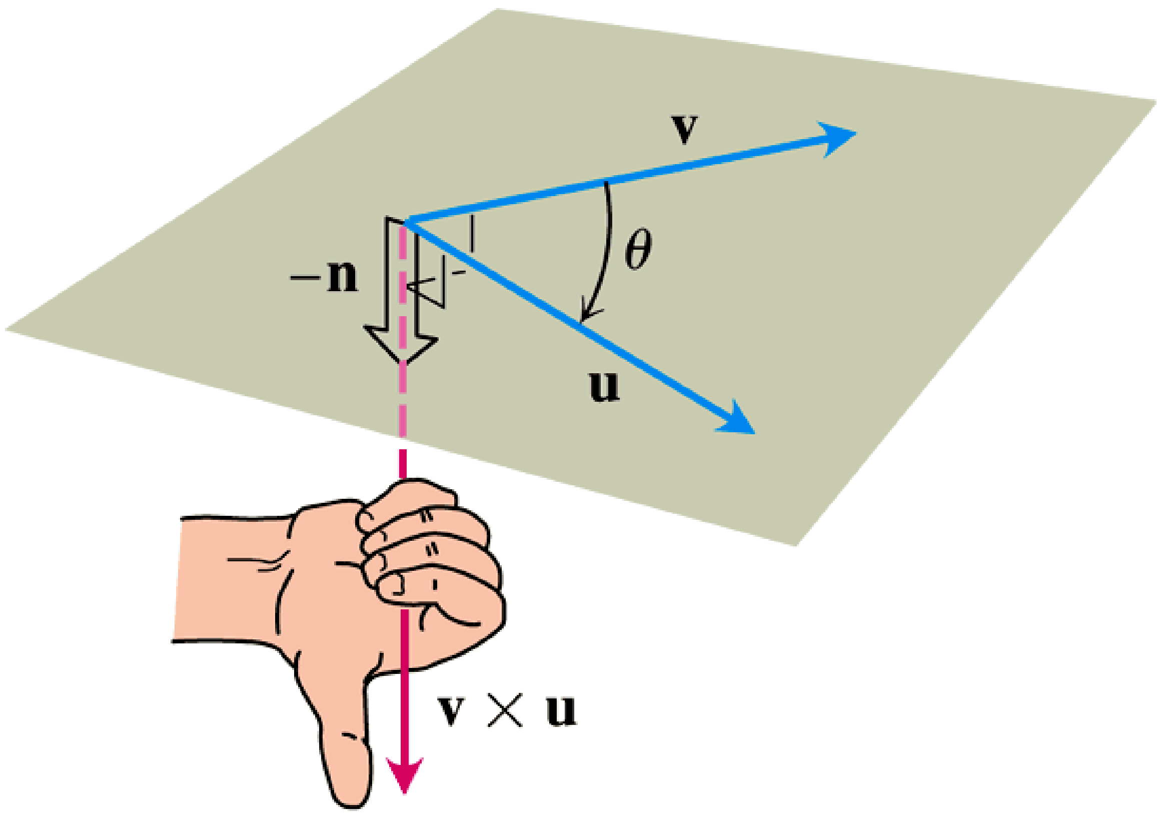


FIGURE 12.28 The construction of $\mathbf{v} \times \mathbf{u}$.

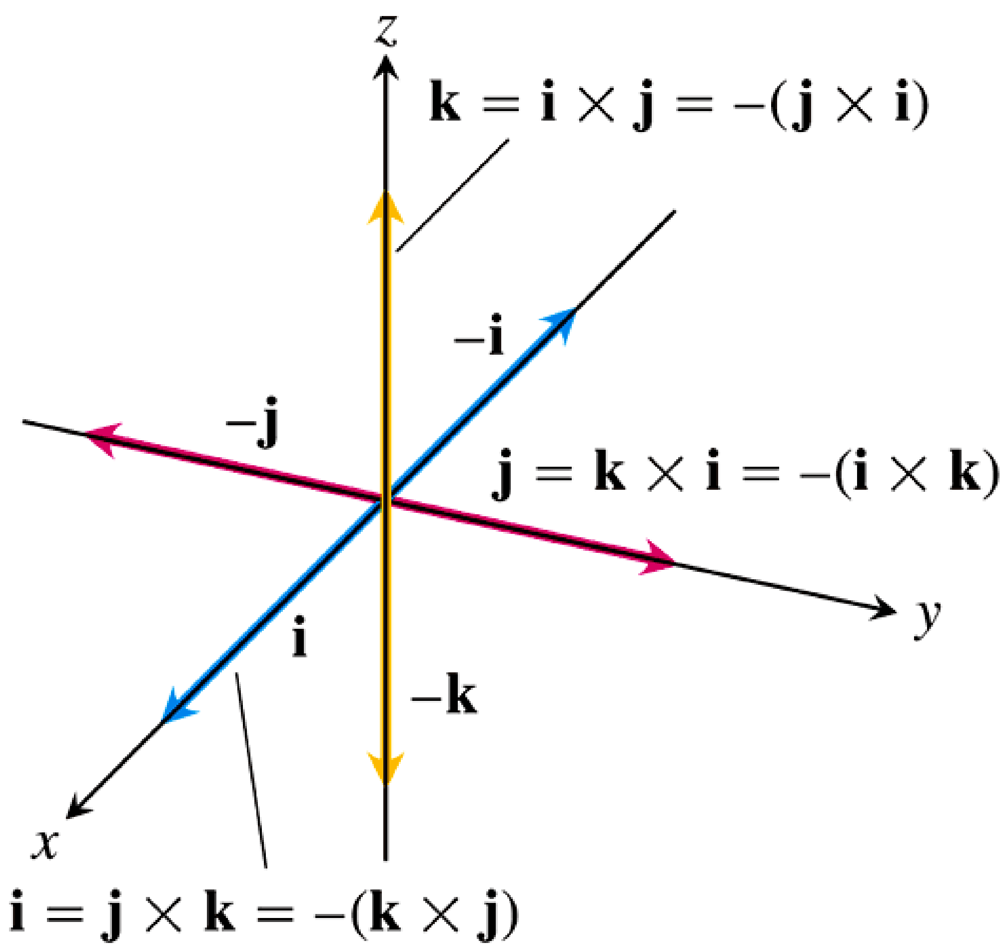


FIGURE 12.29 The pairwise cross products of **i**, **j**, and **k**.

Area of the parallelogram determined by ***u*** and ***v***

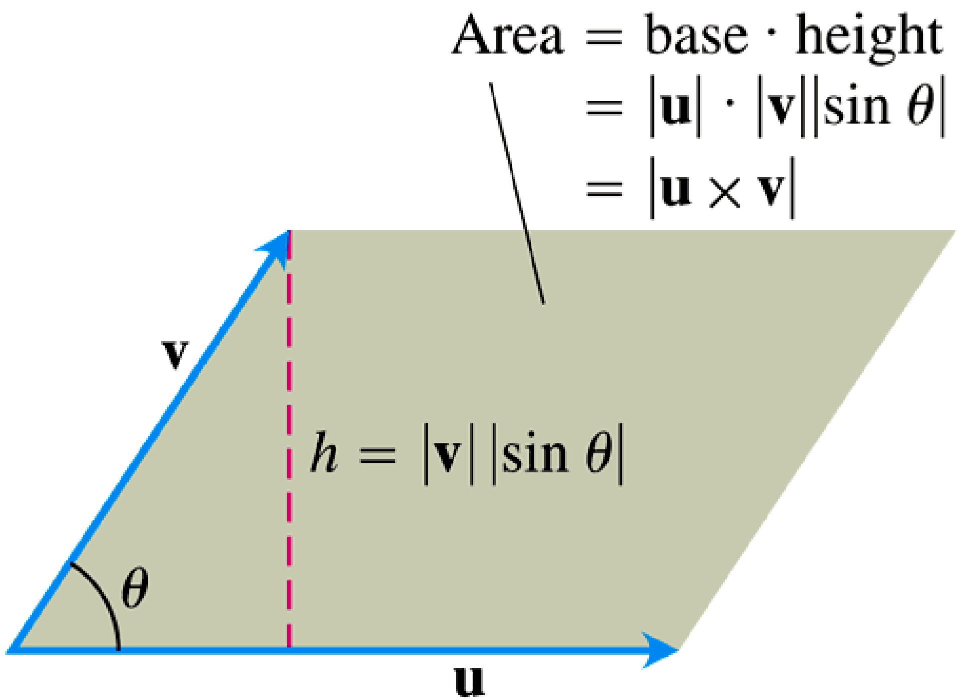


FIGURE 12.30 The parallelogram determined by **u** and **v**.

The area of the parallelogram determined by $\boldsymbol{u}$ and $\boldsymbol{v}$ is

$$\text{Area} = |\boldsymbol{u} \times \boldsymbol{v}|.$$

This area can be computed using that

$$\boxed{|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}| sin\theta}$$

Calculating the Cross Product as a Determinant

If $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, then

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

EXAMPLE 1 Find $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ if $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

Solution

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix}\mathbf{k}$$

$$= -2\mathbf{i} - 6\mathbf{j} + 10\mathbf{k}$$

$$\mathbf{v} \times \mathbf{u} = -(\mathbf{u} \times \mathbf{v}) = 2\mathbf{i} + 6\mathbf{j} - 10\mathbf{k}$$

To find the area of the parallelogram determined by $\boldsymbol{u} = \langle 2, 1,1\rangle$ and $\boldsymbol{v} = \langle -4,3,1\rangle$ we use that

$$\text{Area} = |\boldsymbol{u} \times \boldsymbol{v}|.$$

We found in Example 1 that $\boldsymbol{u} \times \boldsymbol{v} = 2\boldsymbol{i} + 6\boldsymbol{j} - 10\boldsymbol{k}$.

Then,

$$\text{Area} = |\boldsymbol{u} \times \boldsymbol{v}| = \sqrt{(2)^2+(6)^2+(-10)^2} = \sqrt{140}.$$

EXAMPLE 3 Find the area of the triangle with vertices *P(1, -1, 0)*, *Q(2, 1, -1)*, and *R(-1, 1, 2)*.

Solution $\overrightarrow{PQ} = \langle 1,2,-1 \rangle$, $\overrightarrow{PR} = \langle -2,2,2 \rangle$,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2 & -1 \\ -2 & 2 & 2 \end{vmatrix} = 6\boldsymbol{i} + 6\boldsymbol{k}.$$

The area of the parallelogram determined by *P*, *Q*, and *R* is

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{(6)^2+(0)^2+(6)^2} = \sqrt{72}.$$

The triangle's area is half of this, or $\sqrt{72}/2$.

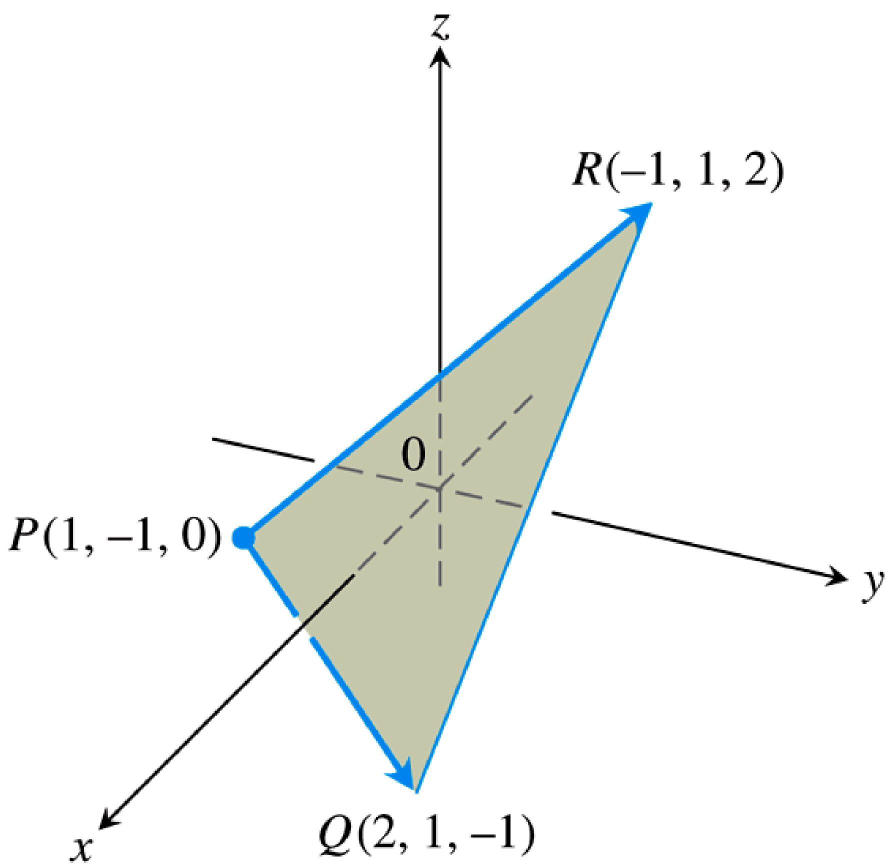


FIGURE 12.31 The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane of triangle PQR (Example 2). The area of triangle PQR is half of $|\overrightarrow{PQ} \times \overrightarrow{PR}|$ (Example 3).

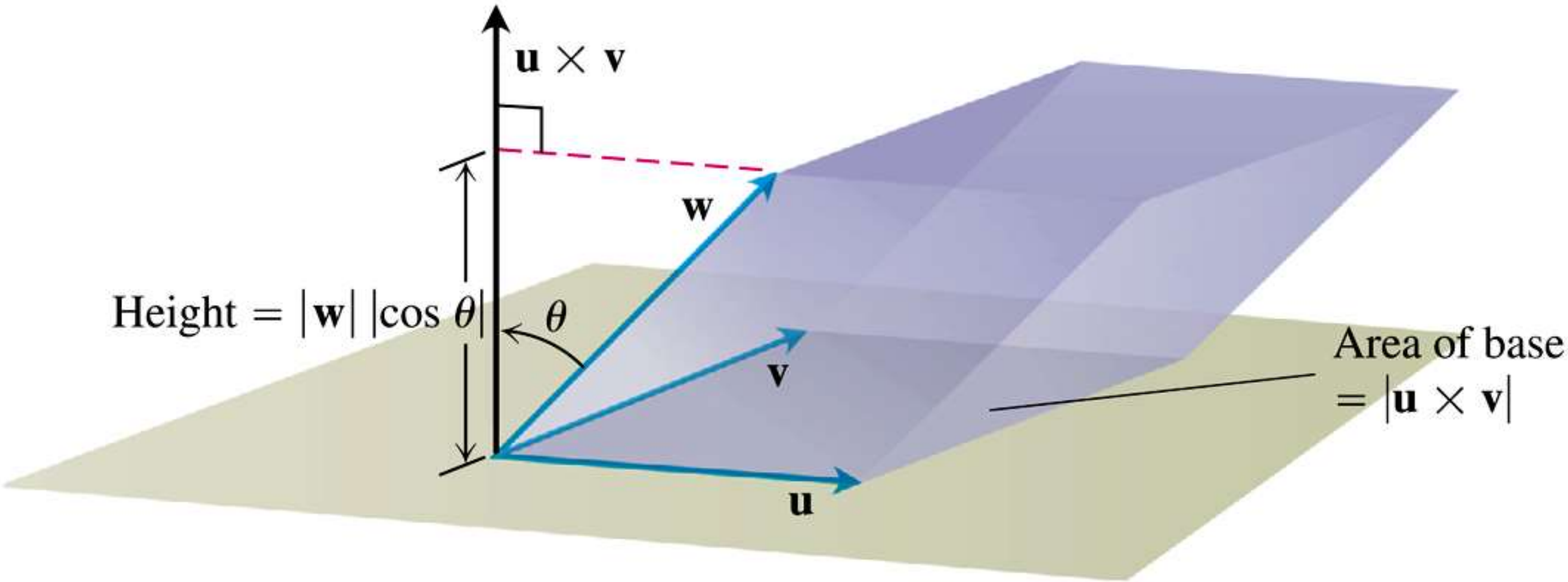
The product $(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \cdot \boldsymbol{w}$ is called the triple scalar product of $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ (in that order).

As you can see from the formula

$$|(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}) \cdot \boldsymbol{w}| = |\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}||\boldsymbol{w}||cos\theta|,$$

the absolute value of this product is the volume of the parallelepiped determined by $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$ (Figure 12.34).

The number $|\vec{\boldsymbol{u}} \times \vec{\boldsymbol{v}}|$ is the area of the base parallelogram. The number $|\boldsymbol{w}||cos\theta|$ is the parallelepiped's height.



$$\text{Volume} = \text{area of base} \cdot \text{height} = |\mathbf{u} \times \mathbf{v}|\,|\mathbf{w}|\,|\cos\theta| = |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$$

FIGURE 12.34 The number $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}|$ is the volume of a parallelepiped.

Calculating the Triple Scalar Product as a Determinant

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

EXAMPLE 6 Find the volume of the box (parallelepiped) determined by $\mathbf{u} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{k}$, and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution Using the rule for calculating determinants, we find

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = \begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix} = -23.$$

The volume is $|(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| = 23$ units cubed.

12.5

Lines and Planes in Space

Lines in Space

Suppose that L is a line in space passing through a point $P_0(x_0, y_0, z_0)$ parallel to a vector $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$. Then L is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is parallel to $\boldsymbol{v}$ (Figure 12.35). Thus, $\overrightarrow{P_0P} = t\boldsymbol{v}$ for some scalar parameter t. The value of t depends on the location of the point P along the line, and the domain of t is $(-\infty, \infty)$.

Vector Equation for a Line

A vector equation for the line L through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v}$ is

$$\mathbf{r}(t) = \mathbf{r}_0 + t\mathbf{v}, \qquad -\infty < t < \infty, \tag{2}$$

where $\mathbf{r}$ is the position vector of a point $P(x, y, z)$ on L and $\mathbf{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

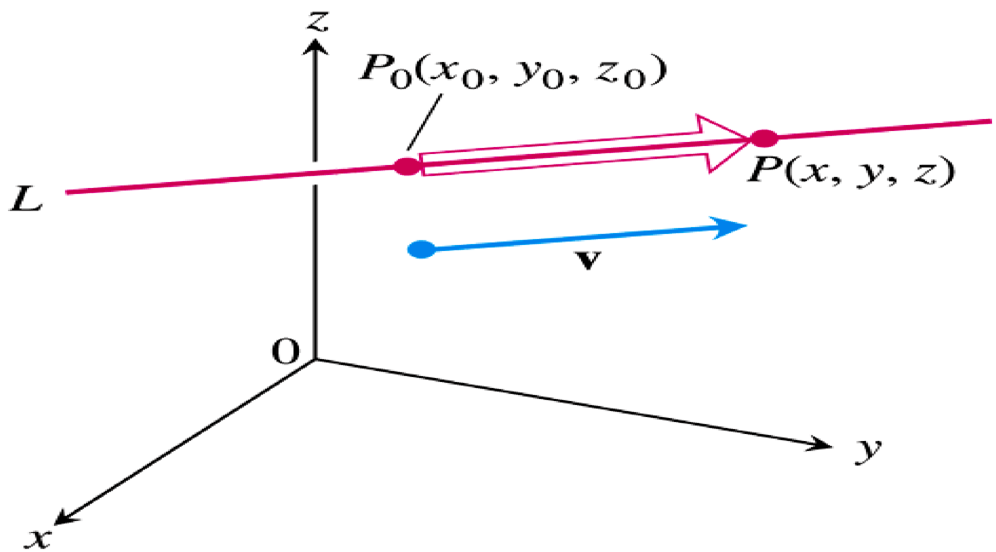


FIGURE 12.35 A point P lies on L through P_0 parallel to $\mathbf{v}$ if and only if $\overrightarrow{P_0P}$ is a scalar multiple of $\mathbf{v}$.

Parametric Equations for a Line

The standard parametrization of the line through $P_0(x_0, y_0, z_0)$ parallel to $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty \tag{3}$$

EXAMPLE 1 Find parametric equations for the line through $(-2, 0, 4)$ parallel to $\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$ (Figure 12.36).

Solution With $P_0(x_0, y_0, z_0)$ equal to $(-2, 0, 4)$ and $v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$ equal to $2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$, Equations (3) become

$$x = -2 + 2t, \qquad y = 4t, \qquad z = 4 - 2t.$$

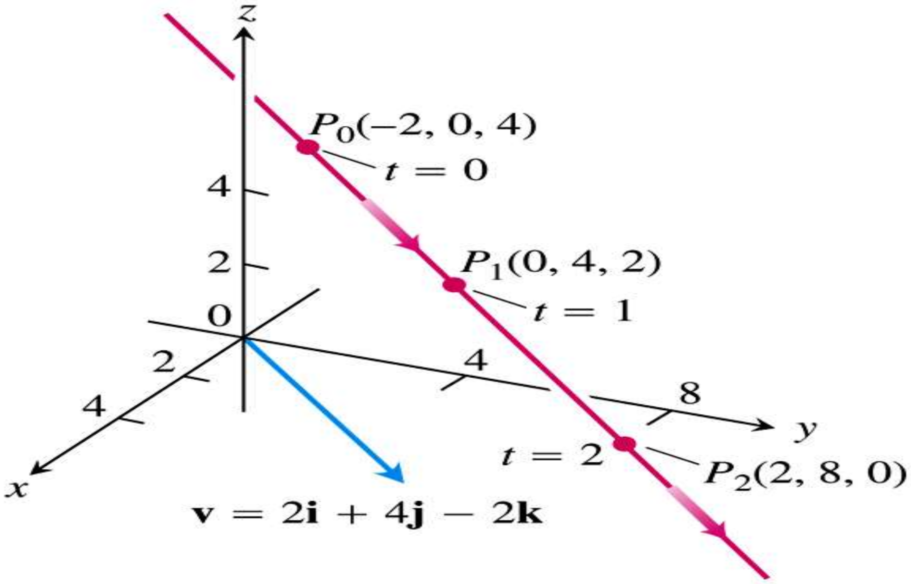


FIGURE 12.36 Selected points and parameter values on the line in Example 1. The arrows show the direction of increasing t.

EXAMPLE 2 Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Solution The vector

$$\begin{aligned}\overrightarrow{PQ} &= (1 - (-3))\mathbf{i} + (-1 - 2)\mathbf{j} + (4 - (-3))\mathbf{k} \\ &= 4\mathbf{i} - 3\mathbf{j} + 7\mathbf{k}\end{aligned}$$

is parallel to the line, and Equations (3) with $(x_0, y_0, z_0) = (-3, 2, -3)$ give

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We could have chosen $Q(1, -1, 4)$ as the "base point" and written

$$x = 1 + 4t, \qquad y = -1 - 3t, \qquad z = 4 + 7t.$$

These equations serve as well as the first; they simply place you at a different point on the line for a given value of t. ■

EXAMPLE 3 Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$ (Figure 12.37).

Solution We begin with equations for the line through P and Q, taking them, in this case, from Example 2:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t.$$

We observe that the point

$$(x, y, z) = (-3 + 4t, 2 - 3t, -3 + 7t)$$

on the line passes through $P(-3, 2, -3)$ at $t = 0$ and $Q(1, -1, 4)$ at $t = 1$. We add the restriction $0 \leq t \leq 1$ to parametrize the segment:

$$x = -3 + 4t, \qquad y = 2 - 3t, \qquad z = -3 + 7t, \qquad 0 \leq t \leq 1.$$

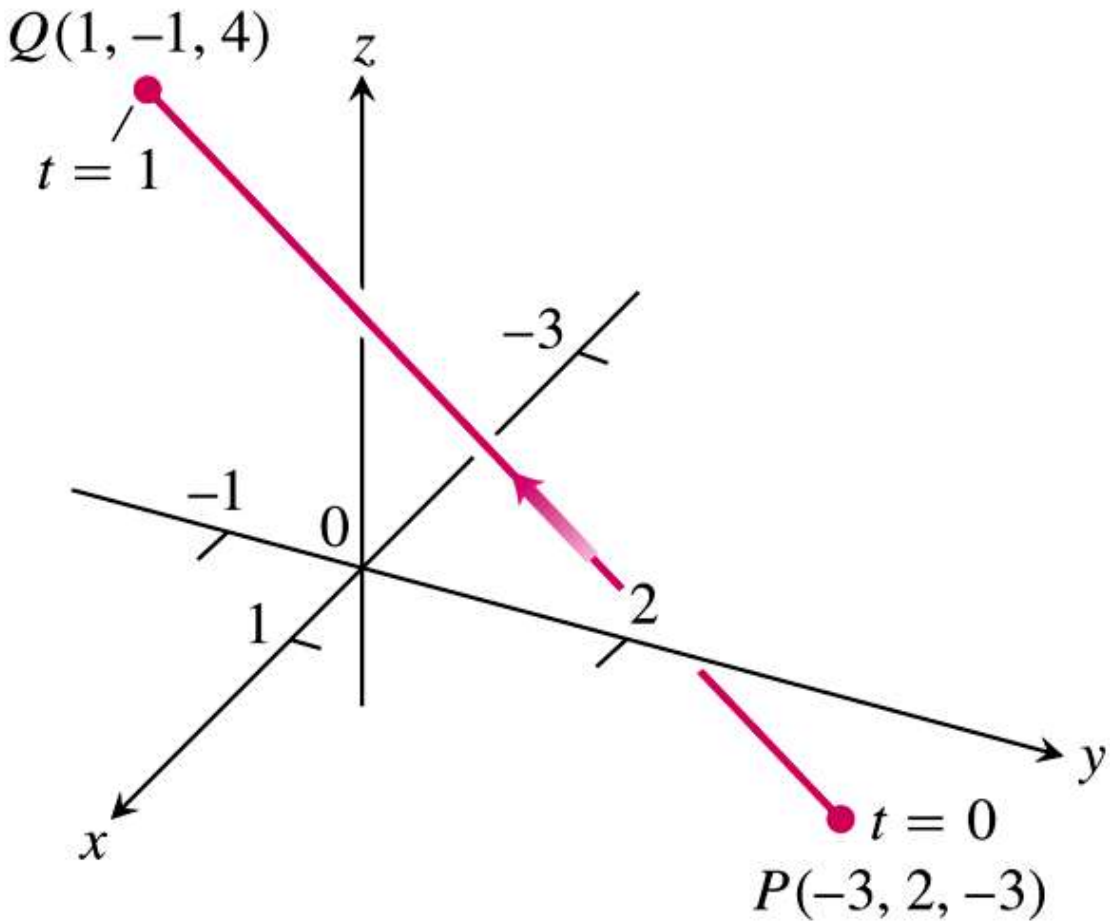


FIGURE 12.37 Example 3 derives a parametrization of line segment PQ. The arrow shows the direction of increasing t.

Distance from a Point S to a Line Through P Parallel to $\mathbf{v}$

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} \tag{5}$$

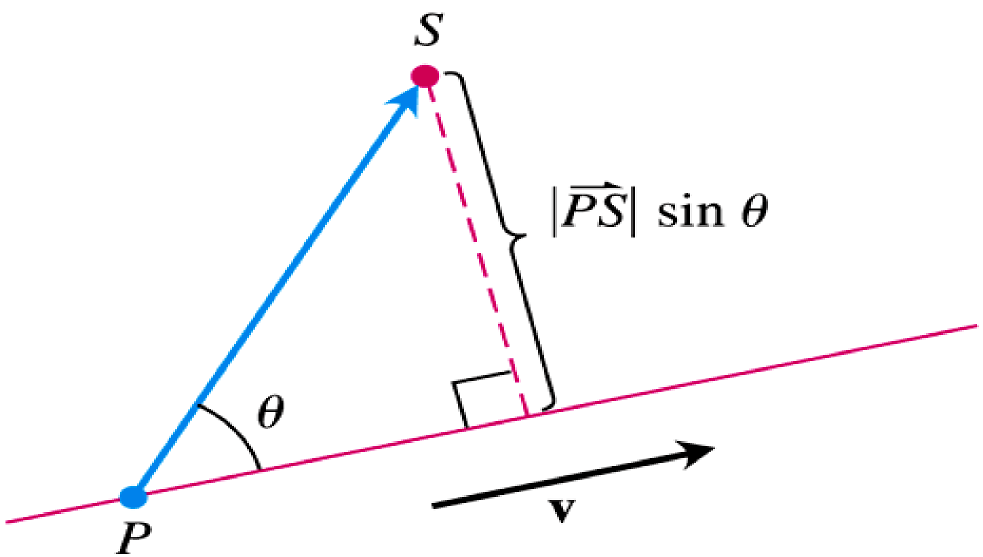


FIGURE 12.38 The distance from S to the line through P parallel to $\mathbf{v}$ is $|\overrightarrow{PS}| \sin \theta$, where θ is the angle between $\overrightarrow{PS}$ and $\mathbf{v}$.

EXAMPLE 5 Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \qquad x = 1 + t, \qquad y = 3 - t, \qquad z = 2t.$$

Solution We see from the equations for L that L passes through $P(1, 3, 0)$ parallel to $\mathbf{v} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$. With

$$\overrightarrow{PS} = (1 - 1)\mathbf{i} + (1 - 3)\mathbf{j} + (5 - 0)\mathbf{k} = -2\mathbf{j} + 5\mathbf{k}$$

and

$$\overrightarrow{PS} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 5 \\ 1 & -1 & 2 \end{vmatrix} = \mathbf{i} + 5\mathbf{j} + 2\mathbf{k},$$

Equation (5) gives

$$d = \frac{|\overrightarrow{PS} \times \mathbf{v}|}{|\mathbf{v}|} = \frac{\sqrt{1 + 25 + 4}}{\sqrt{1 + 1 + 4}} = \frac{\sqrt{30}}{\sqrt{6}} = \sqrt{5}.$$

An Equation for a Plane in Space

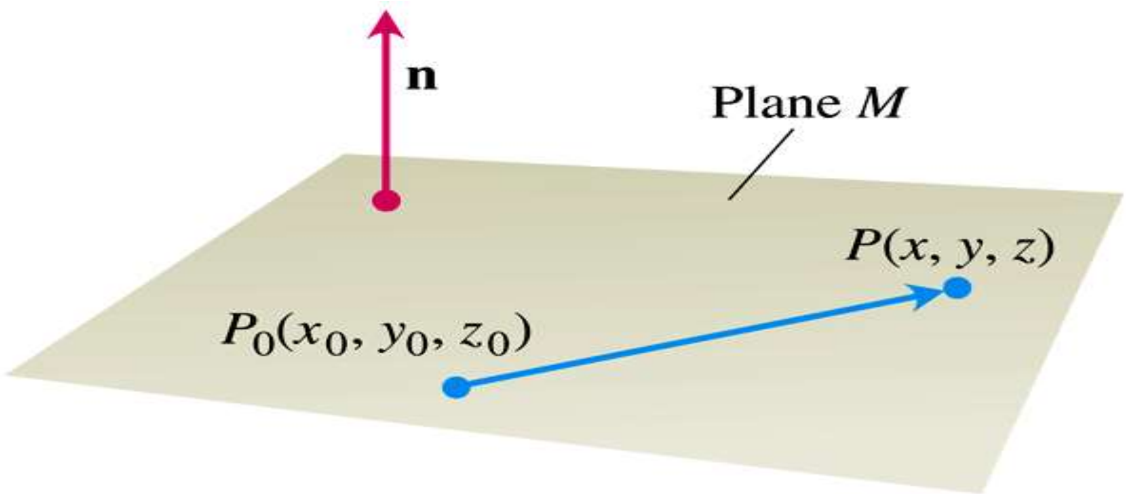


FIGURE 12.39 The standard equation for a plane in space is defined in terms of a vector normal to the plane: A point P lies in the plane through P_0 normal to $\mathbf{n}$ if and only if $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$.

An Equation for a Plane in Space

A plane in space is determined by knowing a point on the plane and a vector that is perpendicular or normal to the plane.

Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the vector $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$. Then M is the set of all points $P(x, y, z)$ for which $\overrightarrow{P_0P}$ is orthogonal to $\boldsymbol{n}$ (Figure 12.39). Thus, the dot product

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0.$$

This equation is equivalent to

$$\mathbf{n} \cdot \overrightarrow{P_0P} = 0$$

$$(A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}) \cdot [(x - x_0)\boldsymbol{i} + (y - y_0)\boldsymbol{j} + (z - z_0)\boldsymbol{k}] = 0$$

or

$$\mathrm{A}(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Equation for a Plane

The plane through $P_0(x_0, y_0, z_0)$ normal to $\mathbf{n} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$ has

Vector equation: $\mathbf{n} \cdot \overrightarrow{P_0P} = 0$

Component equation: $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

Component equation simplified: $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$

Example 6 Find an equation for the plane through $P_0(-3,0,7)$ perpendicular to $\boldsymbol{n} = 5\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$.

Solution The component equation is

$$\mathrm{A}(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$5(x - (-3)) + 2(y - 0) + (-1)(z - 7) = 0$$

Simplifying, we obtain

$$5\mathrm{x} + 2y - z = -22.$$

EXAMPLE 7 Find an equation for the plane through *A(0, 0,1)*, *B(2, 0, 0)*, and *C(0, 3, 0)*.

Solution To write an equation for the plane, we find a vector normal to the plane and use it with one of the points. As a point we choose *A(0, 0,1)* and the vector $\overrightarrow{AB} \times \overrightarrow{AC}$ is normal to the plane

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 0 & -1 \\ 0 & 3 & -1 \end{vmatrix} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}.$$

The component equation for the plane is

$$3(x - 0) + 2(y - 0) + 6(z - 1) = 0$$

EXAMPLE 8 Find parametric equations for the line in which the planes intersect

$$3x - 6y - 2z = 15 \quad \text{and} \quad 2x + y - 2z = 5.$$

Solution It is known from geometry that $\boldsymbol{n_1} \times \boldsymbol{n_2}$ is a vector parallel to the planes' line of intersection. In our case,

$$\boldsymbol{n_1} \times \boldsymbol{n_2} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = 14\boldsymbol{i} + 2\boldsymbol{j} + 15\boldsymbol{k}.$$

To find a point on the line, we can take any point common to the two planes. Substituting $z = 0$ in the plane equations and solving for x and y simultaneously

$$3x - 6y = 15$$

$$2x + y = 5$$

identifies one of these points as $(3, -1, 0)$.

The line is

$$x = 3 + 14t, \quad y = -1 + 2t, \quad z = 15t.$$

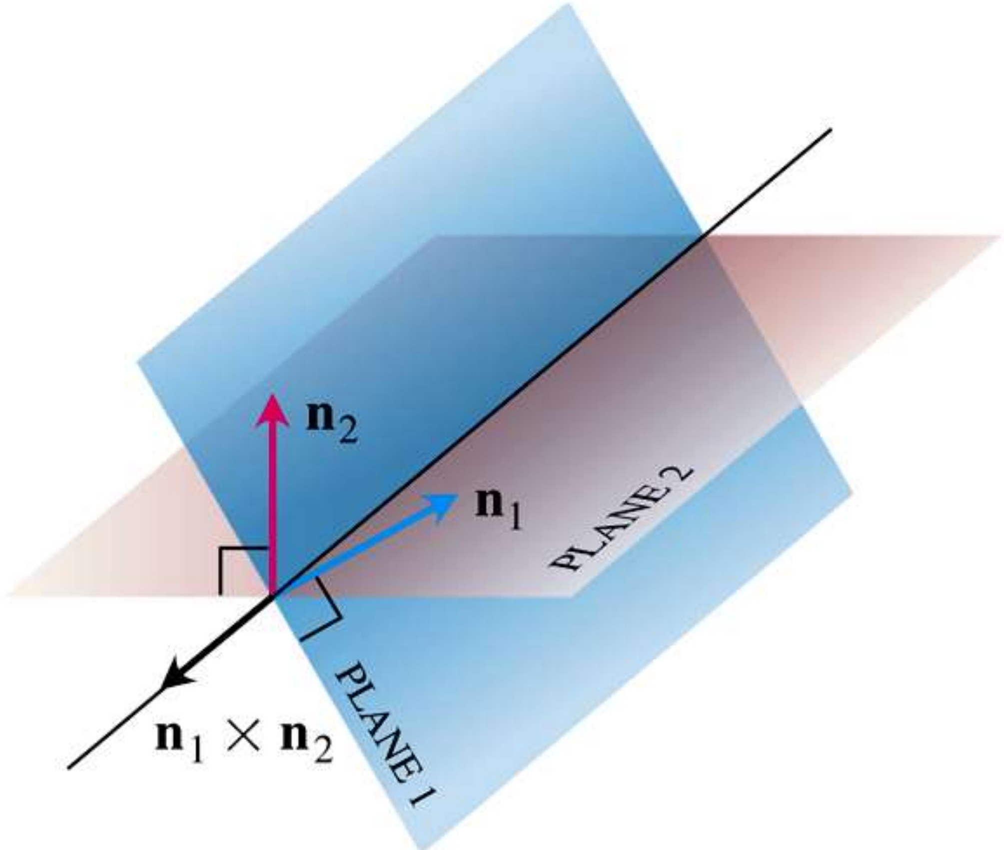


FIGURE 12.40 How the line of intersection of two planes is related to the planes' normal vectors (Example 8).

EXAMPLE 10 Find the point where the line

$$x = 8/3 + 2t, \quad y = -2t, \quad z = 1 + t$$

intersects the plane $3x + 2y + 6z = 6$.

Solution The point $(8/3 + 2t, -2t, 1 + t)$ lies in the plane if its coordinates satisfy the equation of the plane, that is, if

$$3(8/3 + 2t) + 2(-2t) + 6(1 + t) = 6$$

$$t = -1$$

The point of intersection (x,y,z) is

$(8/3 + 2(-1), -2(-1), 1 + (-1)) = (2/3, 2, 0)$.

The Distance from a Point to a Plane

If P is a point on a plane with normal $\boldsymbol{n}$, then the distance from any point S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\boldsymbol{n}$. That is, the distance from S to the plane is

$$d = \left|\overrightarrow{PS} \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right|$$

EXAMPLE 11 Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Solution We find a point P in the plane and calculate the length of the vector projection of $\overrightarrow{PS}$ onto a vector $\boldsymbol{n}$ normal to the plane (Figure 12.41). From the equation of the plane we obtain that

$$\boldsymbol{n} = 3\boldsymbol{i} + 2\boldsymbol{j} + 6\boldsymbol{k}.$$

To find a point P in the plane we set x = 0 and z = 0 in the equation of the plane to get $P(0, 3, 0)$.

$$\overrightarrow{PS} = (1-0)\boldsymbol{i} + (1-3)\boldsymbol{j} + (3-0)\boldsymbol{k}$$
$$= \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$$

$$|\boldsymbol{n}| = \sqrt{(3)^2+(2)^2+(6)^2} = 7$$

The distance from S to the plane is

$$d = \left|\overrightarrow{PS} \cdot \frac{\boldsymbol{n}}{|\boldsymbol{n}|}\right| = \left|(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) \cdot (\frac{3}{7}\boldsymbol{i} + \frac{2}{7}\boldsymbol{j} + \frac{6}{7}\boldsymbol{k})\right|$$

$$= \left|\frac{3}{7} - \frac{4}{7} + \frac{18}{7}\right| = \frac{17}{7}$$

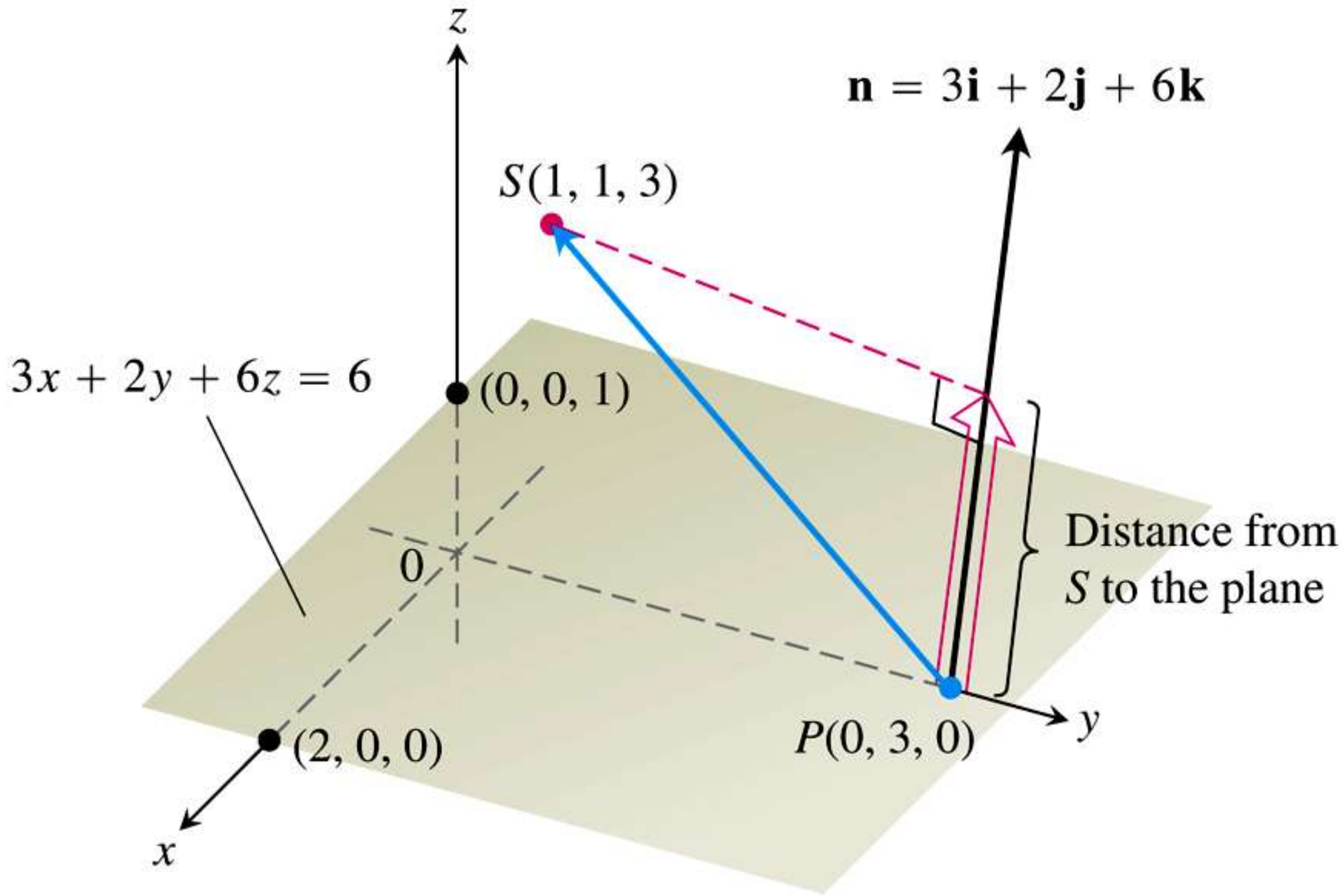


FIGURE 12.41 The distance from S to the plane is the length of the vector projection of $\overrightarrow{PS}$ onto $\mathbf{n}$ (Example 11).

