Calculus III

Study concepts, example questions & explanations

Question #1: 3D Space

Describe in words the region of R^3 represented by the equations or inequalities.

(a) $x^2 + y^2 + z^2 \leq 4.$
(b) $x^2 + z^2 \leq 9.$
(c) $x = z.$
(d) $y^2 + z^2 = 4, \quad x = 1.$

Correct Answer:

(a) All points on or inside a sphere with radius 2 and center (0,0,0).
(b) All points on or inside a circular cylinder of radius 3 centered on the y-axis.
(c) All points on the plane $z = x$.
(d) All points on a circle with radius 2 with center on the x-axis.

Explanation:

(a) All points on or inside a sphere with radius 2 and center (0,0,0).

(b) All points on or inside a circular cylinder of radius 3 centered on the y-axis.
(c) All points on the plane $z = x$.
(d) All points on a circle with radius 2 with center on the x-axis.

Question #2: Vectors in Space

Given the space vectors $\vec{u} = \langle 2,1,2 \rangle$, $\vec{v} = \langle 0,3,4 \rangle$ **and** $\vec{w} = \langle 3,-2,5 \rangle$**. Find**

(a) the projection of $\vec{\boldsymbol{u}}$ onto $\vec{\boldsymbol{v}}$,
(b) the vector component of $\vec{\boldsymbol{u}}$ orthogonal to $\vec{\boldsymbol{v}}$,
(c) the scalar component of $\vec{\boldsymbol{u}}$ in the direction of $\vec{\boldsymbol{v}}$.
(d) a vector in the direction of $\vec{u}$ with length 3.
(e) the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.
(f) the volume of the parallelepiped determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Correct Answer:

(a) $proj_{\vec{u}}\vec{v} = \langle 0, \frac{33}{25}, \frac{44}{25} \rangle$.
(b) $\vec{u} - proj_{\vec{u}}\vec{v} = \langle 2, \frac{-8}{25}, \frac{6}{25} \rangle$.
(c) $|proj_{\vec{u}}\vec{v}| = \frac{11}{5} = 2.2$.
(d) $\frac{4}{3}\vec{\boldsymbol{u}} = \langle \frac{8}{3}, \frac{4}{3}, \frac{8}{3} \rangle$.
(e) Area = $|\vec{u} \times \vec{v}| = \sqrt{104}$.
(f) Volume = 40

Explanation:

$$\vec{u} \cdot \vec{v} = (2)(0) + (1)(3) + (2)(4) = 11, \qquad \vec{u} \times \vec{v} = \langle -2, -8, 6 \rangle,$$

$$|\vec{v}| = \sqrt{(0)^2 + (3)^2 + (4)^2} = 5, \quad |\vec{u}| = \sqrt{(2)^2 + (1)^2 + (2)^2} = 3.$$

a) $proj_{\vec{u}}\vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{11}{25}\langle 0,3,4 \rangle = \langle 0, \frac{33}{25}, \frac{44}{25} \rangle$
b) $\vec{u} - proj_{\vec{u}}\vec{v} = \langle 2,1,2 \rangle - \langle 0, \frac{33}{25}, \frac{44}{25} \rangle = \langle 2, \frac{-8}{25}, \frac{6}{25} \rangle$.
c) $|proj_{\vec{u}}\vec{v}| = \sqrt{(0)^2 + (\frac{33}{25})^2 + (\frac{44}{25})^2} = \frac{11}{5} = 2.2$
d) The required vector is $\frac{4}{3}\vec{\boldsymbol{u}} = \langle \frac{8}{3}, \frac{4}{3}, \frac{8}{3} \rangle$.
e) Area = $|\vec{u} \times \vec{v}| = |\vec{u} \times \vec{v}| = \sqrt{(-2)^2 + (-8)^2 + (6)^2} = \sqrt{104}$.
f) Volume = $|\vec{u} \cdot (\vec{v} \times \vec{w})| = \begin{vmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 3 & -2 & 5 \end{vmatrix} = 40$.

Question #3: Vectors in Space

Find the angle (in degrees) between the two vectors $\vec{u} = \vec{i} - 3\vec{j} + 7\vec{k}$ and $\vec{v} = -2\vec{i} + \vec{j} + 4\vec{k}$.

Correct Answer:

$\theta \approx 49.2^{\circ}$

Explanation:

The angle between any two vectors $\vec{u}$ and $\vec{v}$ is $\theta = cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$.

For this problem, $\vec{u} \cdot \vec{v} = (1)(-2) + (-3)(1) + (7)(4) = 23$,
$|\vec{u}| = \sqrt{(1)^2 + (-3)^2 + (7)^2} = \sqrt{59}$,
$|\vec{v}| = \sqrt{(-2)^2 + (1)^2 + (4)^2} = \sqrt{21}$.

Substituting, we have

$\theta = cos^{-1}\left(\frac{23}{\sqrt{59}\sqrt{21}}\right) \approx 49.2^{\circ}$.

Question #4: Vector Functions

Find a parametric representation of the curve of intersection of the cylinder $9x^2 + y^2 = 9$ and the plane $x + y + z = 7$.

Correct Answer:

$\vec{r}(t) = \langle \text{cost}, 3\text{sint}, 7 - \text{cost} - 3\text{sint} \rangle$

Explanation:

We can begin by rewriting the expression for the cylinder as follows

$9x^2 + y^2 = 9 \Rightarrow x^2 + (y/3)^2 = 1.$

This tells us that $x = \cos t$, $y = 3\sin t$. Plugging this back into the equation for the plane $x + y + z = 7$ to find $z = 7 - x - y = 7 - \cos t - 3\sin t$.

This gives us the representation of the curve of intersection as

$\vec{r}(t) = \langle \cos t, 3\sin t, 7 - \cos t - 3\sin t \rangle.$

Question #5: Vectors in Space

Given that $\vec{u} = \langle 3,2,-1 \rangle$ and $\vec{v} = \langle 6,k,-2 \rangle$ are orthogonal. Find the value of *k*.

Correct Answer:

$k = -10$

Explanation:

If $\vec{u}$ and $\vec{v}$ are orthogonal, then $\vec{v} \cdot \vec{v} = 0$. In this case

$$\vec{v} \cdot \vec{v} = 18 + 2k + 2 = 0.$$

So, $k = -10$.

Report an Error

Question #6: Vectors in Space

The angle (in degrees) between the two unit vectors $\vec{u}$ and $\vec{v}$ is $\mathbf{65^\circ}$**. Find $|\vec{u} \times \vec{v}|$.**

Correct Answer:

$|\vec{u} \times \vec{v}| = \mathbf{0.906.}$

Explanation:

To find $|\vec{u}\times\vec{v}|$ **we apply the formula** $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\boldsymbol{sin\theta}$.

As $\vec{\boldsymbol{u}}$ **and** $\vec{\boldsymbol{v}}$ are unit vectors we have $|\vec{\boldsymbol{u}}| = \mathbf{1}$ and $|\vec{\boldsymbol{v}}| = \mathbf{1}$.

So, $|\vec{u}\times\vec{v}| = (\mathbf{1})(\mathbf{1})\boldsymbol{sin}(\mathbf{65}^{\circ}) = \mathbf{0.906}$.

Question #7: Lines in Space

Write the vector equation of the line that passes through the points $P(-3, 4, 14)$ **and** $Q(2, 10, -6)$**.**

Correct answer:

$\vec{r}(t) = \langle 2 + 5t, 10 + 6t, -6 - 20t \rangle$

Explanation:

Remember the general equation of a line in vector form:

$\vec{\boldsymbol{r}}(t) = \boldsymbol{r}_0 + t\vec{\boldsymbol{v}} = \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle$, where $\langle x_0, y_0, z_0 \rangle$ is the starting point, and $\vec{\boldsymbol{v}}$ is a vector in the direction of the line. We can take $\vec{\boldsymbol{v}} = \overrightarrow{PQ} = \langle 2 - (-3), 10 - 4, -6 - 14 \rangle = \langle 5, 6, -20 \rangle$.

Let's apply this to our problem.

$\vec{\boldsymbol{r}}(t) = \langle 2, 10, -6 \rangle + t\,\langle 5, 6, -20 \rangle$.

This can be written as

$\vec{\boldsymbol{r}}(t) = \langle 2 + 5t, 10 + 6t, -6 - 20t \rangle$.

Question #8: Planes

Find the approximate angle (in radians) between the planes $4x-3y-2z=1$**, and** $12x+2y-7z=16$**.**

Correct Answer:

0.736

Explanation:

Finding the angle between two planes requires us to find the angle between their normal vectors.

To obtain normal vectors, we simply take the coefficients in front of x, y, z.