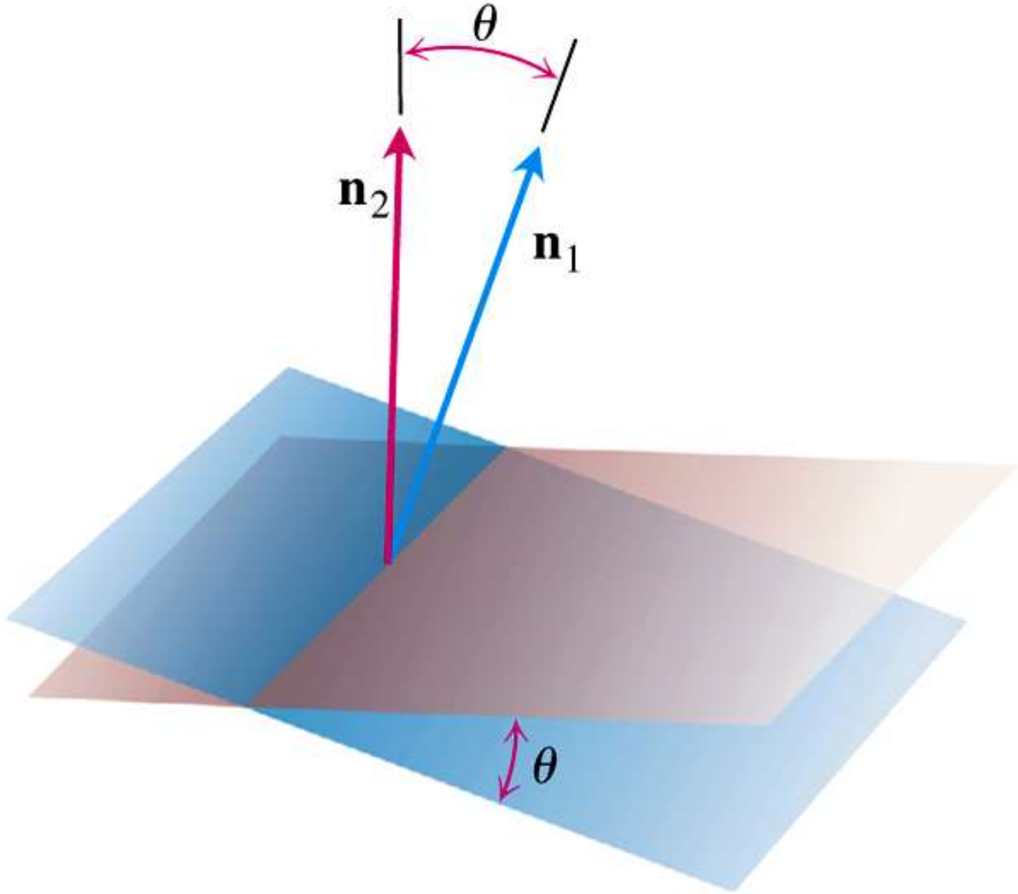


FIGURE 12.42 The angle between two planes is obtained from the angle between their normals.

EXAMPLE 12 Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Solution The vectors

$$\boldsymbol{n}_1 = 3\boldsymbol{i} - 6\boldsymbol{j} - 2\boldsymbol{k}, \quad \boldsymbol{n}_2 = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$$

are normals to the planes. The angle between them is

$$\theta = cos^{-1}\left(\frac{\boldsymbol{n}_1 \cdot \boldsymbol{n}_2}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}\right) = cos^{-1}\left(\frac{4}{21}\right)$$

$$\approx 1.38 \text{ radians}$$

12.6

Cylinders and Quadric Surfaces

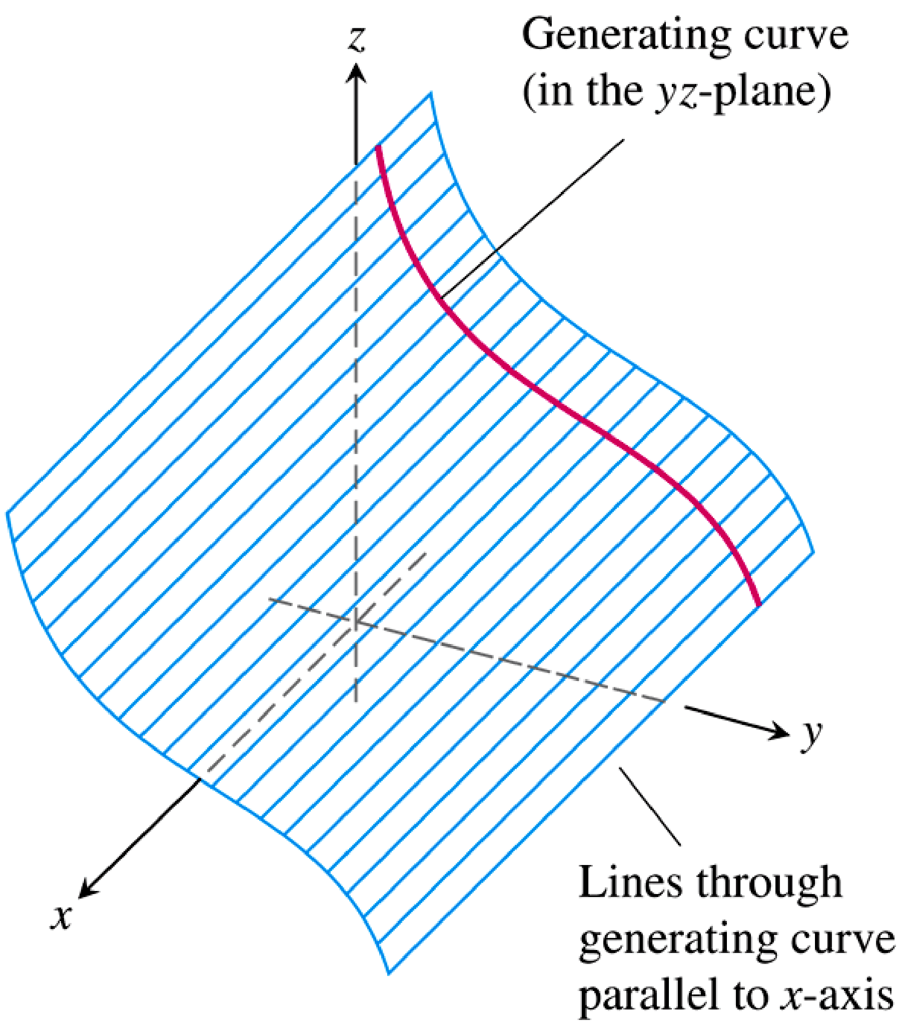


FIGURE 12.43 A cylinder and generating curve.

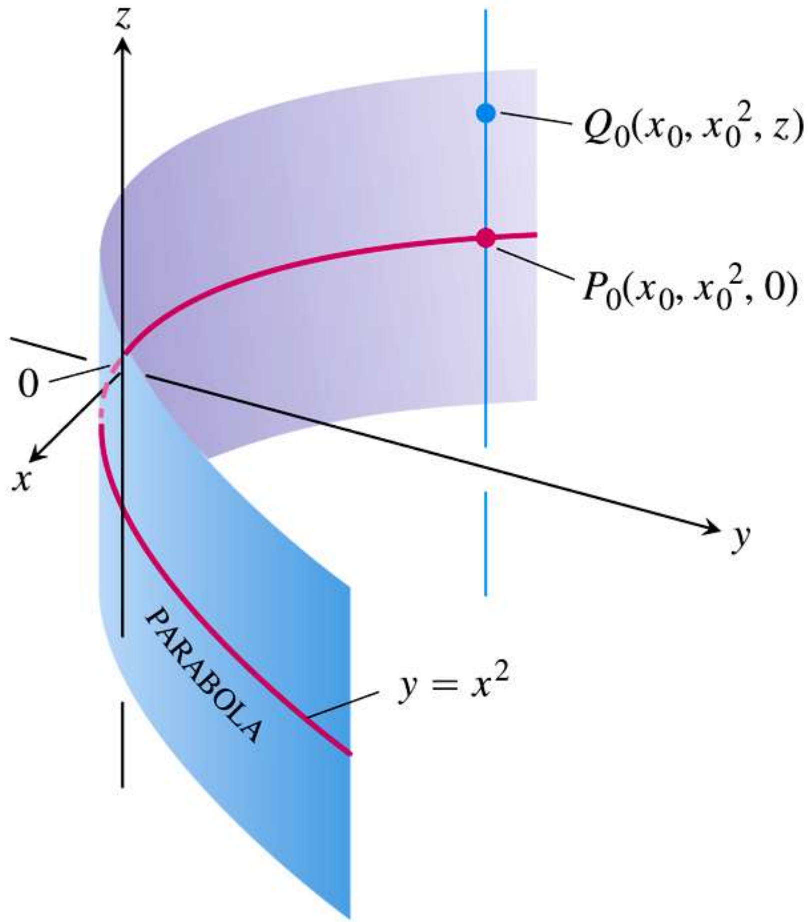


FIGURE 12.44 Every point of the cylinder in Example 1 has coordinates of the form (x_0, x_0^2, z). We call it "the cylinder $y = x^2$."

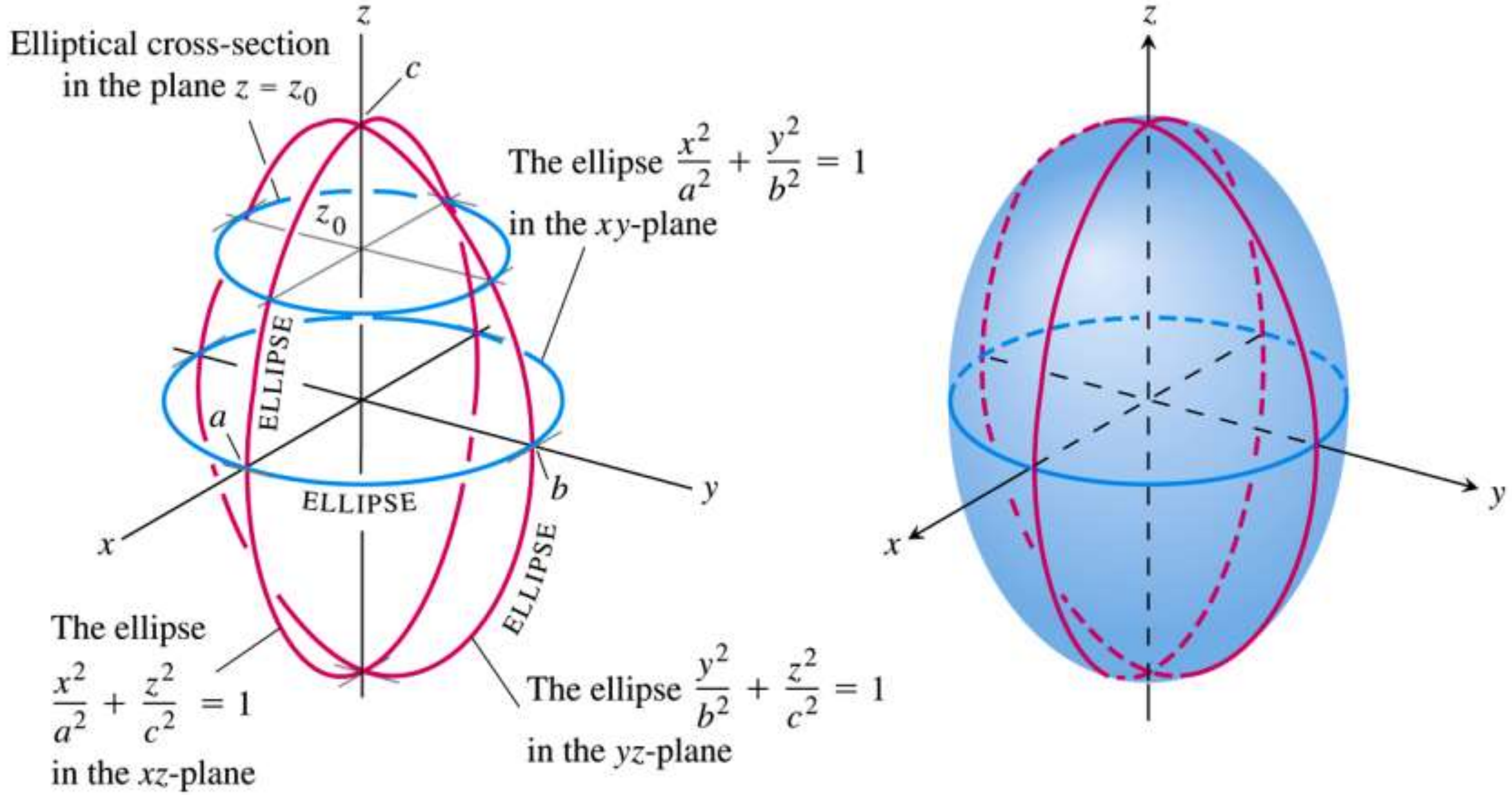


FIGURE 12.45 The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in Example 2 has elliptical cross-sections in each of the three coordinate planes.

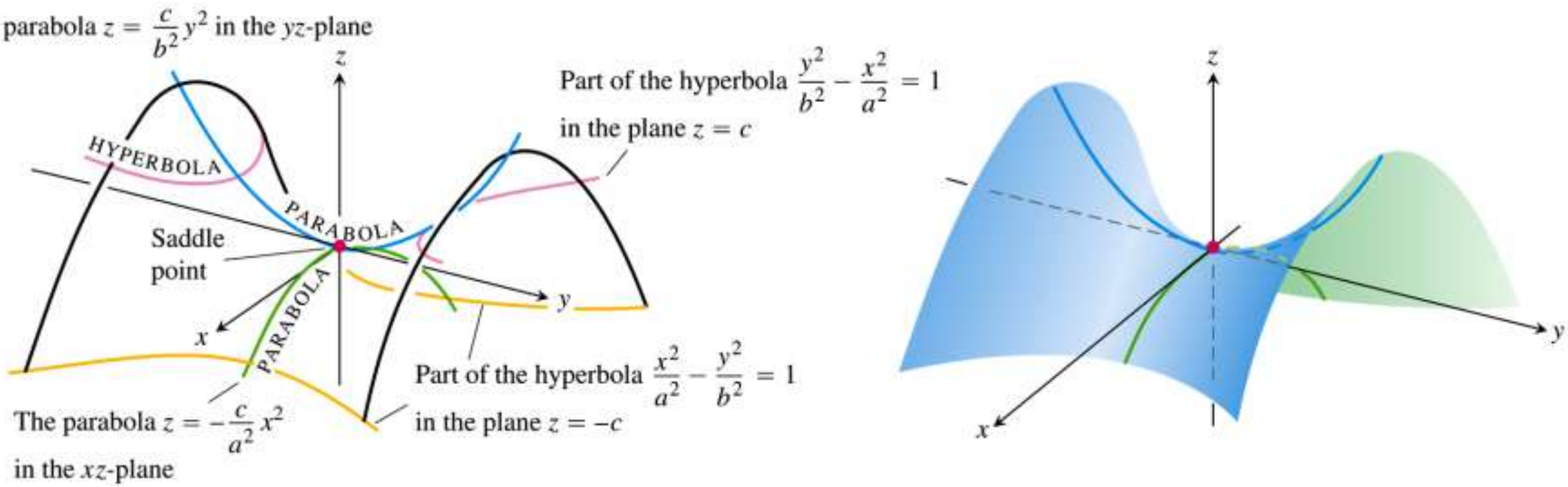
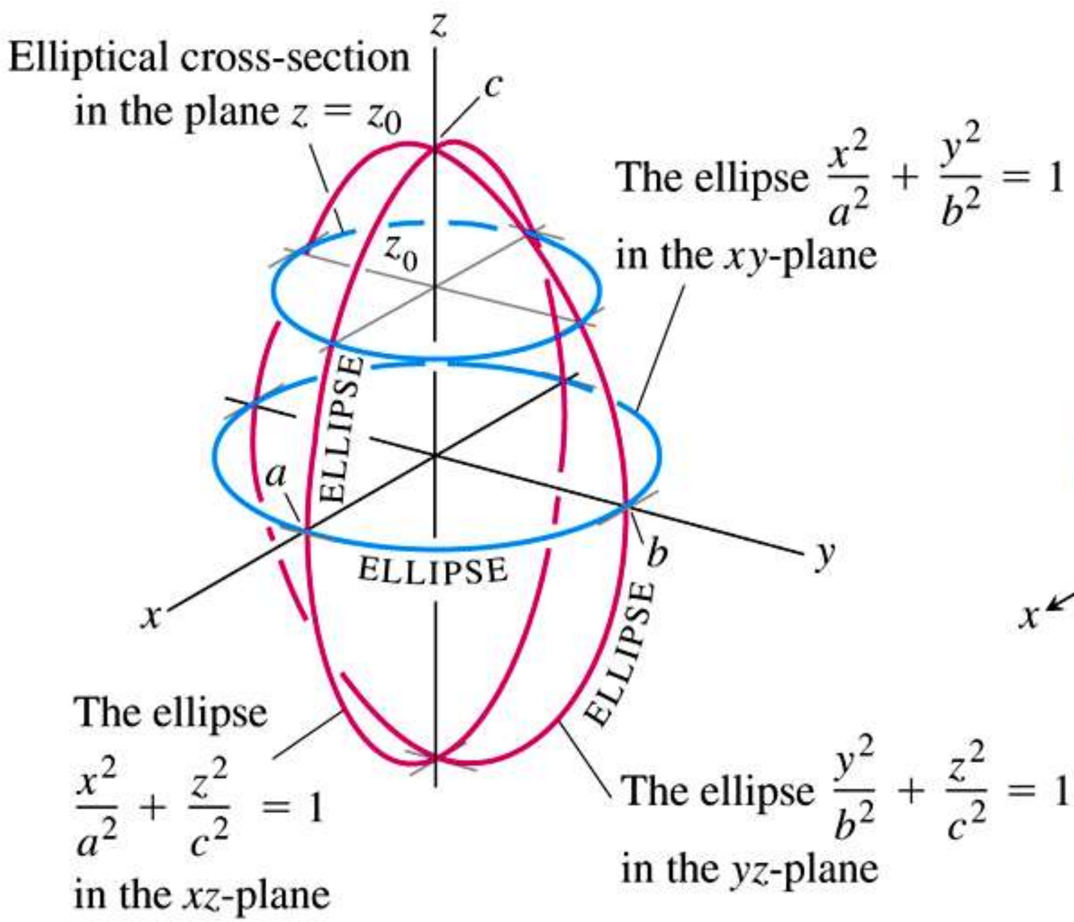
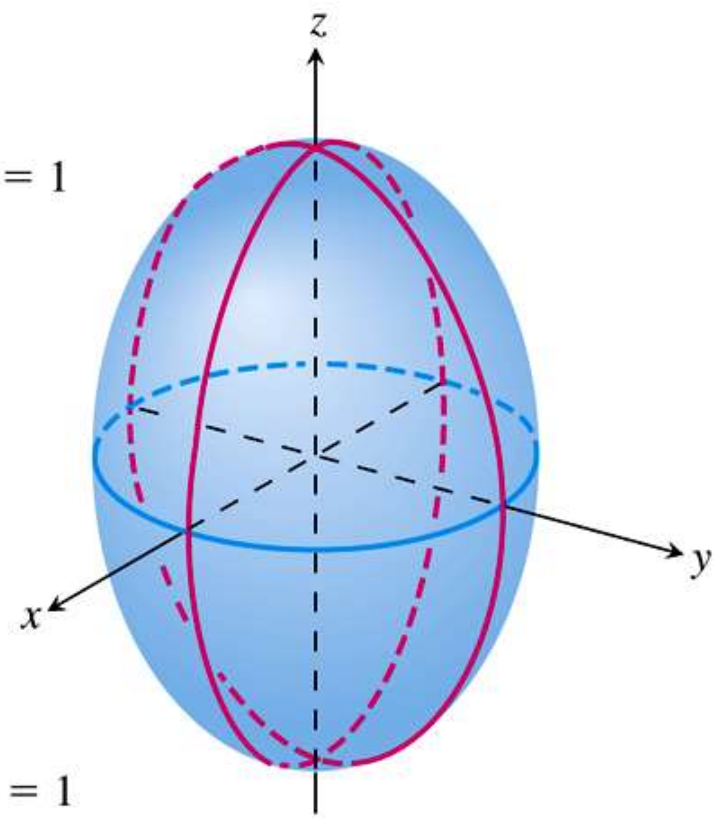


FIGURE 12.46 The hyperbolic paraboloid $(y^2/b^2) - (x^2/a^2) = z/c$, $c > 0$. The cross-sections in planes perpendicular to the z-axis above and below the xy-plane are hyperbolas. The cross-sections in planes perpendicular to the other axes are parabolas.

Table 12.1 Graphs of Quadric Surfaces

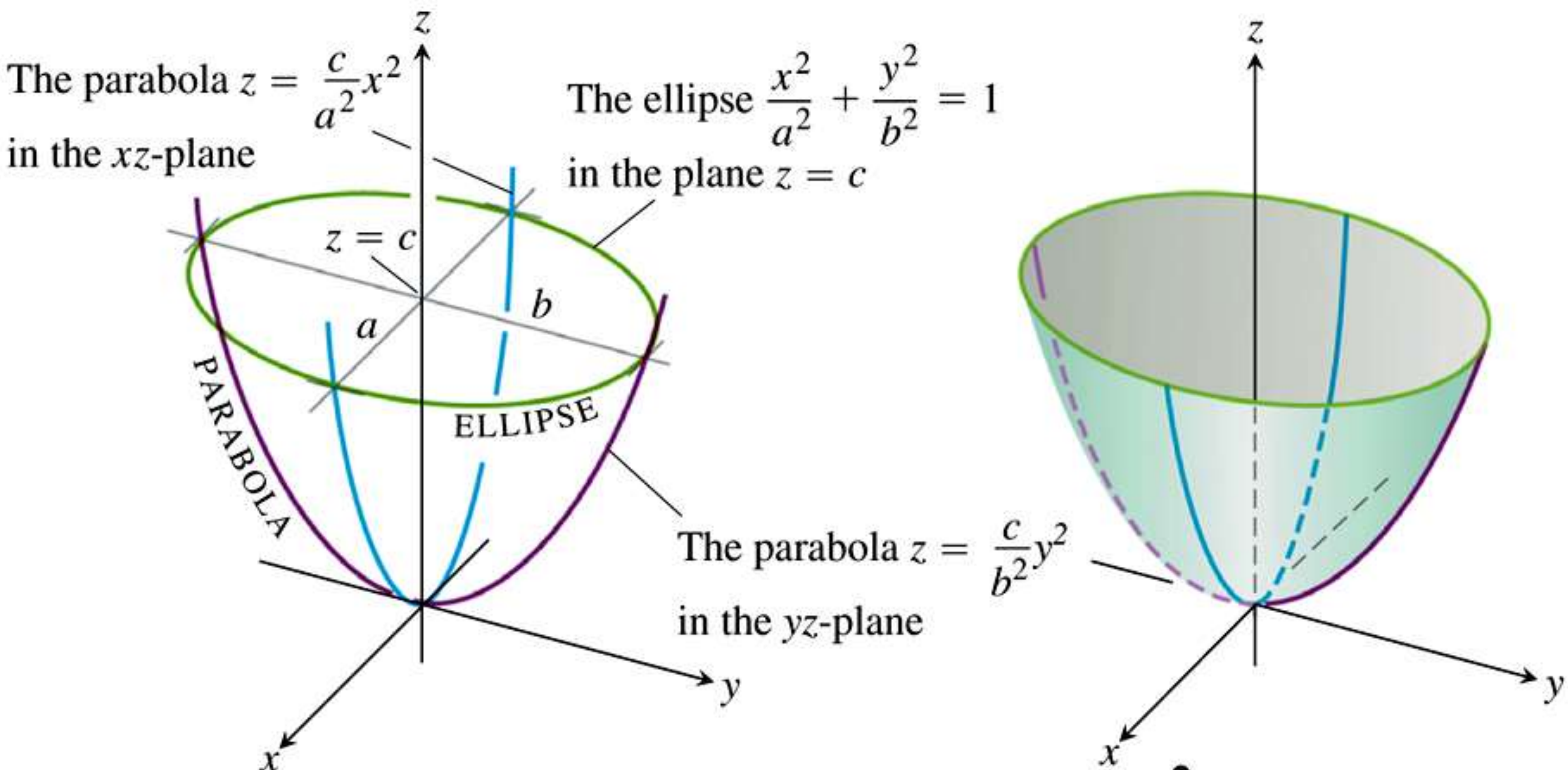




ELLIPSOID

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

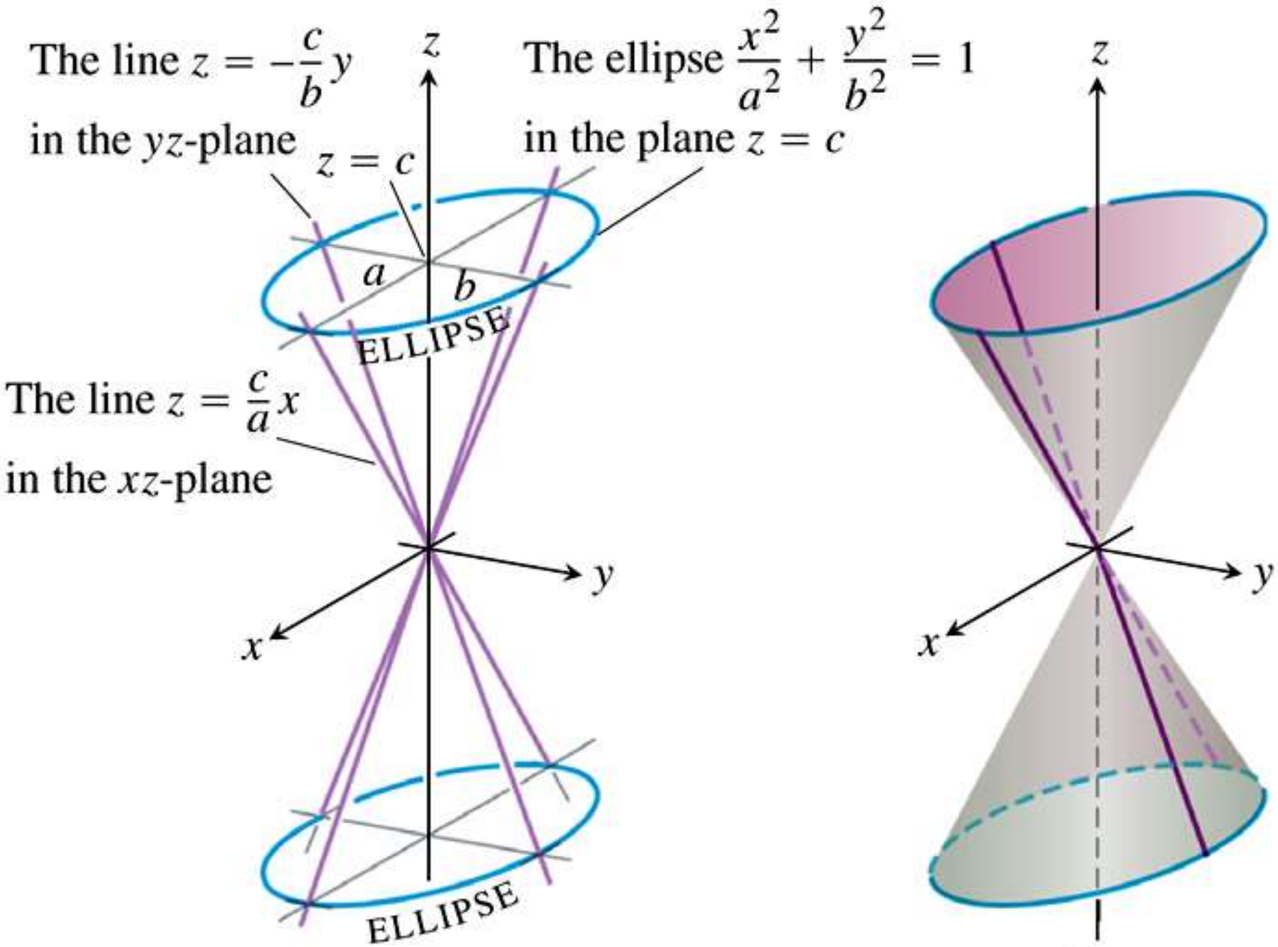
Table 12.1 Graphs of Quadric Surfaces (cont)



ELLIPTICAL PARABOLOID

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

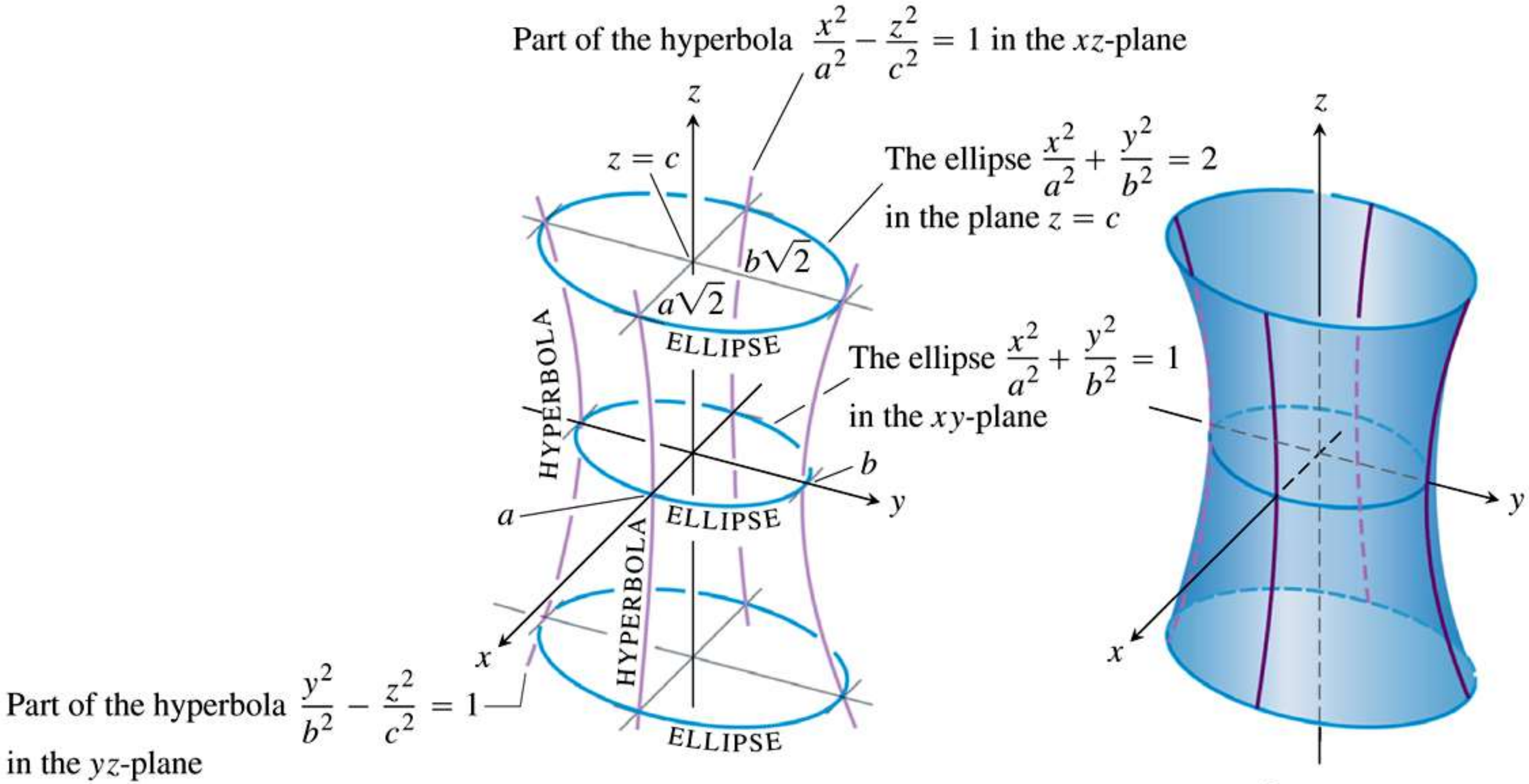
Table 12.1 Graphs of Quadric Surfaces (cont)



ELLIPTICAL CONE

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

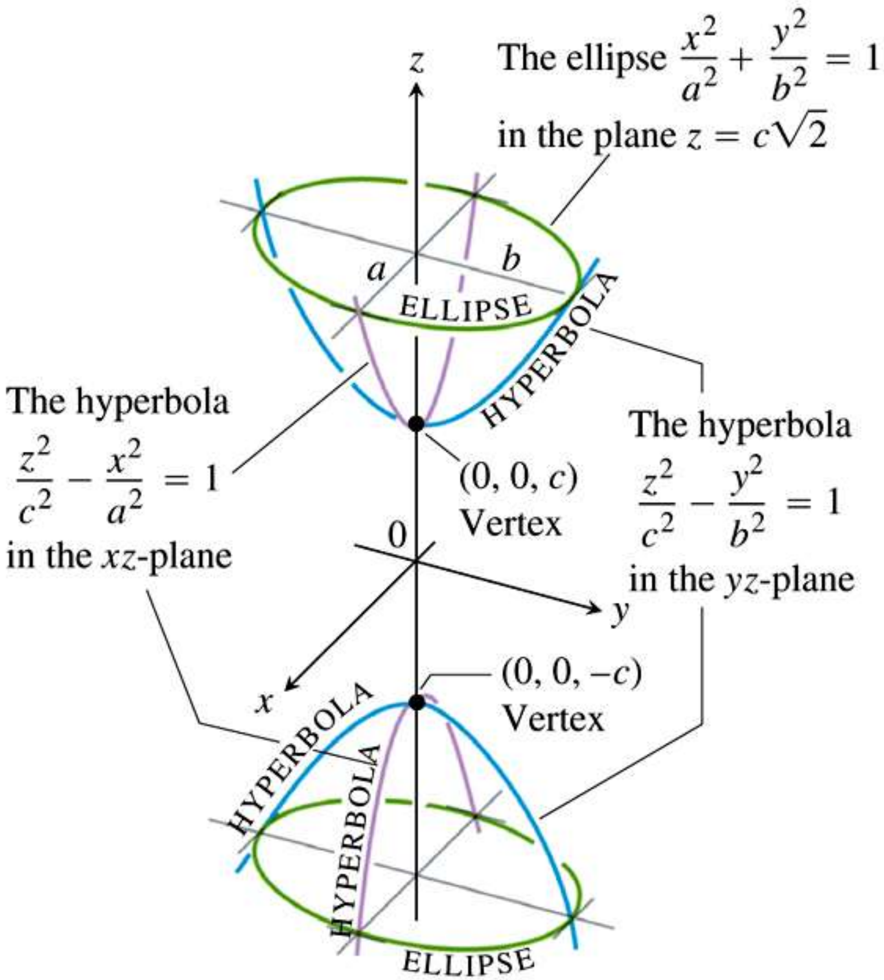
Table 12.1 Graphs of Quadric Surfaces (cont)

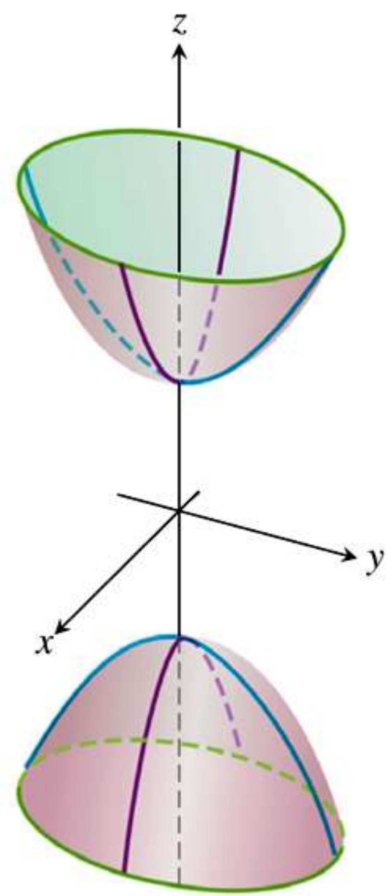


HYPERBOLOID OF ONE SHEET

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Table 12.1 Graphs of Quadric Surfaces (cont)

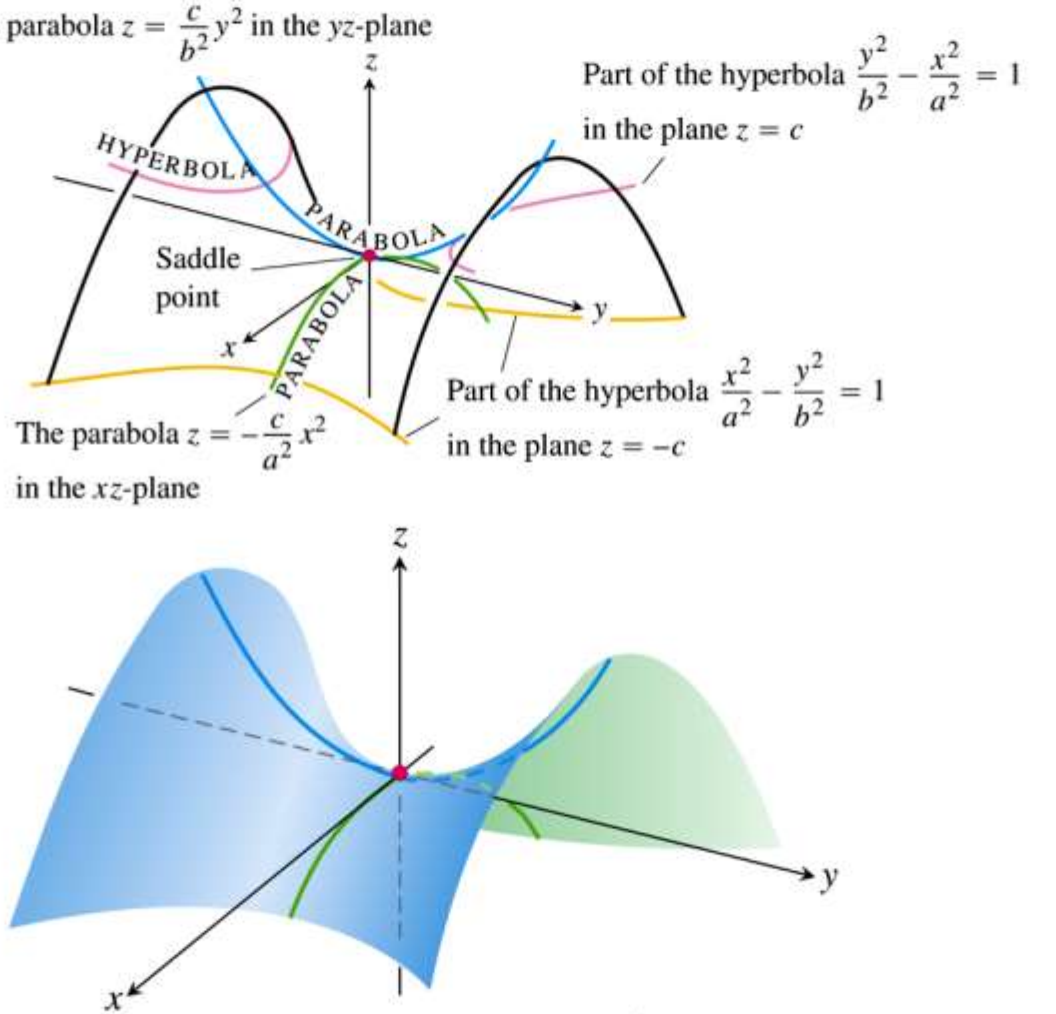




HYPERBOLOID OF TWO SHEETS

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Table 12.1 Graphs of Quadric Surfaces (cont)



HYPERBOLIC PARABOLOID $\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$

13 Vector Functions

13.1 Vector Functions and Space Curves

Vector Functions and Space Curves

In general, a function is a rule that assigns to each element in the domain an element in the range.

A **vector-valued function**, or **vector function**, is simply a function whose domain is a set of real numbers and whose range is a set of vectors.

We are most interested in vector functions **r** whose values are three-dimensional vectors.

This means that for every number t in the domain of **r** there is a unique vector in V_3 denoted by $\mathbf{r}(t)$.

Vector Functions and Space Curves

If $f(t)$, $g(t)$, and $h(t)$ are the components of the vector $\boldsymbol{r}(t)$, then f, g, and h are real-valued functions called the **component functions** of $\boldsymbol{r}$ and we can write

$$\boldsymbol{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

We use the letter t to denote the independent variable because it represents time in most applications of vector functions.

Example 1 – *Domain of a vector function*

Find the domain of the vector function

$$\boldsymbol{r}(t) = \left\langle e^{-3t}, \sqrt{4 - t^2}, \ln(t+1) \right\rangle$$

Solution: The component functions are

$$f(t) = e^{-3t} \qquad g(t) = \sqrt{4 - t^2} \qquad h(t) = \ln(t+1)$$

By our usual convention, the domain of ***r*** consists of all values of *t* for which the expression for ***r***(*t*) is defined.

e^{-3t} is defined for all $t \in \mathbb{R}$, $\sqrt{4 - t^2}$ is defined when $4 - t^2 \geq 0$ or $-2 \leq t \leq 2$, and $\ln(t+1)$ is defined when $t + 1 > 0$ or $t > -1$.

Therefore the domain of ***r*** is the interval (-1, 2].

Example 2 – *Domain of a vector function*

Find the domain of the vector function

$$\boldsymbol{r}(t) = \langle t^3, \ln(3 - t), \sqrt{t} \rangle$$

Solution: The component functions are

$$f(t) = t^3 \qquad g(t) = \ln(3 - t) \qquad h(t) = \sqrt{t}$$

By our usual convention, the domain of ***r*** consists of all values of t for which the expression for $\boldsymbol{r}(t)$ is defined.

t^3 is defined for all $t \in \mathbb{R}$, $\ln(3 - t)$ is defined when $3 - t > 0$ or $t < 3$, and $\sqrt{t}$ is defined when $t \geq 0$.

Therefore the domain of ***r*** is the interval $[0, 3)$.

Vector Functions and Space Curves

The **limit** of a vector function **r** is defined by taking the limits of its component functions as follows.

$\boxed{1}$ If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$$

provided the limits of the component functions exist.

Limits of vector functions obey the same rules as limits of real-valued functions.

Example 3 – Limit *of a vector function*

EXAMPLE 1 If $\boldsymbol{r}(t) = \left\langle t^3, \frac{t}{t^2-t}, 3\frac{sint}{t} \right\rangle$, then

$$\lim_{t\to 0} \boldsymbol{r}(t) = \left\langle \lim_{t\to 0} t^3, \lim_{t\to 0} \frac{t}{t^2-t}, \lim_{t\to 0} 3\frac{sint}{t} \right\rangle = \langle 0, -1, 3 \rangle$$

EXAMPLE 2 If $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}$, then

$$\lim_{t\to\pi/4} \mathbf{r}(t) = \left(\lim_{t\to\pi/4} \cos t\right)\mathbf{i} + \left(\lim_{t\to\pi/4} \sin t\right)\mathbf{j} + \left(\lim_{t\to\pi/4} t\right)\mathbf{k}$$

$$= \frac{\sqrt{2}}{2}\mathbf{i} + \frac{\sqrt{2}}{2}\mathbf{j} + \frac{\pi}{4}\mathbf{k}.$$

Vector Functions and Space Curves

A vector function **r** is **continuous at *a*** if

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$$

In view of Definition 1, we see that **r** is continuous at *a* if and only if its component functions *f*, *g*, and *h* are continuous at *a*.

There is a close connection between continuous vector functions and space curves.

Vector Functions and Space Curves

Suppose that *f*, *g*, and *h* are continuous real-valued functions on an interval *I*.

Then the set *C* of all points (x, y, z) in space, where

$$\boxed{2} \qquad x = f(t) \qquad y = g(t) \qquad z = h(t)$$

and *t* varies throughout the interval *I*, is called a **space curve**.

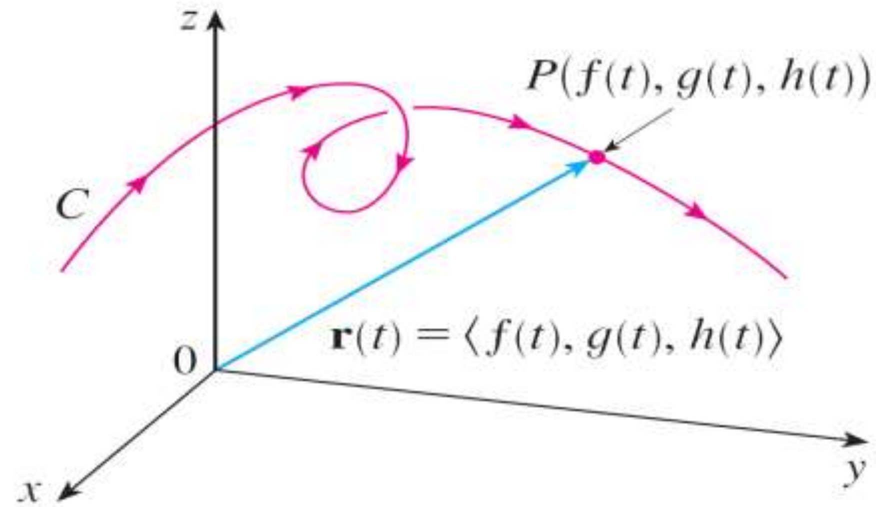
The equations in $\boxed{2}$ are called **parametric equations of C** and *t* is called a **parameter**.

We can think of *C* as being traced out by a moving particle whose position at time *t* is $(f(t), g(t), h(t))$.

Vector Functions and Space Curves

If we now consider the vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then $\mathbf{r}(t)$ is the position vector of the point $P(f(t), g(t), h(t))$ on C.

Thus any continuous vector function $\mathbf{r}$ defines a space curve C that is traced out by the tip of the moving vector $\mathbf{r}(t)$, as shown in Figure 1.



C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

Figure 1

Example 4 – *Sketching a helix*

Sketch the curve whose vector equation is

$$\mathbf{r}(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j} + t\ \mathbf{k}$$

Solution:

The parametric equations for this curve are

$$x = \cos t \qquad y = \sin t \qquad z = t$$

Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the curve must lie on the circular cylinder $x^2 + y^2 = 1$.

The point (x, y, z) lies directly above the point $(x, y, 0)$, which moves counterclockwise around the circle $x^2 + y^2 = 1$ in the xy-plane.

Example 4 – *Solution*

cont'd

(The projection of the curve onto the *xy*-plane has vector equation $\mathbf{r}(t) = \langle \cos t, \sin t, 0 \rangle$.) Since $z = t$, the curve spirals upward around the cylinder as t increases. The curve, shown in Figure 2, is called a **helix**.

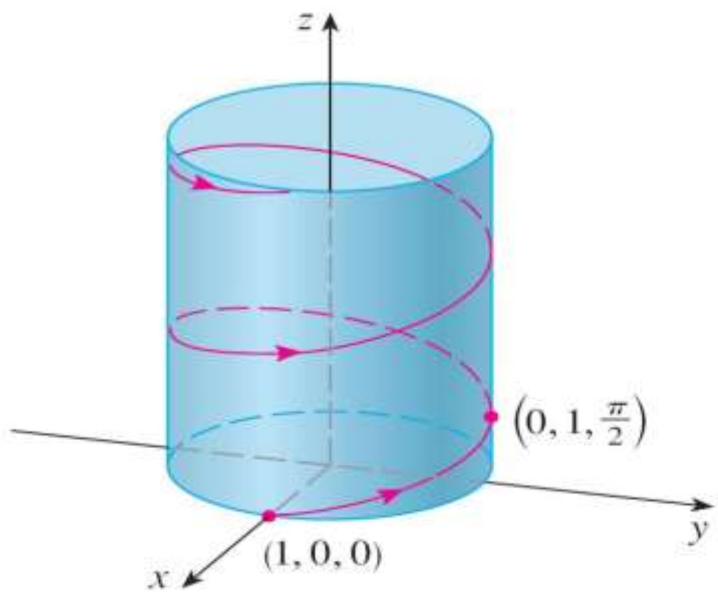


Figure 2

Vector Functions and Space Curves

The corkscrew shape of the helix in Example 4 is familiar from its occurrence in coiled springs.

It also occurs in the model of DNA (deoxyribonucleic acid, the genetic material of living cells).

In 1953 James Watson and Francis Crick showed that the structure of the DNA molecule is that of two linked, parallel helixes that are intertwined as in Figure 3.

A double helix

Figure 3

Example 5 – Curve of intersection

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z + y = 2$.

Solution: The curve of intersection *C* is an ellipse.

The projection of *C* onto the *xy*-plane is the circle $x^2 + y^2 = 1,\ z = 0$. So we can

$$x = \cos t \qquad y = \sin t \qquad 0 \le t \le 2\pi.$$

From the equation of the plane, we have

$$z = 2 - y = 2 - sint$$

So we can write parametric equations for *C* as

$$x = \cos t \qquad y = \sin t \qquad z + y = 2 \qquad 0 \le t \le 2\pi.$$

The corresponding vector equation is

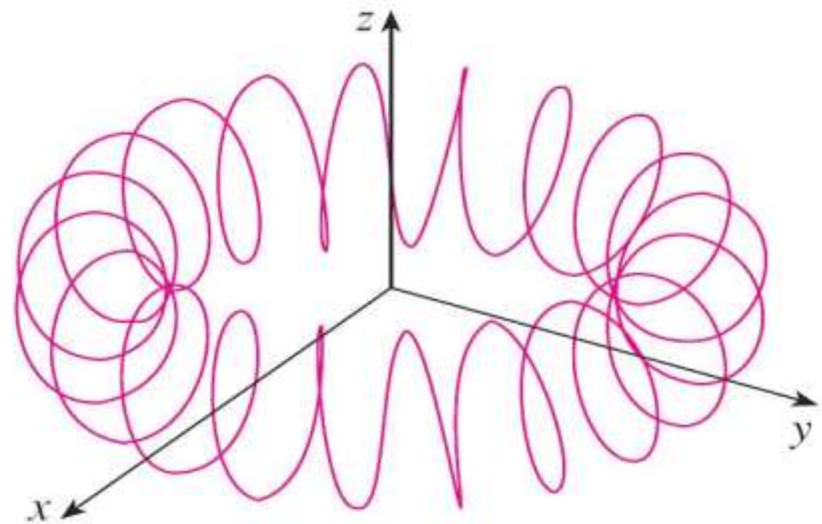
$$\boldsymbol{r}(t) = cost\ \boldsymbol{i} + sint\ \boldsymbol{j} + (2 - sint)\ \boldsymbol{k}$$

Using Computers to Draw Space Curves

Space curves are inherently more difficult to draw by hand than plane curves; for an accurate representation we need to use technology.

For instance, Figure 7 shows a computer-generated graph of the curve with parametric equations

$x = (4 + \sin 20t) \cos t$
$y = (4 + \sin 20t) \sin t$
$z = \cos 20t$



A toroidal spiral

Figure 7

It's called a **toroidal spiral** because it lies on a torus.

13 Vector Functions

13.2 Derivatives and Integrals of Vector Functions

Derivatives

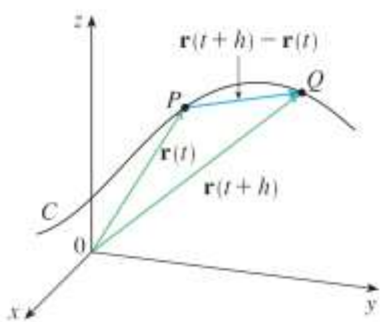
Derivatives

The **derivative r′** of a vector function **r** is defined in much the same way as for real valued functions:

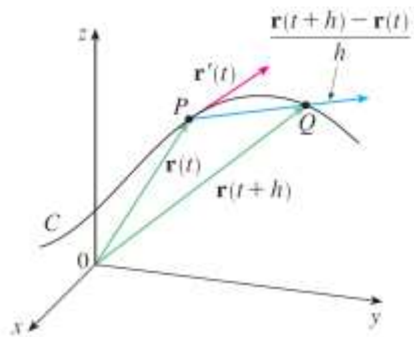
1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if this limit exists. The geometric significance of this definition is shown in Figure 1.



(a) The secant vector $\overrightarrow{PQ}$



(b) The tangent vector r′(t)

Figure 1

Derivatives

If the points P and Q have position vectors $\mathbf{r}(t)$ and $\mathbf{r}(t + h)$, then $\overrightarrow{PQ}$ represents the vector $\mathbf{r}(t + h) - \mathbf{r}(t)$, which can therefore be regarded as a secant vector.

If $h > 0$, the scalar multiple $(1/h)(\mathbf{r}(t + h) - \mathbf{r}(t))$ has the same direction as $\mathbf{r}(t + h) - \mathbf{r}(t)$. As $h \to 0$, it appears that this vector approaches a vector that lies on the tangent line.

For this reason, the vector $\mathbf{r}'(t)$ is called the **tangent vector** to the curve defined by $\mathbf{r}$ at the point P, provided that $\mathbf{r}'(t)$ exists and $\mathbf{r}'(t) \neq \mathbf{0}$.

Derivatives

The **tangent line** to *C* at *P* is defined to be the line through *P* parallel to the tangent vector $\mathbf{r}'(t)$.

We will also have occasion to consider the **unit tangent vector**, which is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Derivatives

The following theorem gives us a convenient method for computing the derivative of a vector function **r**: just differentiate each component of **r**.

$\boxed{2}$ **Theorem** If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$, where f, g, and h are differentiable functions, then

$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

Example 1

(a) Find the derivative of $\mathbf{r}(t) = (1 + t^3)\mathbf{i} + te^{-t}\,\mathbf{j} + \sin 2t\,\mathbf{k}$.

(b) Find the unit tangent vector at the point where $t = 0$.

Solution:

(a) According to Theorem 2, we differentiate each component of $\mathbf{r}$:

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + (1 - t)e^{-t}\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$$

Example 1 – *Solution*

cont'd

(b) Since $\mathbf{r}(0) = \mathbf{i}$ and $\mathbf{r}'(0) = \mathbf{j} + 2\mathbf{k}$, the unit tangent vector at the point (1, 0, 0) is

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|}$$

$$= \frac{\mathbf{j} + 2\mathbf{k}}{\sqrt{1 + 4}}$$

$$= \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

Example 2: Tangent line

Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t, \qquad y = \sin t, \qquad z = t,$$

at the point $(0,1,\frac{\pi}{2})$.

SOLUTION The vector equation of the helix is

$$\boldsymbol{r}(t) = \langle 2\cos t, \sin t, t\rangle, \text{ so } \boldsymbol{r}'(t) = \langle -2\sin t, \cos t, 1\rangle.$$

The parameter value corresponding to the point $(0,1,\frac{\pi}{2})$ is $t = \frac{\pi}{2}$, so the tangent vector there is $\boldsymbol{r}'\left(\frac{\pi}{2}\right) = \langle -2,0,1\rangle$. The tangent line is the line through $P(0,1,\frac{\pi}{2})$ parallel to the vector $\vec{v} = \langle -2,0,1\rangle$, so its parametric equations are

$$x = -2t, \qquad y = 1, \qquad z = \frac{\pi}{2} + t$$

Derivatives

Just as for real-valued functions, the **second derivative** of a vector function $\mathbf{r}$ is the derivative of $\mathbf{r}'$, that is, $\mathbf{r}'' = (\mathbf{r}')'$.

For instance, the second derivative of the function, $\mathbf{r}(t) = \langle 2 \cos t, \sin t, t \rangle$, is

$$\mathbf{r}''(t) = \langle -2 \cos t, -\sin t, 0 \rangle.$$

Differentiation Rules

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Example 3

Show that if $|\mathbf{r}(t)| = c$ (a constant), then $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$ for all t.

Solution:
Since

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = c^2$$

and c^2 is a constant, Formula 4 of Theorem 3 gives

$$0 = \frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{r}(t)] = \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 2\mathbf{r}'(t) \cdot \mathbf{r}(t)$$

Thus $\mathbf{r}'(t) \cdot \mathbf{r}(t) = 0$, which says that $\mathbf{r}'(t)$ is orthogonal to $\mathbf{r}(t)$.

Example 3 – *Solution*

cont'd

Geometrically, this result says that if a curve lies on a sphere with center the origin, then the tangent vector $\mathbf{r}'(t)$ is always perpendicular to the position vector $\mathbf{r}(t)$.

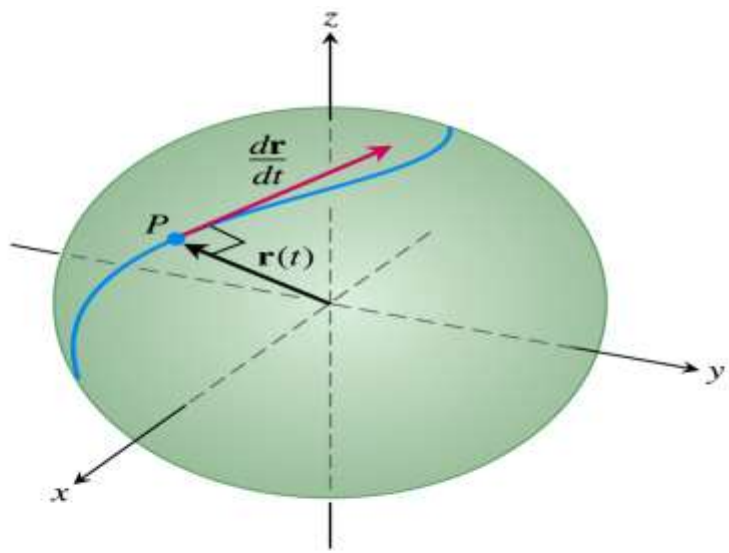


FIGURE 13.8 If a particle moves on a sphere in such a way that its position $\mathbf{r}$ is a differentiable function of time, then $\mathbf{r} \cdot (d\mathbf{r}/dt) = 0$.

Integrals

Integrals

The **definite integral** of a continuous vector function **r**(t) can be defined in much the same way as for real-valued functions except that the integral is a vector.

But then we can express the integral of **r** in terms of the integrals of its component functions f, g, and h as follows.

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \mathbf{r}(t_i^*)\,\Delta t$$

$$= \lim_{n\to\infty} \left[\left(\sum_{i=1}^{n} f(t_i^*)\,\Delta t \right) \mathbf{i} + \left(\sum_{i=1}^{n} g(t_i^*)\,\Delta t \right) \mathbf{j} + \left(\sum_{i=1}^{n} h(t_i^*)\,\Delta t \right) \mathbf{k} \right]$$

Integrals

and so

$$\boxed{\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}}$$

This means that we can evaluate an integral of a vector function by integrating each component function.

Integrals

We can extend the Fundamental Theorem of Calculus to continuous vector functions as follows:

$$\int_a^b \mathbf{r}(t)\,dt = \mathbf{R}(t)\Big]_a^b = \mathbf{R}(b) - \mathbf{R}(a)$$

where **R** is an antiderivative of **r**, that is, $\mathbf{R}'(t) = \mathbf{r}(t)$.

We use the notation $\int \mathbf{r}(t)\,dt$ for indefinite integrals (antiderivatives).

Example 4

If $\mathbf{r}(t) = 2 \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + 2t\, \mathbf{k}$, then

(1) Indefinite integral:

$$\int \mathbf{r}(t)\, dt = \left(\int 2 \cos t\, dt \right) \mathbf{i} + \left(\int \sin t\, dt \right) \mathbf{j} + \left(\int 2t\, dt \right) \mathbf{k}$$

$$= 2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k} + \mathbf{C}$$

where $\mathbf{C}$ is a vector constant of integration.

(2) Definite Integral:

$$\int_0^{\pi/2} \mathbf{r}(t)\, dt = \left[2 \sin t\, \mathbf{i} - \cos t\, \mathbf{j} + t^2\, \mathbf{k} \right]_0^{\pi/2}$$

$$= 2\,\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$$

13 Vector Functions

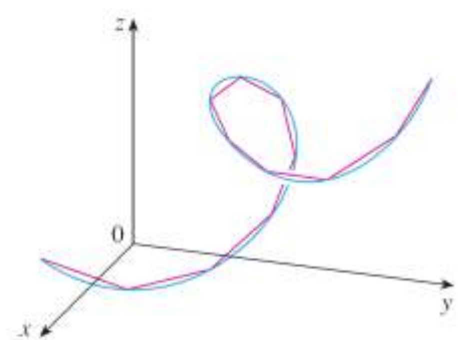
13.3 Arc Length and Curvature

Arc Length and Curvature

We have defined the length of a plane curve with parametric equations $x = f(t)$, $y = g(t)$, $a \le t \le b$, as the limit of lengths of inscribed polygons and, for the case where f' and g' are continuous, we arrived at the formula

$$\boxed{1} \qquad L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

The length of a space curve is defined in exactly the same way (see Figure 1).



The length of a space curve is the limit of lengths of inscribed polygons.

Figure 1

Arc Length and Curvature

Suppose that the curve has the vector equation, $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, $a \le t \le b$, or, equivalently, the parametric equations $x = f(t)$, $y = g(t)$, $z = h(t)$, where f', g', and h' are continuous.

If the curve is traversed exactly once as t increases from a to b, then it can be shown that its length is

2

$$\boxed{\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \end{aligned}}$$

Arc Length and Curvature

Notice that both of the arc length formulas $\boxed{1}$ and $\boxed{2}$ can be put into the more compact form

$$\boxed{3} \qquad L = \int_a^b |\mathbf{r}'(t)|\, dt$$

because, for plane curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j}| = \sqrt{[f'(t)]^2 + [g'(t)]^2}$$

and for space curves $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$,

$$|\mathbf{r}'(t)| = |f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}$$

Example 1

Find the length of the arc of the circular helix with vector equation **r**(t) = cos t **i** + sin t **j** + t **k** from the point (1, 0, 0) to the point (1, 0, 2π).

Solution:

Since **r**′(t) = –sin t **i** + cos t **j** + **k**, we have

$$|\mathbf{r}'(t)| = \sqrt{(-\sin t)^2 + \cos^2 t + 1} = \sqrt{2}$$

The arc from (1, 0, 0) to (1, 0, 2π) is described by the parameter interval $0 \le t \le 2\pi$ and so, from Formula 3, we have

$$L = \int_0^{2\pi} |\mathbf{r}'(t)|\,dt = \int_0^{2\pi} \sqrt{2}\,dt = 2\sqrt{2}\,\pi$$

Curvature

Curvatures

A parametrization **r**(*t*) is called **smooth** on an interval *I* if **r′** is continuous and **r′**(*t*) $\neq$ **0** on *I*.

A curve is called **smooth** if it has a smooth parametrization. A smooth curve has no sharp corners or cusps; when the tangent vector turns, it does so continuously.

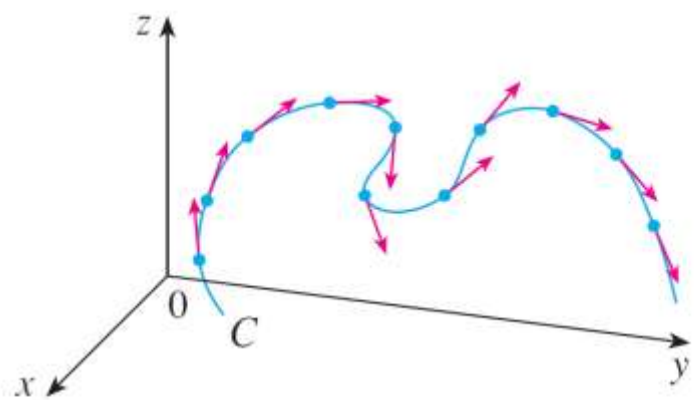
If *C* is a smooth curve defined by the vector function **r**, recall that the unit tangent vector **T**(*t*) is given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

and indicates the direction of the curve.

Curvatures

From Figure 4 you can see that **T**(t) changes direction very slowly when C is fairly straight, but it changes direction more quickly when C bends or twists more sharply.



Unit tangent vectors at equally spaced points on C

Figure 4

Curvatures

The curvature of C at a given point is a measure of how quickly the curve changes direction at that point.

Specifically, we define it to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. (We use arc length so that the curvature will be independent of the parametrization.)

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

Curvatures

The curvature is easier to compute if it is expressed in terms of the parameter *t* instead of *s*, so we use the Chain Rule to write

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} \qquad \text{and} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right|$$

But $ds/dt = |\mathbf{r}'(t)|$ from Equation 7, so

9

$$\boxed{\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}}$$

Example 3

Show that the curvature of a circle of radius a is $1/a$.

Solution:

We can take the circle to have center the origin, and then a parametrization is

$$\mathbf{r}(t) = a \cos t\, \mathbf{i} + a \sin t\, \mathbf{j}$$

Therefore $\quad \mathbf{r}'(t) = -a \sin t\, \mathbf{i} + a \cos t\, \mathbf{j} \quad$ and $\quad |\mathbf{r}'(t)| = a$

so

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = -\sin t\, \mathbf{i} + \cos t\, \mathbf{j}$$

and

$$\mathbf{T}'(t) = -\cos t\, \mathbf{i} - \sin t\, \mathbf{j}$$

Example 3 – *Solution*

cont'd

This gives |**T**′(t)| = 1, so using Equation 9, we have

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1}{a}$$

Curvature

The result of Example 3 shows that small circles have large curvature and large circles have small curvature, in accordance with our intuition.

We can see directly from the definition of curvature that the curvature of a straight line is always 0 because the tangent vector is constant.

Although Formula 9 can be used in all cases to compute the curvature, the formula given by the following theorem is often more convenient to apply.

10 Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

Example 4: Curvature of a Curve

Find the curvature of the twisted cubic $\boldsymbol{r}(t) = \langle t, t^2, t^3 \rangle$ at (0, 0, 0).

Solution: The point (0, 0, 0) corresponds to t = 0. So we need to find $\kappa(0)$. To do so, we first compute the required ingredients:

$$\boldsymbol{r}'(t) = \langle 1,2t, 3t^2 \rangle, \quad \boldsymbol{r}''(t) = \langle 0,2,6t \rangle,$$

$$|\boldsymbol{r}'(t)| = \sqrt{1 + 4t^2 + 9t^4}.$$

$$\boldsymbol{r}'(t) \times \boldsymbol{r}''(t) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 6t^2\boldsymbol{i} - 6t\boldsymbol{j} + 2\boldsymbol{k},$$

$$|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)| = 2\sqrt{1 + 9t^2 + 9t^4}.$$

Thus, $\kappa(t) = \frac{|\boldsymbol{r}'(t) \times \boldsymbol{r}''(t)|}{|\boldsymbol{r}'(t)|^3} = \frac{2\sqrt{1+9t^2+9t^4}}{(1+4t^2+9t^4)^{3/2}}$, and so $\kappa(0) = 2$.

The Normal and Binormal Vectors

The Normal and Binormal Vectors

At a given point on a smooth space curve $\mathbf{r}(t)$, there are many vectors that are orthogonal to the unit tangent vector $\mathbf{T}(t)$.

We single out one by observing that, because $|\mathbf{T}(t)| = 1$ for all t, we have $\mathbf{T}(t) \cdot \mathbf{T}'(t) = 0$, so $\mathbf{T}'(t)$ is orthogonal to $\mathbf{T}(t)$.

Note that $\mathbf{T}'(t)$ is itself not a unit vector.

But at any point where $\kappa \neq 0$ we can define the **principal unit normal vector N**(t) (or simply **unit normal**) as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

The Normal and Binormal Vectors

The vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ is called the **binormal vector**.

It is perpendicular to both **T** and **N** and is also a unit vector. (See Figure 6.)

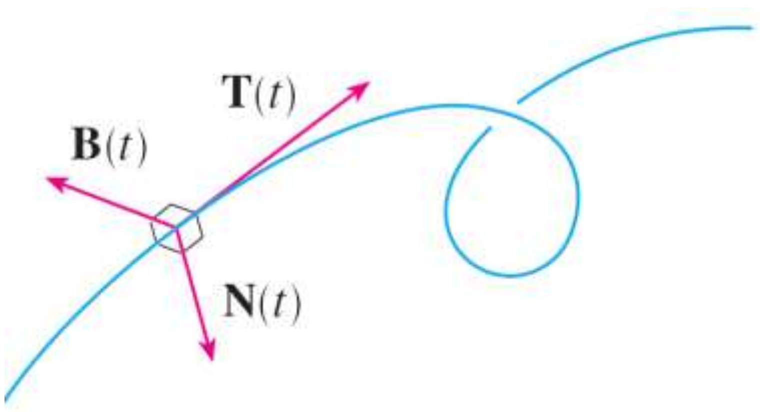


Figure 6

Example 6

Find the unit normal and binormal vectors for the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

Solution:

We first compute the ingredients needed for the unit normal vector:

$$\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \qquad |\mathbf{r}'(t)| = \sqrt{2}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k})$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i} - \sin t\,\mathbf{j}) \qquad |\mathbf{T}'(t)| = \frac{1}{\sqrt{2}}$$

Example 6 – *Solution*

cont'd

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j} = \langle -\cos t, -\sin t, 0 \rangle$$

This shows that the normal vector at a point on the helix is horizontal and points toward the *z*-axis.

The binormal vector is

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = \frac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 1 \\ -\cos t & -\sin t & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \langle \sin t, -\cos t, 1 \rangle$$

The Normal and Binormal Vectors

The plane determined by the normal and binormal vectors **N** and **B** at a point P on a curve C is called the **normal plane** of C at P.

It consists of all lines that are orthogonal to the tangent vector **T**.

The plane determined by the vectors **T** and **N** is called the **osculating plane** of C at P.

The name comes from the Latin *osculum*, meaning "kiss." It is the plane that comes closest to containing the part of the curve near P. (For a plane curve, the osculating plane is simply the plane that contains the curve.)

The Normal and Binormal Vectors

The circle that lies in the osculating plane of C at P, has the same tangent as C at P, lies on the concave side of C (toward which **N** points), and has radius $\rho = 1/\kappa$ (the reciprocal of the curvature) is called the **osculating circle** (or the **circle of curvature**) of C at P.

It is the circle that best describes how C behaves near P; it shares the same tangent, normal, and curvature at P.

The Normal and Binormal Vectors

We summarize here the formulas for unit tangent, unit normal and binormal vectors, and curvature.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

14 Partial Derivatives

14.1 Functions of Several Variables

Functions of Several Variables

In this section we study functions of two or more variables from four points of view:

- verbally (by a description in words)
- numerically (by a table of values)
- algebraically (by an explicit formula)
- visually (by a graph or level curves)

Functions of Two Variables

Functions of Several Variables

The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point.

We can think of T as being a function of the two variables x and y , or as a function of the pair (x, y). We indicate this functional dependence by writing $T = f(x, y)$.

The volume V of a circular cylinder depends on its radius r and its height h. In fact, we know that $V = \pi r^2 h$. We say that V is a function of r and h, and we write $V(r, h) = \pi r^2 h$.

Functions of Several Variables

Definition A **function f of two variables** is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) \mid (x, y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y).

The variables x and y are **independent variables** and z is the **dependent variable**.
[Compare this with the notation $y = f(x)$ for functions of a single variable.]

Example 1

Find the domain of *f* if

$$f(x, y) = \ln(x - y) + xy + 1$$

Solution:

The expression for $f(x, y)$ is defined as long as $x - y > 0$ or y < x, so the domain of *f* is

$$D = \{(x, y) \in R^2 \mid y < x\}$$

The domain consists of all points that lie below the line $y = x$.

Example 2

Consider the function $g(x, y) = \sqrt{9 - x^2 - y^2}$.

(1) The expression for *g* is defined as long as
$9 - x^2 - y^2 \geq 0$, so the domain of *g* is

$$D = \{(x, y) \in R^2 \mid 9 - x^2 - y^2 \geq 0\}$$

(2) Points in the domain satisfy

$$9 - x^2 - y^2 \geq 0 \quad \text{or} \quad x^2 + y^2 \leq 9$$

So, the domain consists of all points that lie on and inside the circle $x^2 + y^2 = 9$.

Example 2

The graph has equation $z = g(x, y) = \sqrt{9 - x^2 - y^2}$.
We square both sides of this equation to obtain

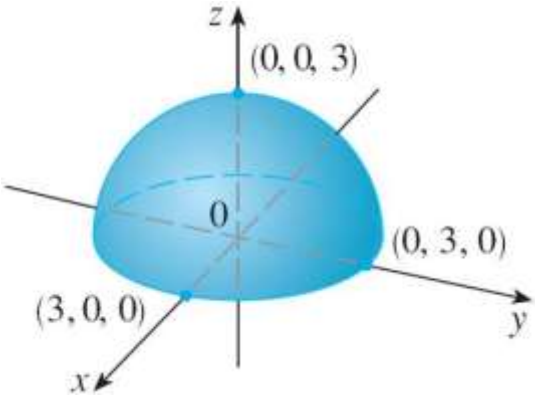
$$z^2 = 9 - x^2 - y^2 \quad \text{or} \quad x^2 + y^2 + z^2 = 9,$$

which we recognize as an equation of the sphere with center the origin and radius 3.

But, since $z \geq 0$, the graph of g is just the top half of this sphere (see the figure to the right).

From the graph it's clear the range of g is

Range of $g = \{z \mid 0 \leq z \leq 3\} = [0,3]$



Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Example 3

In 1928 Charles Cobb and Paul Douglas published a study in which they modeled the growth of the American economy during the period 1899–1922.

They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the amount of capital invested.

While there are many other factors affecting economic performance, their model proved to be remarkably accurate.

Example 3

cont'd

The function they used to model production was of the form

$$\boxed{1} \qquad P(L, K) = bL^{\alpha}K^{1-\alpha}$$

where P is the total production (the monetary value of all goods produced in a year), L is the amount of labor (the total number of person-hours worked in a year), and K is the amount of capital invested (the monetary worth of all machinery, equipment, and buildings).

Example 3

cont'd

Cobb and Douglas used economic data published by the government to obtain Table 1.

Year	P	L	K
1899	100	100	100
1900	101	105	107
1901	112	110	114
1902	122	117	122
1903	124	122	131
1904	122	121	138
1905	143	125	149
1906	152	134	163
1907	151	140	176
1908	126	123	185
1909	155	143	198
1910	159	147	208

Year	P	L	K
1911	153	148	216
1912	177	155	226
1913	184	156	236
1914	169	152	244
1915	189	156	266
1916	225	183	298
1917	227	198	335
1918	223	201	366
1919	218	196	387
1920	231	194	407
1921	179	146	417
1922	240	161	431

Table 1

Example 3

cont'd

They took the year 1899 as a baseline and P, L, and K for 1899 were each assigned the value 100.

The values for other years were expressed as percentages of the 1899 figures.

Cobb and Douglas used the method of least squares to fit the data of Table 1 to the function

$$\boxed{2} \qquad P(L, K) = 1.01L^{0.75}K^{0.25}$$

Example 3

cont'd

If we use the model given by the function in Equation 2 to compute the production in the years 1910 and 1920, we get the values

$$P(147, 208) = 1.01(147)^{0.75}(208)^{0.25} \approx 161.9$$

$$P(194, 407) = 1.01(194)^{0.75}(407)^{0.25} \approx 235.8$$

which are quite close to the actual values, 159 and 231.

The production function $\boxed{1}$ has subsequently been used in many settings, ranging from individual firms to global economics. It has become known as the **Cobb-Douglas production function**.

Example 3

cont'd

Its domain is $\{(L, K) \mid L \geq 0, K \geq 0\}$ because L and K represent labor and capital and are therefore never negative.

Graphs

Graphs

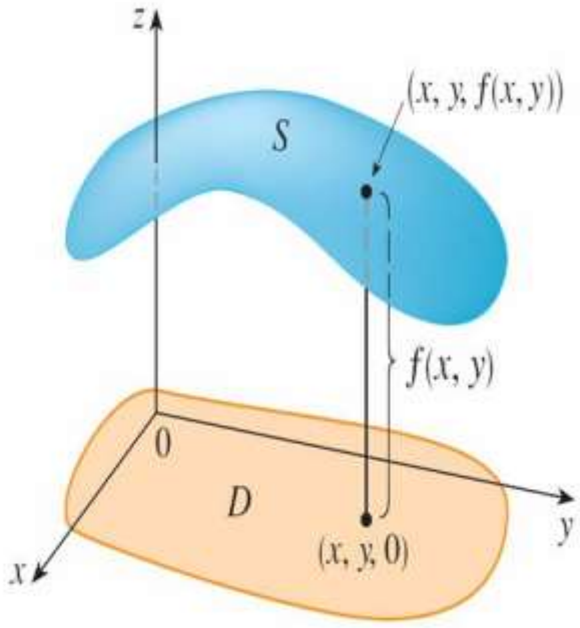
Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

Just as the graph of a function f of one variable is a curve C with equation $y = f(x)$, so the graph of a function f of two variables is a surface S with equation $z = f(x, y)$.

Graphs

We can visualize the graph S of f as lying directly above or below its domain D in the xy-plane (see the figure below).



Graphs

The function $f(x, y) = ax + by + c$ is called as a **linear function**.

The graph of such a function has the equation

$$z = ax + by + c \qquad \text{or} \qquad ax + by - z + c = 0$$

so it is a plane. In much the same way that linear functions of one variable are important in single-variable calculus, we will see that linear functions of two variables play a central role in multivariable calculus.

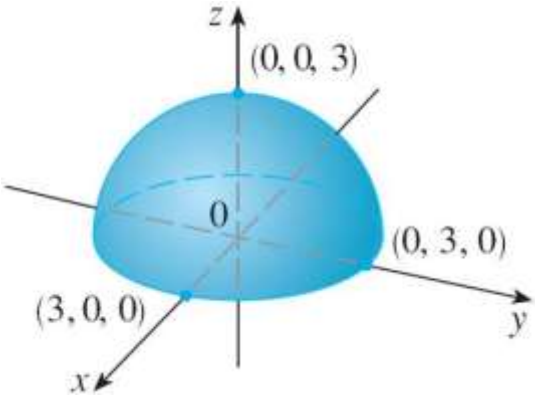
Example 4

Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution:

The graph has equation $z = \sqrt{9 - x^2 - y^2}$. We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + z^2 = 9$, which we recognize as an equation of the sphere with center the origin and radius 3.

But, since $z \geq 0$, the graph of g is just the top half of this sphere (see the figure to the right).



Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$

Level Curves

Level Curves

So far we have two methods for visualizing functions: arrow diagrams and graphs. A third method, borrowed from mapmakers, is a contour map on which points of constant elevation are joined to form *contour lines*, or *level curves*.

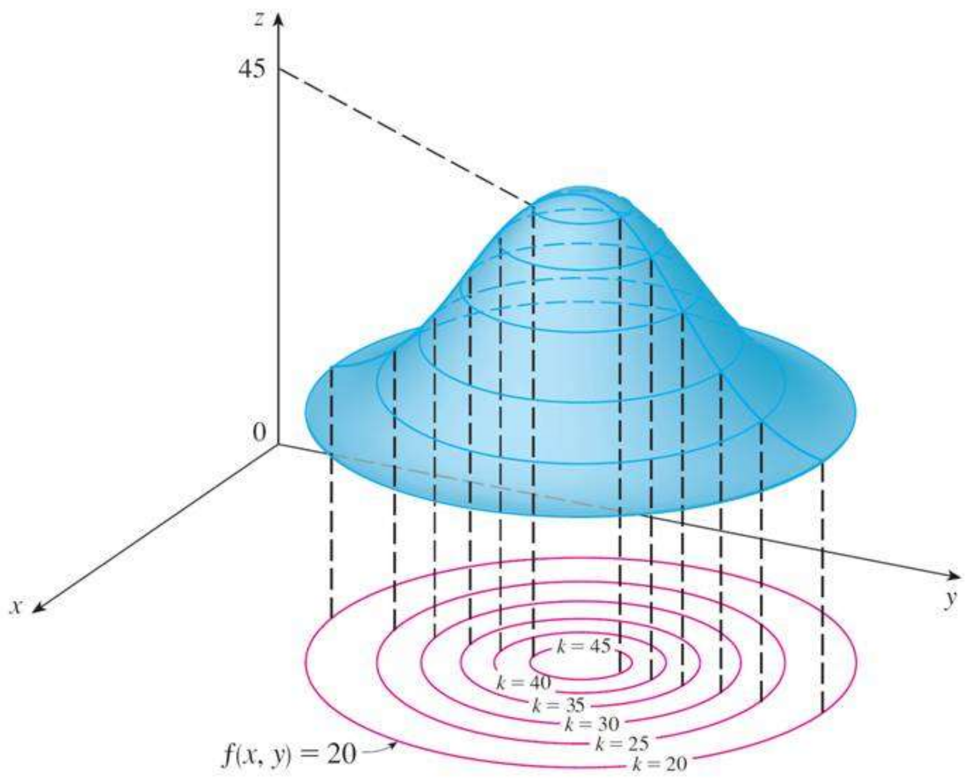
Definition The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant (in the range of f).

A level curve $f(x, y) = k$ is the set of all points in the domain of f at which f takes on a given value k.

In other words, it shows where the graph of f has height k.

Level Curves

You can see from the figure below the relation between level curves and horizontal traces.



Level Curves

The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.

So if you draw the level curves of a function and visualize them being lifted up to the surface at the indicated height, then you can mentally piece together a picture of the graph.

The surface is steep where the level curves are close together. It is somewhat flatter where they are farther apart.

Example

Sketch the level curves of the function

$$f(x, y) = 100 - x^2 - y^2$$

for k = 51, 75.

Solution: The level curves are

$$100 - x^2 - y^2 = k \quad \text{or} \quad x^2 + y^2 = 100 - k$$

This is a family of concentric circles with center (0, 0) and radius $\sqrt{100 - k}$. The case $k = 51$ gives the level curve $x^2 + y^2 = 49$ which is a circle of center (0, 0) and radius 7.

The 75-level curve is the circle $x^2 + y^2 = 25$ of center (0, 0) and radius 5. See the figures in the next slide.

Example

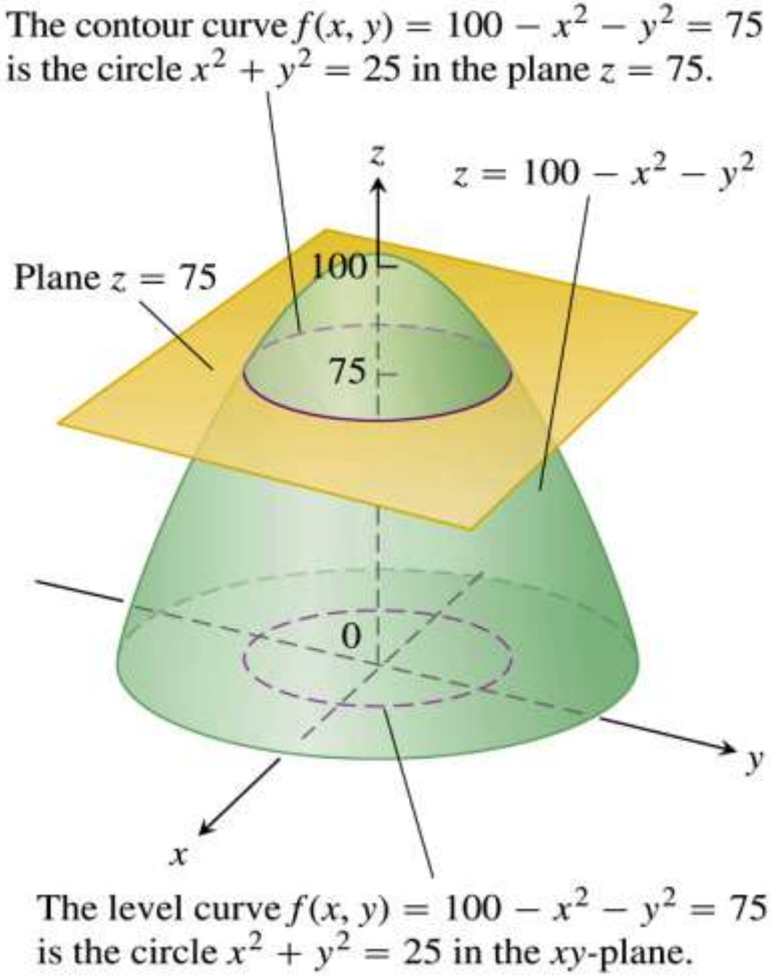


FIGURE 14.6 A plane $z = c$ parallel to the xy-plane intersecting a surface $z = f(x, y)$ produces a contour curve.

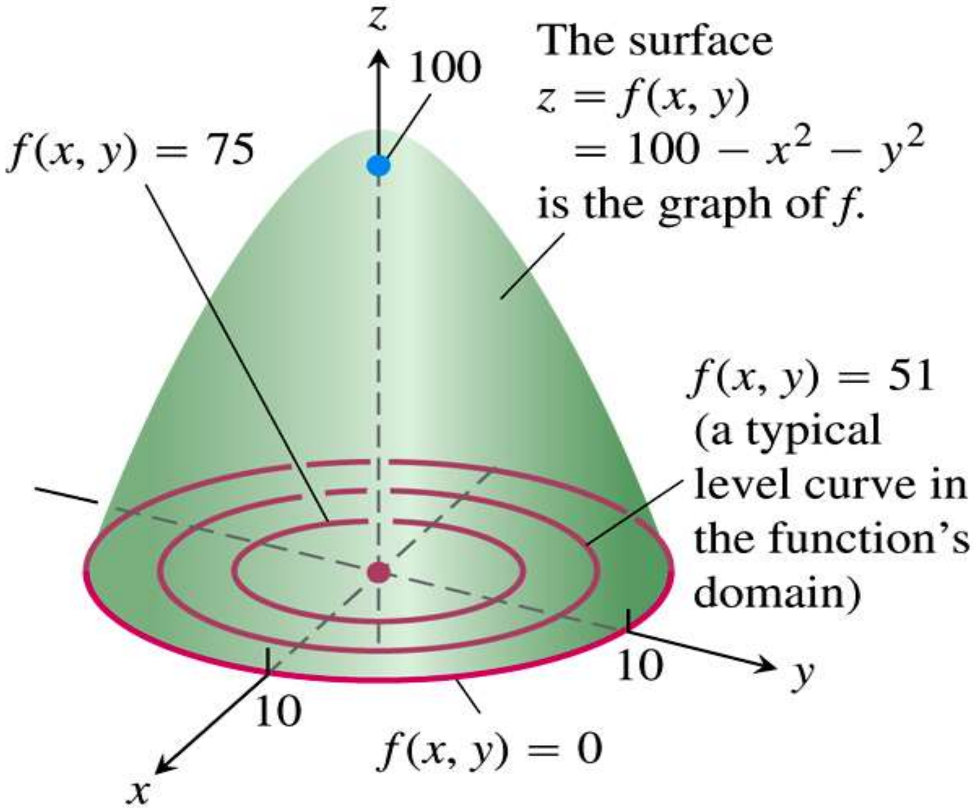
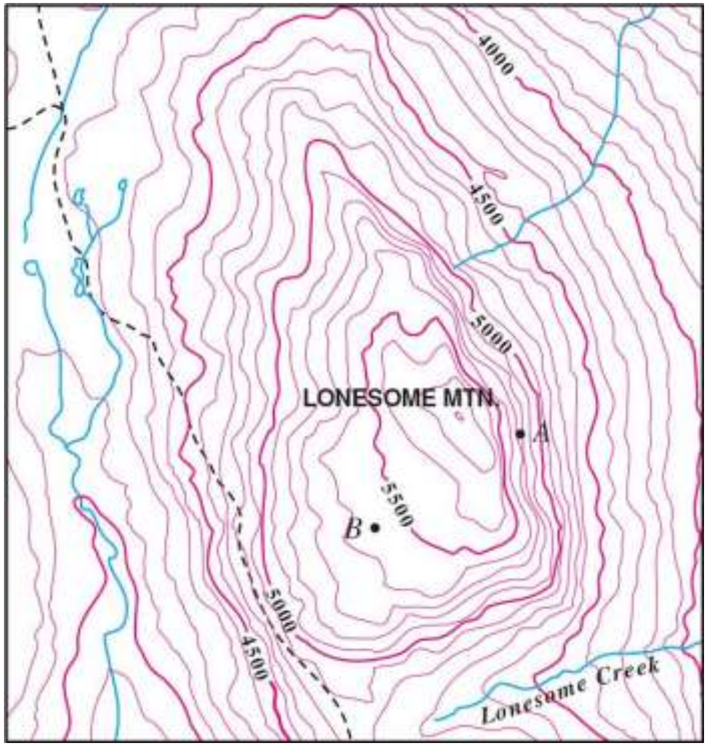


FIGURE 14.5 The graph and selected level curves of the function $f(x, y)$ in Example 3.