$\overrightarrow{\boldsymbol{n_1}} = \langle 4, -3, -2 \rangle, \quad \overrightarrow{\boldsymbol{n_2}} = \langle 12, 2, -7 \rangle.$

The (acute) angle between any two vectors is

$$\theta = cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right).$$

Here, $\overrightarrow{\boldsymbol{n_1}} \cdot \overrightarrow{\boldsymbol{n_2}} = (4)(12) + (-3)(2) + (-2)(-7) = 56,$
$|\overrightarrow{\boldsymbol{n_1}}| = \sqrt{(4)^2 + (-3)^2 + (-2)^2} = \sqrt{29},$
$|\overrightarrow{\boldsymbol{n_2}}| = \sqrt{(12)^2 + (2)^2 + (-7)^2} = \sqrt{197}.$

Substituting, we have

$$\theta = cos^{-1}\left(\frac{56}{\sqrt{29}\sqrt{197}}\right) \approx 0.736.$$

Question #9: Equations Of Lines And Planes

Find the point of intersection of the plane $2x + y + z = 9$ **and the line described by** $\vec{r}(t) = \langle 2t + 4, t - 1, -t \rangle$**.**

Correct Answer:

$(5, -12, -12)$

Explanation:

Substituting the components of the line into those of the plane, we have

$2(2t + 4) + (t - 1) + (-t) = 9 \;\Rightarrow\; 4t + 8 + t - 1 - t = 9 \;\Rightarrow\; t = 12.$

Substituting this value of t back into the components of the line gives us $(5, -12, -12)$.

Question #10: Planes

Show that the two planes are parallel $x + y + z = 2, \;\; x + y + z = 0$**.**

Correct Answer:

Two planes are parallel if their normal vectors are parallel.

Explanation:

Two planes are parallel if their normal vectors are parallel.

A vector normal to the first plane is $\overrightarrow{\boldsymbol{n_1}} = \langle 1,1,1 \rangle$

A vector normal to the second plane is $\overrightarrow{\boldsymbol{n_2}} = \langle 1,1,1 \rangle$.

We can verify that $\overrightarrow{\boldsymbol{n_1}} \times \overrightarrow{\boldsymbol{n_2}} = \overrightarrow{\boldsymbol{0}}$ and so $\overrightarrow{\boldsymbol{n_1}}$ and $\overrightarrow{\boldsymbol{n_2}}$ are parallel.

Another way to solve this problem is to find the angle between the normal vectors, we have

$$\theta = cos^{-1}\left(\frac{\overrightarrow{\boldsymbol{n_1}} \cdot \overrightarrow{\boldsymbol{n}_2}}{|\overrightarrow{\boldsymbol{n_1}}||\overrightarrow{\boldsymbol{n}_2}|}\right) = cos^{-1}\left(\frac{3}{\sqrt{3}\sqrt{3}}\right) = cos^{-1}(1) = 0.$$

This means that the two planes are parallel.

Question #11: Planes

Determine the equation of the plane that contains the points $O(0,0,0)$**,** $P(5,4,0)$**, and** $Q(-3,-7,0)$**.**

Correct Answer:

z = 0

Explanation:

The equation of a plane is defined as

$$A(x - x_0) + B(x - y_0) + C(x - z_0) = 0$$

where $\vec{\boldsymbol{n}} = \langle A, B, C \rangle$ is the normal vector of the plane and (x_0, y_0, z_0) is a point on the plane.

To find the normal vector, we first get two vectors on the plane

$\overrightarrow{\boldsymbol{OP}} = \langle 5,4,0 \rangle$ and $\overrightarrow{\boldsymbol{OQ}} = \langle -3, -7,0 \rangle$ and find their cross product.

So, $\vec{\boldsymbol{n}} = \overrightarrow{\boldsymbol{OP}} \times \overrightarrow{\boldsymbol{OQ}} = -23\vec{\boldsymbol{k}} = \langle 0,0, -23 \rangle$.

Using the point O and the normal vector to find the equation of the plane yields

$$0(x - 0) + 0(y - 0) - 23(z - 0) = 0.$$

Simplified gives the equation of the plane $z = 0$.

Question #12: Planes

Find the equation of the plane containing the point $(3, -2, 1)$**, and is parallel to the plane with the equation** $2x + 5y + z = 20$.

Correct Answer:

2x + 5y + z = −3

Explanation:

We were given a point on the plane, and we need the normal vector to the plane. It is known that two planes that are parallel to each other have the same normal vector, so in this case $\vec{\boldsymbol{n}} = \langle 2,5,1 \rangle$ (given by the equation of the other plane). To complete the problem, we use the equation
$A(x - x_0) + B(x - y_0) + C(x - z_0) = 0.$

Using the information we have, we get:

$2(x - 3) + 5(y + 2) + (z - 1) = 0.$

Through algebraic manipulation, we then get:

$2x + 5y + z = -3.$

Question #13: Equations Of Lines And Planes

Write the equation of the plane passing through $(2, 4, 2)$ **and orthogonal to the line through** $(3, -1, 2)$ **and** $(4, 6, 1)$**.**

Correct Answer:

$-13x+5y+22z=40$

Explanation:

The vector $\vec{\boldsymbol{v}} = \overrightarrow{PQ} = \langle 1,7,-1 \rangle$ is in the direction of the line through $P(3, -1, 2)$ and $Q(4, 6, 1)$ which in turn is normal to the plane in question. The equation of the plane is found by taking the point (2, 4, 2), and the normal vector $\vec{\boldsymbol{n}} = \langle 1,7,-1 \rangle$ and plugging them into the equation

$$A(x - x_0) + B(x - y_0) + C(x - z_0) = 0.$$

We get, $(x - 2) + 7(y - 4) - (z - 2) = 0,$

which simplified becomes $-13x + 5y + 22z = 40.$

Question #14: Lines in Space

Give the parametric equations of the line through the point (−7, 2, 4) and parallel to the line given by *x* = 5 − 8t, *y* = 6 + t, *z* = −12t.

Correct Answer:

$\boldsymbol{x = -7 - 8t, \qquad y = 2 + t, \qquad z = 4 - 12t.}$

Explanation:

We know that the coefficients of the t's in the equation of the line forms a vector parallel to the line. So, $\vec{\boldsymbol{v}} = \langle -8, 1, -12 \rangle$ is a vector parallel to the given line. Now, if $\vec{\boldsymbol{v}}$ is parallel to the given line and the requested line must be parallel to the given line then $\vec{\boldsymbol{v}}$ must also be parallel to the requested line.

Using the point (-7, 2, 4) and the directional vector $\vec{\boldsymbol{v}} = \langle -8, 1, -12 \rangle$ we can write the parametric equations of the new line as

$$\mathrm{x} = -7 - 8\mathrm{t}, \ \mathrm{y} = 2 + \mathrm{t}, \ \mathrm{z} = 4 - 12\mathrm{t}.$$

Question #15: Lines in Space

Is the line through the points *P*(2, 0, 9) and *Q*(−4, 1, −5) parallel, orthogonal or skew to the line given by $\vec{\boldsymbol{r}}(t) = \langle 5, 1 - 9t, -8 - 4t \rangle$.

Correct Answer:

The two lines are skew.

Explanation:

A directional vector to the first line is $\overrightarrow{\boldsymbol{d_1}} = \overrightarrow{\boldsymbol{PQ}} = \langle -6,1,-14 \rangle$ and a directional vector to the second line is $\overrightarrow{\boldsymbol{d_2}} = \langle 0,-9,-4 \rangle$.

Now, $\overrightarrow{\boldsymbol{d_1}} \times \overrightarrow{\boldsymbol{d_2}} \neq \vec{\boldsymbol{0}}$. This in turn means that **the two lines can't possibly be parallel**. Next, $\overrightarrow{\boldsymbol{d_1}} \cdot \overrightarrow{\boldsymbol{d_2}} = -47$. The dot product $\overrightarrow{\boldsymbol{d_1}} \cdot \overrightarrow{\boldsymbol{d_2}} \neq 0$ and so these vectors aren't orthogonal and this in turn means that **the two lines are not orthogonal**. We arrive to the conclusion that the two lines are skew.

Question #16: Planes

Show that the two planes are orthogonal

$p_1: 4x - 9y - z = 2, \quad p_2: x + 2y - 14z = -6$

Correct Answer:

The two planes are orthogonal.

Explanation:

Two planes are orthogonal if their normal vectors are orthogonal.

A normal vector to the first plane is $\overrightarrow{\boldsymbol{n_1}} = \langle 4,-9,-1 \rangle$ and a normal vector to the second plane is $\overrightarrow{\boldsymbol{n_2}} = \langle 1,2,-14 \rangle$.

Now, $\overrightarrow{\boldsymbol{n_1}} \cdot \overrightarrow{\boldsymbol{n_2}} = (4)(1) + (-9)(2) + (-1)(-14) = 0$ and so these vectors are orthogonal and this in turn means that **the two planes are orthogonal**.