Level Curves

One common example of level curves occurs in topographic maps of mountainous regions, such as the map in the figure below.



Level Curves

The level curves are curves of constant elevation above sea level.

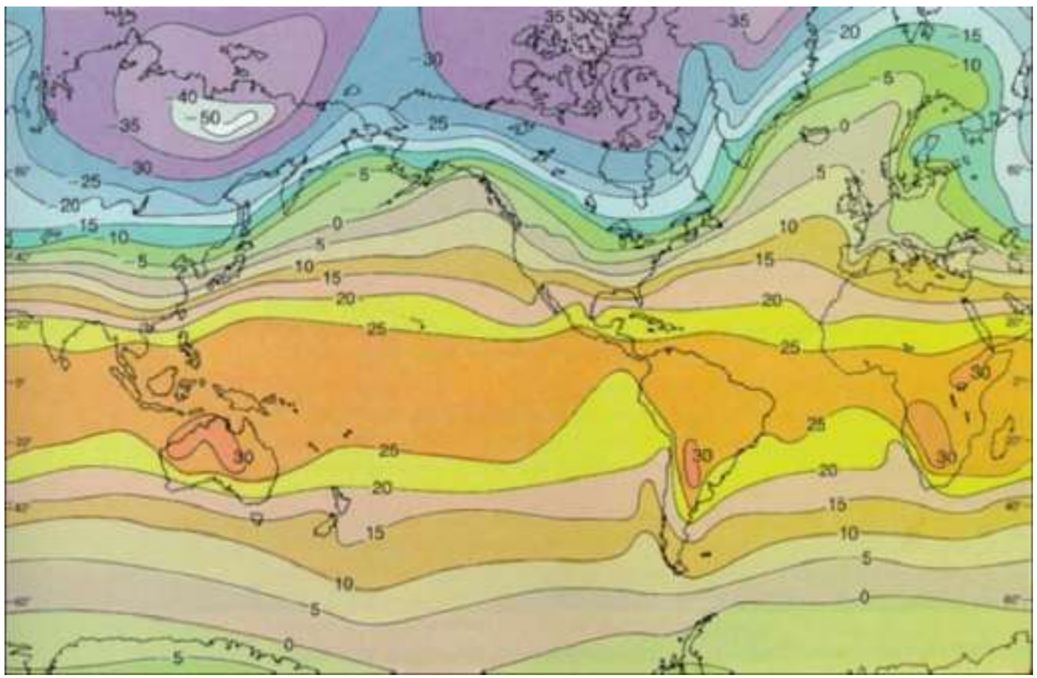
If you walk along one of these contour lines, you neither ascend nor descend.

Another common example is the temperature function introduced in the opening paragraph of this section.

Here the level curves are called **isothermals** and join locations with the same temperature.

Level Curves

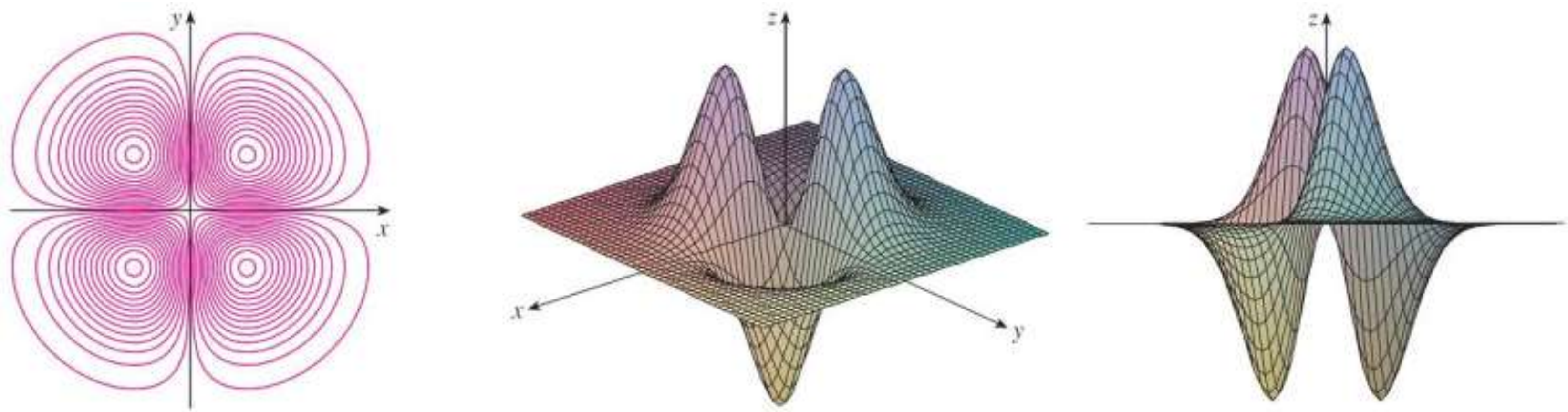
The figure below shows a weather map of the world indicating the average January temperatures. The isothermals are the curves that separate the colored bands.



World mean sea-level temperatures in January in degrees Celsius

Level Curves

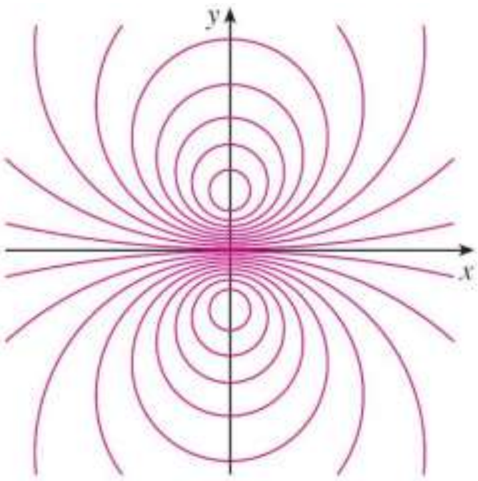
For some purposes, a contour map is more useful than a graph. It is true in estimating function values. The accompanying figures show some computer-generated level curves together with the corresponding computer-generated graphs.



(a) Level curves of $f(x, y) = -xye^{-x^2-y^2}$

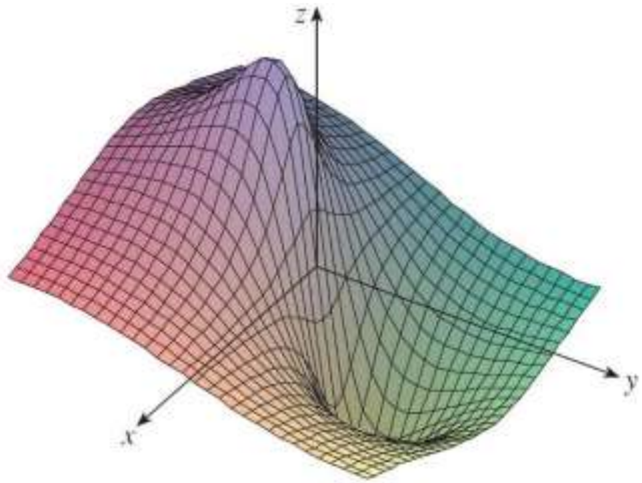
(b) Two views of $f(x, y) = -xye^{-x^2-y^2}$

Level Curves



Level curves of $f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

Figure c



$f(x, y) = \dfrac{-3y}{x^2 + y^2 + 1}$

Figure d

Notice that the level curves in Figure (c) crowd together near the origin. That corresponds to the fact that the graph in Figure (d) is very steep near the origin.

Functions of Three or More Variables

Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

Example 14

Find the domain of f if

$$f(x, y, z) = \ln(z - y) + xy \sin z$$

Solution:

The expression for $f(x, y, z)$ is defined as long as $z - y > 0$, so the domain of f is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid z > y\}$$

This is a **half-space** consisting of all points that lie above the plane $z = y$.

Functions of Three or More Variables

It's very difficult to visualize a function f of three variables by its graph, since that would lie in a four-dimensional space.

However, we do gain some insight into f by examining its **level surfaces**, which are the surfaces with equations $f(x, y, z) = k$, where k is a constant. If the point (x, y, z) moves along a level surface, the value of $f(x, y, z)$ remains fixed.

$\mathbb{R}^n$

Example: level surfaces

Find the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

Solution:

The level surfaces are $\sqrt{x^2 + y^2 + z^2} = k$, where $k \geq 0$. These form a family of concentric spheres, $x^2 + y^2 + z^2 = k^2$, with radius *k*. Thus, as (x, y, z) varies over any sphere with center *O*, the value of $f(x, y, z)$ remains fixed.

See the figure in the next slide.

Example: Level Surfaces

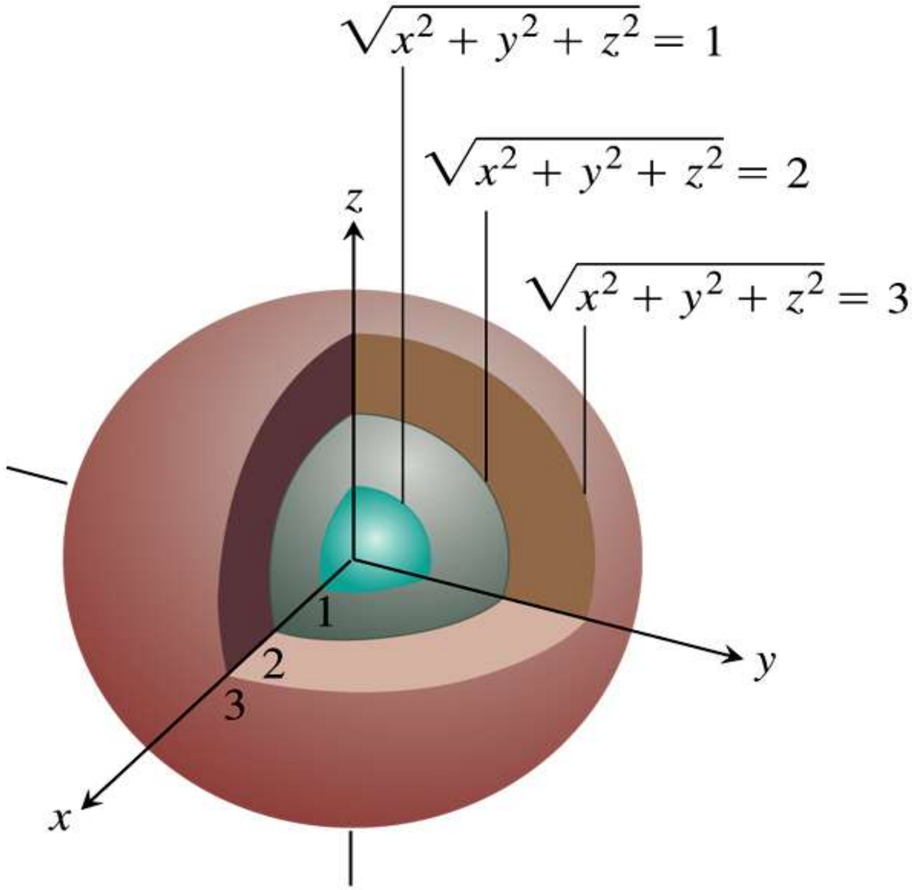


FIGURE 14.8 The level surfaces of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ are concentric spheres (Example 4).

Functions of Three or More Variables

Functions of any number of variables can be considered.

A **function of *n* variables** is a rule that assigns a number $z = f(x_1, x_2, \ldots, x_n)$ to an *n*-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers. We denote by the set of all such *n*-tuples.

$$\mathbb{R}^3$$

Functions of Three or More Variables

For example, if a company uses n different ingredients in making a food product, c_i is the cost per unit of the ith ingredient, and x_i units of the ith ingredient are used, then the total cost C of the ingredients is a function of the n variables $x_1, x_2, \ldots, x_n$:

$$\boxed{3} \qquad C = f(x_1, x_2, \ldots, x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The function f is a real-valued function whose domain is a subset of $\mathbb{R}^n$.

Functions of Three or More Variables

Sometimes we will use vector notation to write such functions more compactly: If $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$, we often write $f(\mathbf{x})$ in place of $f(x_1, x_2, \ldots, x_n)$.

With this notation we can rewrite the function defined in Equation 3 as

$$f(\mathbf{x}) = \mathbf{c} \cdot \mathbf{x}$$

where $\mathbf{c} = \langle c_1, c_2, \ldots, c_n \rangle$ and $\mathbf{c} \cdot \mathbf{x}$ denotes the dot product of the vectors $\mathbf{c}$ and $\mathbf{x}$ in V_n.

Functions of Three or More Variables

In view of the one-to-one correspondence between points $(x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$ and their position vectors

$\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ in V_n, we have three ways of looking at a function f defined on a subset of $\mathbb{R}^n$:

1. As a function of n real variables $x_1, x_2, \ldots, x_n$

2. As a function of a single point variable $(x_1, x_2, \ldots, x_n)$

3. As a function of a single vector variable $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$

14 Partial Derivatives

14.2 Limits and Continuity

Limits and Continuity

Let's compare the behavior of the functions

$$f(x, y) = \frac{\sin(x^2 + y^2)}{x^2 + y^2}$$

and

$$g(x, y) = \frac{x^2 - y^2}{x^2 + y^2}$$

as *x* and *y* both approach 0 [and therefore the point (*x*, *y*) approaches the origin].

Limits and Continuity

Tables 1 and 2 show values of $f(x, y)$ and $g(x, y)$, correct to three decimal places, for points (x, y) near the origin. (Notice that neither function is defined at the origin.)

$x \backslash y$	-1.0	-0.5	-0.2	0	0.2	0.5	1.0
-1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455
-0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
-0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0	0.841	0.990	1.000		1.000	0.990	0.841
0.2	0.829	0.986	0.999	1.000	0.999	0.986	0.829
0.5	0.759	0.959	0.986	0.990	0.986	0.959	0.759
1.0	0.455	0.759	0.829	0.841	0.829	0.759	0.455

Values of $f(x, y)$

Table 1

Limits and Continuity

x \ y	−1.0	−0.5	−0.2	0	0.2	0.5	1.0
−1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000
−0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
−0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0	−1.000	−1.000	−1.000		−1.000	−1.000	−1.000
0.2	−0.923	−0.724	0.000	1.000	0.000	−0.724	−0.923
0.5	−0.600	0.000	0.724	1.000	0.724	0.000	−0.600
1.0	0.000	0.600	0.923	1.000	0.923	0.600	0.000

Values of $g(x, y)$

Table 2

Limits and Continuity

It appears that as (x, y) approaches (0, 0), the values of $f(x, y)$ are approaching 1 whereas the values of $g(x, y)$ aren't approaching any number. It turns out that these guesses based on numerical evidence are correct, and we write

$$\lim_{(x, y)\to(0, 0)} \frac{\sin(x^2 + y^2)}{x^2 + y^2} = 1 \quad \text{and}$$

$$\lim_{(x, y)\to(0, 0)} \frac{x^2 - y^2}{x^2 + y^2} \quad \text{does not exist}$$

Limits and Continuity

In general, we use the notation

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that stays within the domain of f.

Limits and Continuity

In other words, we can make the values of $f(x, y)$ as close to L as we like by taking the point (x, y) sufficiently close to the point (a, b), but not equal to (a, b). A more precise definition follows.

$\boxed{1}$ **Definition** Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x, y) \to (a, b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad (x, y) \in D \quad \text{and} \quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad \text{then} \quad |f(x, y) - L| < \varepsilon$$

Limits and Continuity

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L \qquad \text{and}$$

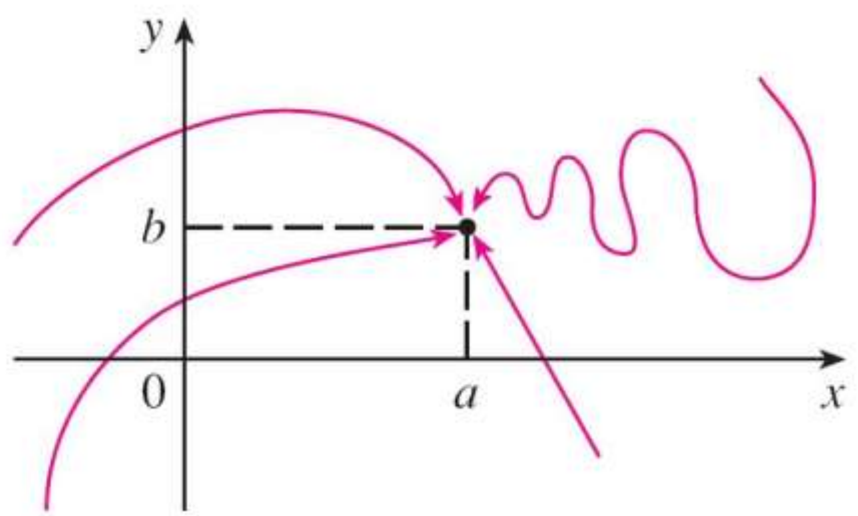
$$f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

For functions of a single variable, when we let x approach a, there are only two possible directions of approach, from the left or from the right.

We recall that if $\lim_{x\to a^-} f(x) \neq \lim_{x\to a^+} f(x)$, then $\lim_{x\to a} f(x)$ does not exist.

Limits and Continuity

For functions of two variables the situation is not as simple because we can let (x, y) approach (a, b) from an infinite number of directions in any manner whatsoever (see the figure below) as long as (x, y) stays within the domain of f.



Limits and Continuity

Definition 1 says that the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) to (a, b) sufficiently small (but not 0).

The definition refers only to the *distance* between (x, y) and (a, b). It does not refer to the direction of approach.

Therefore, if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches (a, b).

Limits and Continuity

Thus, if we can find two different paths of approach along which the function $f(x, y)$ has different limits, then it follows that $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x, y) \to (a, b)} f(x, y)$ does not exist.

Example: Limits

Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution:

Let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$.

First let's approach (0, 0) along the x-axis.

Then $y = 0$ gives $f(x, 0) = x^2/x^2 = 1$ for all $x \neq 0$, so

$$f(x, y) \to 1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } x\text{-axis}$$

Example – *Solution*

cont'd

We now approach along the y-axis by putting $x = 0$.

Then $f(0, y) = \dfrac{-y^2}{y^2} = -1$ for all $y \neq 0$, so

$$f(x, y) \to -1 \quad \text{as} \quad (x, y) \to (0, 0) \text{ along the } y\text{-axis}$$

(See Figure 4.)

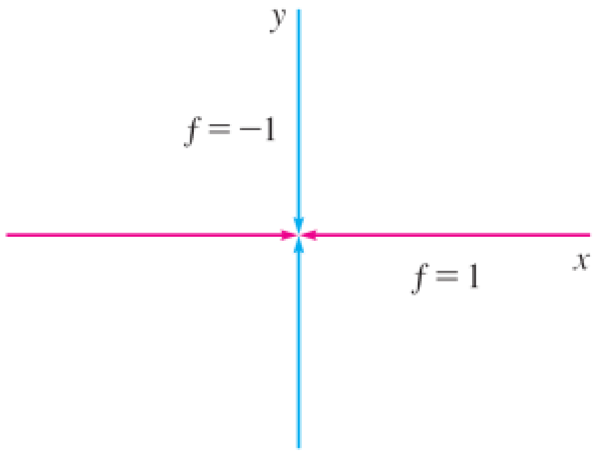


Figure 4

Example – *Solution*

cont'd

Since *f* has two different limits along two different lines, the given limit does not exist. (This confirms the conjecture we made on the basis of numerical evidence at the beginning of this section.)

Example: Limits

EXAMPLE 6 Show that the function

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

(Figure 14.14) has no limit as (x, y) approaches $(0, 0)$.

Solution The limit cannot be found by direct substitution, which gives the indeterminate form 0/0. We examine the values of f along curves that end at $(0, 0)$. Along the curve $y = kx^2$, $x \neq 0$, the function has the constant value

$$f(x, y)\bigg|_{y=kx^2} = \frac{2x^2y}{x^4 + y^2}\bigg|_{y=kx^2} = \frac{2x^2(kx^2)}{x^4 + (kx^2)^2} = \frac{2kx^4}{x^4 + k^2x^4} = \frac{2k}{1 + k^2}.$$

Therefore,

$$\lim_{\substack{(x, y)\to(0,0) \\ \text{along } y=kx^2}} f(x, y) = \lim_{(x, y)\to(0,0)} \left[f(x, y)\bigg|_{y=kx^2} \right] = \frac{2k}{1 + k^2}.$$

This limit varies with the path of approach. If (x, y) approaches $(0, 0)$ along the parabola $y = x^2$, for instance, $k = 1$ and the limit is 1. If (x, y) approaches $(0, 0)$ along the x-axis, $k = 0$ and the limit is 0. By the two-path test, f has no limit as (x, y) approaches $(0, 0)$.

It can be shown that the function in Example 6 has limit 0 along every path $y = mx$

Limits and Continuity

Now let’s look at limits that *do* exist. Just as for functions of one variable, the calculation of limits for functions of two variables can be greatly simplified by the use of properties of limits.

The Limit Laws can be extended to functions of two variables: The limit of a sum is the sum of the limits, the limit of a product is the product of the limits, and so on.

In particular, the following equations are true.

2 $$\lim_{(x, y)\to(a, b)} x = a \qquad \lim_{(x, y)\to(a, b)} y = b \qquad \lim_{(x, y)\to(a, b)} c = c$$

The Squeeze Theorem also holds.

Limits and Continuity

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$$
2. *Difference Rule:* $$\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$$
3. *Constant Multiple Rule:* $$\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL \qquad \text{(any number } k\text{)}$$
4. *Product Rule:* $$\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$$
5. *Quotient Rule:* $$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \qquad M \neq 0$$
6. *Power Rule:* $$\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n, \text{ } n \text{ a positive integer}$$
7. *Root Rule:* $$\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n},$$
   n a positive integer, and if n is even, we assume that $L > 0$.

Example: Limits

EXAMPLE 1 In this example, we can combine the three simple results following the limit definition with the results in Theorem 1 to calculate the limits. We simply substitute the x and y values of the point being approached into the functional expression to find the limiting value.

(a) $$\lim_{(x,y)\to(0,1)} \frac{x - xy + 3}{x^2y + 5xy - y^3} = \frac{0 - (0)(1) + 3}{(0)^2(1) + 5(0)(1) - (1)^3} = -3$$

(b) $$\lim_{(x,y)\to(3,-4)} \sqrt{x^2 + y^2} = \sqrt{(3)^2 + (-4)^2} = \sqrt{25} = 5$$

Example: Limits

EXAMPLE 2 Find

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Solution Since the denominator $\sqrt{x} - \sqrt{y}$ approaches 0 as $(x, y) \to (0, 0)$, we cannot use the Quotient Rule from Theorem 1. If we multiply numerator and denominator by $\sqrt{x} + \sqrt{y}$, however, we produce an equivalent fraction whose limit we *can* find:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}} &= \lim_{(x,y)\to(0,0)} \frac{(x^2 - xy)(\sqrt{x} + \sqrt{y})}{(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y})} \\
&= \lim_{(x,y)\to(0,0)} \frac{x(x - y)(\sqrt{x} + \sqrt{y})}{x - y} && \text{Algebra} \\
&= \lim_{(x,y)\to(0,0)} x(\sqrt{x} + \sqrt{y}) && \text{Cancel the nonzero factor } (x - y). \\
&= 0(\sqrt{0} + \sqrt{0}) = 0 && \text{Known limit values}
\end{aligned}$$

We can cancel the factor $(x - y)$ because the path $y = x$ (along which $x - y = 0$) is *not* in the domain of the function

$$\frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}.$$

Example: Limits

Evaluate $\lim\limits_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}$

Solution:

Using polar coordinates $x = rcos\theta,\ y = rsin\theta$, we have that as $(x, y) \to (0,0)$, then $r \to 0$. So we can find the limit as follows:

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{r\to 0} \frac{3(rcos\theta)^2(rsin\theta)}{(rcos\theta)^2+(rsin\theta)^2} = \lim_{r\to 0} \frac{3r^3(cos\theta)^2(sin\theta)}{r^2(cos^2\theta+sin^2\theta)}$$

$$= \lim_{r\to 0} 3r(cos\theta)^2(sin\theta) = 0.$$

Continuity

Continuity

Recall that evaluating limits of *continuous* functions of a single variable is easy.

It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \to a} f(x) = f(a)$.

Continuous functions of two variables are also defined by the direct substitution property.

$\boxed{4}$ **Definition** A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

Continuity

The intuitive meaning of continuity is that if the point (x, y) changes by a small amount, then the value of $f(x, y)$ changes by a small amount.

This means that a surface that is the graph of a continuous function has no hole or break.

Using the properties of limits, you can see that sums, differences, products, and quotients of continuous functions are continuous on their domains.

Let's use this fact to give examples of continuous functions.

Continuity

A **polynomial function of two variables** (or polynomial, for short) is a sum of terms of the form cx^my^n, where c is a constant and m and n are nonnegative integers.

A **rational function** is a ratio of polynomials.

For instance,

$$f(x, y) = x^4 + 5x^3y^2 + 6xy^4 - 7y + 6$$

is a polynomial which is continuous everywhere, whereas

$$g(x, y) = \frac{2xy + 1}{x^2 + y^2}$$

is a rational function which is continuous on

$D = \{(x, y) | (x, y) \neq (0,0)\}$.

Example: Continuity

Let $f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & if \;\; (x,y) \neq (0,0) \\ 0 & if \;\; (x,y) = (0,0) \end{cases}$

Here *f* is defined at (0, 0) but is still discontinuous there because $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. To show this,

$$\underbrace{\lim_{(x,y)\to(0,0)}}_{Along\ x-axis} f(x,y) = \lim_{y\to 0} \frac{0}{y^2} = 0,$$

$$\underbrace{\lim_{(x,y)\to(0,0)}}_{Along\ y=x} f(x,y) = \lim_{x\to 0} \frac{2x^2}{2x^2} = 1.$$

Since *f* has two different limits along two different paths, the given limit does not exist.

Continuity

The limits in $\boxed{2}$ show that the functions $f(x, y) = x$, $g(x, y) = y$, and $h(x, y) = c$ are continuous.

Since any polynomial can be built up out of the simple functions f, g, and h by multiplication and addition, it follows that *all polynomials are continuous on* $\mathbb{R}^2$.

Likewise, any rational function is continuous on its domain because it is a quotient of continuous functions.

Example

Evaluate $\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution:

Since $f(x, y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, it is continuous everywhere, so we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)} (x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2$$

$$= 11$$

Continuity

Just as for functions of one variable, composition is another way of combining two continuous functions to get a third.

In fact, it can be shown that if f is a continuous function of two variables and g is a continuous function of a single variable that is defined on the range of f, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is also a continuous function.

Functions of Three or More Variables

Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables.

The notation

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) along any path in the domain of f.

Functions of Three or More Variables

Because the distance between two points (x, y, z) and (a, b, c) in $\mathbb{R}^3$ is given by $\sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}$, we can write the precise definition as follows: For every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if (x, y, z) is in the domain of f and

$0 < \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} < \delta$

then $|f(x, y, z) - L| < \varepsilon$

Functions of Three or More Variables

The function *f* is **continuous** at (*a*, *b*, *c*) if

$$\lim_{(x, y, z) \to (a, b, c)} f(x, y, z) = f(a, b, c)$$

For instance, the function

$$f(x, y, z) = \frac{1}{x^2 + y^2 + z^2 - 1}$$

is a rational function of three variables and so is continuous at every point in $\mathbb{R}^3$ except where $x_2 + y_2 + z_2 = 1$. In other words, it is discontinuous on the sphere with center the origin and radius 1.

Functions of Three or More Variables

We can write the definitions of a limit for functions of two or three variables in a single compact form as follows.

$\boxed{5}$ If f is defined on a subset D of $\mathbb{R}^n$, then $\lim_{\mathbf{x} \to \mathbf{a}} f(\mathbf{x}) = L$ means that for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

$$\text{if} \quad \mathbf{x} \in D \quad \text{and} \quad 0 < |\mathbf{x} - \mathbf{a}| < \delta \quad \text{then} \quad |f(\mathbf{x}) - L| < \varepsilon$$

14 Partial Derivatives

14.3 Partial Derivatives

Partial Derivatives

On a hot day, extreme humidity makes us think the temperature is higher than it really is, whereas in very dry air we perceive the temperature to be lower than the thermometer indicates.

The National Weather Service has devised the *heat index* (also called the temperature-humidity index, or humidex, in some countries) to describe the combined effects of temperature and humidity.

The heat index I is the perceived air temperature when the actual temperature is T and the relative humidity is H.
So I is a function of T and H and we can write $I = f(T, H)$.

Partial Derivatives

The following table of values of *I* is an excerpt from a table compiled by the National Weather Service.

Relative humidity (%)

Actual temperature (°F)

T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Heat index *I* as a function of temperature and humidity

Table 1

Partial Derivatives

If we concentrate on the highlighted column of the table, which corresponds to a relative humidity of $H = 70\%$, we are considering the heat index as a function of the single variable T for a fixed value of H. Let's write $g(T) = f(T, 70)$.

Then $g(T)$ describes how the heat index I increases as the actual temperature T increases when the relative humidity is 70%.

The derivative of g when $T = 96°\mathrm{F}$ is the rate of change of I with respect to T when $T = 96°\mathrm{F}$:

$$g'(96) = \lim_{h \to 0} \frac{g(96 + h) - g(96)}{h} = \lim_{h \to 0} \frac{f(96 + h, 70) - f(96, 70)}{h}$$

Partial Derivatives

We can approximate $g'(96)$ using the values in Table 1 by taking $h = 2$ and -2:

$$g'(96) \approx \frac{g(98) - g(96)}{2} = \frac{f(98, 70) - f(96, 70)}{2} = \frac{133 - 125}{2} = 4$$

$$g'(96) \approx \frac{g(94) - g(96)}{-2} = \frac{f(94, 70) - f(96, 70)}{-2} = \frac{118 - 125}{-2} = 3.5$$

Averaging these values, we can say that the derivative $g'(96)$ is approximately 3.75.

Partial Derivatives

This means that, when the actual temperature is 96°F and the relative humidity is 70%, the apparent temperature (heat index) rises by about 3.75°F for every degree that the actual temperature rises!

Partial Derivatives

Now let's look at the highlighted row in Table 1, which corresponds to a fixed temperature of T = 96°F.

Relative humidity (%)

Actual temperature (°F) T \ H	50	55	60	65	70	75	80	85	90
90	96	98	100	103	106	109	112	115	119
92	100	103	105	108	112	115	119	123	128
94	104	107	111	114	118	122	127	132	137
96	109	113	116	121	125	130	135	141	146
98	114	118	123	127	133	138	144	150	157
100	119	124	129	135	141	147	154	161	168

Heat index I as a function of temperature and humidity

Table 1

Partial Derivatives

The numbers in this row are values of the function $G(H) = f(96, H)$, which describes how the heat index increases as the relative humidity H increases when the actual temperature is $T = 96°F$.

The derivative of this function when $H = 70\%$ is the rate of change of I with respect to H when $H = 70\%$:

$$G'(70) = \lim_{h \to 0} \frac{G(70 + h) - G(70)}{h} = \lim_{h \to 0} \frac{f(96, 70 + h) - f(96, 70)}{h}$$

Partial Derivatives

By taking $h = 5$ and -5, we approximate $G'(70)$ using the tabular values:

$$G'(70) \approx \frac{G(75) - G(70)}{5} = \frac{f(96, 75) - f(96, 70)}{5} = \frac{130 - 125}{5} = 1$$

$$G'(70) \approx \frac{G(65) - G(70)}{-5} = \frac{f(96, 65) - f(96, 70)}{-5} = \frac{121 - 125}{-5} = 0.8$$

By averaging these values we get the estimate $G'(70) \approx 0.9$. This says that, when the temperature is 96°F and the relative humidity is 70%, the heat index rises about 0.9°F for every percent that the relative humidity rises.

Partial Derivatives

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of *f* with respect to *x* at (*a*, *b*)** and denote it by $f_x(a, b)$. Thus

1 $$\boxed{f_x(a, b) = g'(a) \qquad \text{where} \qquad g(x) = f(x, b)}$$

Partial Derivatives

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a + h) - g(a)}{h}$$

and so Equation 1 becomes

$$\boxed{2} \qquad \boxed{f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}}$$

Partial Derivatives

Similarly, the **partial derivative of *f* with respect to *y* at (*a*, *b*)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h} \tag{3}$$

With this notation for partial derivatives, we can write the rates of change of the heat index I with respect to the actual temperature T and relative humidity H when $T = 96°\text{F}$ and $H = 70\%$ as follows:

$$f_T(96, 70) \approx 3.75 \qquad f_H(96, 70) \approx 0.9$$

Partial Derivatives

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

$\boxed{4}$ If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Partial Derivatives

There are many alternative notations for partial derivatives.

For instance, instead of f_x we can write f_1 or $D_1 f$ (to indicate differentiation with respect to the *first* variable) or $\partial f/\partial x$.

But here $\partial f/\partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed.

Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example: Partial derivatives

If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution:

Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so $$f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Example: Partial derivatives

EXAMPLE 2 Find $\partial f/\partial y$ as a function if $f(x, y) = y \sin xy$.

Solution We treat x as a constant and f as a product of y and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y}(y)$$

$$= (y \cos xy) \frac{\partial}{\partial y}(xy) + \sin xy = xy \cos xy + \sin xy.$$

Example: Partial derivatives

If $f(x,y) = sin\left(\frac{x}{1+y}\right)$, calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

Solution:

$$\frac{\partial f}{\partial x} = cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial x}\left(\frac{x}{1+y}\right) = cos\left(\frac{x}{1+y}\right) \cdot \frac{1}{1+y}$$

$$\frac{\partial f}{\partial y} = cos\left(\frac{x}{1+y}\right) \cdot \frac{\partial}{\partial y}\left(\frac{x}{1+y}\right) = cos\left(\frac{x}{1+y}\right) \cdot \frac{-x}{(1+y)^2}$$

Example: Implicit differentiation

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if *z* is defined implicitly as a function of *x* and *y* by the equation $x^3 + y^3 + z^3 + 6xyz = 1$.

Solution: To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to *x*, being careful to treat *y* as a constant:

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 6xy \frac{\partial z}{\partial x} + 6yz = 0$$

Solving this equation for $\frac{\partial z}{\partial x}$, we obtain

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}$$

Example: implicit Differentiation

To find $\frac{\partial z}{\partial y}$, we differentiate implicitly with respect to *y*, being careful to treat *x* as a constant:

$$0 + 3z^2 \frac{\partial z}{\partial y} + 3y^2 + 6xy \frac{\partial z}{\partial y} + 6xz = 0$$

Solving this equation for $\frac{\partial z}{\partial y}$, we obtain

$$\frac{\partial z}{\partial y} = \frac{-3y^2 - 6xz}{3z^2 + 6xy}$$

Interpretations of Partial Derivatives

Interpretations of Partial Derivatives

To give a geometric interpretation of partial derivatives, we recall that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.

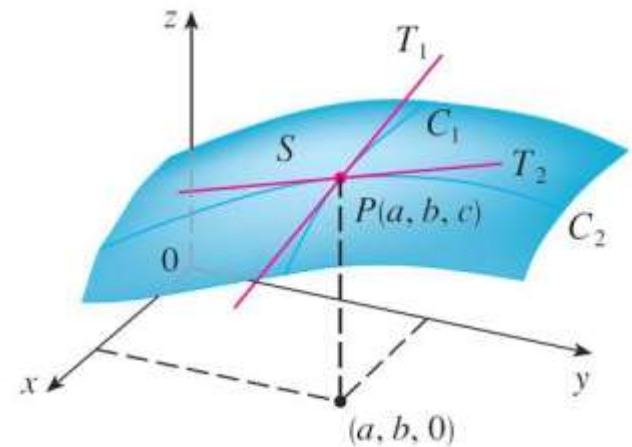
By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.)

Interpretations of Partial Derivatives

Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See the figure to the right)

Notice that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$.

The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Interpretations of Partial Derivatives

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function, partial derivatives can also be interpreted as *rates of change*.

If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Example: Interpretations of Partial Derivatives

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:

We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

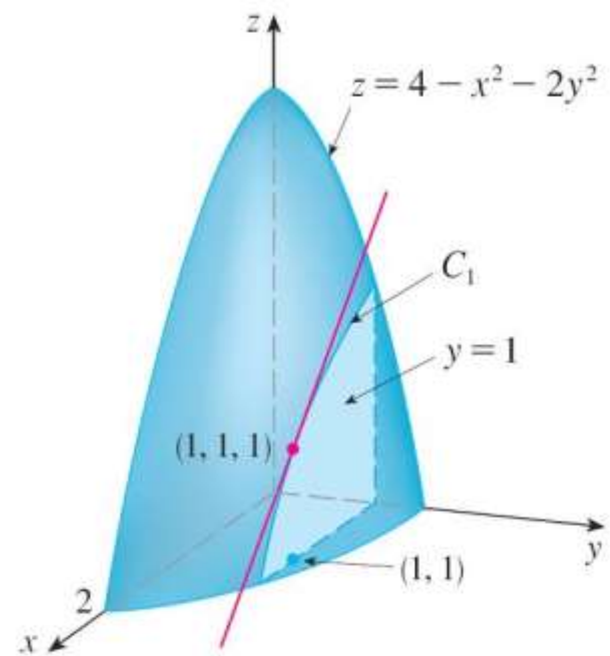
$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

Example – *Solution*

cont'd

The graph of f is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in the given figure.)

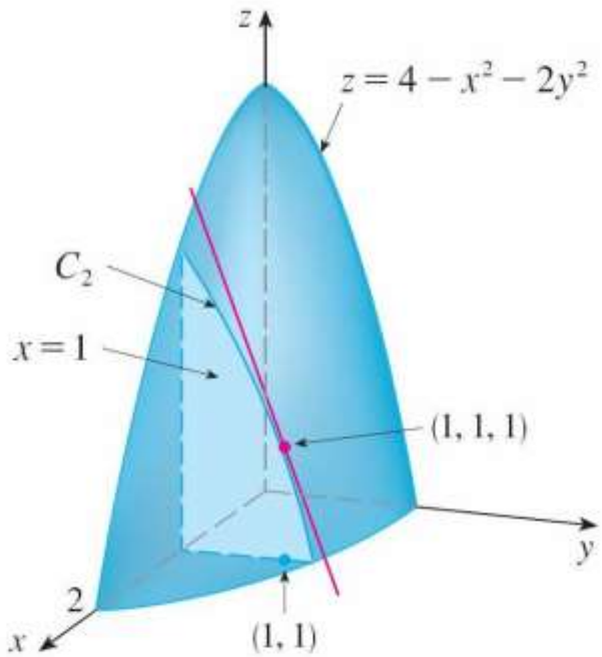
The slope of the tangent line to this parabola at the point (1, 1, 1) is $f_x(1, 1) = -2$.



Example 2 – *Solution*

cont'd

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at (1, 1, 1) is $f_y(1, 1) = -4$. (See the figure below.)



Functions of More Than Two Variables

Functions of More Than Two Variables

Partial derivatives can also be defined for functions of three or more variables. For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding *y* and *z* as constants and differentiating *f*(*x*, *y*, *z*) with respect to *x*.

Functions of More Than Two Variables

If $w = f(x, y, z)$, then $f_x = \partial w/\partial x$ can be interpreted as the rate of change of w with respect to x when y and z are held fixed. But we can't interpret it geometrically because the graph of f lies in four-dimensional space.

In general, if u is a function of n variables, $u = f(x_1, x_2, \ldots, x_n)$, its partial derivative with respect to the ith variable x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_{i-1}, x_i + h, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_i, \ldots, x_n)}{h}$$

and we also write

$$\frac{\partial u}{\partial x_i} = \frac{\partial f}{\partial x_i} = f_{x_i} = f_i = D_i f$$

Example

Find f_x, f_y, and f_z if $f(x, y, z) = e^{xy} \ln z$.

Solution:

Holding y and z constant and differentiating with respect to x, we have

$$f_x = ye^{xy} \ln z$$

Similarly,

$$f_y = xe^{xy} \ln z \qquad \text{and} \qquad f_z = \frac{e^{xy}}{z}$$

Higher Derivatives

Higher Derivatives

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f.

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$

Higher Derivatives

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} (or $\partial^2 f/\partial y\, \partial x$) means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example: Second partial derivatives

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution:

In Example 1 we found that

$$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3)$$

$$= 6x + 2y^3$$

Example – *Solution*

cont'd

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3)$$

$$= 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y)$$

$$= 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y)$$

$$= 6x^2y - 4$$

Higher Derivatives

Notice that $f_{xy} = f_{yx}$ in the last example. This is not just a coincidence.

It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Example: Mixed Partial Derivatives

Let $f(x, y) = \begin{cases} \frac{x^3y - xy^3}{x^2+y^2} & if \quad (x, y) \neq (0,0) \\ 0 & if \quad (x, y) = (0,0) \end{cases}$

(1) Find $f_x(0,0)$ and $f_y(0,0)$.

(2) Show that $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$.

(3) Does the result of part (2) contradict Clairaut's Theorem?

SOLUTION: Note that for $(x, y) \neq (0,0)$, we have

$$f_x = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \quad \text{and} \quad f_y = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}.$$

Example: Mixed Partial Derivatives

(1) $f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0,$

$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0\,.$

(2) $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,0+h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{-h-0}{h} = -1,$

$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1.$

(3) No, since f_{xy} and f_{xy} are not continuous.

Higher Derivatives

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\,\partial x}\right) = \frac{\partial^3 f}{\partial y^2\,\partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Example: Higher Derivatives

EXAMPLE 9 If $f(x, y) = x\cos y + ye^x$, find the second-order derivatives

$$\frac{\partial^2 f}{\partial x^2}, \qquad \frac{\partial^2 f}{\partial y \partial x}, \qquad \frac{\partial^2 f}{\partial y^2}, \qquad \text{and} \qquad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution The first step is to calculate both first partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x\cos y + ye^x) \qquad\qquad \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x\cos y + ye^x)$$

$$= \cos y + ye^x \qquad\qquad = -x\sin y + e^x$$

Now we find both partial derivatives of each first partial:

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x. \qquad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x\cos y.$$

Partial Differential Equations

Partial Differential Equations

Partial derivatives occur in *partial differential equations* that express certain physical laws.