Question #17: Lines in Space

Find the distance from the point (1, -3, 2) to the line

$$x = 1 + t, \quad y = 2 - t, \quad z = -1 + 2t.$$

Correct Answer:

$D = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{83}}{\sqrt{6}} = 3.719.$

Explanation:

Distance from a point S to a line through P parallel to $\vec{v}$ is

$$D = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|}.$$

Here, $S(1, -3, 2)$, $P(1,2,-1)$, $\vec{v} = \langle 1, -1, 2 \rangle$, $\overrightarrow{PS} = \langle 0, -5, 3 \rangle$, $\overrightarrow{PS} \times \vec{v} = \langle -7, 3, 5 \rangle$, $|\overrightarrow{PS} \times \vec{v}| = \sqrt{(-7)^2 + (3)^2 + (5)^2} = \sqrt{83}$, $|\vec{v}| = \sqrt{6}$.

Then plugging into the formula, we get

$$D = \frac{|\overrightarrow{PS} \times \vec{v}|}{|\vec{v}|} = \frac{\sqrt{83}}{\sqrt{6}} = 3.719.$$

Question #18: Planes

Find the distance between the given parallel planes

p_1: $2x - 3y + z = 4$, $\quad$ p_2: $4x - 6y + 2z = 3$.

Correct Answer:

$D = 0.668.$

Explanation:

Distance from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is

$$D = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

First we find a point on the first plane by letting x = 0 and y = 0 in the plane equation to get z = 4 and so (0, 0, 4) is a point on the first plane. Next, we find the distance from *P*(0, 0, 4) to the second plane **4*x* - 6*y* + 2*z* – 3 = 0**. Finally, plugging into the formula, we get

$$D = \frac{|\mathbf{4(0) - 6(0) + 2(4) - 3}|}{\sqrt{(4)^2 + (-6)^2 + (2)^2}} = \frac{5}{\sqrt{56}} = 0.668.$$

Question #19: Vector Functions

Find the unit tangent vector for the curve at (e, 1, 1)

$$\vec{\boldsymbol{r}}(t) = (te^t)\vec{\boldsymbol{\imath}} + (t^2)\vec{\boldsymbol{\jmath}} + (t^3)\vec{\boldsymbol{k}}.$$

Possible Answer:

$$\boldsymbol{T}(t) = \frac{(e^t+te^t)\vec{\boldsymbol{\imath}}+(2t)\vec{\boldsymbol{\jmath}}+(3t^2)\vec{\boldsymbol{k}}.}{\sqrt{e^{2t}(1+2t+t^2)+4t^2+9t^4}}$$

Explanation:

To find the unit tangent vector $\boldsymbol{T}(t)$ for a given curve, we use that

$$\boldsymbol{T}(t) = \frac{\vec{r}\prime(t)}{|\vec{r}\prime(t)|} = \frac{(e^t+te^t)\vec{\boldsymbol{\imath}}+(2t)\vec{\boldsymbol{\jmath}}+(3t^2)\vec{\boldsymbol{k}}.}{\sqrt{e^{2t}(1+2t+t^2)+4t^2+9t^4}}.$$

At the point (e, 1, 1) we have *t* = 1. So,

$$\boldsymbol{T}(1) = \left(\frac{2e}{\sqrt{4e^2+13}}\right)\vec{\boldsymbol{\imath}} + \left(\frac{2}{\sqrt{4e^2+13}}\right)\vec{\boldsymbol{\jmath}} + \left(\frac{3}{\sqrt{4e^2+13}}\right)\vec{\boldsymbol{k}}$$

Question #20: Arc Length

Find the length of the curve $\vec{\boldsymbol{r}}(t) = (e^{2t})\vec{\boldsymbol{\imath}} + (e^{-2t})\vec{\boldsymbol{\jmath}} + \left(2\sqrt{2}t\right)\vec{\boldsymbol{k}}$**,**
from $t = 0$ **to** $t = 5$**.**

Correct Answer:

$e^{10} - e^{-10}$

Explanation:

The formula for the length of a parametric curve in 3-dimensional space is $L = \int_a^b |\vec{r}'(t)| dt$.

Taking derivatives, we have

$\vec{r}'(t) = (2e^{2t})\vec{\boldsymbol{\imath}} + (-2e^{-2t})\vec{\boldsymbol{\jmath}} + \left(2\sqrt{2}\right)\vec{\boldsymbol{k}}.$

So,

$$|\vec{r}'(t)| = \sqrt{(2e^{2t})^2 + (-2e^{-2t})^2 + \left(2\sqrt{2}\right)^2} = \sqrt{4e^{4t} + 4e^{-4t} + 8}.$$

Substituting leads to

$$L = \int_0^5 \sqrt{4e^{4t} + 4e^{-4t} + 8}\ dt = \int_0^5 \sqrt{4(e^{4t} + e^{-4t} + 2)}\ dt$$

$$= 2\int_0^5 \sqrt{(e^{2t} + e^{-2t})^2}\ dt$$

$$= 2\int_0^5 (e^{2t} + e^{-2t})\ dt = 2\left(\frac{1}{2}e^{2t} - \frac{1}{2}e^{-2t}\right)\Big]_0^5 = e^{10} - e^{-10}$$

Question #21 : Curvature

Determine the curvature of the vector $\vec{\boldsymbol{r}}(t) = \langle 3t^2, 5t, 0\rangle$ at *t* = 0.

Correct Answer:

$$\kappa = \frac{30}{(36t^2 + 25)^{\frac{3}{2}}}$$

Explanation:

To find the curvature, we use the formula $\kappa = \frac{|\vec{\boldsymbol{r}}\prime(t) \times \vec{\boldsymbol{r}}\prime\prime(t)|}{|\vec{\boldsymbol{r}}\prime(t)|^3}$.

We compute $\vec{\boldsymbol{r}}'(t) = \langle 6t, 5,0\rangle$, $\vec{\boldsymbol{r}}''(t) = \langle 6,0,0\rangle$,

$\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t) = \langle 0,0,-30\rangle$, $|\vec{\boldsymbol{r}}'(t) \times \vec{\boldsymbol{r}}''(t)| = 30$, and

$|\vec{\boldsymbol{r}}'(t)| = \sqrt{36t^2 + 25}$. Then plugging into the formula, we get

$\kappa = \frac{30}{(36t^2+25)^{\frac{3}{2}}}$. At *t* = 0, $\kappa = \frac{6}{25} = 0.24$.

Question #22: Curvature

Determine the curvature of the ellipse $\frac{\boldsymbol{x^2}}{\boldsymbol{9}} + \frac{\boldsymbol{y^2}}{\boldsymbol{16}} = 1$ at the point (3, 0, 0).

Correct Answer:

$$\kappa = \frac{3}{16}$$

Explanation:

Parametrizing the ellipse in space gives

$$x = 3cost, \;\; y = 4sint, \;\; z = 0, \quad 0 \le t \le 2\pi.$$

The vector function can be written as $\vec{\boldsymbol{r}}(t) = \langle 3cost, 4sint, 0 \rangle$**.**

To find the curvature, we use the formula $\kappa = \frac{|\vec{r}\prime(t) \times \vec{r}\prime\prime(t)|}{|\vec{r}\prime(t)|^3}$.

We compute $\vec{r}'(t) = \langle -3sint, 4cost, 0 \rangle$, $\vec{r}''(t) = \langle -3cost, -4sint, 0 \rangle$,

$\vec{r}'(t) \times \vec{r}''(t) = \langle 0,0,12 \rangle$, $|\vec{r}'(t) \times \vec{r}''(t)| = 12$, and

$|\vec{r}'(t)| = \sqrt{9sin^2t + 16cos^2t}$. Then plugging into the formula, we get

$$\kappa(t) = \frac{12}{(9sin^2t+16cos^2t)^{\frac{3}{2}}}.$$

The point (3, 0, 0) can be achieved when *t* = 0 and so

$$\kappa(0) = \frac{3}{16}.$$

Question #23: Normal Vectors

Find the unit normal vector of $\vec{\boldsymbol{r}}(t) = (5cost)\vec{\boldsymbol{\imath}} + (5sint)\vec{\boldsymbol{\jmath}} + (2)\vec{\boldsymbol{k}}$**.**

Correct Answer:

$\boldsymbol{N}(t) = (-cost)\vec{\boldsymbol{\imath}} + (-sint)\vec{\boldsymbol{\jmath}}$

Explanation:

To find the unit normal vector, you must first find the unit tangent vector. The equation for the unit tangent vector, is $\boldsymbol{T}(t) = \frac{\vec{\boldsymbol{r}}\prime(t)}{|\vec{\boldsymbol{r}}\prime(t)|}$.

Then, the equation for the unit normal vector, is $\boldsymbol{N}(t) = \frac{T\prime(t)}{|T\prime(t)|}$.

For this problem

$\vec{\boldsymbol{r}}(t) = (-5sint)\vec{\boldsymbol{\imath}} + (5cost)\vec{\boldsymbol{\jmath}} + (0)\vec{\boldsymbol{k}}$, $|\vec{\boldsymbol{r}}'(t)| = \sqrt{25sin^2t + 25cos^2t} = 5.$

$\boldsymbol{T}(t) = \frac{\vec{\boldsymbol{r}}\prime(t)}{|\vec{\boldsymbol{r}}\prime(t)|} = (-sint)\vec{\boldsymbol{\imath}} + (cost)\vec{\boldsymbol{\jmath}} + (0)\vec{\boldsymbol{k}},$

$\boldsymbol{T}'(t) = (-cost)\vec{\boldsymbol{\imath}} + (-sint)\vec{\boldsymbol{\jmath}} + (0)\vec{\boldsymbol{k}}, \quad |\boldsymbol{T}'(t)| = \sqrt{cos^2t + sin^2t} = 1.$

$\boldsymbol{N}(t) = \frac{T'(t)}{|T'(t)|} = (-cost)\vec{\boldsymbol{\imath}} + (-sint)\vec{\boldsymbol{\jmath}} + (0)\vec{\boldsymbol{k}}.$

Question #24: Normal Vectors

If $\vec{\boldsymbol{r}}(t) \neq 0$**, show that** $\frac{d}{dt}|\vec{\boldsymbol{r}}(t)| = \frac{1}{|\vec{\boldsymbol{r}}(t)|}\vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}'(t)$

Explanation:

We know that $|\vec{\boldsymbol{r}}(t)|^2 = \vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}(t).$

Differentiate both sides with respect to *t* to get

$$\frac{d}{dt}|\vec{\boldsymbol{r}}(t)|^2 = \frac{d}{dt}\left(\vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}(t)\right)$$

$$2|\vec{\boldsymbol{r}}(t)|\frac{d}{dt}|\vec{\boldsymbol{r}}(t)| = \vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}'(t) + \vec{\boldsymbol{r}}'(t) \cdot \vec{\boldsymbol{r}}(t)$$

But $\vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}'^{(t)} = \vec{\boldsymbol{r}}'(t) \cdot \vec{\boldsymbol{r}}(t)$, so

$$2|\vec{\boldsymbol{r}}(t)|\frac{d}{dt}|\vec{\boldsymbol{r}}(t)| = \vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}'(t) + \vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}'(t)$$

$$2|\vec{\boldsymbol{r}}(t)|\frac{d}{dt}|\vec{\boldsymbol{r}}(t)| = 2\left(\vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}'(t)\right)$$

$$\frac{d}{dt}|\vec{\boldsymbol{r}}(t)| = \frac{1}{|\vec{\boldsymbol{r}}(t)|}\vec{\boldsymbol{r}}(t) \cdot \vec{\boldsymbol{r}}'(t)$$

Question 1 (6 points): Let $\vec{u} = \langle 2,0,-3 \rangle$ and $\vec{v} = \langle 1,4,-1 \rangle$. Find

a. $|2\vec{u} - 3\vec{v}|$

$$2\vec{u} - 3\vec{v} = \langle 4, 0, -6 \rangle - \langle 3, 12, -3 \rangle$$

$$= \langle 1, -12, -3 \rangle$$

$$|2\vec{u} - 3\vec{v}| = \sqrt{1^2 + (-12)^2 + (-3)^2} = \sqrt{154} \approx 12.4$$

b. $proj_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$

$$\vec{u} \cdot \vec{v} = (2)(1) + (0)(4) + (-3)(-1) = 5$$

$$|\vec{v}|^2 = 1^2 + 4^2 + (-1)^2 = 18$$

$$Proj_{\vec{v}}\vec{u} = \frac{5}{18} \langle 1, 4, -1 \rangle = \left\langle \frac{5}{18}, \frac{20}{18}, \frac{-5}{18} \right\rangle$$

c. a unit vector $\vec{w}$ in the direction opposite to the direction of $\vec{u}$.

$$|\vec{u}| = \sqrt{2^2 + 0^2 + (-3)^2} = \sqrt{13}$$

$\dfrac{-1}{\sqrt{13}} \vec{u} = \left\langle -\dfrac{2}{\sqrt{13}}, 0, \dfrac{3}{\sqrt{13}} \right\rangle$ is a unit vector in the direction opposite to the direction of $\vec{u}$

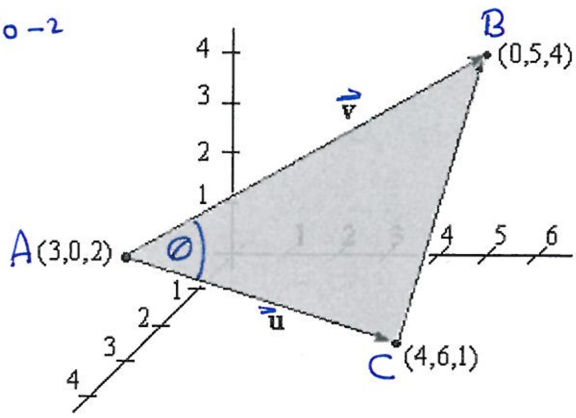
Question 2 (3 points): Find the angle θ in the accompanying figure.

$\vec{v} = \overrightarrow{AB} = \langle -3, 5, 2 \rangle$

$\vec{u} = \overrightarrow{AC} = \langle 1, 6, -1 \rangle$

$\vec{u} \cdot \vec{v} = -3 + 30 - 2$

$= 25$



$|\vec{v}| = \sqrt{(-3)^2 + 5^2 + 2^2} = \sqrt{38}$

$|\vec{u}| = \sqrt{1^2 + 6^2 + (-1)^2} = \sqrt{38}$

$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right) = \cos^{-1}\left(\frac{25}{38}\right) = 48.8^\circ$ (0.85 in radian)

Question 3 (3 points): Find parametric equations for the line through $P(-1,3,-4)$ and $Q(7,-2,6)$.

As a point on the line we take $P(-1,3,-4)$

As a directional vector we take $\overrightarrow{PQ}$

$\vec{d} = \overrightarrow{PQ} = \langle 8, -5, 10 \rangle$

Parametric Eqs for the line are

$x = -1 + 8t$

$y = 3 - 5t \qquad -\infty < t < \infty$

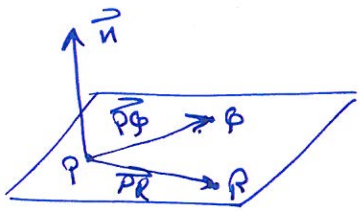
$z = -4 + 10t$

Question 6 (4 points): Find an equation of the plane containing the points $P(1,-1,7)$, $Q(3,2,-5)$ and $R(0,2,1)$.

As a point we take $P(1,-1,7)$

A vector $\vec{n}$ normal to the plane can be

$$\vec{n} = \overrightarrow{PQ} \times \overrightarrow{PR}$$



$$= \langle 2, 3, -12 \rangle \times \langle -1, 3, -6 \rangle$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -12 \\ -1 & 3 & -6 \end{vmatrix}$$

$$= 18\vec{i} + 24\vec{j} + 9\vec{k}$$

The equation of the plane is

$$18(x-1) + 24(y+1) + 9(z-7) = 0$$

$$18x + 24y + 9z = 57$$

Question 7 (3 points): **Find the distance from the point $P(2,2,3)$ to the plane $2x + y + 2z = 4$.**

$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right|$$

To find a point S on the plane we set x=y=0

We get the point $S(0,0,2)$

$\vec{PS} = \langle -2, -2, -1 \rangle$

From the equation of the plane we find $\vec{n} = \langle 2, 1, 2 \rangle$

$|\vec{n}| = \sqrt{2^2 + 1^2 + 2^2} = 3$

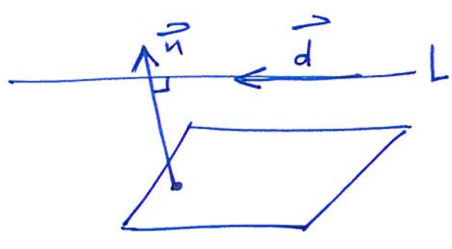
$$d = \left| \vec{PS} \cdot \frac{\vec{n}}{|\vec{n}|} \right| = \left| \langle -2,-2,-1 \rangle \cdot \langle \tfrac{2}{3}, \tfrac{1}{3}, \tfrac{2}{3} \rangle \right| = \left| -\frac{4}{3} - \frac{2}{3} - \frac{2}{3} \right|$$

$$= \frac{8}{3}$$

Question 8 (2 points): **Show that the line**

$$x = 3 + 5t, \qquad y = 1 + t, \qquad z = 1 - 2t$$

is parallel to the plane $x + 3y + 4z = 5$.