For instance, the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is called **Laplace's equation** after Pierre Laplace (1749–1827).

Solutions of this equation are called **harmonic functions**; they play a role in problems of heat conduction, fluid flow, and electric potential.

Example 8

Show that the function $u(x, y) = e^x \sin y$ is a solution of Laplace's equation.

Solution:

We first compute the needed second-order partial derivatives:

$$u_x = e^x \sin y \qquad u_y = e^x \cos y$$

$$u_{xx} = e^x \sin y \qquad u_{yy} = -e^x \sin y$$

So $\qquad u_{xx} + u_{yy} = e^x \sin y - e^x \sin y = 0$

Therefore u satisfies Laplace's equation.

Partial Differential Equations

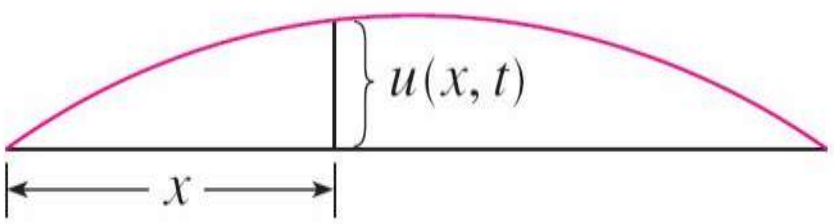
The **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

describes the motion of a waveform, which could be an ocean wave, a sound wave, a light wave, or a wave traveling along a vibrating string.

Partial Differential Equations

For instance, if $u(x, t)$ represents the displacement of a vibrating violin string at time t and at a distance x from one end of the string (as in the figure below), then $u(x, t)$ satisfies the wave equation.



Here the constant a depends on the density of the string and on the tension in the string.

14 Partial Derivatives

14.4 Tangent Planes and Linear Approximations

Tangent Planes

Tangent Planes

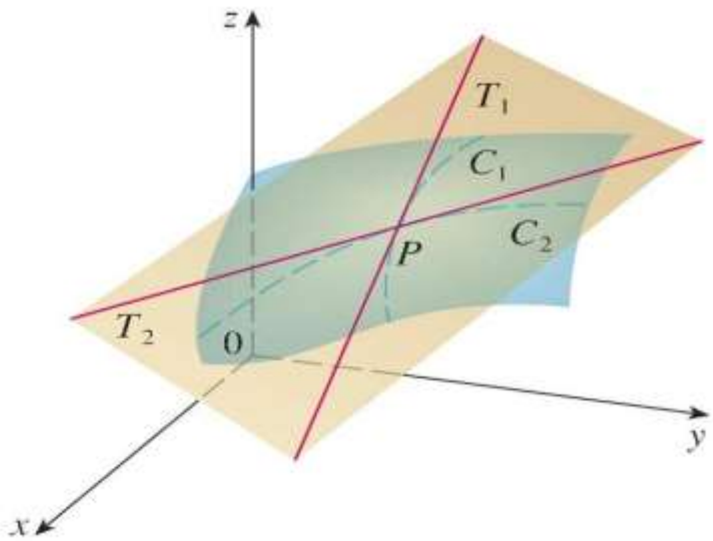
Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2.

Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Tangent Planes

Then the **tangent plane** to the surface *S* at the point *P* is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)



The tangent plane contains the tangent lines T_1 and T_2.

Figure 1

Tangent Planes

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P. We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes

By dividing this equation by C and letting $a = –A/C$ and $b = –B/C$, we can write it in the form

$$\boxed{1} \qquad z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a.

Tangent Planes

But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$.

Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

$\boxed{2}$ Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 1

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

Solution:

Let $f(x, y) = 2x^2 + y^2$.

Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then $\boxed{2}$ gives the equation of the tangent plane at (1, 1, 3) as

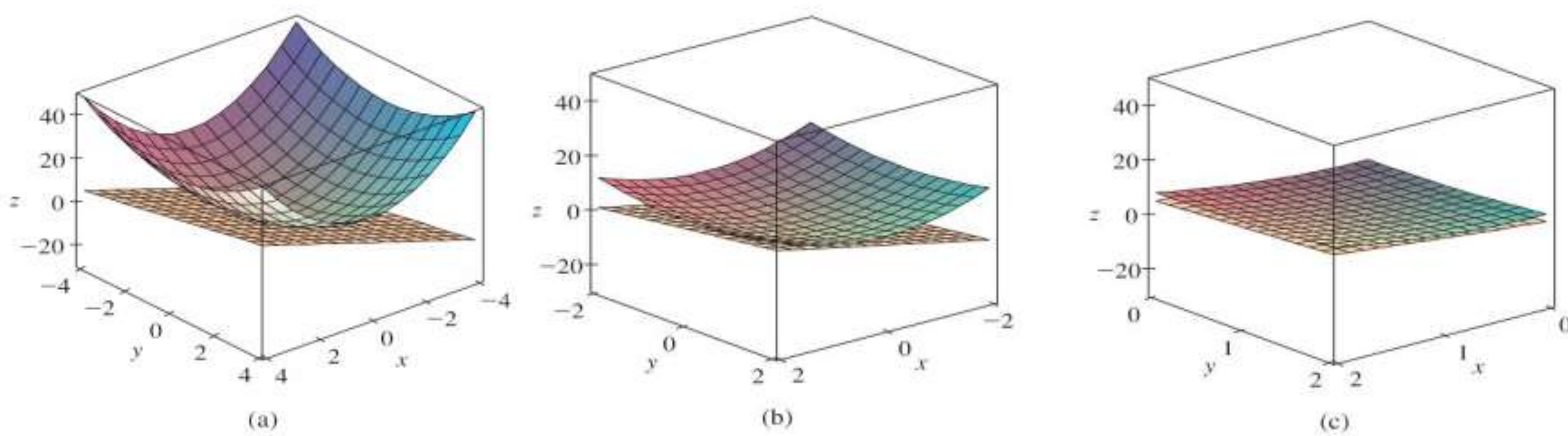
$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$

Tangent Planes

Figure 2(a) shows the elliptic paraboloid and its tangent plane at (1, 1, 3) that we found in Example 1. In parts (b) and (c) we zoom in toward the point (1, 1, 3) by restricting the domain of the function $f(x, y) = 2x^2 + y^2$.

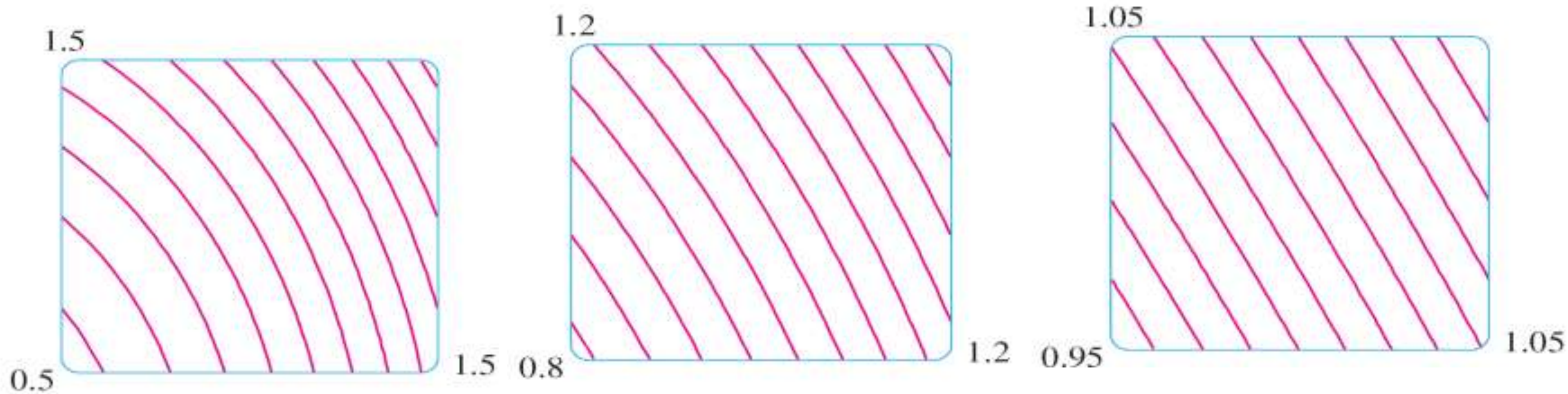


The elliptic paraboloid $z = 2x^2 + y^2$ appears to coincide with its tangent plane as we zoom in toward (1, 1, 3).

Figure 2

Tangent Planes

Notice that the more we zoom in, the flatter the graph appears and the more it resembles its tangent plane.
In Figure 3 we corroborate this impression by zooming in toward the point (1, 1) on a contour map of the function $f(x, y) = 2x^2 + y^2$.



Zooming in toward (1, 1) on a contour map of $f(x, y) = 2x^2 + y^2$

Figure 3

Tangent Planes

Notice that the more we zoom in, the more the level curves look like equally spaced parallel lines, which is characteristic of a plane.

Differentiability

THEOREM 3—The Increment Theorem for Functions of Two Variables Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

DEFINITION A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

8 **Theorem** If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

14

Partial Derivatives

14.5 The Chain Rule

The Chain Rule

Recall that the Chain Rule for functions of a single variable gives the rule for differentiating a composite function:
If $y = f(x)$ and $x = g(t)$, where f and g are differentiable functions, then y is indirectly a differentiable function of t and

$$\boxed{1} \qquad \frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

For functions of more than one variable, the Chain Rule has several versions, each of them giving a rule for differentiating a composite function.

The Chain Rule

The first version (Theorem 2) deals with the case where $z = f(x, y)$ and each of the variables x and y is, in turn, a function of a variable t.

This means that z is indirectly a function of t, $z = f(g(t), h(t))$, and the Chain Rule gives a formula for differentiating z as a function of t. We assume that f is differentiable.

The Chain Rule

Recall that this is the case when f_x and f_y are continuous.

2 The Chain Rule (Case 1) Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Since we often write $\partial z/\partial x$ in place of $\partial f/\partial x$, we can rewrite the Chain Rule in the form

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Example 1

If $z = x^2y + 3xy^4$, where $x = \sin 2t$ and $y = \cos t$, find dz/dt when $t = 0$.

Solution:

The Chain Rule gives

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

It's not necessary to substitute the expressions for x and y in terms of t.

Example 1 – *Solution*

cont'd

We simply observe that when $t = 0$, we have $x = \sin 0 = 0$ and $y = \cos 0 = 1$.

Therefore

$$\left.\frac{dz}{dt}\right|_{t=0} = (0 + 3)(2 \cos 0) + (0 + 0)(-\sin 0) = 6$$

The Chain Rule

We now consider the situation where $z = f(x, y)$ but each of x and y is a function of two variables s and t:
$x = g(s, t)$, $y = h(s, t)$.

Then z is indirectly a function of s and t and we wish to find $\partial z/\partial s$ and $\partial z/\partial t$.

Recall that in computing $\partial z/\partial t$ we hold s fixed and compute the ordinary derivative of z with respect to t.

Therefore we can apply Theorem 2 to obtain

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

The Chain Rule

A similar argument holds for $\partial z/\partial s$ and so we have proved the following version of the Chain Rule.

$\boxed{3}$ **The Chain Rule (Case 2)** Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Case 2 of the Chain Rule contains three types of variables: *s* and *t* are **independent** variables, *x* and *y* are called **intermediate** variables, and *z* is the **dependent** variable.

The Chain Rule

Notice that Theorem 3 has one term for each intermediate variable and each of these terms resembles the one-dimensional Chain Rule in Equation 1.

To remember the Chain Rule, it's helpful to draw the **tree diagram** in Figure 2.

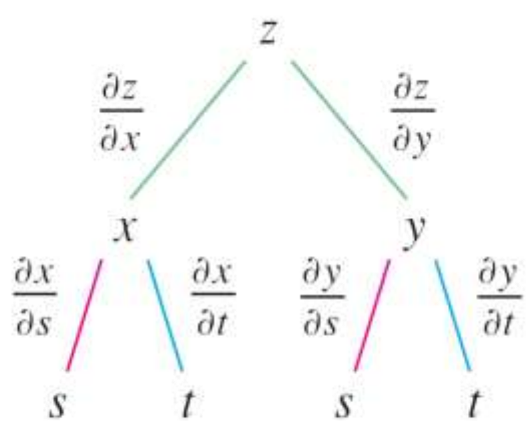
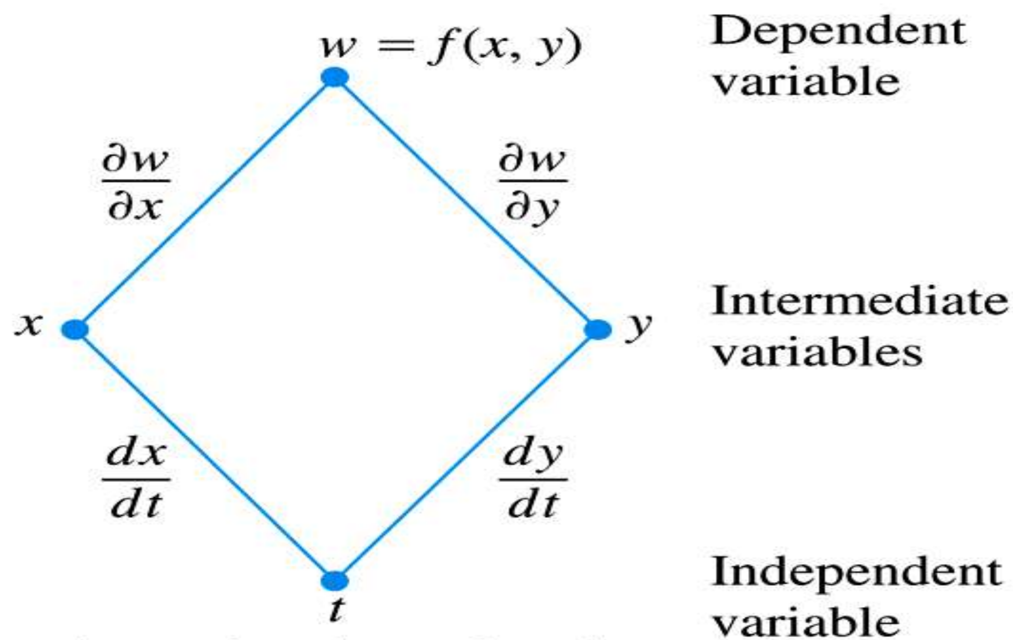


Figure 2

The Chain Rule: Tree Diagram

To remember the Chain Rule picture the diagram below. To find dw/dt, start at w and read down each route to t, multiplying derivatives along the way. Then add the products.

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

The Chain Rule

We draw branches from the dependent variable *z* to the intermediate variables *x* and *y* to indicate that *z* is a function of *x* and *y*. Then we draw branches from *x* and *y* to the independent variables *s* and *t*.

On each branch we write the corresponding partial derivative. To find $\partial z/\partial s$, we find the product of the partial derivatives along each path from *z* to *s* and then add these products:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

The Chain Rule

Similarly, we find $\partial z/\partial t$ by using the paths from z to t.

Now we consider the general situation in which a dependent variable u is a function of n intermediate variables $x_1, \ldots, x_n$, each of which is, in turn, a function of m independent variables $t_1, \ldots, t_m$.

Notice that there are n terms, one for each intermediate variable. The proof is similar to that of Case 1.

The Chain Rule

4 **The Chain Rule (General Version)** Suppose that u is a differentiable function of the n variables $x_1, x_2, \ldots, x_n$ and each x_j is a differentiable function of the m variables $t_1, t_2, \ldots, t_m$. Then u is a function of $t_1, t_2, \ldots, t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

for each $i = 1, 2, \ldots, m$.

The Chain Rule

EXAMPLE 2 Find dw/dt if

$$w = xy + z, \qquad x = \cos t, \qquad y = \sin t, \qquad z = t.$$

In this example the values of $w(t)$ are changing along the path of a helix (Section 13.1) as t changes. What is the derivative's value at $t = 0$?

Solution Using the Chain Rule for three independent variables, we have

$$\begin{aligned} \frac{dw}{dt} &= \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \\ &= (y)(-\sin t) + (x)(\cos t) + (1)(1) \\ &= (\sin t)(-\sin t) + (\cos t)(\cos t) + 1 \\ &= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t, \end{aligned}$$

Substitute for the intermediate variables.

so

$$\left(\frac{dw}{dt}\right)_{t=0} = 1 + \cos(0) = 2.$$

The Chain Rule

EXAMPLE 3 Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x + 2y + z^2, \qquad x = \frac{r}{s}, \qquad y = r^2 + \ln s, \qquad z = 2r.$$

Solution Using the formulas in Theorem 7, we find

$$\begin{aligned}
\frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r} \\
&= (1)\left(\frac{1}{s}\right) + (2)(2r) + (2z)(2) \\
&= \frac{1}{s} + 4r + (4r)(2) = \frac{1}{s} + 12r \qquad \text{Substitute for intermediate variable } z. \\
\frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s} \\
&= (1)\left(-\frac{r}{s^2}\right) + (2)\left(\frac{1}{s}\right) + (2z)(0) = \frac{2}{s} - \frac{r}{s^2}
\end{aligned}$$

Implicit Differentiation

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation.

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x.

Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Implicit Differentiation

But $dx/dx = 1$, so if $\partial F/\partial x \neq 0$ we solve for dy/dx and obtain

6
$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x.

Implicit Differentiation

The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid:

It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

Example 8

Find y' if $x^3 + y^3 = 6xy$.

Solution:

The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Implicit Differentiation

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Implicit Differentiation

But $\dfrac{\partial}{\partial x}(x) = 1$ and $\dfrac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in Equations 7.

The formula for $\partial z/\partial y$ is obtained in a similar manner.

Implicit Differentiation

$$\boxed{7} \qquad \frac{\partial z}{\partial x} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\dfrac{\partial F}{\partial y}}{\dfrac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid:

If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by $\boxed{7}$.

14 Partial Derivatives

14.6 Directional Derivatives and the Gradient Vector

Directional Derivatives and the Gradient Vector

In this section we introduce a type of derivative, called a *directional derivative,* that enables us to find the rate of change of a function of two or more variables in any direction.

Directional Derivatives

Directional Derivatives

Recall that if $z = f(x, y)$, then the partial derivatives f_x and f_y are defined as

$$\boxed{1} \qquad f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

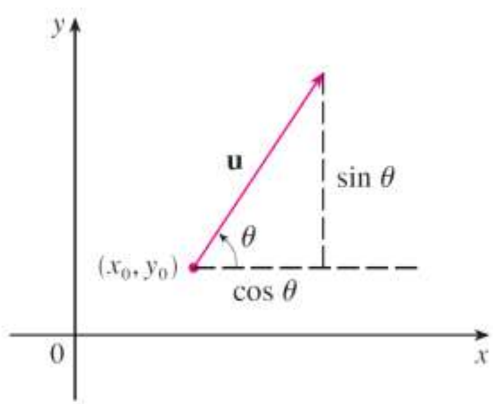
$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

and represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.

Directional Derivatives

Suppose that we now wish to find the rate of change of z at (x_0, y_0) in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$. (See Figure 2.)

To do this we consider the surface S with the equation $z = f(x, y)$ (the graph of f) and we let $z_0 = f(x_0, y_0)$. Then the point $P(x_0, y_0, z_0)$ lies on S.



A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

Figure 2

Directional Derivatives

The vertical plane that passes through P in the direction of $\mathbf{u}$ intersects S in a curve C. (See Figure 3.)

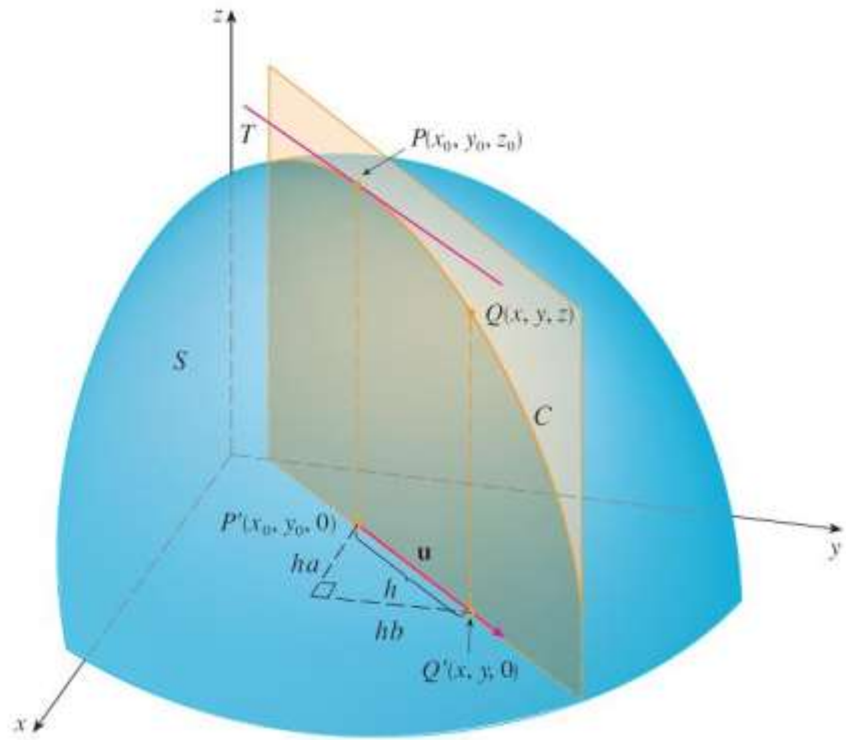


Figure 3

Directional Derivatives

The slope of the tangent line T to C at the point P is the rate of change of z in the direction of **u**. If $Q(x, y, z)$ is another point on C and P', Q' are the projections of P, Q onto the xy-plane, then the vector $\overrightarrow{P'Q'}$ is parallel to **u** and so

$$\overrightarrow{P'Q'} = h\mathbf{u} = \langle ha, hb \rangle$$

for some scalar h. Therefore $x - x_0 = ha$, $y - y_0 = hb$, so $x = x_0 + ha$, $y = y_0 + hb$, and

$$\frac{\Delta z}{h} = \frac{z - z_0}{h} = \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

Directional Derivatives

If we take the limit as $h \to 0$, we obtain the rate of change of z (with respect to distance) in the direction of **u**, which is called the directional derivative of f in the direction of **u**.

$\boxed{2}$ **Definition** The **directional derivative** of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Directional Derivatives

By comparing Definition 2 with Equations $\boxed{1}$, we see that if $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}}f = f_x$ and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}}f = f_y$.

In other words, the partial derivatives of f with respect to x and y are just special cases of the directional derivative.

Example 1

Use the weather map in Figure 1 to estimate the value of the directional derivative of the temperature function at Reno in the southeasterly direction.

Figure 1

Example 1 – *Solution*

The unit vector directed toward the southeast is $\mathbf{u} = (\mathbf{i} - \mathbf{j})/\sqrt{2}$, but we won't need to use this expression.

We start by drawing a line through Reno toward the southeast (see Figure 4).

Figure 4

Example 1 – *Solution*

cont'd

We approximate the directional derivative $D_{\mathbf{u}}T$ by the average rate of change of the temperature between the points where this line intersects the isothermals $T = 50$ and $T = 60$.

The temperature at the point southeast of Reno is $T = 60°\text{F}$ and the temperature at the point northwest of Reno is $T = 50°\text{F}$.

The distance between these points looks to be about 75 miles. So the rate of change of the temperature in the southeasterly direction is

$$D_{\mathbf{u}}T \approx \frac{60 - 50}{75} = \frac{10}{75} \approx 0.13°\text{F/mi}$$

Directional Derivatives

When we compute the directional derivative of a function defined by a formula, we generally use the following theorem.

$\boxed{3}$ **Theorem** If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\, a + f_y(x, y)\, b$$

Directional Derivatives

If the unit vector **u** makes an angle θ with the positive x-axis (as in Figure 2), then we can write $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$ and the formula in Theorem 3 becomes

$$\boxed{6} \qquad D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta$$

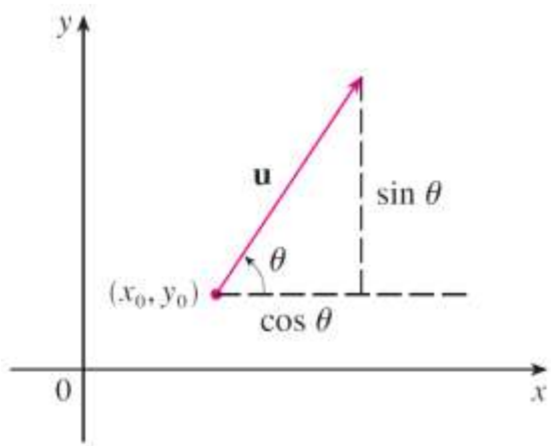


Figure 2

A unit vector $\mathbf{u} = \langle a, b \rangle = \langle \cos\theta, \sin\theta \rangle$

The Gradient Vectors

The Gradient Vectors

Notice from Theorem 3 that the directional derivative of a differentiable function can be written as the dot product of two vectors:

$$\boxed{7} \qquad \begin{aligned} D_{\mathbf{u}}f(x, y) &= f_x(x, y)a + f_y(x, y)b \\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \langle a, b\rangle \\ &= \langle f_x(x, y), f_y(x, y)\rangle \cdot \mathbf{u} \end{aligned}$$

The first vector in this dot product occurs not only in computing directional derivatives but in many other contexts as well.

So we give it a special name (the *gradient* of *f*) and a special notation (**grad** *f* or ∇f, which is read "del *f*").

The Gradient Vectors

$\boxed{8}$ **Definition** If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

Example 3

If $f(x, y) = \sin x + e^{xy}$, then

$$\nabla f(x, y) = \langle f_x, f_y \rangle$$

$$= \langle \cos x + ye^{xy}, xe^{xy} \rangle$$

and $\qquad \nabla f(0, 1) = \langle 2, 0 \rangle$

The Gradient Vectors

With this notation for the gradient vector, we can rewrite the expression (7) for the directional derivative of a differentiable function as

9

$$\boxed{D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}}$$

This expresses the directional derivative in the direction of **u** as the scalar projection of the gradient vector onto **u**.

Example

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution The direction of $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2\cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

(Figure 14.28). The derivative of f at $(2, 0)$ in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)|_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} \qquad \text{Eq. (4)}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

Functions of Three Variables

Functions of Three Variables

For functions of three variables we can define directional derivatives in a similar manner.

Again $D_{\mathbf{u}}f(x, y, z)$ can be interpreted as the rate of change of the function in the direction of a unit vector **u**.

10 Definition The **directional derivative** of f at (x_0, y_0, z_0) in the direction of a unit vector $\mathbf{u} = \langle a, b, c \rangle$ is

$$D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

if this limit exists.

Functions of Three Variables

If we use vector notation, then we can write both definitions (2 and 10) of the directional derivative in the compact form

11
$$D_{\mathbf{u}} f(\mathbf{x}_0) = \lim_{h \to 0} \frac{f(\mathbf{x}_0 + h\mathbf{u}) - f(\mathbf{x}_0)}{h}$$

where $\mathbf{x}_0 = \langle x_0, y_0 \rangle$ if $n = 2$ and $\mathbf{x}_0 = \langle x_0, y_0, z_0 \rangle$ if $n = 3$.

This is reasonable because the vector equation of the line through $\mathbf{x}_0$ in the direction of the vector $\mathbf{u}$ is given by $\mathbf{x} = \mathbf{x}_0 + t\,\mathbf{u}$ and so $f(\mathbf{x}_0 + h\mathbf{u})$ represents the value of f at a point on this line.

Functions of Three Variables

If $f(x, y, z)$ is differentiable and $\mathbf{u} = \langle a, b, c \rangle$, then

12 $$D_{\mathbf{u}} f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c$$

For a function f of three variables, the **gradient vector**, denoted by ∇f or **grad** f, is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

or, for short,

13 $$\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Functions of Three Variables

Then, just as with functions of two variables, Formula 12 for the directional derivative can be rewritten as

14

$$D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$$

Example 5

If $f(x, y, z) = x \sin yz$, (a) find the gradient of f and (b) find the directional derivative of f at (1, 3, 0) in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

Solution:

(a) The gradient of f is

$$\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle$$

$$= \langle \sin yz, xz \cos yz, xy \cos yz \rangle$$

Example 5 – *Solution*

cont'd

(b) At (1, 3, 0) we have $\nabla f(1, 3, 0) = \langle 0, 0, 3 \rangle$.

The unit vector in the direction of $\mathbf{v} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ is

$$\mathbf{u} = \frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}$$

Therefore Equation 14 gives

$$D_{\mathbf{u}}f(1, 3, 0) = \nabla f(1, 3, 0) \cdot \mathbf{u}$$

$$= 3\mathbf{k} \cdot \left(\frac{1}{\sqrt{6}}\mathbf{i} + \frac{2}{\sqrt{6}}\mathbf{j} - \frac{1}{\sqrt{6}}\mathbf{k}\right)$$

$$= 3\left(-\frac{1}{\sqrt{6}}\right) = -\sqrt{\frac{3}{2}}$$

Maximizing the Directional Derivatives

Maximizing the Directional Derivatives

Suppose we have a function *f* of two or three variables and we consider all possible directional derivatives of *f* at a given point.

These give the rates of change of *f* in all possible directions.

We can then ask the questions: In which of these directions does *f* change fastest and what is the maximum rate of change? The answers are provided by the following theorem.

15 Theorem Suppose f is a differentiable function of two or three variables. The maximum value of the directional derivative $D_{\mathbf{u}} f(\mathbf{x})$ is $|\nabla f(\mathbf{x})|$ and it occurs when $\mathbf{u}$ has the same direction as the gradient vector $\nabla f(\mathbf{x})$.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

Example 6

(a) If $f(x, y) = xe^y$, find the rate of change of f at the point $P(2, 0)$ in the direction from P to $Q(\frac{1}{2}, 2)$.

(b) In what direction does f have the maximum rate of change? What is this maximum rate of change?

Solution:

(a) We first compute the gradient vector:

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle e^y, xe^y \rangle$$

$$\nabla f(2, 0) = \langle 1, 2 \rangle$$

Example 6 – *Solution*

cont'd

The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2\rangle$ is $\mathbf{u} = \left\langle -\frac{3}{5}, \frac{4}{5}\right\rangle$, so the rate of change of f in the direction from P to Q is

$$D_{\mathbf{u}}f(2, 0) = \nabla f(2, 0) \cdot \mathbf{u}$$

$$= \langle 1, 2\rangle \cdot \left\langle -\tfrac{3}{5}, \tfrac{4}{5}\right\rangle$$

$$= 1\left(-\tfrac{3}{5}\right) + 2\left(\tfrac{4}{5}\right) = 1$$

(b) According to Theorem 15, f increases fastest in the direction of the gradient vector $\nabla f(2, 0) = \langle 1, 2\rangle$. The maximum rate of change is

$$|\nabla f(2, 0)| = |\langle 1, 2\rangle| = \sqrt{5}$$

Example

Suppose that the temperature at a point in space is given by $T(x, y, z) = 80/(1 + x^2 + 2y^2 + 3z^2)$, where *T* is measured in degrees Celsius and *x*, *y*, *z* in meters. In which direction does the temperature increase fastest at the point (1, 1, -2)? What is the maximum rate of increase?

Solution:

(a) We first compute the gradient vector:

$$\nabla T(x, y, z) = \langle f_x, f_y, f_z \rangle =$$

$$\left\langle \frac{-160x}{(1+x^2+2y^2+3z^2)^2}, \frac{-320y}{(1+x^2+2y^2+3z^2)^2}, \frac{-480z}{(1+x^2+2y^2+3z^2)^2} \right\rangle$$

At the point (1, 1, -2) the gradient vector is

Example: Solution

$$\nabla T(1,1,-2) = \left\langle \frac{-5}{8}, \frac{-10}{8}, \frac{30}{8} \right\rangle.$$

By Theorem 15 the temperature increases fastest in the direction of the gradient vector $\nabla T(1,1,-2) = \left\langle \frac{-5}{8}, \frac{-10}{8}, \frac{30}{8} \right\rangle$.

The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1,1,-2)| = \sqrt{\left(\frac{-5}{8}\right)^2 + \left(\frac{-10}{8}\right)^2 \left(\frac{30}{8}\right)^2} = 4^{\circ} C/\mathrm{m}.$$

Tangent Planes to Level Surfaces

Tangent Planes to Level Surfaces

Suppose S is a surface with equation $F(x, y, z) = k$, that is, it is a level surface of a function F of three variables, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C be any curve that lies on the surface S and passes through the point P. Recall that the curve C is described by a continuous vector function $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$.

Let t_0 be the parameter value corresponding to P; that is, $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$. Since C lies on S, any point $(x(t), y(t), z(t))$ must satisfy the equation of S, that is,

$$F(x(t), y(t), z(t)) = k \tag{16}$$

Tangent Planes to Level Surfaces

If x, y, and z are differentiable functions of t and F is also differentiable, then we can use the Chain Rule to differentiate both sides of Equation 16 as follows:

$$\boxed{17} \qquad \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial F}{\partial z}\frac{dz}{dt} = 0$$

But, since $\nabla F = \langle F_x, F_y, F_z \rangle$ and $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$, Equation 17 can be written in terms of a dot product as

$$\nabla F \cdot \mathbf{r}'(t) = 0$$

Tangent Planes to Level Surfaces

In particular, when $t = t_0$ we have $\mathbf{r}(t_0) = \langle x_0, y_0, z_0 \rangle$, so

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0 \qquad \boxed{18}$$

Equation 18 says that *the gradient vector at P*, $\nabla F(x_0, y_0, z_0)$, *is perpendicular to the tangent vector* $\mathbf{r}'(t_0)$ *to any curve C on S that passes through P*. (See Figure 9.)

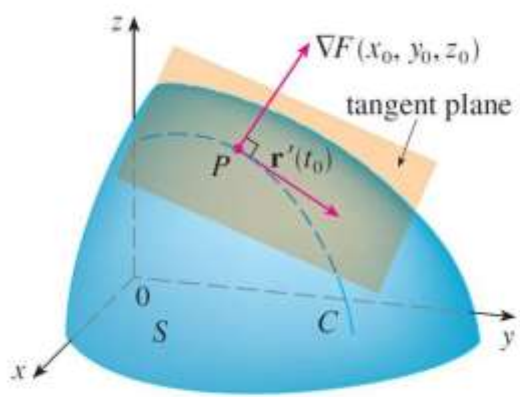


Figure 9

Tangent Planes to Level Surfaces

If $\nabla F(x_0, y_0, z_0) \neq \mathbf{0}$, it is therefore natural to define the **tangent plane to the level surface** $F(x, y, z) = k$ **at** $P(x_0, y_0, z_0)$ as the plane that passes through P and has normal vector $\nabla F(x_0, y_0, z_0)$.

Using the standard equation of a plane, we can write the equation of this tangent plane as

$$\boxed{19} \quad F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$$

Tangent Planes to Level Surfaces

The **normal line** to *S* at *P* is the line passing through *P* and perpendicular to the tangent plane.

The direction of the normal line is therefore given by the gradient vector $\nabla F(x_0, y_0, z_0)$ and so, its symmetric equations are

$$\boxed{20} \qquad \frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$$

Tangent Planes to Level Surfaces

In the special case in which the equation of a surface S is of the form $z = f(x, y)$ (that is, S is the graph of a function f of two variables), we can rewrite the equation as

$$F(x, y, z) = f(x, y) - z = 0$$

and regard S as a level surface (with $k = 0$) of F. Then

$$F_x(x_0, y_0, z_0) = f_x(x_0, y_0)$$
$$F_y(x_0, y_0, z_0) = f_y(x_0, y_0)$$
$$F_z(x_0, y_0, z_0) = -1$$

so Equation 19 becomes

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

Example 8

Find the equations of the tangent plane and normal line at the point (–2, 1, –3) to the ellipsoid

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution:

The ellipsoid is the level surface (with *k* = 3) of the function

$$F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}$$

Significance of the Gradient Vectors

Example 8 – *Solution*

cont'd

Therefore we have

$F_x(x, y, z) = \frac{x}{2}$ $\qquad F_y(x, y, z) = 2y$ $\qquad F_z(x, y, z) = \frac{2z}{9}$

$F_x(-2, 1, -3) = -1$ $\qquad F_y(-2, 1, -3) = 2$ $\qquad F_z(-2, 1, -3) = -\frac{2}{3}$

Then Equation 19 gives the equation of the tangent plane at (–2, 1, –3) as

$$-1(x + 2) + 2(y - 1) - \tfrac{2}{3}(z + 3) = 0$$

which simplifies to $3x - 6y + 2z + 18 = 0$.

By Equation 20, symmetric equations of the normal line are

$$\frac{x+2}{-1} = \frac{y-1}{2} = \frac{z+3}{-\frac{2}{3}}$$

Significance of the Gradient Vectors

We now summarize the ways in which the gradient vector is significant.

We first consider a function *f* of three variables and a point $P(x_0, y_0, z_0)$ in its domain.

On the one hand, we know from Theorem 15 that the gradient vector $\nabla f(x_0, y_0, z_0)$ gives the direction of fastest increase of *f*.

On the other hand, we know that $\nabla f(x_0, y_0, z_0)$ is orthogonal to the level surface *S* of *f* through *P*. (Refer to Figure 9.)

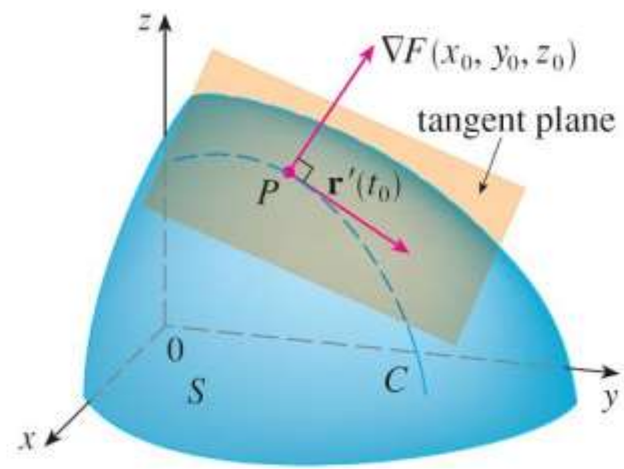


Figure 9

Significance of the Gradient Vectors

These two properties are quite compatible intuitively because as we move away from P on the level surface S, the value of f does not change at all.

So it seems reasonable that if we move in the perpendicular direction, we get the maximum increase.

In like manner we consider a function f of two variables and a point $P(x_0, y_0)$ in its domain.

Again the gradient vector $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f. Also, by considerations similar to our discussion of tangent planes, it can be shown that $\nabla f(x_0, y_0)$ is perpendicular to the level curve $f(x, y) = k$ that passes through P.

Significance of the Gradient Vectors

Again this is intuitively plausible because the values of *f* remain constant as we move along the curve.
(See Figure 11.)

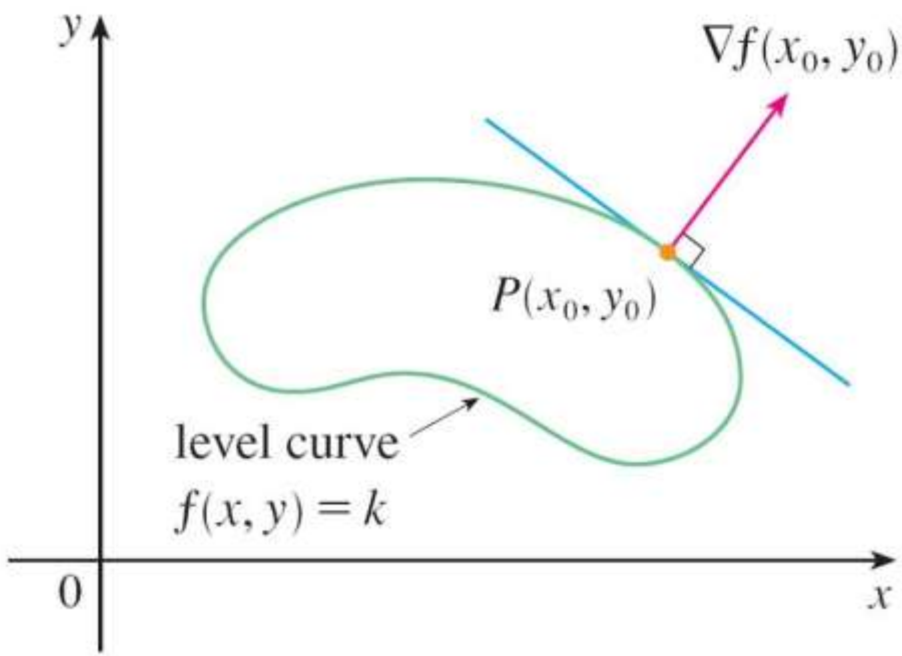


Figure 11

Significance of the Gradient Vectors

If we consider a topographical map of a hill and let $f(x, y)$ represent the height above sea level at a point with coordinates (x, y), then a curve of steepest ascent can be drawn as in Figure 12 by making it perpendicular to all of the contour lines.

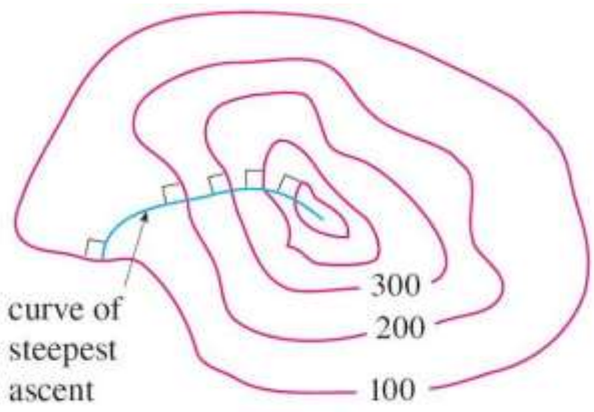


Figure 12

Significance of the Gradient Vectors

Computer algebra systems have commands that plot sample gradient vectors.

Each gradient vector $\nabla f(a, b)$ is plotted starting at the point (a, b). Figure 13 shows such a plot (called a *gradient vector field*) for the function $f(x, y) = x^2 - y^2$ superimposed on a contour map of f.

As expected, the gradient vectors point "uphill" and are perpendicular to the level curves.

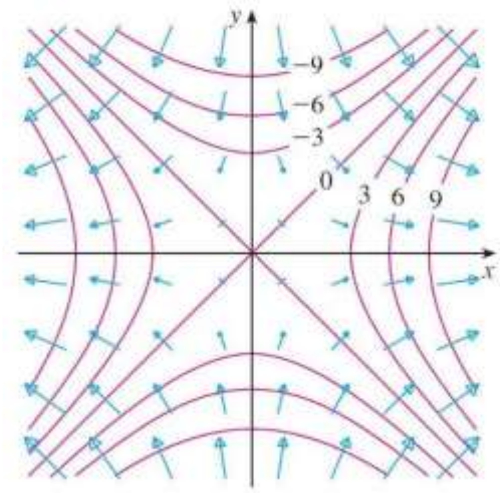


Figure 13

14

Partial Derivatives

14.7 Maximum and Minimum Values

Maximum and Minimum Values

In this section we see how to use partial derivatives to locate maxima and minima of functions of two variables.

Look at the hills and valleys in the graph of *f* shown in Figure 1.