It is enough to show that

$\vec{d} = \langle 5, 1, -2 \rangle$ and $\vec{n} = \langle 1, 3, 4 \rangle$

are orthogonal. To do that we find

$\vec{d} \cdot \vec{n} = 5 + 3 - 8 = 0.$

Question 1. (2+1+2+2=7 marks)

If $\vec{u} = \vec{i} + 2\vec{j}$, $\vec{v} = \vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{w} = \vec{i} + 4\vec{j} - 3\vec{k}$. Find

(i) $|2\vec{u} - 3\vec{w}|$ $= |2(i+2j) - 3(i+4j-3K)|$ $\frac{1}{2}$

$= |2i+4j-3i-12j+9K|$ $\frac{1}{2}$

$= |-i-8j+9K|$ $\frac{1}{2}$

$= \sqrt{(-1)^2+(-8)^2+(9)^2} = \sqrt{146}$ $\frac{1}{2}$

(ii) $\vec{w} \cdot \vec{u}$

$= (i+4j-3K) \cdot (i+2j+0K)$ $\frac{1}{2}$

$= (1)(1) + (4)(2) + (-3)(0)$ $\frac{1}{2}$

$= 1+8 = 9$

(iii) A vector orthogonal to both $\vec{v}$ and $\vec{w}$

A vector orthogonal to both $\vec{v}$ and $\vec{w}$ is $\vec{v} \times \vec{w}$

So $\vec{v} \times \vec{w} = \begin{vmatrix} i & j & K \\ 1 & 3 & -2 \\ 1 & 4 & -3 \end{vmatrix} = i\begin{vmatrix} 3 & -2 \\ 4 & -3 \end{vmatrix} - j\begin{vmatrix} 1 & -2 \\ 1 & -3 \end{vmatrix} + K\begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$ $\frac{1}{2}$ $\frac{1}{2}$

$= i(-9+8) - j(-3+2) + K(4-3)$ $\frac{1}{2}$

$= \cancel{-i-5j+K} = -i+j+K$ $\frac{1}{2}$

(iv) $proj_{\vec{w}} \vec{v}$

$= \left(\frac{V \cdot W}{|W|^2}\right) W$ $\frac{1}{2}$

$= \left(\frac{(i+3j-2K)\cdot(i+4j-3K)}{\left(\sqrt{(1)^2+(4)^2+(-3)^2}\right)^2}\right)(i+4j-3K)$ $\frac{1}{2}$

$$= \left(\frac{(1)(1)+(3)(4)+(-2)(-3)}{26}\right)(i+4j-3k) \quad \frac{1}{2}$$

$$= \frac{19}{26}(i+4j-3k) = \frac{19}{26}i + \frac{38}{13}j - \frac{57}{26}k \quad \frac{1}{2}$$

Question 2. (3 marks)

Find the volume of the parallelepiped determined by the vectors $\vec{u} = \langle 1,0,6\rangle$, $\vec{v} = \langle 2,3,-8\rangle$ and $\vec{w} = \langle 8,-5,6\rangle$.

$$\begin{array}{ccc} + & - & + \end{array}$$

$$\text{Volume} = \left|\begin{vmatrix} 1 & 0 & 6 \\ 2 & 3 & -8 \\ 8 & -5 & 6 \end{vmatrix}\right| \quad 1$$

$$= 1\begin{vmatrix} 3 & -8 \\ -5 & 6 \end{vmatrix} - 0\begin{vmatrix} 2 & -8 \\ 8 & 6 \end{vmatrix} + 6\begin{vmatrix} 2 & 3 \\ 8 & -5 \end{vmatrix} \quad 1$$

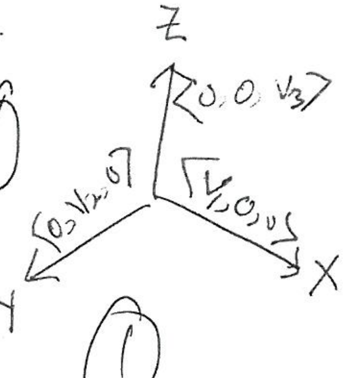
$$= 1(18-40) - 0 + 6(-10-24)$$

$$= -22 - 204 = -226 \quad 1$$

Volume is always positive quantity, so $|-226| = 226$ cubic un.

Question 3. (3 marks)

Find parametric equations for the line through $P(1,-2,3)$ and parallel to the y-axis.



$P_0(x_0, y_0, z_0) = P(1,-2,3)$ 1

A vector parallel to Y-axis $= \langle 0, V_2, 0\rangle$ 1

So parametric equations for lines are

$x = 1 + 0t$, $y = -2 + V_2 t$, and $z = 3 + 0t$ 1

Question 4. (4 marks)

Find an equation of the plane passing through the points $A(2,4,5)$, $B(1,5,7)$ and $C(-1,6,8)$.

$\vec{AB} = \langle -1, 1, 2 \rangle$ $\frac{1}{2}$

$\vec{AC} = \langle -3, 2, 3 \rangle$ $\frac{1}{2}$

So normal vector $\vec{n} = \vec{AB} \times \vec{AC}$

$$= \begin{vmatrix} i & j & k \\ -1 & 1 & 2 \\ -3 & 2 & 3 \end{vmatrix} \quad 1$$

$$= i\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} - j\begin{vmatrix} -1 & 2 \\ -3 & 3 \end{vmatrix} + k\begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix}$$

$$= i(3-4) - j(-3+6) + k(-2+3)$$

$$= -i - 3j + k \quad \frac{1}{2}$$

But normal vector to any plane is given by

$n = Ai + Bj + Ck$ $\frac{1}{2}$

$\Rightarrow A = -1$, $B = -3$ and $C = 1$, Also we choose

PB. 4

$P_0(x_0, y_0, z_0) = A(2, 4, 5)$. Thus required

equation of plane is

$\boxed{A(x-x_0)+B(y-y_0)+C(z-z_0)=0}$ 1

$-1(x-2)-3(y-4)+1(z-5)=0$

$\Rightarrow -x+2-3y+12+z-5=0 \Rightarrow -x-3y+z=-9$

Question 5. (3 marks)

Find the angle between the planes $x+z=2$ and $2x-3y+2z=4$.

From given planes, we can find the normal vector n_1 and n_2 respectively. $\frac{1}{2}$

$n_1=\langle 1,0,1\rangle$, $n_2=\langle 2,-3,2\rangle$ $\frac{1}{2}$

Angle between the planes is in fact an angle between their normal vectors, so

$$\theta=\cos^{-1}\left(\frac{n_1\cdot n_2}{|n_1||n_2|}\right) \quad \frac{1}{2}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{\langle 1,0,1\rangle\cdot\langle 2,-3,2\rangle}{\sqrt{(1)^2+(0)^2+(1)^2}\sqrt{(2)^2+(-3)^2+(2)^2}}\right) \quad \frac{1}{2}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{2-0+2}{\sqrt{2}\sqrt{17}}\right) \quad \frac{1}{2}$$

$$\Rightarrow \theta=\cos^{-1}\left(\frac{4}{\sqrt{34}}\right) \quad \frac{1}{2}$$

$$\Rightarrow \theta=46.68^\circ$$

Answer Key

Problem 1. (3 marks) Find a vector of magnitude 2 in the direction opposite to the direction of the vector $\vec{u} = 2\vec{i} + 2\vec{j} - \vec{k}$

$$|\vec{u}| = \sqrt{(2)^2 + (2)^2 + (-1)^2} = \sqrt{9} = 3$$

A unit vector in the direction of $\vec{u}$ is $\frac{1}{|\vec{u}|}\vec{u}$

A vector of magnitude 2 in the direction opposite to the direction of $\vec{u}$ is

$$\vec{V} = -2\frac{1}{\vec{u}}\vec{u} = \frac{-2}{3}\langle 2, 2, -1\rangle = \langle \frac{-4}{3}, \frac{-4}{3}, \frac{2}{3}\rangle$$

Problem 2. (3 marks) For the given points $A(2,0,-2),\ B(-1,2,-3)$ *and* $C(1,5,-4)$, are the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ orthogonal? Justify your answer.

The components of the vector $\overrightarrow{AB}$ are $\overrightarrow{AB} = \langle -3, 2, -1\rangle$

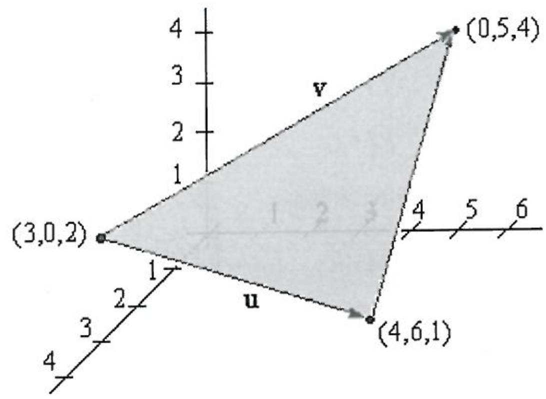
The component of the vector $\overrightarrow{AC}$ are $\overrightarrow{AC} = \langle -1, 5, -2\rangle$

$$\overrightarrow{AB} \cdot \overrightarrow{AC} = (-3)(-1) + (2)(5) + (-1)(-2) = 3 + 10 + 2 = 15$$

Since $\overrightarrow{AB} \cdot \overrightarrow{AC} \neq 0$ then the two vectors are

not orthogonal.

Problem 3. (4 marks) **Find the area of the shaded tringle in the accompanying figure.**



$\vec{u} = \langle 1, 6, -1 \rangle$

$\vec{v} = \langle -3, 5, 2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & -1 \\ -3 & 5 & 2 \end{vmatrix}$$

$$= (12 - (-5))\vec{i} - (2 - 3)\vec{j} + (5 - (-18))\vec{k}$$

$$= 17\vec{i} + \vec{j} + 23\vec{k}$$

Area of the tringle $= \frac{1}{2}$ Area of the parallelogram determined by $\vec{u}$ and $\vec{v}$

$$= \frac{1}{2} |\vec{u} \times \vec{v}|$$

$$= \frac{1}{2} \sqrt{17^2 + 1^2 + 23^2}$$

$$= \frac{3}{2}\sqrt{91} = 14.30$$

Answer Key

Problem 1. **(3 marks)** Find a vector of magnitude 3 in the direction opposite to the direction of the vector $\vec{u} = \vec{\imath} + 4\vec{\jmath} - 2\vec{k}$

$$|\vec{u}| = \sqrt{(1)^2 + (4)^2 + (-2)^2} = \sqrt{21}$$

A unit vector in the direction of $\vec{u}$ is $\frac{1}{|\vec{u}|}\vec{u} = \frac{1}{\sqrt{21}}\vec{u}$

The required vector is

$$\vec{v} = -3\frac{1}{\sqrt{21}}\vec{u} = \frac{-3}{\sqrt{21}}\langle 1, 4, -2\rangle = \left\langle \frac{-3}{\sqrt{21}}, \frac{-12}{\sqrt{21}}, \frac{6}{\sqrt{21}} \right\rangle$$

Problem 2. **(3 marks)** For the given points $A(1,1,0)$, $B(1,-2,3)$ *and* $C(0,-5,1)$, are the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ parallel? Justify your answer.

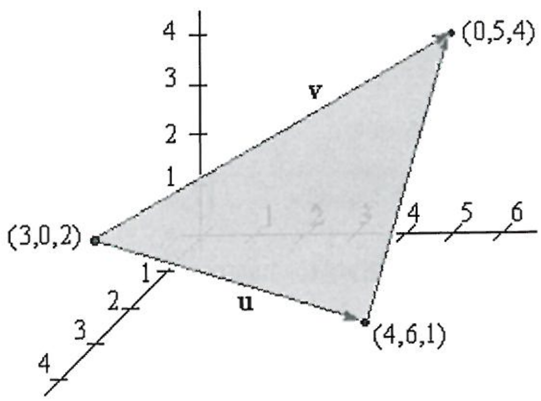
$$\overrightarrow{AB} = \langle 0, -3, 3\rangle$$

$$\overrightarrow{AC} = \langle -1, -6, 1\rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -3 & 3 \\ -1 & -6 & 1 \end{vmatrix} = 15\vec{i} - 3\vec{j} - 3\vec{k}$$

Since $\overrightarrow{AB} \times \overrightarrow{AC} \neq \vec{0}$ the two vectors are <u>not</u> parallel

Problem 3. (4 marks) Find the area of the shaded tringle in the accompanying figure.



$\vec{u} = \langle 1, 6, -1 \rangle$

$\vec{v} = \langle -3, 5, 2 \rangle$

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 6 & -1 \\ -3 & 5 & 2 \end{vmatrix}$$

$$= 17\vec{i} + \vec{j} + 23\vec{k}$$

Area of the tringle $= \frac{1}{2}$ Area of the parallelogram determined by $\vec{u}$ and $\vec{v}$

$$= \frac{1}{2} |\vec{u} \times \vec{v}|$$

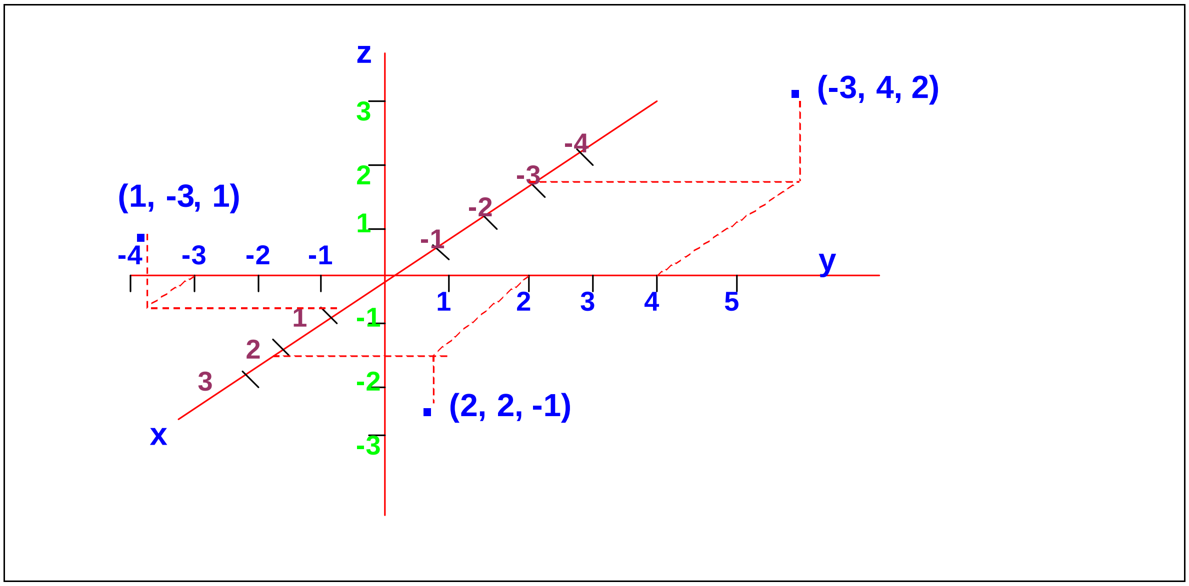
$$= \frac{1}{2}\sqrt{17^2 + 1^2 + 23^2}$$

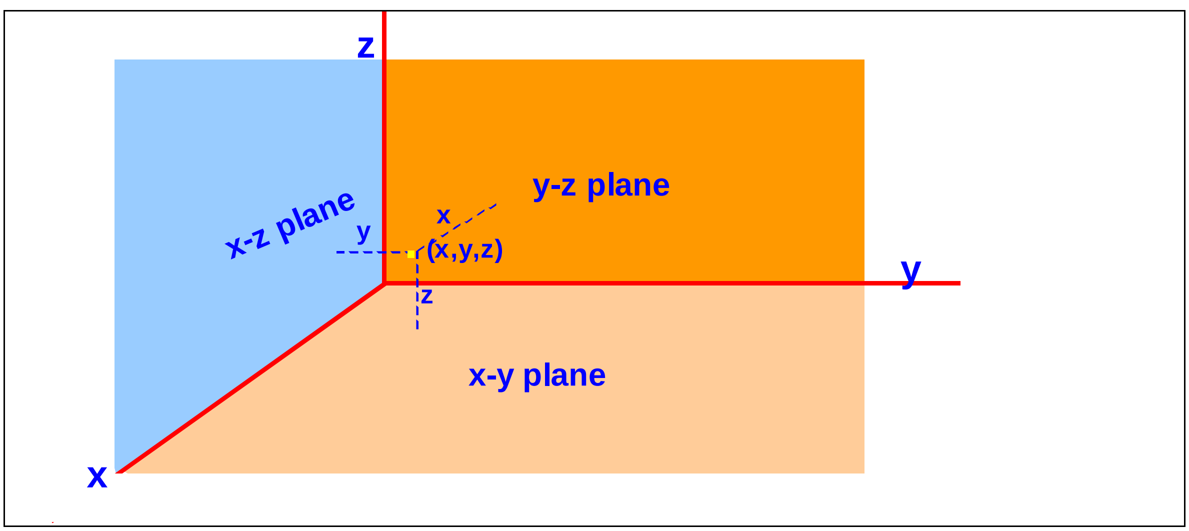
$$= \frac{3}{2}\sqrt{91}$$

$$= 14.30$$

Lesson 1.2
Vectors in Space

In this lesson we will look at vectors in a three dimensional space. So far we looked at vectors in the x-y plane defined by x and y axes. A space is defined by three axes, x, y and z. In a right handed system, the three axes will look like the diagram below.