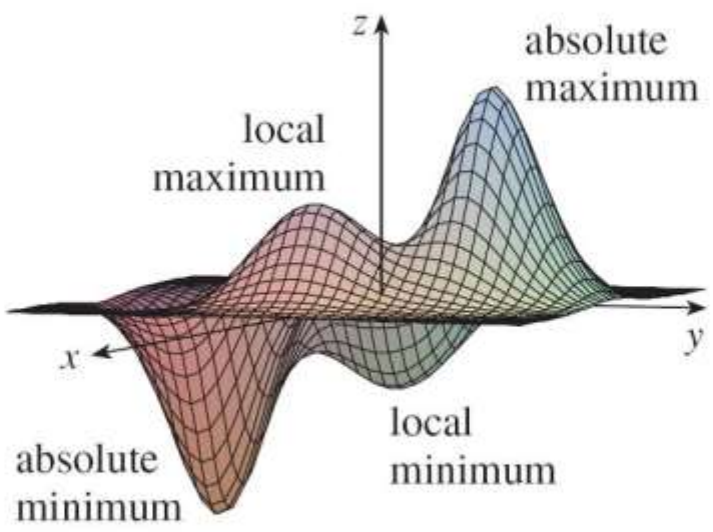


Figure 1

Maximum and Minimum Values

There are two points (a, b) where f has a *local maximum*, that is, where $f(a, b)$ is larger than nearby values of $f(x, y)$.

The larger of these two values is the *absolute maximum*.

Likewise, f has two *local minima*, where $f(a, b)$ is smaller than nearby values.

The smaller of these two values is the *absolute minimum*.

Maximum and Minimum Values

$\boxed{1}$ **Definition** A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

If the inequalities in Definition 1 hold for *all* points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).

$\boxed{2}$ **Fermat's Theorem for Functions of Two Variables** If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Maximum and Minimum Values

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist.

Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f.

However, as in single-variable calculus, not all critical points give rise to maxima or minima.

At a critical point, a function could have a local maximum or a local minimum or neither.

Example 1

Let $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Then

$$f_x(x, y) = 2x - 2 \qquad f_y(x, y) = 2y - 6$$

These partial derivatives are equal to 0 when $x = 1$ and $y = 3$, so the only critical point is (1, 3).

By completing the square, we find that

$$f(x, y) = 4 + (x - 1)^2 + (y - 3)^2$$

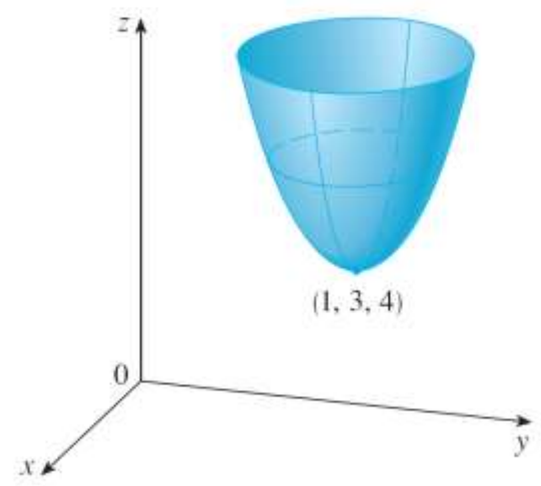
Example 1

cont'd

Since $(x - 1)^2 \geq 0$ and $(y - 3)^2 \geq 0$, we have $f(x, y) \geq 4$ for all values of x and y.

Therefore $f(1, 3) = 4$ is a local minimum, and in fact it is the absolute minimum of f.

This can be confirmed geometrically from the graph of f, which is the elliptic paraboloid with vertex (1, 3, 4) shown in Figure 2.



$z = x^2 + y^2 - 2x - 6y + 14$

Figure 2

EXAMPLE 1 Find the local extreme values of $f(x, y) = x^2 + y^2 - 4y + 9$.

Solution The domain of f is the entire plane (so there are no boundary points) and the partial derivatives $f_x = 2x$ and $f_y = 2y - 4$ exist everywhere. Therefore, local extreme values can occur only where

$$f_x = 2x = 0 \qquad \text{and} \qquad f_y = 2y - 4 = 0.$$

The only possibility is the point $(0, 2)$, where the value of f is 5. Since $f(x, y) = x^2 + (y - 2)^2 + 5$ is never less than 5, we see that the critical point $(0, 2)$ gives a local minimum (Figure 14.43).

Maximum and Minimum Values

The following test, is analogous to the Second Derivative Test for functions of one variable.

3 **Second Derivatives Test** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

In case (c) the point (a, b) is called a **saddle point** of *f* and the graph of *f* crosses its tangent plane at (a, b).

EXAMPLE 3 Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4.$$

Solution The function is defined and differentiable for all x and y and its domain has no boundary points. The function therefore has extreme values only at the points where f_x and f_y are simultaneously zero. This leads to

$$f_x = y - 2x - 2 = 0, \qquad f_y = x - 2y - 2 = 0,$$

or

$$x = y = -2.$$

Therefore, the point $(-2, -2)$ is the only point where f may take on an extreme value. To see if it does so, we calculate

$$f_{xx} = -2, \qquad f_{yy} = -2, \qquad f_{xy} = 1.$$

The discriminant of f at $(a, b) = (-2, -2)$ is

$$f_{xx}f_{yy} - f_{xy}^2 = (-2)(-2) - (1)^2 = 4 - 1 = 3.$$

The combination

$$f_{xx} < 0 \qquad \text{and} \qquad f_{xx}f_{yy} - f_{xy}^2 > 0$$

tells us that f has a local maximum at $(-2, -2)$. The value of f at this point is $f(-2, -2) = 8$.

Example

Find the shortest distance from the point (1, 0, -2) to the plane $x + 2y + z = 4$.

SOLUTION The distance from any point (x, y, z) to the point (1, 0, -2) is

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (z+2)^2}$$

but if (x, y, z) lies on the plane $x + 2y + z = 4$, then $\mathrm{z} = 4 - x - 2y$ and so we have

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (4 - x - 2y + 2)^2}$$

We can minimize *d* by minimizing the simpler expression

$$d^2 = f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$$

By solving the equations

Example

$$f_x = 2(x-1) - 2(6-x-2y) = 4x + 4y - 14 = 0,$$
$$f_x = 2y - 4(6-x-2y) = 4x + 10y - 24 = 0,$$

we find that the only critical point is $\left(\frac{11}{6}, \frac{5}{3}\right)$. Since $f_{xx} = 4$, $f_{xy} = 4$, $f_{yy} = 10$, we have $D = f_{xx}f_{yy} - \left(f_{xy}\right)^2 = 24 > 0$ and $f_{xx} > 0$, so by the Second Derivatives Test f has a local minimum at $\left(\frac{11}{6}, \frac{5}{3}\right)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to (1, 0, -2). At this point we get $d = \frac{5}{6}\sqrt{6}$. So, the shortest distance from the point (1, 0, -2) to the plane $x + 2y + z = 4$ is $\frac{5}{6}\sqrt{6}$.

Absolute Maximum and Minimum Values

Absolute Maximum and Minimum Values

For a function *f* of one variable, the Extreme Value Theorem says that if *f* is continuous on a closed interval $[a, b]$, then *f* has an absolute minimum value and an absolute maximum value.

According to the Closed Interval Method, we found these by evaluating *f* not only at the critical numbers but also at the endpoints *a* and *b*.

There is a similar situation for functions of two variables. Just as a closed interval contains its endpoints, a **closed set** in $\mathbb{R}^2$ is one that contains all its boundary points.

Absolute Maximum and Minimum Values

[A boundary point of D is a point (a, b) such that every disk with center (a, b) contains points in D and also points not in D.]

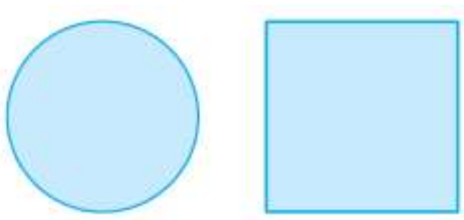
For instance, the disk

$$D = \{(x, y) \mid x^2 + y^2 \leq 1\}$$

which consists of all points on and inside the circle $x^2 + y^2 = 1$, is a closed set because it contains all of its boundary points (which are the points on the circle $x^2 + y^2 = 1$).

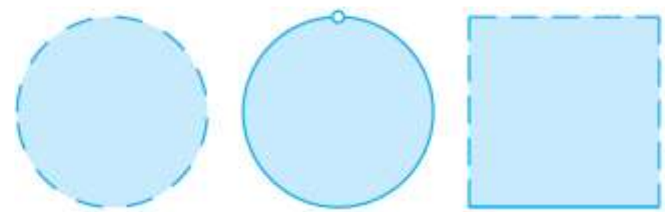
Absolute Maximum and Minimum Values

But if even one point on the boundary curve were omitted, the set would not be closed. (See Figure 11.)



Closed sets

Figure 11(a)



Sets that are not closed

Figure 11(b)

A **bounded set** in $\mathbb{R}^2$ is one that is contained within some disk.

Absolute Maximum and Minimum Values

In other words, it is finite in extent.

Then, in terms of closed and bounded sets, we can state the following counterpart of the Extreme Value Theorem in two dimensions.

$\boxed{8}$ **Extreme Value Theorem for Functions of Two Variables** If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum value $f(x_1, y_1)$ and an absolute minimum value $f(x_2, y_2)$ at some points (x_1, y_1) and (x_2, y_2) in D.

Absolute Maximum and Minimum Values

To find the extreme values guaranteed by Theorem 8, we note that, by Theorem 2, if *f* has an extreme value at (x_1, y_1), then (x_1, y_1) is either a critical point of *f* or a boundary point of *D*.

Thus we have the following extension of the Closed Interval Method.

$\boxed{9}$ To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example 7

Find the absolute maximum and minimum values of the function $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 2\}$.

Solution:

Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum.

According to step 1 in $\boxed{9}$, we first find the critical points. These occur when

$$f_x = 2x - 2y = 0 \qquad f_y = -2x + 2 = 0$$

Example 7 – *Solution*

cont'd

so the only critical point is (1, 1), and the value of f there is $f(1, 1) = 1$.

In step 2 we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in Figure 12.

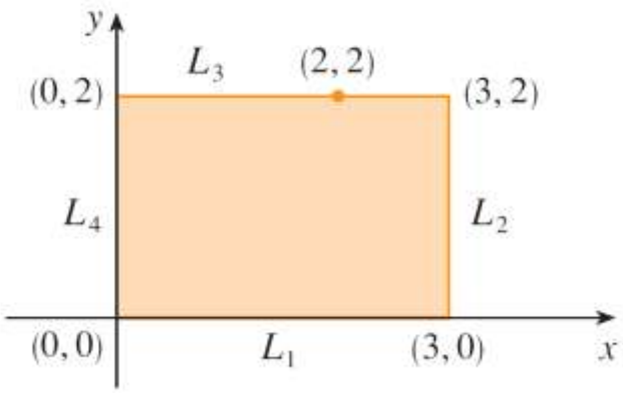


Figure 12

Example 7 – *Solution*

cont'd

On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \le x \le 3$$

This is an increasing function of x, so its minimum value is $f(0, 0) = 0$ and its maximum value is $f(3, 0) = 9$.

On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad 0 \le y \le 2$$

This is a decreasing function of y, so its maximum value is $f(3, 0) = 9$ and its minimum value is $f(3, 2) = 1$.

Example 7 – *Solution*

cont'd

On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad 0 \le x \le 3$$

Simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2, 2) = 0$ and the maximum value is $f(0, 2) = 4$.

Example 7 – *Solution*

cont'd

Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad 0 \le y \le 2$$

with maximum value $f(0, 2) = 4$ and minimum value $f(0, 0) = 0$.

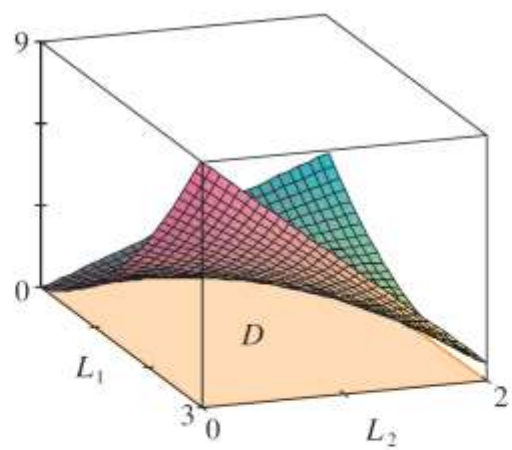
Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Example 7 – *Solution*

cont'd

In step 3 we compare these values with the value $f(1, 1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0, 0) = f(2, 2) = 0$.

Figure 13 shows the graph of f.



$f(x, y) = x^2 - 2xy + 2y$

Figure 13

Example 8

Find the absolute maximum and minimum values of

$$f(x,y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.

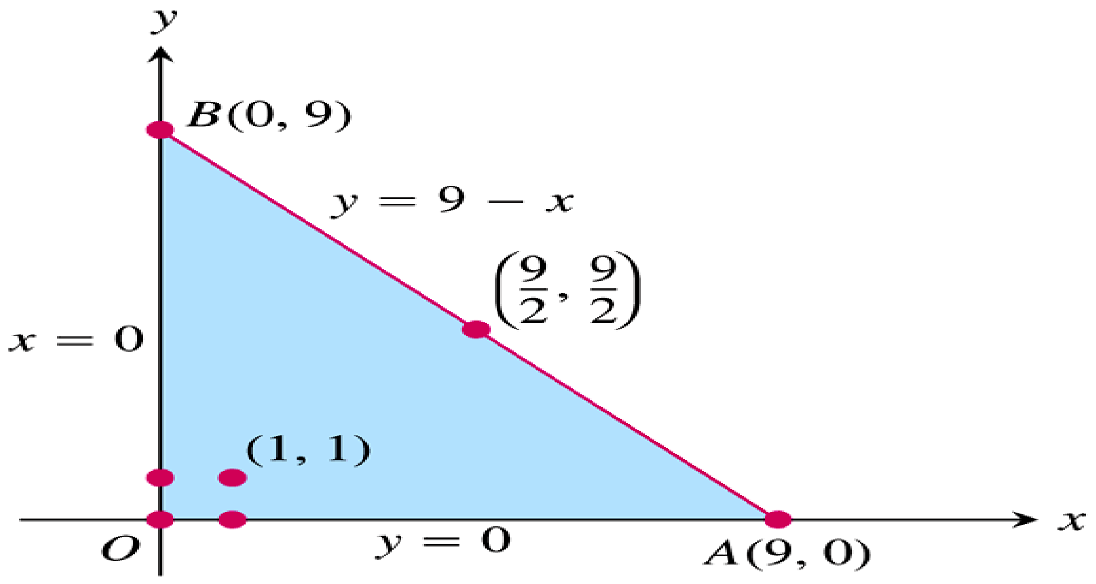


FIGURE 14.46 This triangular region is the domain of the function in Example 5.

Example 8 – *Solution*

cont'd

Since *f* is differentiable, the only places where *f* can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary.

(a) Interior points. For these we have

$$f_x = 2 - 2x = 0,\ f_y = 2 - 2y = 0,$$

yielding the single point (x, y) = (1, 1). The value of *f* there is *f*(1,1) = 4.

(b) Boundary points. We take the triangle one side at a time:

i) On the segment OA, y = 0. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

Example 8 – *Solution*

cont'd

may now be regarded as a function of *x* defined on the closed interval [0, 9]. Its extreme values may occur at the endpoints

$$x = 0 \quad\quad where\;\; f(0,0) = 2$$
$$x = 9 \quad\quad where\;\; f(9,0) = -61$$

and at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where

$f'(x, 0) = 0$ is $x = 1$, where $f(1,0) = 3$.

ii) On the segment *OB*, $x = 0$ and $f(x,y) = f(0,y) = 2 + 2y - y^2$. We know from the symmetry of *f* in *x* and *y* and from the analysis we just carried out that the candidates on this segment are: $f(0,0) = 2$, $f(0,9) = -61$, $f(0,1) = 3$.

Example 8 – *Solution*

cont'd

iii) We have already accounted for the values of *f* at the endpoints of *AB,* so we need only look at the interior points of *AB.* With $y = 9 - x$, we have

$$f(x, 9 - x) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2$$
$$= -61 + 18x - 2x^2.$$

Setting $f'(x, 9 - x) = 18 - 4x = 0$ gives $x = 9/2$.

At this value of *x*, $y = 9 - x = 9 - \frac{9}{2} = 9/2$ and

$$f\left(\frac{9}{2}, \frac{9}{2}\right) = -41/2.$$

We list all the candidates: 4, 2, – 61, 3, – 41/2. The maximum is 4, which *f* assumes at (1, 1). The minimum is – 61, which *f* assumes at (0, 9) and (9, 0).

15 Multiple Integrals

15.2 Iterated Integrals

Iterated Integrals

Suppose that *f* is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

We use the notation $\int_c^d f(x, y)\, dy$ to mean that *x* is held fixed and $f(x, y)$ is integrated with respect to *y* from $y = c$ to $y = d$. This procedure is called *partial integration with respect to y*. (Notice its similarity to partial differentiation.)

Now $\int_c^d f(x, y)\, dy$ is a number that depends on the value of *x*, so it defines a function of *x*:

$$A(x) = \int_c^d f(x, y)\, dy$$

Iterated Integrals

If we now integrate the function *A* with respect to *x* from $x = a$ to $x = b$, we get

$$\boxed{1} \qquad \int_a^b A(x)\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\boxed{2} \qquad \int_a^b \int_c^d f(x, y)\,dy\,dx = \int_a^b \left[\int_c^d f(x, y)\,dy \right] dx$$

means that we first integrate with respect to *y* from *c* to *d* and then with respect to *x* from *a* to *b*.

Iterated Integrals

Similarly, the iterated integral

$$\boxed{3} \qquad \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy$$

means that we first integrate with respect to x (holding y fixed) from $x = a$ to $x = b$ and then we integrate the resulting function of y with respect to y from $y = c$ to $y = d$.

Notice that in both Equations 2 and 3 we work *from the inside out*.

Example 1

Evaluate the iterated integrals.

(a) $\int_0^3 \int_1^2 x^2 y \, dy \, dx$ (b) $\int_1^2 \int_0^3 x^2 y \, dx \, dy$

Solution:

(a) Regarding *x* as a constant, we obtain

$$\int_1^2 x^2 y \, dy = \left[x^2 \frac{y^2}{2} \right]_{y=1}^{y=2}$$

$$= x^2 \left(\frac{2^2}{2} \right) - x^2 \left(\frac{1^2}{2} \right)$$

$$= \tfrac{3}{2} x^2$$

Example 1 – *Solution*

cont'd

Thus the function *A* in the preceding discussion is given by $A(x) = \frac{3}{2}x^2$ in this example.

We now integrate this function of *x* from 0 to 3:

$$\int_0^3 \int_1^2 x^2 y \, dy \, dx = \int_0^3 \left[\int_1^2 x^2 y \, dy \right] dx$$

$$= \int_0^3 \tfrac{3}{2} x^2 \, dx$$

$$= \frac{x^3}{2} \Big]_0^3$$

$$= \frac{27}{2}$$

Example 1 – *Solution*

cont'd

(b) Here we first integrate with respect to *x*:

$$\int_1^2 \int_0^3 x^2 y \, dx \, dy = \int_1^2 \left[\int_0^3 x^2 y \, dx \right] dy$$

$$= \int_1^2 \left[\frac{x^3}{3} y \right]_{x=0}^{x=3} dy$$

$$= \int_1^2 9y \, dy$$

$$= 9 \frac{y^2}{2} \bigg]_1^2 = \frac{27}{2}$$

Iterated Integrals

Notice that in Example 1 we obtained the same answer whether we integrated with respect to *y* or *x* first.

In general, it turns out (see Theorem 4) that the two iterated integrals in Equations 2 and 3 are always equal; that is, the order of integration does not matter.

$\boxed{4}$ **Fubini's Theorem** If f is continuous on the rectangle $R = \{(x, y) \mid a \leqslant x \leqslant b,\, c \leqslant y \leqslant d\}$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$$

More generally, this is true if we assume that f is bounded on R, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 2

Evaluate $\iint_R y\sin(xy)dA$, where $R = [1,2] \times [0,\pi]$.

SOLUTION: It is easier to integrate with respect to *x* first

$$\iint_R y\sin(xy)dA = \int_0^\pi \int_1^2 y\sin(xy)dxdy = \int_0^\pi \left[\int_1^2 y\sin(xy)dx\right] dy$$

$$= \int_0^\pi \left(y\frac{-\cos(xy)}{y}\Big]_1^2 \right) dx = -\int_0^\pi (\cos(2x) - \cos(x))dx$$

$$= -\left(\frac{\sin(2x)}{2} - \sin(x) \Big]_0^\pi \right) = 0$$

If we reverse the order of integration, $\int_1^2 \int_0^\pi y\sin(xy)dydx$,

we get hard integrals.

Iterated Integrals

In the special case where $f(x, y)$ can be factored as the product of a function of x only and a function of y only, the double integral of f can be written in a particularly simple form.

To be specific, suppose that $f(x, y) = g(x)h(y)$ and $R = [a, b] \times [c, d]$.

Then Fubini's Theorem gives

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b g(x)h(y)\, dx\, dy = \int_c^d \left[\int_a^b g(x)h(y)\, dx \right] dy$$

Iterated Integrals

In the inner integral, y is a constant, so $h(y)$ is a constant and we can write

$$\int_c^d \left[\int_a^b g(x)h(y)\,dx \right] dy = \int_c^d \left[h(y) \left(\int_a^b g(x)\,dx \right) \right] dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

since $\int_a^b g(x)\,dx$ is a constant.

Therefore, in this case, the double integral of f can be written as the product of two single integrals:

$$\boxed{5} \quad \iint_R g(x)\,h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy \qquad \text{where } R = [a, b] \times [c, d]$$

Volumes and Double Integrals

Volumes and Double Integrals

In a similar manner we consider a function *f* of two variables defined on a closed rectangle

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ c \leq y \leq d\}$$

and we first suppose that $f(x, y) \geq 0$.

The graph of *f* is a surface with equation $z = f(x, y)$.

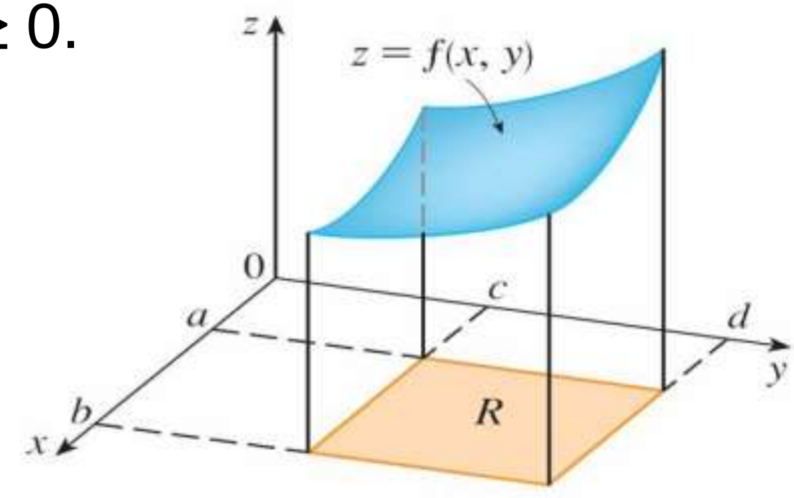


Figure 2

Let *S* be the solid that lies above *R* and under the graph of *f*, that is,

$$S = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq f(x, y),\ (x, y) \in R\}$$

(See Figure 2.)

Volumes and Double Integrals

Our goal is to find the volume of *S*.

volume can be written as a double integral

If $f(x, y) \geqslant 0$, then the volume V of the solid that lies above the rectangle R and below the surface $z = f(x, y)$ is

$$V = \iint_R f(x, y)\, dA$$

Example 1

Find the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$.

SOLUTION:

Volume = $\iint_R (16 - x^2 - 2y^2)dA$

$$= \int_0^2 \int_0^2 (16 - x^2 - 2y^2)dydx = \int_0^2 \left[\int_0^2 (16 - x^2 - 2y^2)dy\right] dx$$

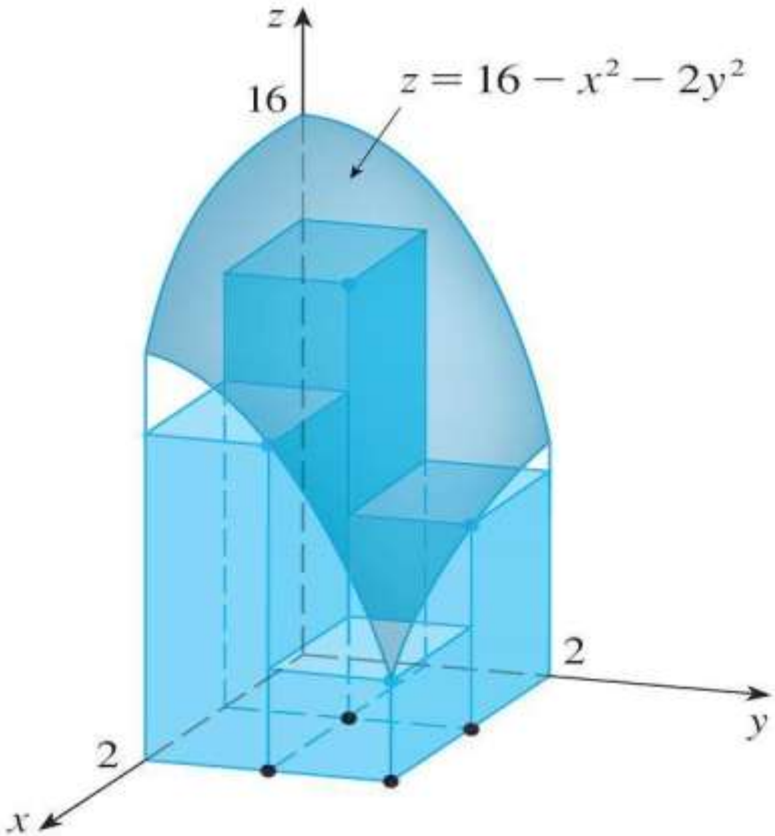
$$= \int_0^2 [16y - x^2y - \frac{2}{3}y^3]_0^2 dx = \int_0^2 \left(\frac{80}{3} - 2x^2\right) dx = \frac{80}{3}x - \frac{2}{3}x^3]_0^2$$

$$= 48.$$

Example 1 – *Solution*

cont'd

The solid in Example 1 is shown in the Figure below.



Example 2

Find the volume of the solid that lies above the rectangular region $R = [0, 2] \times [0, 1]$ and below the plane $z = 4 - x - y$.

SOLUTION:

Volume = $\iint_R (4 - x - y)\,dA$

$$= \int_0^2 \int_0^1 (4 - x - y)\,dy\,dx = \int_0^2 \left[\int_0^1 (4 - x - y)\,dy\right] dx$$

$$= \int_0^2 [4y - xy - \frac{1}{2}y^2]_0^1\,dx$$

$$= \int_0^2 \left(\frac{7}{2} - x\right) dx = \frac{7}{2}x - \frac{1}{2}x^2]_0^2 = 5.$$

Example 2 – *Solution*

cont'd

The solid in Example 2 is shown in the Figure below.

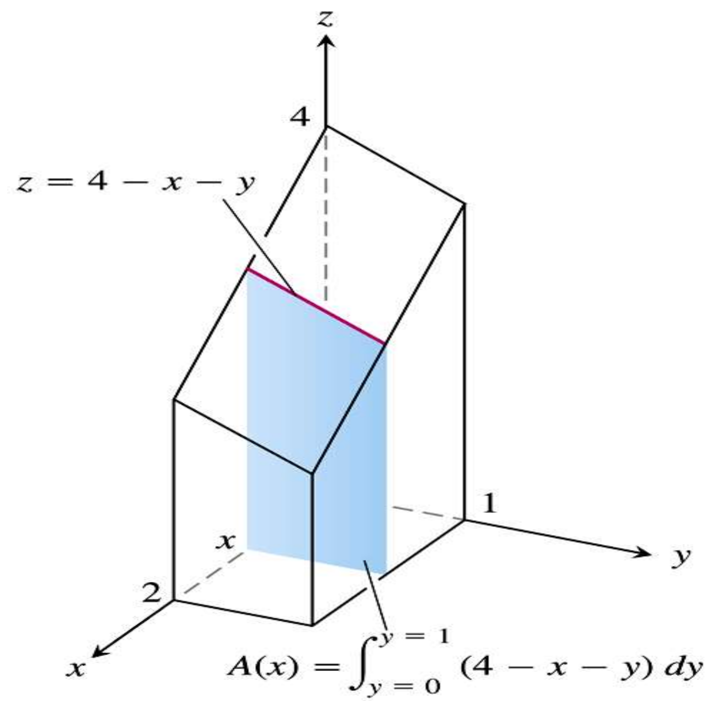


FIGURE 15.4 To obtain the cross-sectional area $A(x)$, we hold x fixed and integrate with respect to y.

Example 3

Find the volume of the solid in the first octant bounded by the cylinder $z = 16 - x^2$ and the plane $y = 5$.

SOLUTION: The cylinder intersects the xy-plane when $0 = 16 - x^2$ or $x = \pm 4$. Since the solid lies in the first octant we have $0 \le x \le 4$ and $0 \le y \le 5$.

Volume = $\iint_R (16 - x^2)dA = \int_0^4 \int_0^5 (16 - x^2)dydx$

$$= \int_0^4 \left[\int_0^5 (16 - x^2)dy\right] dx = \int_0^4 [(16 - x^2)y]_0^5 dx$$

$$= \int_0^4 5(16 - x^2)dx = 5(16x - \frac{1}{3}x^3)]_0^4 = \frac{640}{3}.$$

Properties of Double Integrals

Properties of Double Integrals

We list here three properties of double integrals. We assume that all of the integrals exist. Properties 7 and 8 are referred to as the *linearity* of the integral.

$$\boxed{7} \qquad \iint_R [f(x, y) + g(x, y)]\, dA = \iint_R f(x, y)\, dA + \iint_R g(x, y)\, dA$$

$$\boxed{8} \qquad \iint_R c\, f(x, y)\, dA = c \iint_R f(x, y)\, dA \quad \text{where } c \text{ is a constant}$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in R, then

$$\boxed{9} \qquad \iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA$$

15 Multiple Integrals

15.3 Double Integrals over General Regions

Double Integrals over General Regions

For single integrals, the region over which we integrate is always an interval.

But for double integrals, we want to be able to integrate a function *f* not just over rectangles but also over regions *D* of more general shape, such as the one illustrated in Figure 1.

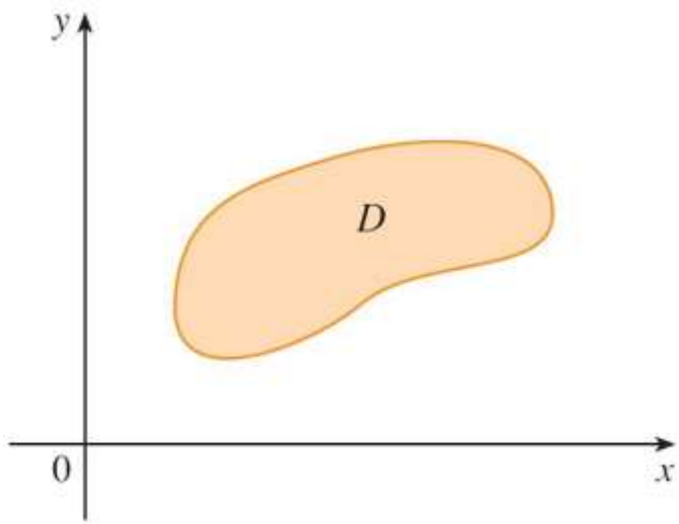


Figure 1

Double Integrals over General Regions

We suppose that D is a bounded region, which means that D can be enclosed in a rectangular region R as in Figure 2.

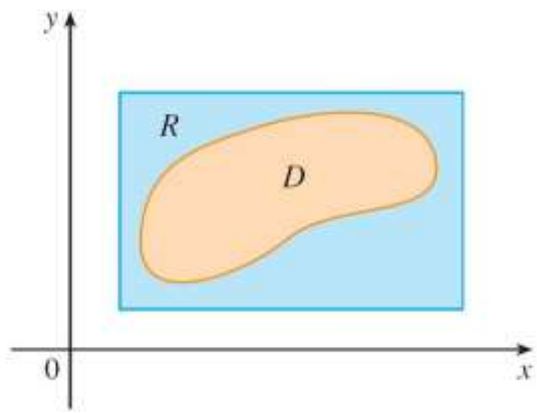


Figure 2

Then we define a new function F with domain R by

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

Double Integrals over General Regions

If *F* is integrable over *R*, then we define the **double integral of *f* over *D*** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

Definition 2 makes sense because *R* is a rectangle and so $\iint_R F(x, y)\, dA$ has been previously defined.

Double Integrals over General Regions

The procedure that we have used is reasonable because the values of $F(x, y)$ are 0 when (x, y) lies outside D and so they contribute nothing to the integral.

This means that it doesn't matter what rectangle R we use as long as it contains D.

In the case where $f(x, y) \geq 0$, we can still interpret $\iint_D f(x, y)\, dA$ as the volume of the solid that lies above D and under the surface $z = f(x, y)$ (the graph of f).

Double Integrals over General Regions

You can see that this is reasonable by comparing the graphs of *f* and *F* in Figures 3 and 4 and remembering that $\iint_R F(x, y)\, dA$ is the volume under the graph of *F*.

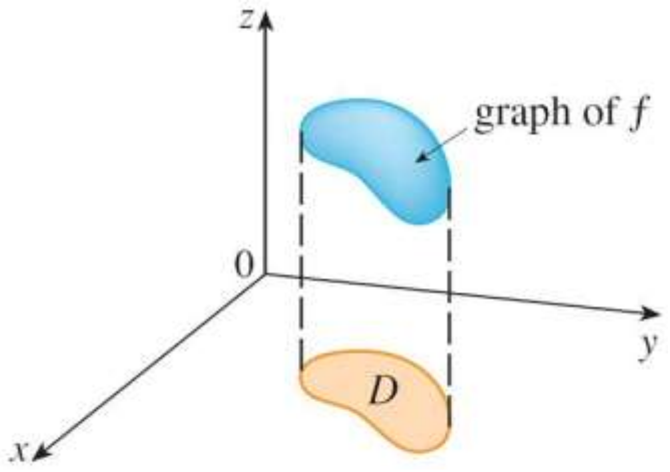


Figure 3

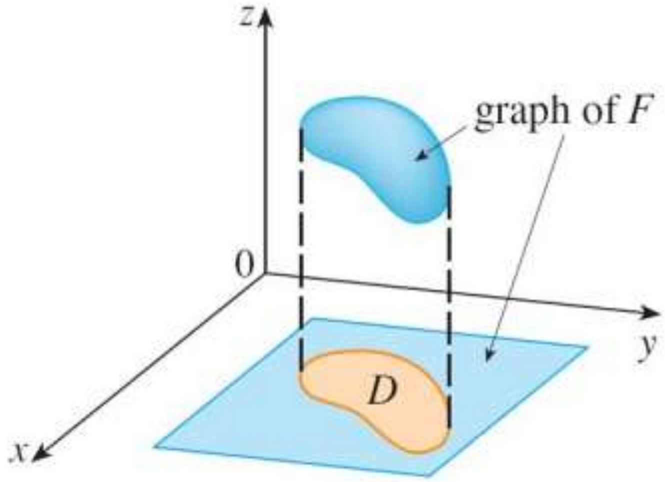


Figure 4

Double Integrals over General Regions

Figure 4 also shows that F is likely to have discontinuities at the boundary points of D.

Nonetheless, if f is continuous on D and the boundary curve of D is “well behaved”, then it can be shown that $\iint_R F(x, y)\, dA$ exists and therefore $\iint_D f(x, y)\, dA$ exists.

In particular, this is the case for **type I** and **type II** regions.

Double Integrals over General Regions

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

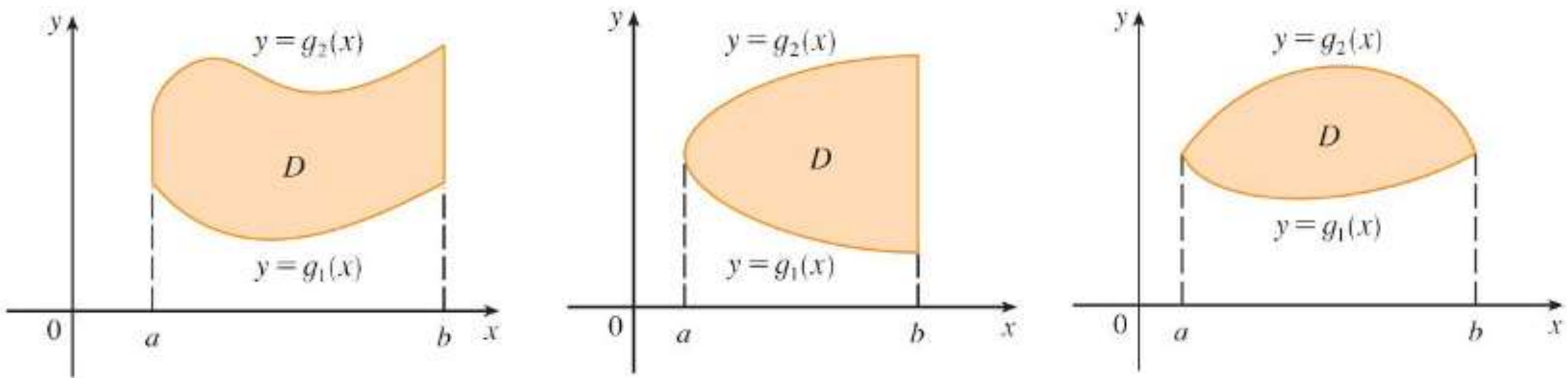


Figure 5

Some type I regions

Double Integrals over General Regions

In order to evaluate $\iint_D f(x, y)\, dA$ when D is a region of type I, we choose a rectangle $R = [a, b] \times [c, d]$ that contains D, as in Figure 6, and we let F be the function given by Equation 1; that is, F agrees with f on D and F is 0 outside D.

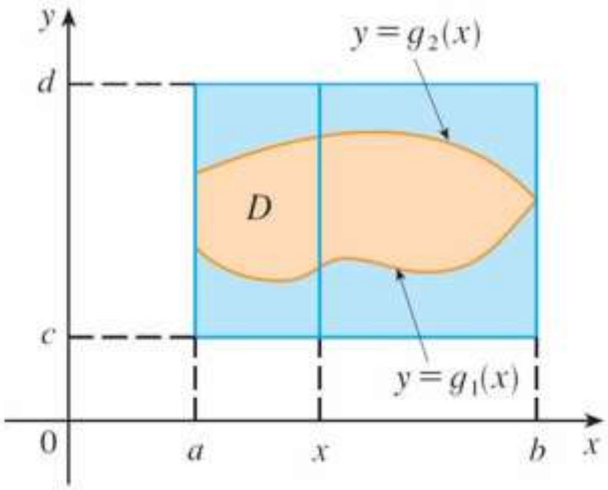


Figure 6

Double Integrals over General Regions

Then, by Fubini's Theorem,

$$\iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA = \int_a^b \int_c^d F(x, y)\, dy\, dx$$

Observe that $F(x, y) = 0$ if $y < g_1(x)$ or $y > g_2(x)$ because (x, y) then lies outside D. Therefore

$$\int_c^d F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} F(x, y)\, dy = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy$$

because $F(x, y) = f(x, y)$ when $g_1(x) \le y \le g_2(x)$.

Double Integrals over General Regions

Thus we have the following formula that enables us to evaluate the double integral as an iterated integral.

$\boxed{3}$ If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

The integral on the right side of $\boxed{3}$ is an iterated integral, except that in the inner integral we regard x as being constant not only in $f(x, y)$ but also in the limits of integration, $g_1(x)$ and $g_2(x)$.

Double Integrals over General Regions

We also consider plane regions of **type II**, which can be expressed as

$$\boxed{4} \qquad D = \{(x, y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

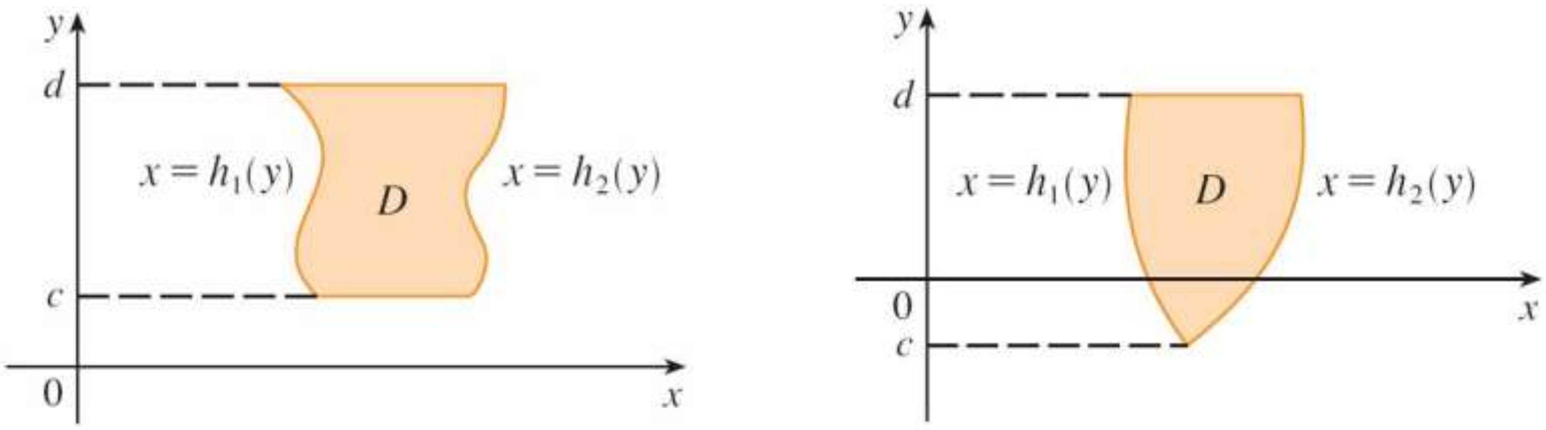


Figure 7

Some type II regions

Double Integrals over General Regions

Using the same methods that were used in establishing $\boxed{3}$, we can show that

$$\boxed{5} \qquad \iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

EXAMPLE 1 Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

Solution See Figure 15.12. For any x between 0 and 1, y may vary from $y = 0$ to $y = x$ (Figure 15.12b). Hence,

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$

When the order of integration is reversed (Figure 15.12c), the integral for the volume is

$$V = \int_0^1 \int_y^1 (3 - x - y)\, dx\, dy = \int_0^1 \left[3x - \frac{x^2}{2} - xy \right]_{x=y}^{x=1} dy$$

$$= \int_0^1 \left(3 - \frac{1}{2} - y - 3y + \frac{y^2}{2} + y^2 \right) dy$$

$$= \int_0^1 \left(\frac{5}{2} - 4y + \frac{3}{2} y^2 \right) dy = \left[\frac{5}{2} y - 2y^2 + \frac{y^3}{2} \right]_{y=0}^{y=1} = 1.$$

The two integrals are equal, as they should be.