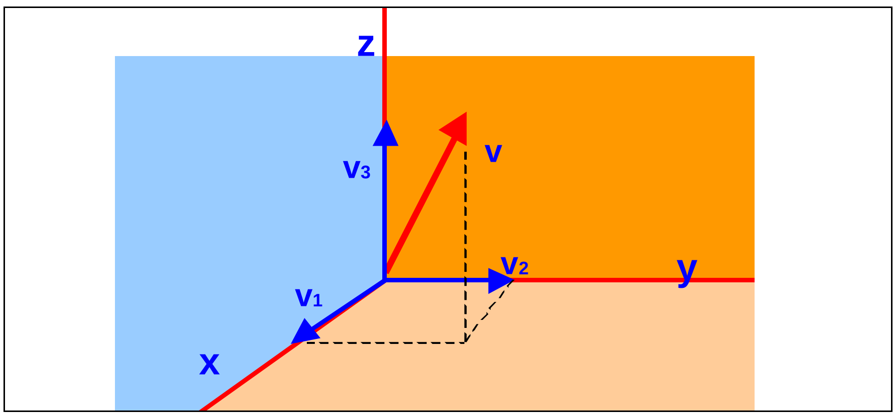
The x and y axes define the x-y plane, the y and z axes define the y-z plane and the x and z axes define the x-z plane. A point P in space is determined by an ordered triple (x,y,z) where x is the directed distance of P from the y-z

plane, y is the directed distance of P from the x-z plane and z is the directed distance of P from the x-y plane.
The three planes namely x-y plane, y-z plane and the x-z plane divide the space into eight octants. In the first octant, all three coordinates are positive.
The distance between two point P(x_1,y_1,z_1) and Q(x_2,y_2,z_2) in space is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

A sphere in space with center (x_o,y_o,z_o) has the standard equation

$$(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2 = r^2$$



A vector in space has three components, an x component ($\mathbf{v}_1$), a y component ($\mathbf{v}_2$) and a z component ($\mathbf{v}_3$). A vector v in the component form can be written as

$$\mathbf{v} = \langle \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \rangle = v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}$$

where **i**, **j**, **k** are unit vectors in the x, y and z directions respectively.
The magnitude of vector v is given by:

$$\|v\| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$$

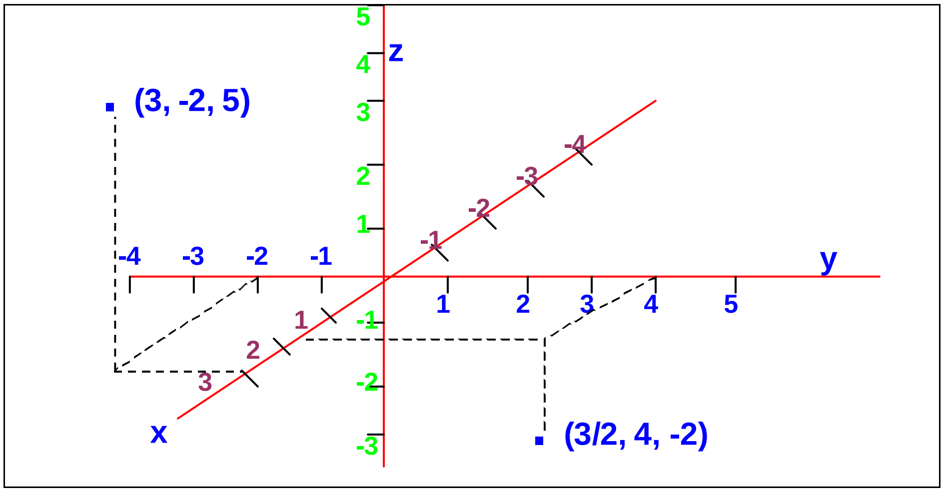
The unit vector in the direction of **v** is given by:

$$n = \frac{v}{\|v\|} = \frac{\langle v_1, v_2, v_3 \rangle}{\sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}}$$

Example 1:

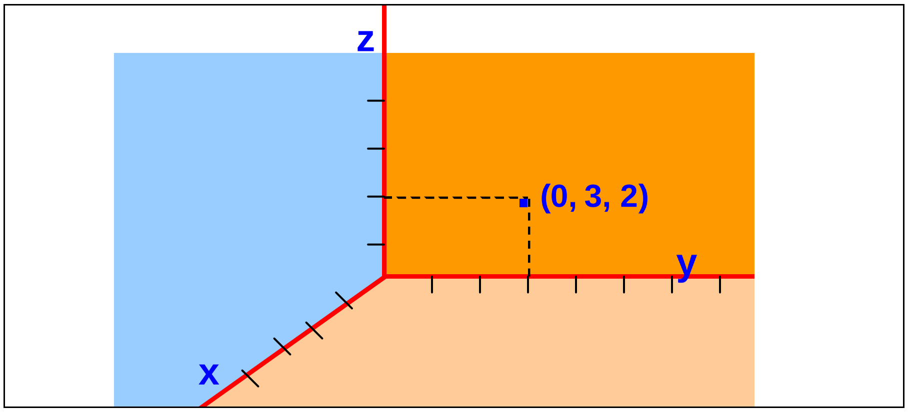
Plot the point (a) (3, -2, 5) and (b) (3/2, 4, -2) on the same three dimensional coordinate system.

Solution:



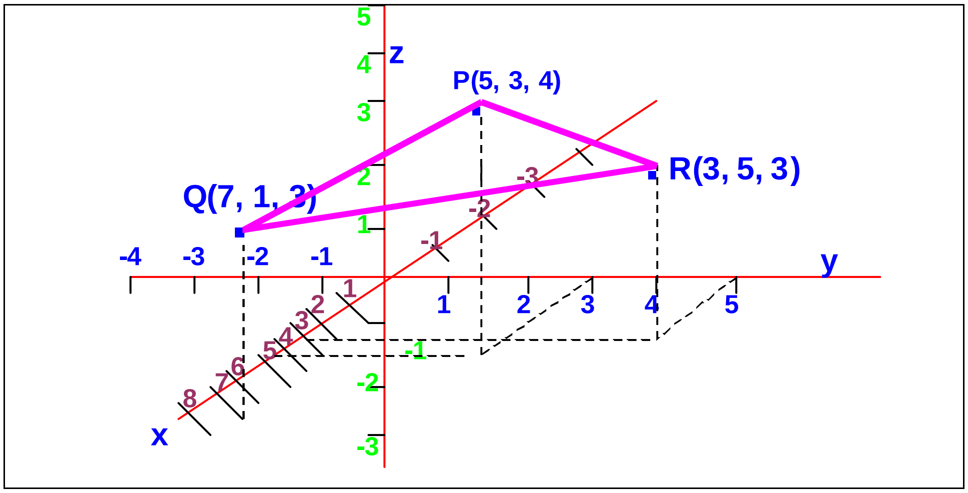
Example 2:
Find the coordinates of the point located on the y-z plane, three units to the right of the x-z plane and two units above the x-y plane.

Solution:



Example 3:
Find the lengths of the sides of the triangle with the vertices P(5, 3, 4), Q(7, 1, 3), R(3, 5, 3) and determine whether the triangle is a right triangle, an isosceles or neither.

Solution:



$$PQ = \sqrt{(7-5)^2 + (1-3)^2 + (3-4)^2} = \sqrt{9} = 3$$
$$QR = \sqrt{(3-7)^2 + (5-1)^2 + (3-3)^2} = \sqrt{32}$$
$$PR = \sqrt{(3-5)^2 + (5-3)^2 + (3-4)^2} = \sqrt{9} = 3$$

Since the lengths of two sides are equal, this is an isosceles triangle.

Example 4:
Find the midpoint of the line segment joining P(4, 0, -6) and Q(8, 8, 20).

Solution:
Midpoint is given by:
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$$
$$= \left(\frac{8+4}{2}, \frac{8+0}{2}, \frac{20-6}{2}\right) = (6, 4, 7)$$

Example 5:
Find the general form of the equation of the sphere with its center at (4, -1, 1) and radius r = 5 units.