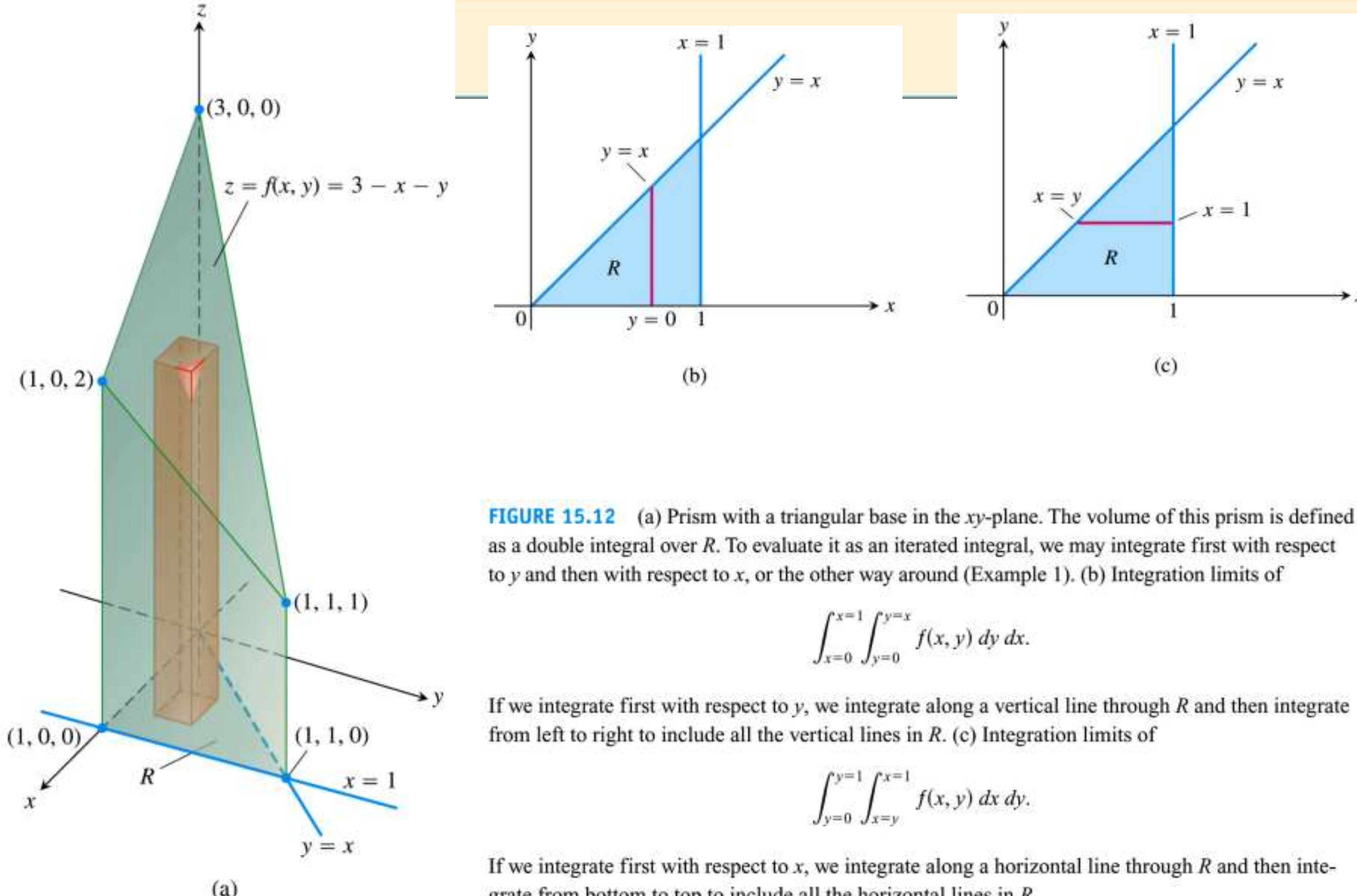


FIGURE 15.12 (a) Prism with a triangular base in the xy-plane. The volume of this prism is defined as a double integral over R. To evaluate it as an iterated integral, we may integrate first with respect to y and then with respect to x, or the other way around (Example 1). (b) Integration limits of

$$\int_{x=0}^{x=1}\int_{y=0}^{y=x} f(x, y)\, dy\, dx.$$

If we integrate first with respect to y, we integrate along a vertical line through R and then integrate from left to right to include all the vertical lines in R. (c) Integration limits of

$$\int_{y=0}^{y=1}\int_{x=y}^{x=1} f(x, y)\, dx\, dy.$$

If we integrate first with respect to x, we integrate along a horizontal line through R and then integrate from bottom to top to include all the horizontal lines in R.

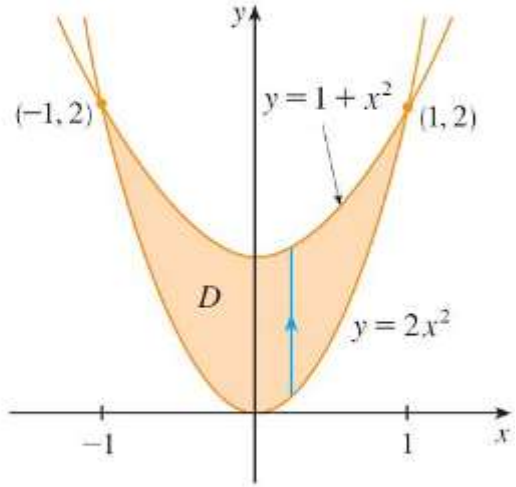
Example 2

Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution:

The parabolas intersect when $2x^2 = 1 + x^2$, that is, $x^2 = 1$, so $x = \pm 1$.

We note that the region D, sketched in the Figure, is a type I region but not a type II region and we can write



$D = \{(x, y) \mid -1 \le x \le 1,\ 2x^2 \le y \le 1 + x^2\}$

Example 2 – *Solution*

cont'd

Since the lower boundary is $y = 2x^2$ and the upper boundary is $y = 1 + x^2$, Equation 3 gives

$$\iint_D (x + 2y)\, dA = \int_{-1}^{1} \int_{2x^2}^{1+x^2} (x + 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy + y^2\right]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} [x(1 + x^2) + (1 + x^2)^2 - x(2x^2) - (2x^2)^2]\, dx$$

Example 1 – *Solution*

cont'd

$$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= -3\,\frac{x^5}{5} - \frac{x^4}{4} + 2\,\frac{x^3}{3} + \frac{x^2}{2} + x\,\bigg]_{-1}^{1}$$

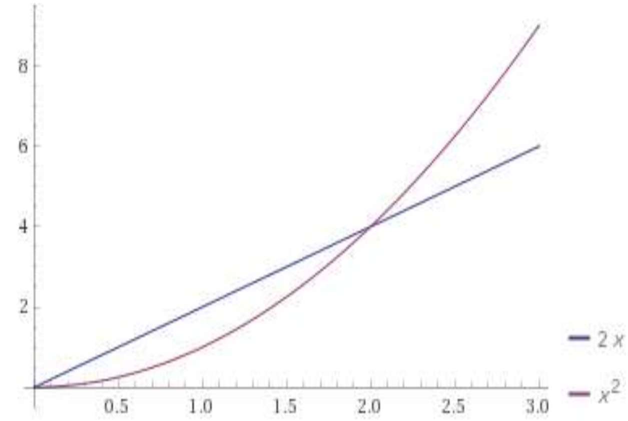
$$= \frac{32}{15}$$

Example 3

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region in the *xy*-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

From the Figure below we see that D is a type I region and

$$D = \{(x, y)\ |0 \le x \le 2, x^2 \le y \le 2x\}$$



Example 3 - Solution

Thus,

Volume = $\iint_D (x^2 + y^2)\, dA = \int_0^2 \int_{x^2}^{2x} (x^2 + y^2) dy dx$

$$= \int_0^2 \left[\int_{x^2}^{2x} (x^2 + y^2) dy\right] dx = \int_0^2 x^2 y + \frac{1}{3} y^3]_{x^2}^{2x} dx$$

$$= \int_0^2 \left(-\frac{x^6}{3} - x^4 + \frac{14}{3} x^3\right) dx$$

$$= -\frac{x^7}{21} - \frac{x^5}{5} + \frac{14}{12} x^4]_0^2 = \frac{216}{35}$$

Example 4

Evaluate the iterated integral $\int_0^1 \int_x^1 sin(y^2)dydx$.

If we try to evaluate the integral as it stands, we are faced with the task of first evaluating $\int sin(y^2)dy$. But it's impossible to do so in finite terms since is not an elementary function. So we must change the order of integration. This is accomplished by first expressing the given iterated integral as a double integral. We have

$$\int_0^1 \int_x^1 sin(y^2)dydx = \iint_D sin(y^2)\, dA$$

where $D = \{(x, y)\ |0 \le x \le 1, x \le y \le 1\}$.

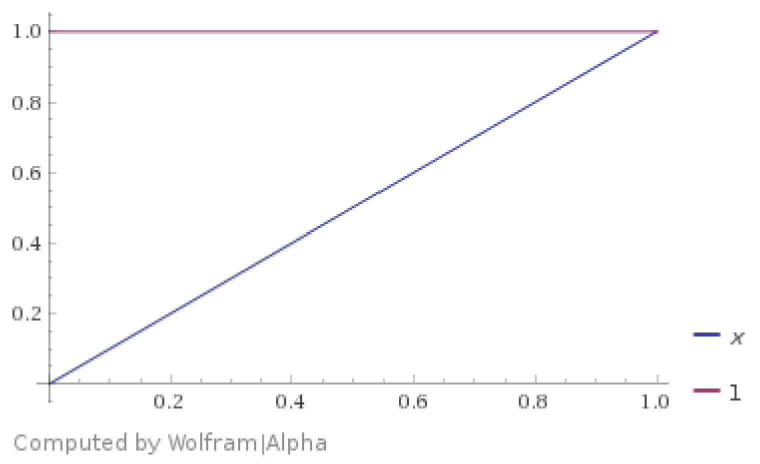
Example 4

We sketch this region. Then from the Figure below we see that an alternative description of D as type II region is

$$D = \{(x, y)\ |0 \le y \le 1, 0 \le x \le y\}.$$

$$\iint_D \ sin(y^2)\, dA = \int_0^1 \int_0^y sin(y^2) dxdy = \int_0^1 xsin(y^2)]_0^y dy$$

$$= \int_0^1 ysin(y^2) dy = -\frac{1}{2} cos(y^2)]_0^1 = \frac{1}{2}(1 - cos1)$$



Properties of Double Integrals

Properties of Double Integrals

We assume that all of the following integrals exist. The first three properties of double integrals over a region *D* follow immediately from Definition 2.

6 $$\iint_D [f(x, y) + g(x, y)]\, dA = \iint_D f(x, y)\, dA + \iint_D g(x, y)\, dA$$

7 $$\iint_D cf(x, y)\, dA = c \iint_D f(x, y)\, dA$$

If $f(x, y) \geq g(x, y)$ for all (x, y) in D, then

8 $$\iint_D f(x, y)\, dA \geq \iint_D g(x, y)\, dA$$

Properties of Double Integrals

The next property of double integrals is similar to the property of single integrals given by the equation

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

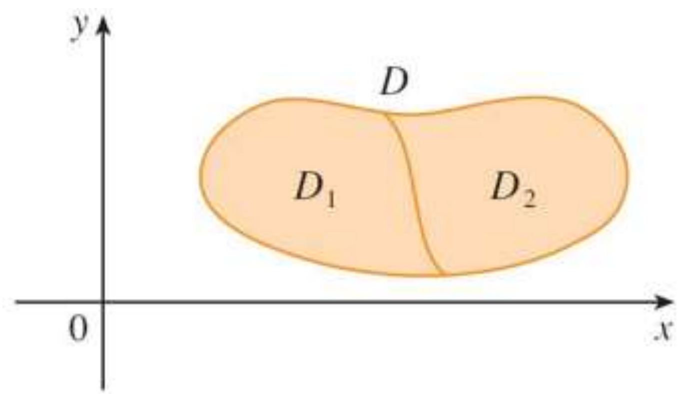


Figure 17

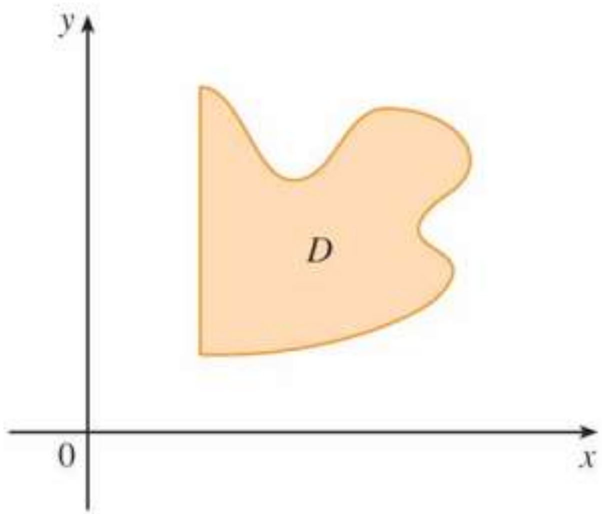
If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

9

$$\boxed{\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA}$$

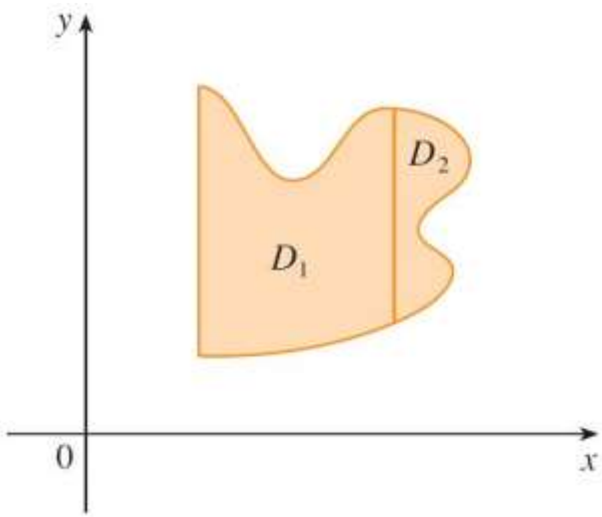
Properties of Double Integrals

Property 9 can be used to evaluate double integrals over regions *D* that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure.



D is neither type I nor type II.

Figure 18(a)



$D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

Figure 18(b)

Properties of Double Integrals

The next property of integrals says that if we integrate the constant function *f*(*x*, *y*) = 1 over a region *D*, we get the area of *D*:

10

$$\iint_D 1 \, dA = A(D)$$

Properties of Double Integrals

Figure 19 illustrates why Equation 10 is true: A solid cylinder whose base is D and whose height is 1 has volume $A(D) \cdot 1 = A(D)$, but we know that we can also write its volume as $\iint_D 1 \, dA$.

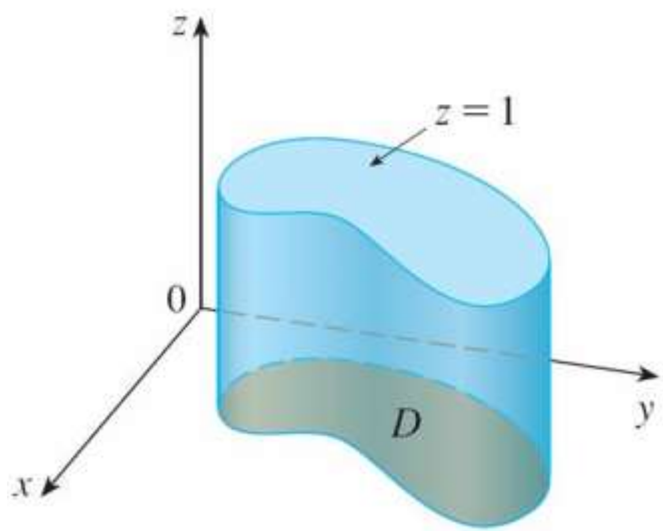


Figure 19

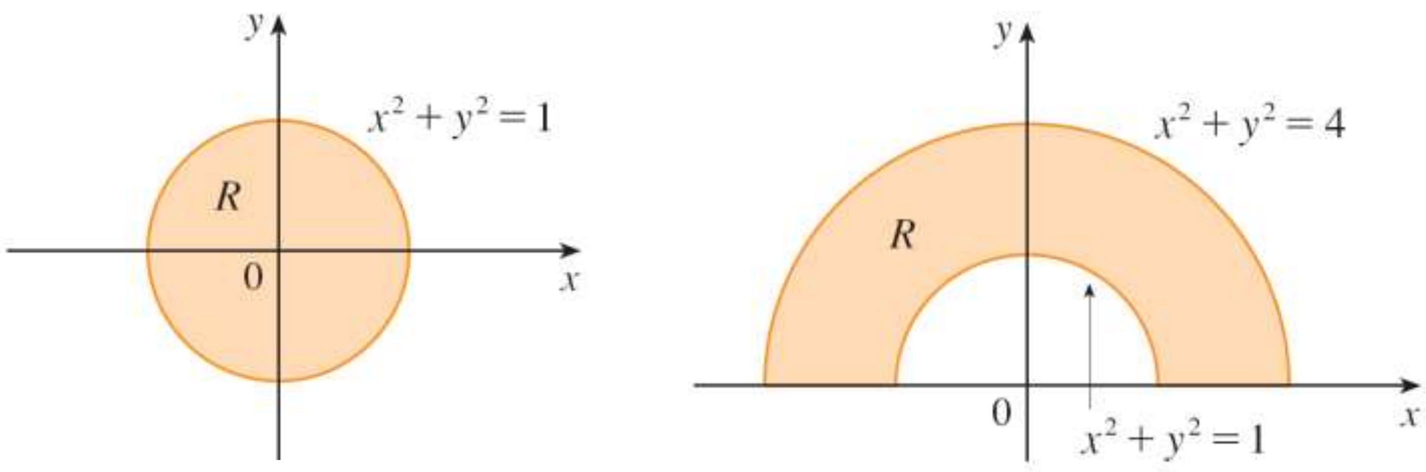
Cylinder with base D and height 1

15 Multiple Integrals

15.4 Double Integrals in Polar Coordinates

Double Integrals in Polar Coordinates

Suppose that we want to evaluate a double integral $\iint_R f(x, y)\, dA$, where R is one of the regions shown in Figure 1. In either case the description of R in terms of rectangular coordinates is rather complicated, but R is easily described using polar coordinates.



(a) $R = \{(r, \theta) \mid 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \leqslant r \leqslant 2, 0 \leqslant \theta \leqslant \pi\}$

Figure 1

Double Integrals in Polar Coordinates

Recall from Figure 2 that the polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

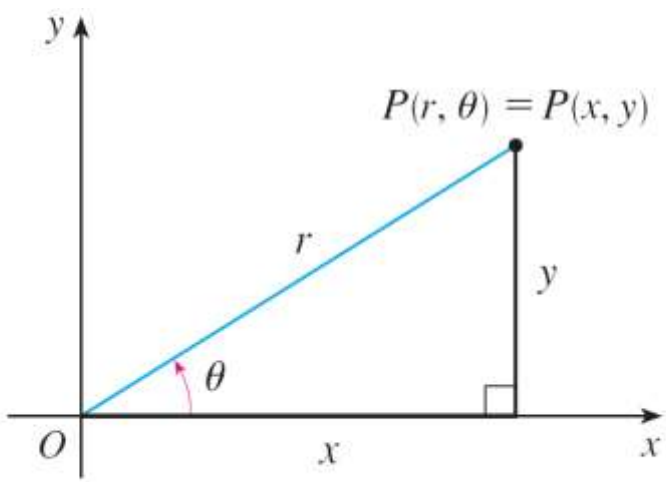


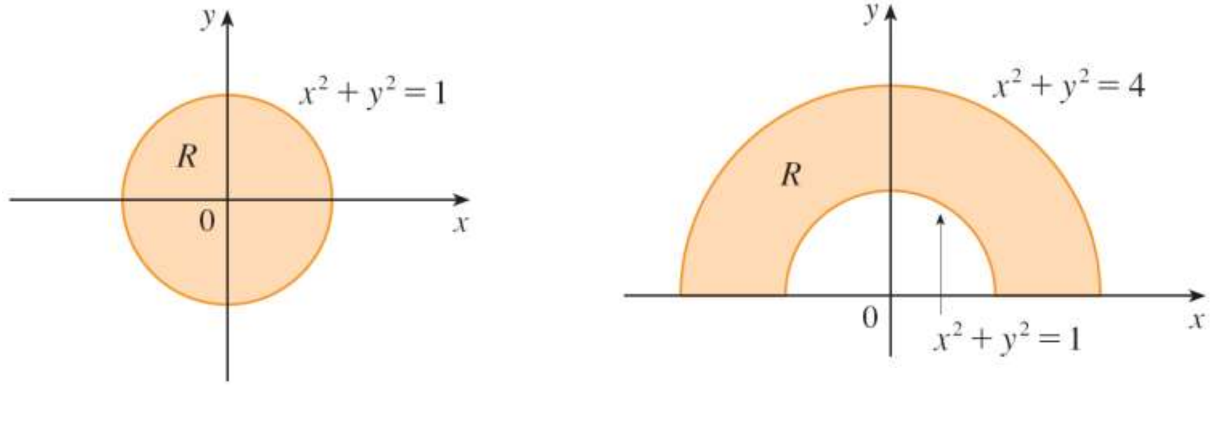
Figure 2

Double Integrals in Polar Coordinates

The regions in Figure 1 are special cases of a **polar rectangle**

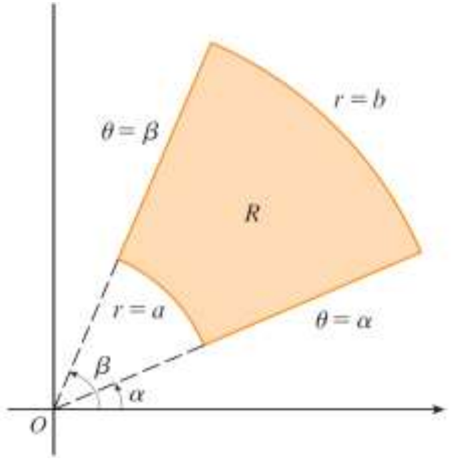
$$R = \{(r, \theta) \mid a \le r \le b, \alpha \le \theta \le \beta\}$$

which is shown in Figure 3.



(a) $R = \{(r, \theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi\}$ (b) $R = \{(r, \theta) \mid 1 \le r \le 2, 0 \le \theta \le \pi\}$

Figure 1



Polar rectangle

Figure 3

Double Integrals in Polar Coordinates

In order to compute the double integral $\iint_R f(x, y)\, dA$, where R is a polar rectangle, we use the following theorem:

2 Change to Polar Coordinates in a Double Integral If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Double Integrals in Polar Coordinates

The formula in $\boxed{2}$ says that we convert from rectangular to polar coordinates in a double integral by writing $x = r\cos\theta$ and $y = r\sin\theta$, using the appropriate limits of integration for r and θ, and replacing dA by $r\,dr\,d\theta$.

Be careful not to forget the additional factor r on the right side of Formula 2.

A classical method for remembering this is shown in Figure 5, where the "infinitesimal" polar rectangle can be thought of as an ordinary rectangle with dimensions $r\,d\theta$ and dr and therefore has "area" $dA = r\,dr\,d\theta$.

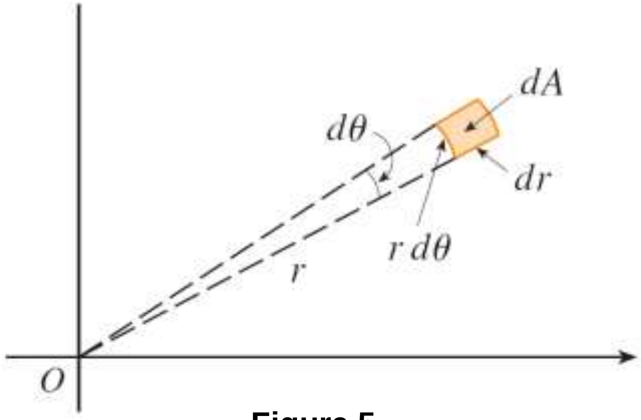


Figure 5

Example 1

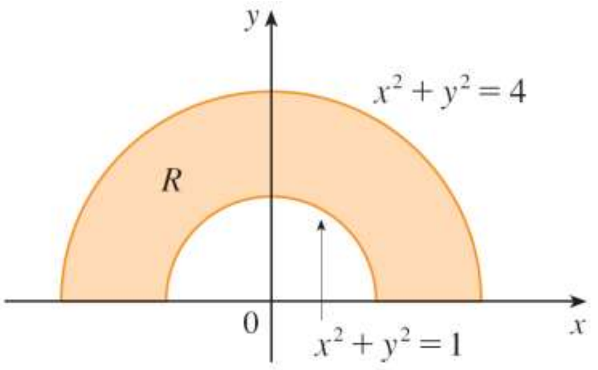
Evaluate $\iint_R (3x + 4y^2)\, dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

The region R can be described as

$$R = \{(x, y) \mid y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}$$

It is the half-ring shown in Figure 1(b), and in polar coordinates it is given by $1 \leq r \leq 2$, $0 \leq \theta \leq \pi$.



$$R = \{(r, \theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq \pi\}$$

Figure 1(b)

Example 1 – *Solution*

cont'd

Therefore, by Formula 2,

$$
\begin{aligned}
\iint_R (3x + 4y^2)\, dA &= \int_0^{\pi} \int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\, r\, dr\, d\theta \\
&= \int_0^{\pi} \int_1^2 (3r^2\cos\theta + 4r^3\sin^2\theta)\, dr\, d\theta \\
&= \int_0^{\pi} \Big[r^3\cos\theta + r^4\sin^2\theta\Big]_{r=1}^{r=2} d\theta \\
&= \int_0^{\pi} (7\cos\theta + 15\sin^2\theta)\, d\theta \\
&= \int_0^{\pi} \Big[7\cos\theta + \tfrac{15}{2}(1 - \cos 2\theta)\Big]\, d\theta \\
&= 7\sin\theta + \frac{15\theta}{2} - \frac{15}{4}\sin 2\theta\,\Big]_0^{\pi} = \frac{15\pi}{2}
\end{aligned}
$$

Example 2

Find the volume of the solid bounded by the plane z = 0 and the paraboloid z = 1 - x^2 - y^2.

SOLUTION: If we put z = 0 in the equation of the paraboloid, we get $x^2 + y^2 = 1$. This means that the plane intersects the paraboloid in the circle $x^2 + y^2 = 1$, so the solid lies under the paraboloid and above the circular disk *D* given by $x^2 + y^2 \leq 1$. In polar coordinates *D* is given by $0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$. Since z = 1 - x^2 - y^2 = 1 - r^2 the volume is

$$\text{Volume} = \iint_D (1 - x^2 - y^2)\,dA = \int_0^{2\pi}\int_0^1 (1 - r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{4}\Big]_0^1 d\theta = \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{\pi}{2}$$

Double Integrals in Polar Coordinates

What we have done so far can be extended to the more complicated type of region shown in Figure 7. In fact, by combining Formula 2 with

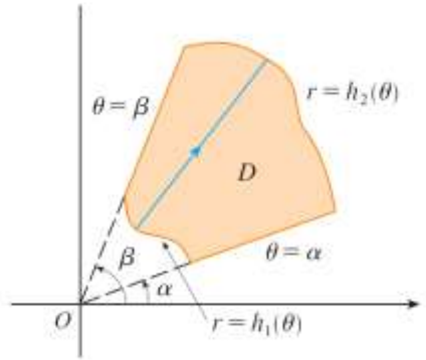


Figure 7

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

where D is a type II region, we obtain the following formula.

$\boxed{3}$ If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then
$$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Double Integrals in Polar Coordinates

In particular, taking $f(x, y) = 1$, $h_1(\theta) = 0$, and $h_2(\theta) = h(\theta)$ in this formula, we see that the area of the region D bounded by $\theta = \alpha$, $\theta = \beta$, and $r = h(\theta)$ is

$$A(D) = \iint_D 1\, dA$$

$$= \int_\alpha^\beta \int_0^{h(\theta)} r\, dr\, d\theta$$

$$= \int_\alpha^\beta \left[\frac{r^2}{2}\right]_0^{h(\theta)} d\theta$$

$$= \int_\alpha^\beta \tfrac{1}{2}[h(\theta)]^2\, d\theta$$

Example 3

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

SOLUTION: The solid lies above the disk *D* whose boundary circle has equation $x^2 + y^2 = 2x$ or, after completing the square $(x-1)^2 + y^2 = 1$, In polar coordinates the boundary circle becomes $r^2 = 2r\cos\theta$, or $r = 2\cos\theta$. Thus the disk *D* is given by

$$D = \left\{(r,\theta) \mid -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le r \le 2\cos\theta\right\}$$

Volume = $\iint_D (x^2 + y^2)\,dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (r^2)\, r\,dr\,d\theta$

$$= \int_{-\pi/2}^{\pi/2} \frac{r^4}{4}\Big]_0^{2\cos\theta} d\theta = 4\int_{-\pi/2}^{\pi/2} (\cos\theta)^4\, d\theta = \frac{3\pi}{2}$$

Example 4

Change the Cartesian integral into an equivalent polar integral.

$$(a)\ \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(x^2+y^2)\,dy\,dx=\int_0^{2\pi}\int_0^{1}(r^2)\,r\,dr\,d\theta$$

$$(b)\ \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}(xy)\,dy\,dx=\int_0^{\pi}\int_0^{2}(r^2\cos\theta\sin\theta)\,r\,dr\,d\theta$$

$$(c)\ \int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\left(\frac{2}{1+\sqrt{x^2+y^2}}\right)dy\,dx=\int_0^{\pi/2}\int_0^{1}\left(\frac{2}{1+r}\right)r\,dr\,d\theta$$

$$(d)\ \int_{0}^{3}\int_{0}^{\sqrt{9-y^2}}e^{-(x^2+y^2)}\,dx\,dy=\int_0^{\pi/2}\int_0^{3}\left(e^{-r^2}\right)r\,dr\,d\theta$$

$$(e)\ \int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}(x+2)\,dx\,dy=\int_{\pi}^{2\pi}\int_0^{1}(r\cos\theta+2)\,r\,dr\,d\theta$$

Example 5

Consider the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy$.

In polar coordinates the boundaries become

$$D = \{(r,\theta) | 0 \le \theta \le \pi/2, 0 \le r \le \infty\}$$

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r dr d\theta$$

Note that, $\int_0^\infty e^{-r^2} r dr = \lim_{a\to\infty} \int_0^a e^{-r^2} r dr$

$$= \lim_{a\to\infty} -\frac{1}{2}\left(e^{-a^2} - 1\right) = \frac{1}{2}$$

So, $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^{\pi/2} \frac{1}{2} d\theta = \frac{\pi}{4}$.

Example 5

cont'd

Let $I = \int_0^{\infty} e^{-x^2} dx$, then

$$I^2 = \left(\int_0^{\infty} e^{-x^2} dx\right)\left(\int_0^{\infty} e^{-y^2} dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)} dx dy = \frac{\pi}{4}$$

Thus, $I = \int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{4}} = \frac{\sqrt{\pi}}{2}$.

15

Multiple Integrals

15.7 Triple Integrals

Triple Integrals

We have defined single integrals for functions of one variable and double integrals for functions of two variables, so we can define triple integrals for functions of three variables.

Let's first deal with the simplest case where f is defined on a rectangular box:

$$\boxed{1} \qquad B = \{(x, y, z) \mid a \leq x \leq b,\ c \leq y \leq d,\ r \leq z \leq s\}$$

The first step is to divide B into sub-boxes. We do this by dividing the interval $[a, b]$ into l subintervals $[x_{i-1}, x_i]$ of equal width Δx, dividing $[c, d]$ into m subintervals of width Δy, and dividing $[r, s]$ into n subintervals of width Δz.

Triple Integrals

The planes through the endpoints of these subintervals parallel to the coordinate planes divide the box B into lmn sub-boxes

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$$

which are shown in Figure 1.

Each sub-box has volume

$\Delta V = \Delta x \, \Delta y \, \Delta z$.

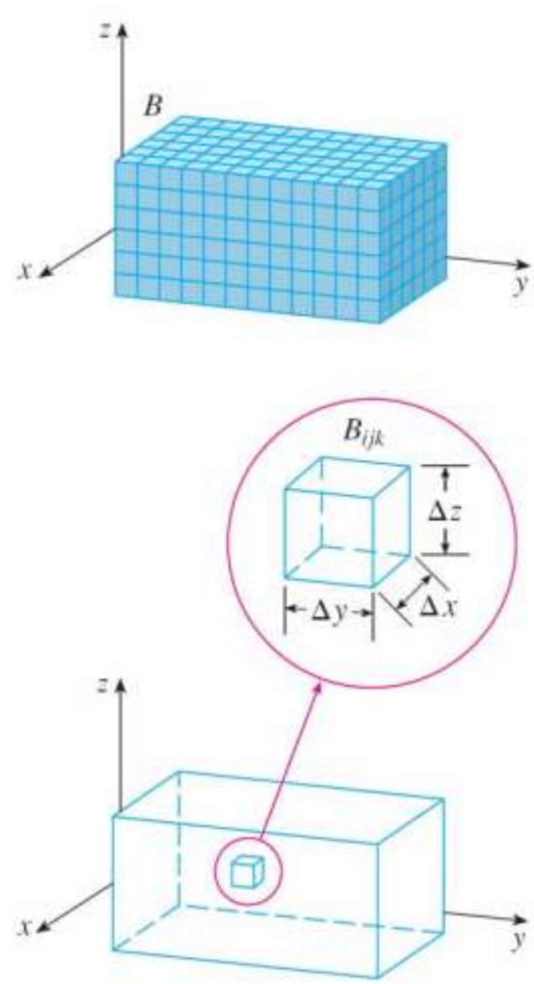


Figure 1

Triple Integrals

Then we form the **triple Riemann sum**

$$\boxed{2} \qquad \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

where the sample point $(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)$ is in B_{ijk}.

By analogy with the definition of a double integral, we define the triple integral as the limit of the triple Riemann sums in $\boxed{2}$.

Triple Integrals

3 Definition The **triple integral** of f over the box B is

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\, \Delta V$$

if this limit exists.

Again, the triple integral always exists if f is continuous. We can choose the sample point to be any point in the sub-box, but if we choose it to be the point (x_i, y_j, z_k) we get a simpler-looking expression for the triple integral:

$$\iiint_B f(x, y, z)\, dV = \lim_{l, m, n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\, \Delta V$$

Triple Integrals

Just as for double integrals, the practical method for evaluating triple integrals is to express them as iterated integrals as follows.

4 Fubini's Theorem for Triple Integrals If f is continuous on the rectangular box $B = [a, b] \times [c, d] \times [r, s]$, then

$$\iiint_B f(x, y, z)\, dV = \int_r^s \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

The iterated integral on the right side of Fubini's Theorem means that we integrate first with respect to x (keeping y and z fixed), then we integrate with respect to y (keeping z fixed), and finally we integrate with respect to z.

Triple Integrals

There are five other possible orders in which we can integrate, all of which give the same value.

For instance, if we integrate with respect to *y*, then *z*, and then *x*, we have

$$\iiint\limits_B f(x, y, z)\, dV = \int_a^b \int_r^s \int_c^d f(x, y, z)\, dy\, dz\, dx$$

Example

Evaluate the triple integral $\iiint_B xyz^2 \, dV$, where B is the rectangular box given by

$$B = \{(x, y, z) \mid 0 \le x \le 1, -1 \le y \le 2, 0 \le z \le 3\}$$

Solution:

We could use any of the six possible orders of integration.

If we choose to integrate with respect to x, then y, and then z, we obtain

$$\iiint_B xyz^2 \, dV = \int_0^3 \int_{-1}^2 \int_0^1 xyz^2 \, dx \, dy \, dz$$

Example – *Solution*

cont'd

$$= \int_0^3 \int_{-1}^2 \left[\frac{x^2yz^2}{2} \right]_{x=0}^{x=1} dy\, dz$$

$$= \int_0^3 \int_{-1}^2 \frac{yz^2}{2}\, dy\, dz$$

$$= \int_0^3 \left[\frac{y^2z^2}{4} \right]_{y=-1}^{y=2} dz$$

$$= \int_0^3 \frac{3z^2}{4}\, dz$$

$$= \frac{z^3}{4} \Big] = \frac{27}{4}$$

Triple Integrals

Now we define the **triple integral over a general bounded region *E*** in three-dimensional space (a solid) by much the same procedure that we used for double integrals.

We enclose E in a box B of the type given by Equation 1. Then we define F so that it agrees with f on E but is 0 for points in B that are outside E.

By definition,

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV$$

This integral exists if f is continuous and the boundary of E is "reasonably smooth."

Triple Integrals

The triple integral has essentially the same properties as the double integral.

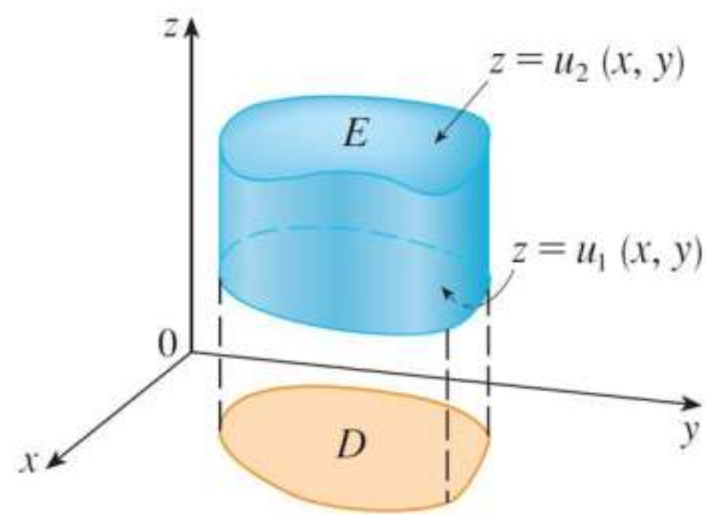
We restrict our attention to continuous functions *f* and to certain simple types of regions.

Triple Integrals

A solid region E is said to be of **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$\boxed{5} \qquad E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is the projection of E onto the xy-plane as shown in Figure 2.



A type 1 solid region

Figure 2

Triple Integrals

Notice that the upper boundary of the solid E is the surface with equation $z = u_2(x, y)$, while the lower boundary is the surface $z = u_1(x, y)$.

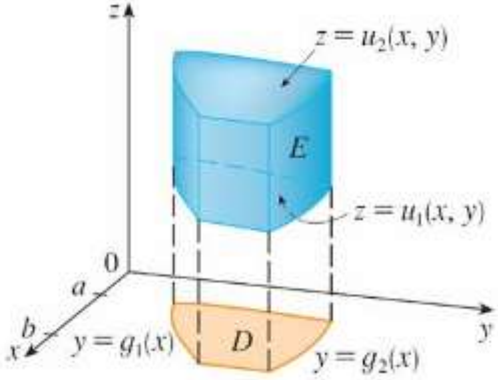
By the same sort of argument, it can be shown that if E is a type 1 region given by Equation 5, then

$$\boxed{6} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

The meaning of the inner integral on the right side of Equation 6 is that x and y are held fixed, and therefore $u_1(x, y)$ and $u_2(x, y)$ are regarded as constants, while $f(x, y, z)$ is integrated with respect to z.

Triple Integrals

In particular, if the projection *D* of *E* onto the *xy*-plane is a type I plane region (as in Figure 3),



A type 1 solid region where the projection *D* is a type I plane region

Figure 3

Triple Integrals

In particular, if the projection D of E onto the xy-plane is a type I plane region (as in Figure 3), then

$E = \{(x, y, z) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x),\ u_1(x, y) \le z \le u_2(x, y)\}$

and Equation 6 becomes

$$\boxed{7} \qquad \iiint_E f(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dy\, dx$$

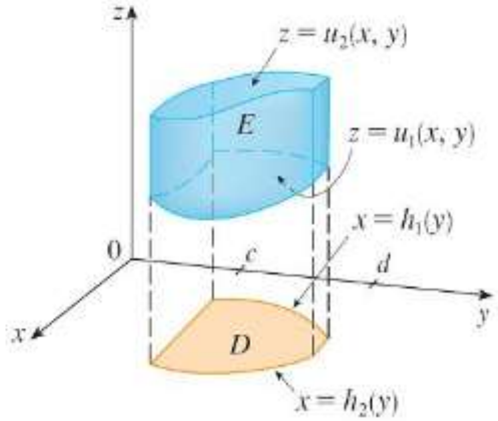
Triple Integrals

If, on the other hand, D is a type II plane region (as in Figure 4), then

$$E = \{(x, y, z) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y),\ u_1(x, y) \le z \le u_2(x, y)\}$$

and Equation 6 becomes

8 $$\iiint_E f(x, y, z)\, dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz\, dx\, dy$$



A type 1 solid region with a type II projection

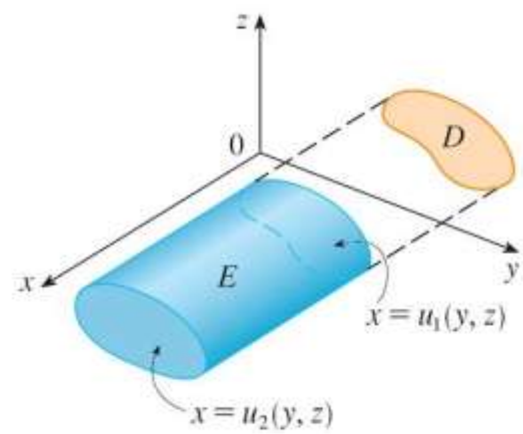
Figure 4

Triple Integrals

A solid region E is of **type 2** if it is of the form

$$E = \{(x, y, z) \mid (y, z) \in D, u_1(y, z) \leq x \leq u_2(y, z)\}$$

where, this time, D is the projection of E onto the yz-plane (see Figure 7).



A type 2 region

Figure 7

The back surface is $x = u_1(y, z)$, the front surface is $x = u_2(y, z)$, and we have

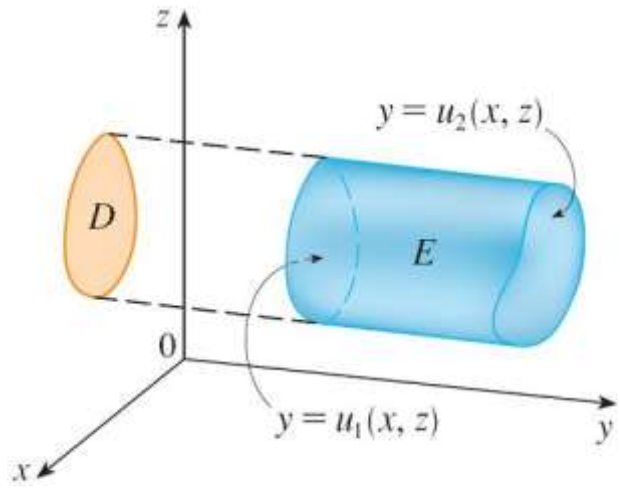
$$\boxed{10} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(y, z)}^{u_2(y, z)} f(x, y, z)\, dx \right] dA$$

Triple Integrals

Finally, a **type 3** region is of the form

$$E = \{(x, y, z) \mid (x, z) \in D,\ u_1(x, z) \le y \le u_2(x, z)\}$$

where D is the projection of E onto the xz-plane, $y = u_1(x, z)$ is the left surface, and $y = u_2(x, z)$ is the right surface (see Figure 8).



A type 3 region

Figure 8

Triple Integrals

For this type of region we have

$$\boxed{11} \qquad \iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, z)}^{u_2(x, z)} f(x, y, z)\, dy \right] dA$$

In each of Equations 10 and 11 there may be two possible expressions for the integral depending on whether *D* is a type I or type II plane region (and corresponding to Equations 7 and 8).

Example

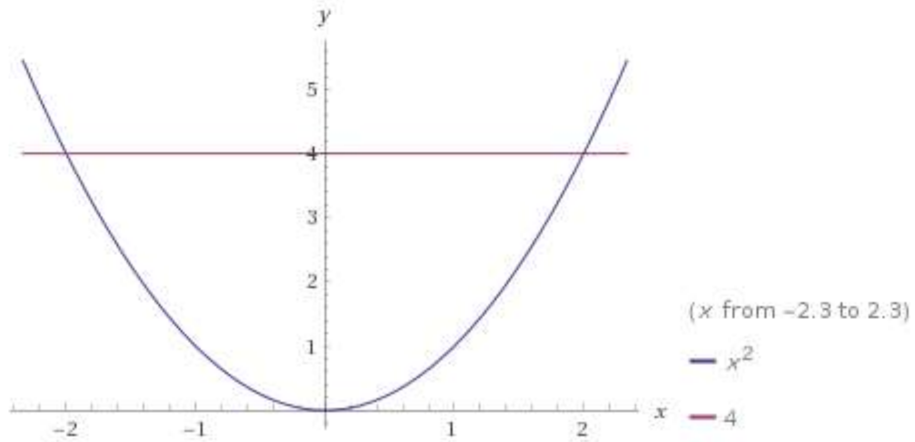
Evaluate $\iiint_E \sqrt{x^2 + z^2}\, dV$ where *E* is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane y = 4.

SOLUTION: If we regard it as a type 1 region, then we need to consider its projection onto the xy-plane:

The trace of $y = x^2 + z^2$ in the *z* = 0 plane is $y = x^2$ and the trace of the plane *y* = 4 is the line *y* = 4.

This is the parabolic region

Example - Solution



From $y = x^2 + z^2$ we obtain $\mathrm{z} = \pm\sqrt{y - x^2}$, so the lower boundary surface of *E* is $\mathrm{z} = -\sqrt{y - x^2}$ and the upper surface is $\mathrm{z} = \sqrt{y - x^2}$. Therefore the description of as a type 1 region is

$$\iiint_E \sqrt{x^2 + z^2}\ dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2 + z^2}\, dzdydx = \frac{128}{15}\pi$$

Applications of Triple Integrals

Applications of Triple Integrals

Recall that if $f(x) \geq 0$, then the single integral $\int_a^b f(x)\,dx$ represents the area under the curve $y = f(x)$ from a to b, and if $f(x, y) \geq 0$, then the double integral $\iint_D f(x, y)\,dA$ represents the volume under the surface $z = f(x, y)$ and above D.

The corresponding interpretation of a triple integral $\iiint_E f(x, y, z)\,dV$, where $f(x, y, z) \geq 0$, is not very useful because it would be the "hypervolume" of a four-dimensional object and, of course, that is very difficult to visualize. (Remember that E is just the *domain* of the function f; the graph of f lies in four-dimensional space.)

Applications of Triple Integrals

Nonetheless, the triple integral $\iiint_E f(x, y, z)\, dV$ can be interpreted in different ways in different physical situations, depending on the physical interpretations of x, y, z and $f(x, y, z)$.

Let's begin with the special case where $f(x, y, z) = 1$ for all points in E. Then the triple integral does represent the volume of E:

12

$$V(E) = \iiint_E dV$$

Applications of Triple Integrals

For example, you can see this in the case of a type 1 region by putting $f(x, y, z) = 1$ in Formula 6:

$$\iiint_E 1\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz \right] dA = \iint_D [u_2(x, y) - u_1(x, y)]\, dA$$

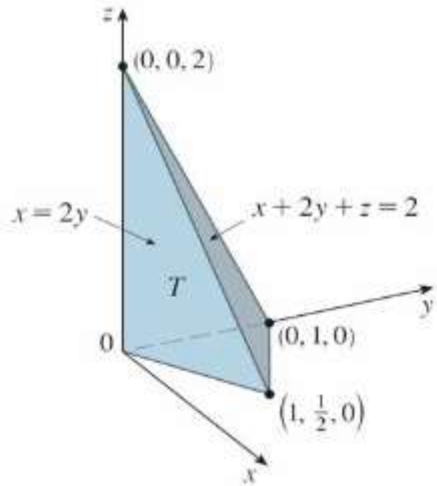
and we know this represents the volume that lies between the surfaces $z = u_1(x, y)$ and $z = u_2(x, y)$.

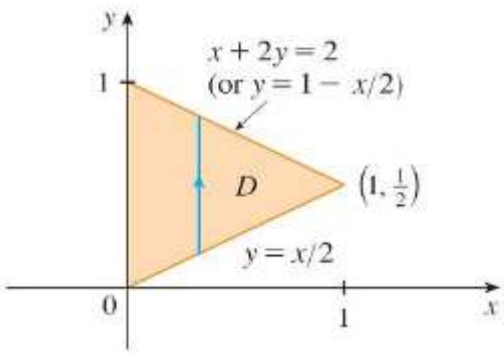
Example

Use a triple integral to find the volume of the tetrahedron *T* bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

Solution:

The tetrahedron *T* and its projection *D* onto the *xy*-plane are shown in the Figures below.





Example – *Solution*

cont'd

The lower boundary of T is the plane $z = 0$ and the upper boundary is the plane $x + 2y + z = 2$, that is, $z = 2 - x - 2y$.

Therefore we have

$$V(T) = \iiint_T dV$$

$$= \int_0^1 \int_{x/2}^{1-x/2} \int_0^{2-x-2y} dz\, dy\, dx$$

$$= \int_0^1 \int_{x/2}^{1-x/2} (2 - x - 2y)\, dy\, dx = \frac{1}{3}$$

(Notice that it is not necessary to use triple integrals to compute volumes. They simply give an alternative method for setting up the calculation.)

Example

Find the volume of the region *D* enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$

Solution The volume is

$$Volume = \iiint_E 1 \ dV$$

To find the limits of integration for evaluating the integral, we first sketch the region. The surfaces intersect on the elliptical cylinder $x^2 + 3y^2 = 8 - x^2 - y^2$ or $x^2 + 2y^2 = 4, \ z \geq 0$. The projection of *E* onto the *xy*-plane, is an ellipse with the same equation: $x^2 + 2y^2 = 4$.

Example - Solution

$$Volume = \iiint_E 1\ dV = \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} 1\, dz dy dx$$

$$= 8\sqrt{2}\pi$$

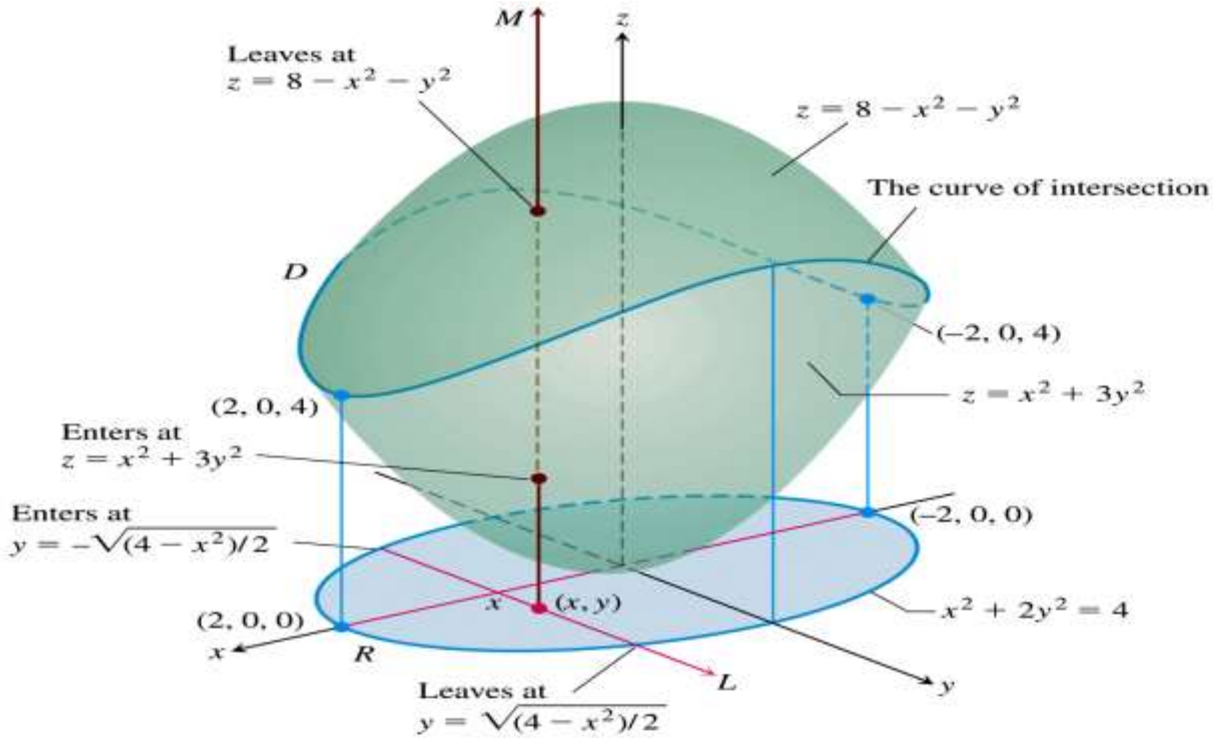


FIGURE 15.30 The volume of the region enclosed by two paraboloids, calculated in Example 1.

Example

Find the volume of the tetrahedron in the accompanying figure.

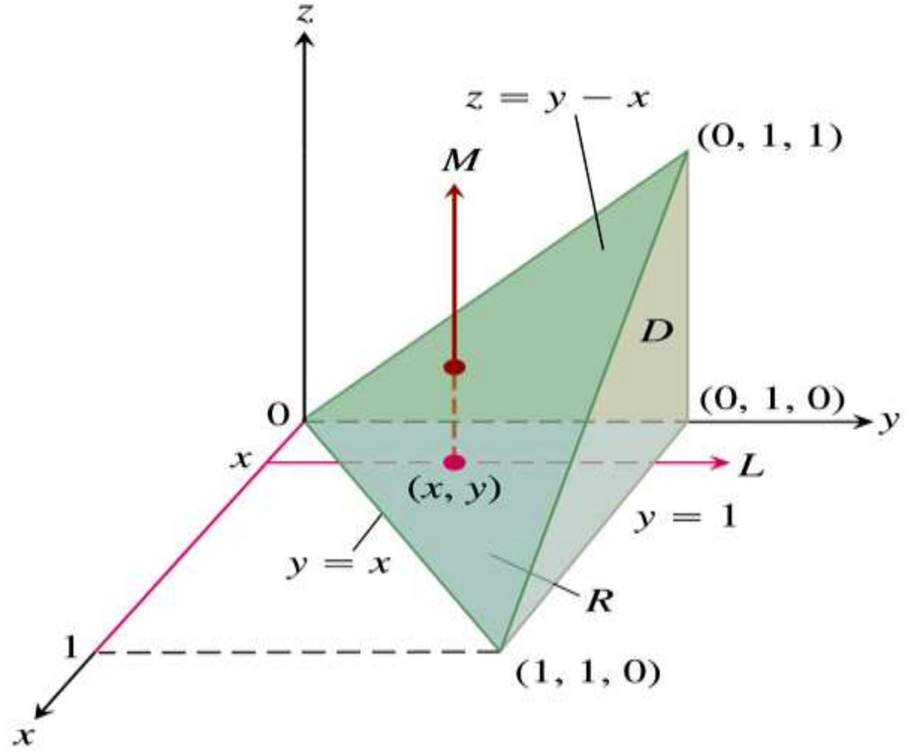


FIGURE 15.32 The tetrahedron in Example 3 showing how the limits of integration are found for the order $dz\, dy\, dx$.

Example - Solution

$$Volume = \iiint_E 1 \; dV$$

First we find the *z*-limits of integration. A line parallel to the *z*-axis through a typical point (*x*, *y*) in the *xy*-plane "shadow" enters the tetrahedron at z = 0 and exits through the upper plane where $z = y - x$.

Next we find the *y*-limits of integration. On the *xy*-plane, where the sloped side of the tetrahedron crosses the plane along the line y = x. A line through (*x*, *y*) parallel to the *y*-axis enters the shadow in the *xy*-plane at y = x and exits at y = 1.

Example - Solution

Finally we find the *x*-limits of integration. As the line parallel to the *y*-axis in the previous step sweeps out the shadow, the value of *x* varies from x = 0 to x = 1.

Thus,

$$Volume = \iiint_E 1 \ dV = \int_0^1 \int_x^1 \int_0^{y-x} 1 \, dzdydx = \frac{1}{6}$$

15 Multiple Integrals

15.8

Triple Integrals in Cylindrical Coordinates

Triple Integrals in Cylindrical Coordinates

In plane geometry the polar coordinate system is used to give a convenient description of certain curves and regions.

Figure 1 enables us to recall the connection between polar and Cartesian coordinates.

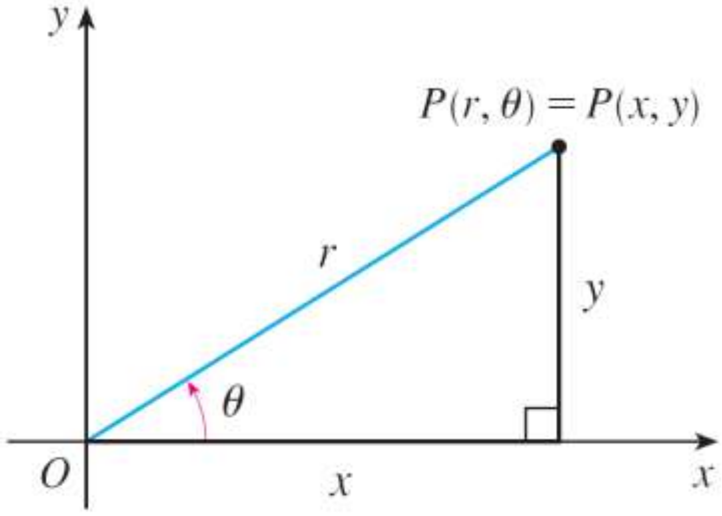


Figure 1

Triple Integrals in Cylindrical Coordinates

If the point P has Cartesian coordinates (x, y) and polar coordinates (r, θ), then, from the figure,

$$x = r \cos \theta \qquad y = r \sin \theta$$

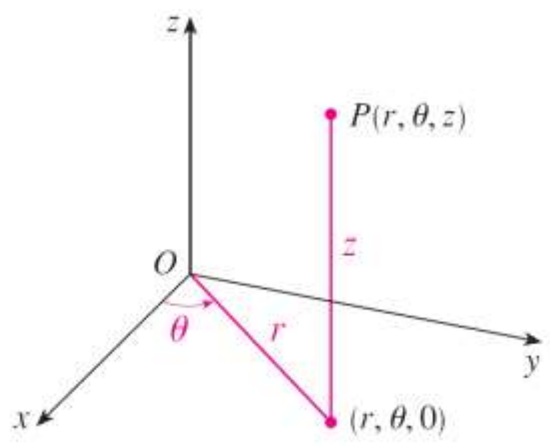
$$r^2 = x^2 + y^2 \qquad \tan\theta \; \frac{y}{x}$$

In three dimensions there is a coordinate system, called *cylindrical coordinates*, that is similar to polar coordinates and gives convenient descriptions of some commonly occurring surfaces and solids. As we will see, some triple integrals are much easier to evaluate in cylindrical coordinates.

Cylindrical Coordinates

Cylindrical Coordinates

In the **cylindrical coordinate system**, a point *P* in three-dimensional space is represented by the ordered triple (r, θ, z) where r and θ are polar coordinates of the projection of *P* onto the *xy*-plane and z is the directed distance from the *xy*-plane to *P*. (See Figure 2.)



The cylindrical coordinates of a point

Figure 2

Cylindrical Coordinates

To convert from cylindrical to rectangular coordinates, we use the equations

1

$$\boxed{x = r\cos\theta \qquad y = r\sin\theta \qquad z = z}$$

whereas to convert from rectangular to cylindrical coordinates, we use

2

$$\boxed{r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z}$$

Example 1

(a) Plot the point with cylindrical coordinates $(2, 2\pi/3, 1)$ and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, -7)$.

Solution:

(a) The point with cylindrical coordinates $(2, 2\pi/3, 1)$ is plotted in Figure 3.

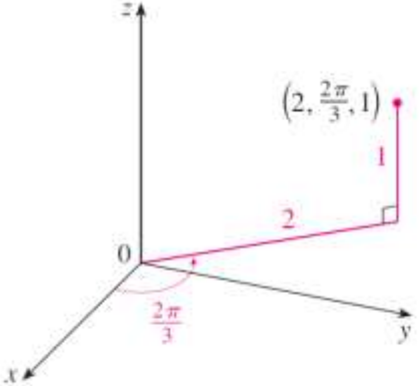


Figure 3

Example 1 – *Solution*

cont'd

From Equations 1, its rectangular coordinates are

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Thus the point is (–1, $\sqrt{3}$, 1) in rectangular coordinates.

Example 1 – *Solution*

cont'd

(b) From Equations 2 we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \text{so} \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

Therefore one set of cylindrical coordinates is $(3\sqrt{2}, 7\pi/4, -7)$. Another is $(3\sqrt{2}, -\pi/4, -7)$.

As with polar coordinates, there are infinitely many choices.

Example 2

Describe the surface whose equation in cylindrical coordinates is

(a) $z = r$.

By converting the equation into rectangular coordinates, we get

$$z = r$$
$$z^2 = r^2$$
$$z^2 = x^2 + y^2$$

This is a circular cone whose axis is the z-axis.

Example 2

(b) r = 2.

Converting the equation into rectangular coordinates, yields

$$r = 2$$
$$r^2 = 4$$
$$x^2 + y^2 = 4$$

This is a circular cylinder whose axis is the z-axis.

Example 2

(c) $z = 4 - r^2$.

Converting the equation into rectangular coordinates, yields

$$z = 4 - (x^2 + y^2)$$

This is a paraboloid whose axis is the z-axis.

(d) $2r^2 + z^2 = 1$.

Converting the equation into rectangular coordinates, yields

$$2(x^2 + y^2) + z^2 = 1$$

$$\frac{x^2}{1/2} + \frac{y^2}{1/2} + z^2 = 1$$

This is an ellipsoid

Evaluating Triple Integrals with Cylindrical Coordinates

Evaluating Triple Integrals with Cylindrical Coordinates

Suppose that *E* is a type 1 region whose projection *D* onto the *xy*-plane is conveniently described in polar coordinates (see Figure 6).

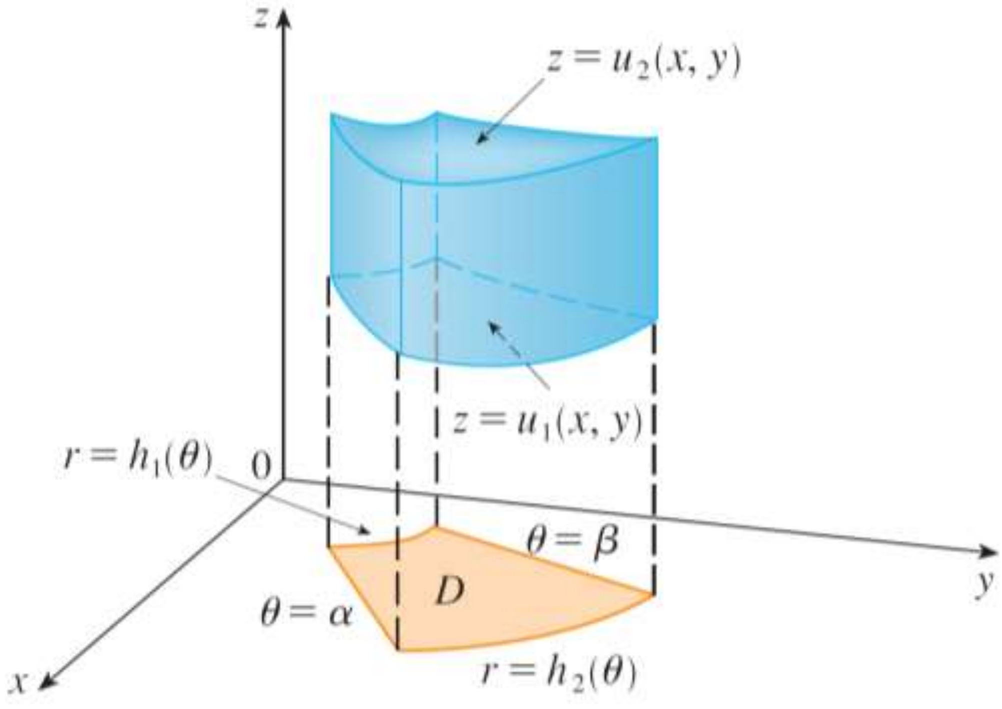


Figure 6

Evaluating Triple Integrals with Cylindrical Coordinates

In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta , h_1(\theta) \le r \le h_2(\theta)\}$$

We know

3
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x, y)}^{u_2(x, y)} f(x, y, z)\, dz \right] dA$$

Evaluating Triple Integrals with Cylindrical Coordinates

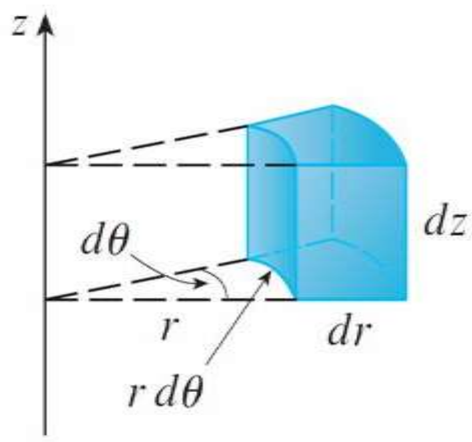
But to evaluate double integrals in polar coordinates, we have the formula

$$\boxed{4} \quad \iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Formula 4 is the **formula for triple integration in cylindrical coordinates.**

Evaluating Triple Integrals with Cylindrical Coordinates

It says that we convert a triple integral from rectangular to cylindrical coordinates by writing $x = r \cos \theta$, $y = r \sin \theta$, leaving z as it is, using the appropriate limits of integration for z, r, and θ, and replacing dV by $r\,dz\,dr\,d\theta$.
(Figure 7 shows how to remember this.)



Volume element in cylindrical coordinates: $dV = r\,dz\,dr\,d\theta$

Figure 7

Evaluating Triple Integrals with Cylindrical Coordinates

It is worthwhile to use this formula when E is a solid region easily described in cylindrical coordinates, and especially when the function $f(x, y, z)$ involves the expression $x^2 + y^2$.

Example 3

Evaluate $\iiint_E \sqrt{x^2 + y^2}\ dV$, where *E* is the solid lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.)

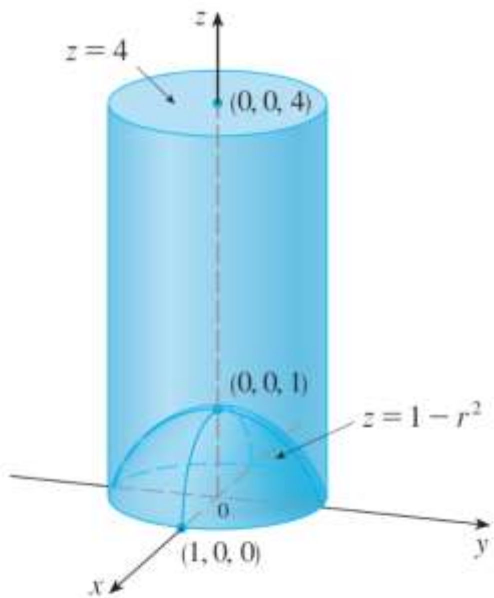


Figure 8

Example 3 – *Solution*

SOLUTION: In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 1 - r^2 \le z \le 4\}$$

$$\iiint_E \sqrt{x^2 + y^2}\; dV = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 \sqrt{r^2}\; r dz dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 r^2\; dz dr d\theta = \int_0^{2\pi} \int_0^1 r^2 z\Big]_{1-r^2}^4\; dr d\theta$$

$$= \int_0^{2\pi} \int_0^1 (3r^2 + r^4)\; dr d\theta = \int_0^{2\pi} \left(r^3 + \frac{r^5}{5}\right)\Bigg]_0^1 d\theta = \int_0^{2\pi} \frac{6}{5}\; d\theta$$

$$= \frac{12\pi}{5}$$

Example 4

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{4} (x^2+y^2)\ dzdydx$.

SOLUTION: This iterated integral is a triple integral over the solid region

$E = \left\{(x, y, z) | -2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}, \sqrt{x^2+y^2} \le z \le 4\right\}$

and the projection of *E* onto the xy-plane is the disk $x^2 + y^2 \le 4$. The lower surface of *E* is the cone $\mathrm{z} = \sqrt{x^2+y^2}$ and its upper surface is the plane z = 4.

This region has a much simpler description in cylindrical coordinates:

$$E = \{(r, \theta, z) | 0 \le \theta \le 2\pi, 0 \le r \le 2, r \le z \le 2\}$$

Example 4

Therefore we have

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{4} (x^2+y^2)\ dzdydx = \iiint_E (x^2+y^2)dV$$

$$= \int_0^{2\pi}\int_0^{2}\int_r^{2} r^2\ rdzdrd\theta = \int_0^{2\pi}\int_0^{2}\int_r^{2} r^3\ dzdrd\theta = \frac{16\pi}{5}$$

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Multiple Integrals

15.9 Triple Integrals in Spherical Coordinates

Triple Integrals in Spherical Coordinates

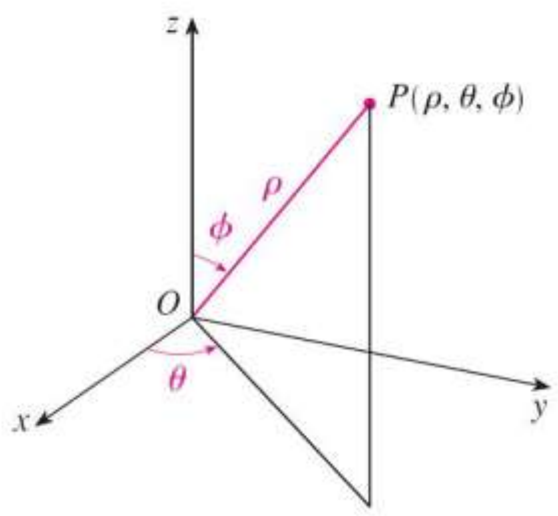
Another useful coordinate system in three dimensions is the *spherical coordinate system*.

It simplifies the evaluation of triple integrals over regions bounded by spheres or cones.

Spherical Coordinates

Spherical Coordinates

The **spherical coordinates** (ρ, θ, ϕ) of a point *P* in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to *P*, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive *z*-axis and the line segment *OP*.



The spherical coordinates of a point

Figure 1

Spherical Coordinates

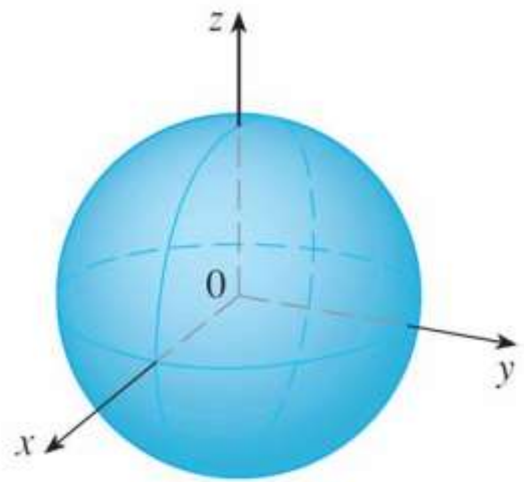
Note that

$$\rho \geq 0 \qquad\qquad 0 \leq \phi \leq \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

Spherical Coordinates

For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name “spherical” coordinates.

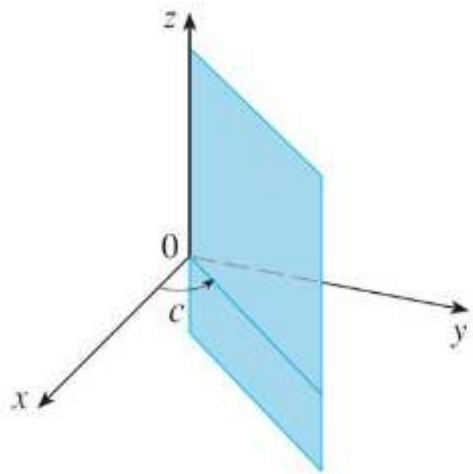


$\rho = c$, a sphere

Figure 2

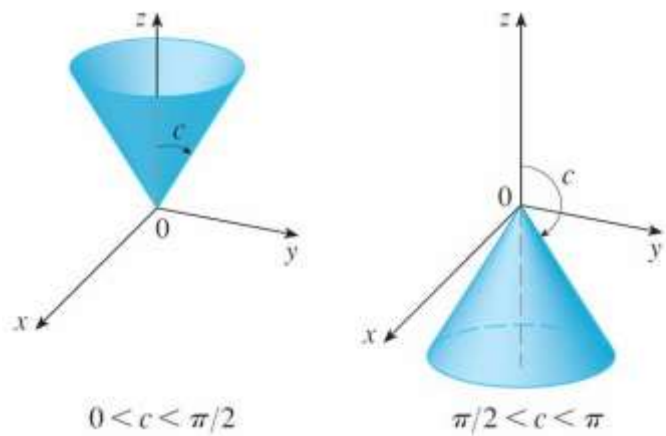
Spherical Coordinates

The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (see Figure 4).



$\theta = c$, a half-plane

Figure 3



$\phi = c$, a helf-cone

Figure 4

Spherical Coordinates

The relationship between rectangular and spherical coordinates can be seen from Figure 5.

From triangles *OPQ* and *OPP′* we have

$$z = \rho \cos \phi \qquad\qquad r = \rho \sin \phi$$

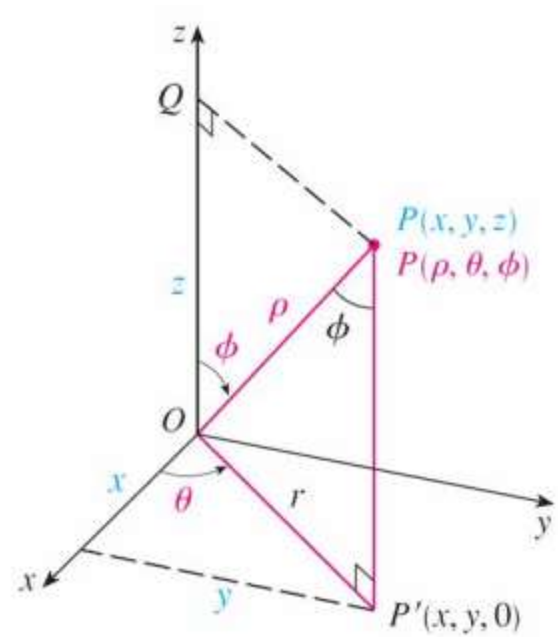


Figure 5

Spherical Coordinates

But $x = r \cos \theta$ and $y = r \sin \theta$, so to convert from spherical to rectangular coordinates, we use the equations

$$\boxed{1} \qquad \boxed{x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi}$$

Also, the distance formula shows that

$$\boxed{2} \qquad \boxed{\rho^2 = x^2 + y^2 + z^2}$$

We use this equation in converting from rectangular to spherical coordinates.

Example 1

The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution:

We plot the point in Figure 6.

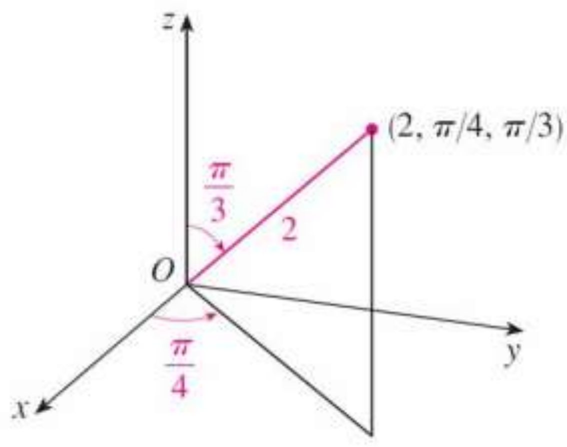


Figure 6

Example 1 – *Solution*

cont'd

From Equations 1 we have

$$x = \rho \sin \phi \cos \theta = 2 \sin \frac{\pi}{3} \cos \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin \phi \sin \theta = 2 \sin \frac{\pi}{3} \sin \frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi = 2 \cos \frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates.

Example 2

Describe the surface whose equation in spherical coordinates is $\rho = sin\phi \; sin\theta$.

SOLUTION: Using that $y = \rho sin\phi \; sin\theta$ we get

$$y = \rho sin\phi \; sin\theta = \rho^2 = x^2 + y^2 + z^2$$

$$x^2 + y^2 - y + z^2 = 0$$

$$x^2 + y^2 - y + \frac{1}{4} - \frac{1}{4} + z^2 = 0$$

$$x^2 + \left(y - \frac{1}{2}\right)^2 + z^2 = \frac{1}{4}$$

This is a sphere with center $\left(0, \frac{1}{2}, 0\right)$ and radius $\frac{1}{2}$.

Evaluating Triple Integrals with Spherical Coordinates

Evaluating Triple Integrals with Spherical Coordinates

In the spherical coordinate system the counterpart of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$$

where $a \ge 0$ and $\beta - \alpha \le 2\pi$, and $d - c \le \pi$. Although we defined triple integrals by dividing solids into small boxes, it can be shown that dividing a solid into small spherical wedges always gives the same result.

Evaluating Triple Integrals with Spherical Coordinates

We have the following **formula for triple integration in spherical coordinates**.

$$\boxed{3} \quad \iiint_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b, \ \alpha \leqslant \theta \leqslant \beta, \ c \leqslant \phi \leqslant d\}$$

Evaluating Triple Integrals with Spherical Coordinates

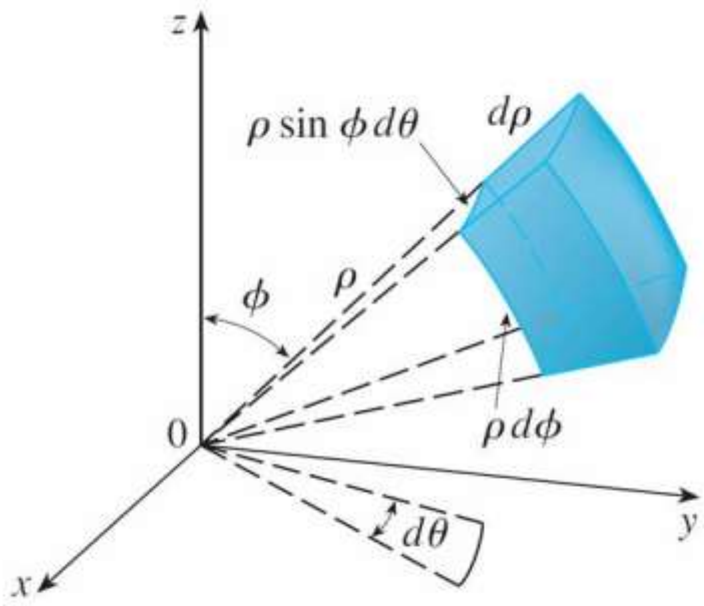
Formula 3 says that we convert a triple integral from rectangular coordinates to spherical coordinates by writing

$$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

using the appropriate limits of integration, and replacing *dv* by $\rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$.

Evaluating Triple Integrals with Spherical Coordinates

This is illustrated in Figure 8.



Volume element in spherical coordinates: $dV = \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$

Figure 8

Evaluating Triple Integrals with Spherical Coordinates

This formula can be extended to include more general spherical regions such as

$$E = \{(\rho, \theta, \phi) \mid \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\}$$

In this case the formula is the same as in $\boxed{3}$ except that the limits of integration for ρ are $g_1(\theta, \phi)$ and $g_2(\theta, \phi)$.

Usually, spherical coordinates are used in triple integrals when surfaces such as cones and spheres form the boundary of the region of integration.

Example 3

Evaluate $\iiint_E e^{(x^2+y^2+z^2)^{3/2}} dV$, where *E* is the unit ball:
$E = \{(x, y, z) | x^2 + y^2 + z^2 \leq 1\}$

SOLUTION: Since the boundary of *E* is a sphere, we use spherical coordinates:

$$E = \{(\rho, \theta, \phi) | 0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi\}.$$

Thus,

$$\iiint_E e^{(x^2+y^2+z^2)^{3/2}} dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 e^{\rho^3} \rho^2 sin\phi \, d\rho d\theta d\phi$$

$$= \frac{4}{3}\pi(e-1).$$

Example 4

Evaluate $\iiint_E (x^2+y^2+z^2)\,dV$, where *E* is the hemisphere

$$E=\{(x,y,z)|\ x^2+y^2+z^2\le 9, z\ge 0\}$$

SOLUTION: Since the boundary of *E* is a part of a sphere, we use spherical coordinates:

$$E=\{(\rho,\theta,\phi)|0\le\rho\le 3, 0\le\theta\le 2\pi, 0\le\phi\le\pi/2\}.$$

Thus,

$$\iiint_E (x^2+y^2+z^2)\,dV=\int_0^{\pi/2}\int_0^{2\pi}\int_0^{3}\rho^2\ \rho^2 sin\phi\ d\rho d\theta d\phi$$

$$=\frac{486}{5}\pi$$

Example 5

Evaluate $\iiint_E (x^2+y^2)\, dV$, where *E* lies between the spheres $x^2+y^2+z^2 \le 9$ and $x^2+y^2+z^2 \le 4$.

SOLUTION: We use spherical coordinates:

$$E = \{(\rho, \theta, \phi) | 2 \le \rho \le 3, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}.$$

Note that

$$x^2+y^2 = \rho^2 sin^2\phi cos^2\theta + \rho^2 sin^2\phi cos^2\theta = \rho^2 sin^2\phi$$

Thus,

$$\iiint_E (x^2+y^2)\, dV = \int_0^{\pi}\int_0^{2\pi}\int_2^{3} \rho^4 sin^3\phi \; d\rho d\theta d\phi = \frac{1688}{5}\pi$$

Example 6

Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$. (See Figure 9.)

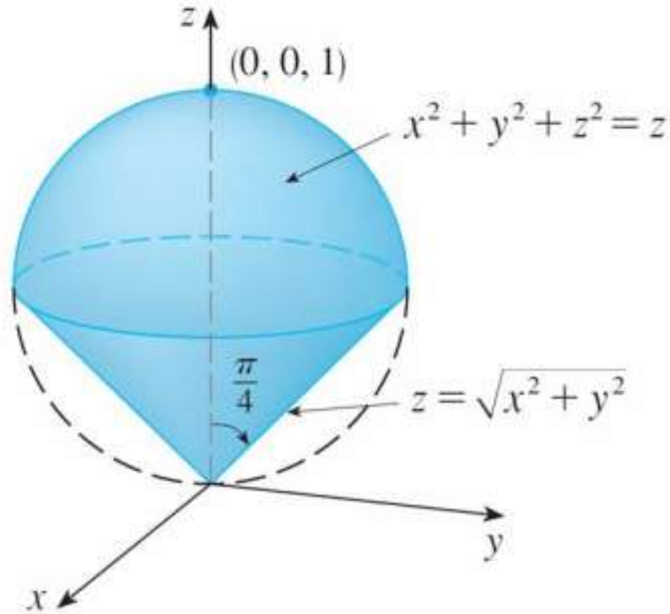


Figure 9

Example 6 – *Solution*

Notice that the sphere passes through the origin and has center $(0, 0, \frac{1}{2})$. We write the equation of the sphere in spherical coordinates as

$$\rho^2 = \rho \cos \phi \qquad \text{or} \qquad \rho = \rho \cos \phi$$

The equation of the cone can be written as

$$\rho \cos \phi = \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta}$$

$$= \rho \sin \phi$$

Example 6 – *Solution*

cont'd

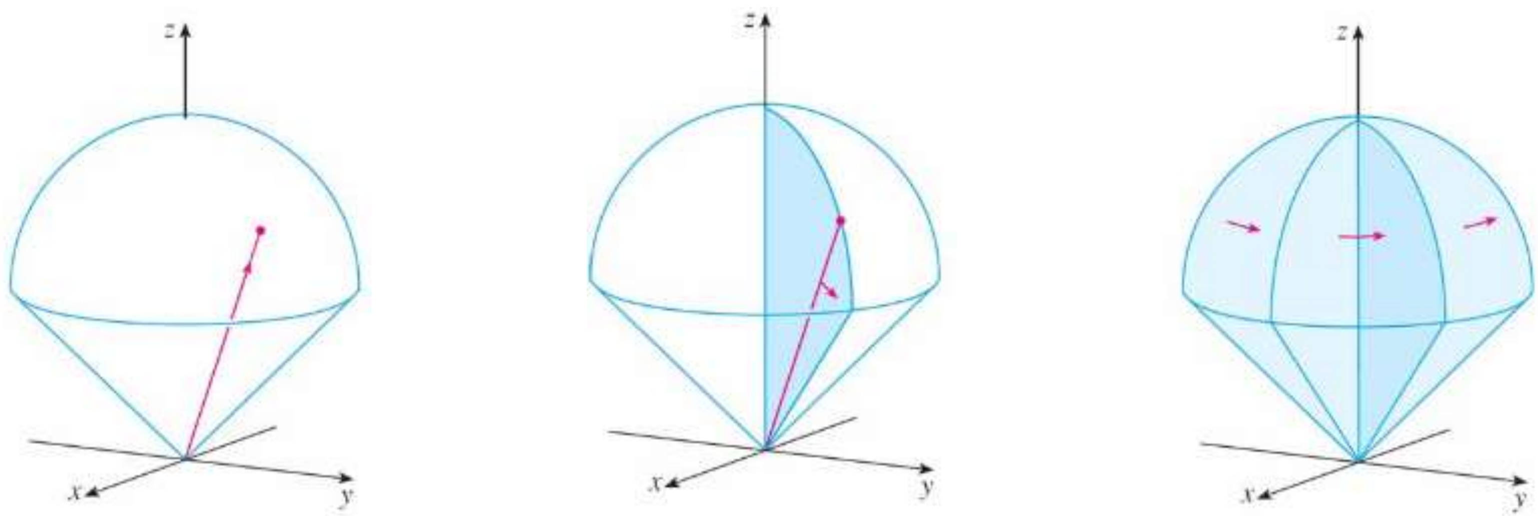
This gives *sin* $\phi = \cos \phi$, or $\phi = \pi/4$. Therefore the description of the solid *E* in spherical coordinates is

$$E = \{(\rho, \theta, \phi) \mid 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/4, 0 \leq \rho \leq \cos \phi\}$$

Example 6 – *Solution*

cont'd

Figure 11 shows how E is swept out if we integrate first with respect to ρ, then ϕ, and then θ.



ρ varies from 0 to cos ϕ while ϕ and θ are constant.

ϕ varies from 0 to $\pi/4$ while θ is constant.

θ varies from 0 to 2π .

Figure 11

Example 6 – *Solution*

cont'd

The volume of *E* is

$$V(E) = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi \left[\frac{\rho^3}{3} \right]_{\rho=0}^{\rho=\cos\phi} d\phi$$

$$= \frac{2\pi}{3} \int_0^{\pi/4} \sin\phi \cos^3\phi \, d\phi = \frac{2\pi}{3} \left[-\frac{\cos^4\phi}{4} \right]_0^{\pi/4} = \frac{\pi}{8}$$