Solution:
The standard equation of the circle is:

$$(x-x_o)^2+(y-y_o)^2+(z-z_o)^2=r^2$$
$$(x-4)^2+(y+1)^2+(z-1)^2=5^2$$
$$x^2-8x+16+y^2+2y+1+z^2-2z+1=25$$
$$x^2+y^2+z^2-8x+2y-2z-7=0$$

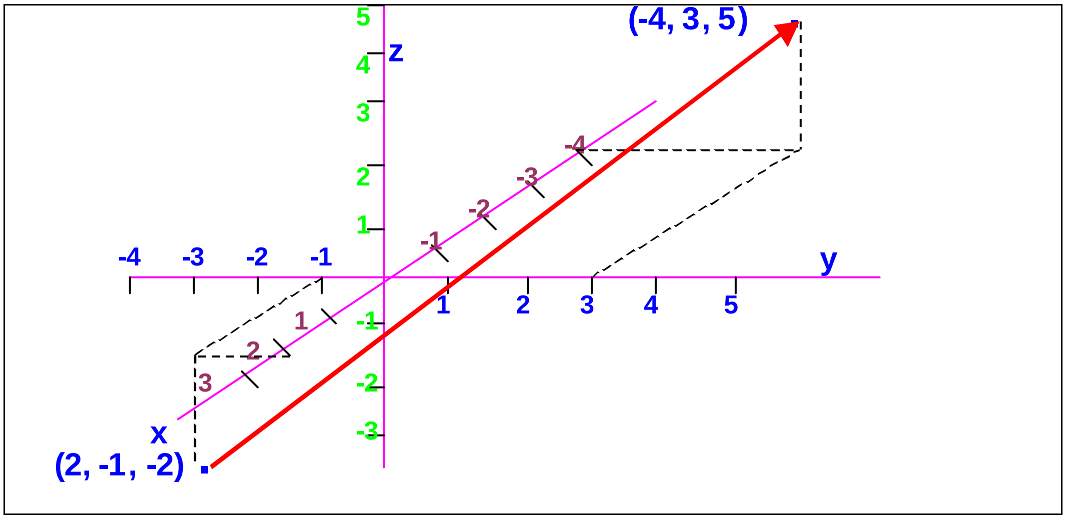
Example 6:
Find the center and the radius of the sphere given by the general equation
$4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 = 0$

Solution:
$4x^2 + 4y^2 + 4z^2 - 4x - 32y + 8z + 33 = 0$
$4x^2 - 4x + 4y^2 - 32y + 4z^2 + 8z = -33$
$4(x^2 - x) + 4(y^2 - 8y) + 4(z^2 + 2z) = -33$
We complete square on this
$4(x^2 - x + (1/2)^2) + 4(y^2 - 8y + 4^2) + 4(z^2 + 2z + 1^2) = -33 + 1 + 64 + 4$
$4(x - ½)^2 + 4(y - 4)^2 + 4(z + 1)^2 = 36$
$(x - ½)^2 + (y - 4)^2 + (z + 1)^2 = 9$
The center is (1/2, 4, -1) and r = 3 units.

Example 7:
The initial point of a vector is (2, -1, -2) and its terminal point is (-4, 3, 5). (a) Sketch the directed line segment, (b) find the component form of the vector, and (c) sketch the vector with its initial point at the origin.

Solution:

(a)



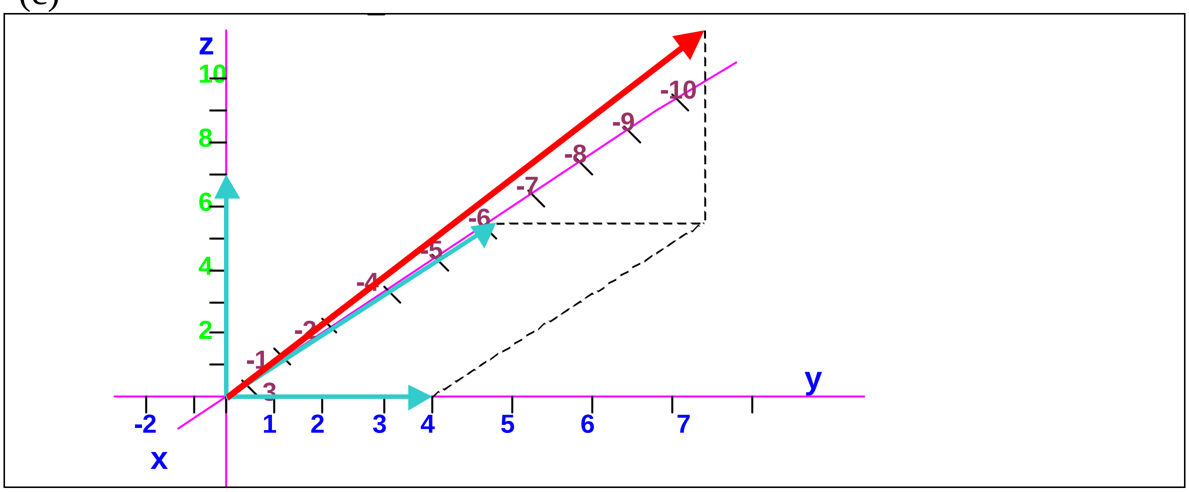
(b) $\mathbf{v}_1 = x_2 - x_1 = -4 - 2 = -6$

$\mathbf{v}_2 = y_2 - y_1 = 3 - (-1) = 4$

$\mathbf{v}_3 = z_2 - z_1 = 5 - (-2) = 7$

The component form of the vector is: $\mathbf{v} = \langle -6, 4, 7 \rangle$

(c)



Example 8:

Find the terminal point of the vector $v = \langle 0, ½, -1/3 \rangle$ having an initial point $(3, 0, -2/3)$

Solution:

Let (x_2, y_2, z_2) be the terminal point.

$0 = x_2 - 3$ or, $x_2 = 3$

$½ = y_2 - 0$ or, $y_2 = ½$

$-1/3 = z_2 - (-2/3)$, or, $z_2 = -1$
The terminal point is **(3, ½, -1)**

Example 9:
If **z** = **u** - **v** + 2**w** and **u** = ⟨1, 2, 3⟩, **v** = ⟨2, 2, -1⟩ and **w** = ⟨4, 0, -4⟩, find the vector **z**.

Solution:

u = ⟨1, 2, 3⟩
-v = ⟨-2, -2, 1⟩, 2**w** = ⟨8, 0, -8⟩
z = u - v + 2w = ⟨7, 0, -4⟩

Example 10:
Which of the following vectors is parallel to:

$$z = \frac{1}{2}i - \frac{2}{3}j + \frac{3}{4}k$$

$$(a) \quad 6i - 4j + 9k$$

$$(b) \quad -i + \frac{4}{3}j - \frac{3}{2}k$$

$$(c) \quad 12i + 9k$$

$$(d) \quad \frac{3}{4}i - j + \frac{9}{8}k$$

Solution:

If a vector is parallel to z, then it must be a scalar multiple of z. If the ratio of all three components is the same, then the vectors are scalar multiples of each other, and hence parallel. So we take the ratios of the components of each of the vectors with the components of z and see in which case all three ratios are the same.

$$(a) \quad 6i - 4j + 9k \qquad z = \langle \frac{1}{2}i - \frac{2}{3}j + \frac{3}{4}k \rangle$$

$$\frac{6}{1/2} = 12; \ \frac{-4}{-2/3} = 6; \ \frac{9}{3/4} = 12$$

This is not parallel to z.

$$(b) \quad -i+\frac{4}{3}j-\frac{3}{2}k \qquad z=\langle\frac{1}{2}i-\frac{2}{3}j+\frac{3}{4}k\rangle$$

$$\frac{-1}{1/2}=-2; \ \frac{4/3}{-2/3}=-2; \ \frac{-3/2}{3/4}=-2$$

$$z=-2b$$

This is parallel to z

$$(c) \quad 12i+9k \qquad z=\langle\frac{1}{2}i-\frac{2}{3}j+\frac{3}{4}k\rangle$$

This is not parallel to z since it has no y component.

$$(d) \quad \frac{3}{4}i-j+\frac{9}{8}k \qquad z=\langle\frac{1}{2}i-\frac{2}{3}j+\frac{3}{4}k\rangle$$

$$\frac{3/4}{1/2}=\frac{3}{2}; \ \frac{-1}{-2/3}=\frac{3}{2}; \ \frac{9/8}{3/4}=\frac{3}{2}$$

$$z=\frac{3}{2}d$$

This vector is parallel to z

Example 11:
Use vectors to determine whether the points P(1, -1, 5), Q(0, -1, 6) and R(3, -1, 3) lie in a straight line.

Solution:
If points P, Q and R lie on a straight line, then vectors $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{QR}$ will be scalar multiples of each other.
P(1, -1, 5), Q(0, -1, 6) and R(3, -1, 3)

$$\overrightarrow{PQ}=\langle 0-1,-1-(-1),6-5\rangle=\langle -1,0,1\rangle$$

$$\overrightarrow{PR}=\langle 3-1,-1-(-1),3-5\rangle=\langle 2,0,-2\rangle=-2\langle -1,0,1\rangle$$

$$\overrightarrow{QR}=\langle 3-0,-1-(-1),3-6\rangle=\langle 3,0,-3\rangle=-3\langle -1,0,1\rangle$$

These vectors are scalar multiples of each other, and so they are collinear.

Example 12:

Find the magnitude of the vector **v** = -4**i** + 3**j** + 7**k**

Solution:

$$\|v\| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}} = \sqrt{(-4)^2 + 3^2 + 7^2} = \sqrt{74}$$

Example 13:
Find a unit vector (a) in the direction of **u**, (b) in the direction opposite to **u** where **u** = ⟨6, 0, 8⟩

Solution:

(a) $$n = \frac{u}{\|u\|} = \frac{\langle 6,0,8\rangle}{\sqrt{6^2+8^2}} = \frac{\langle 6,0,8\rangle}{\sqrt{100}} = \left\langle \frac{3}{5}, 0, \frac{4}{5} \right\rangle$$

(b) $$-\left\langle \frac{3}{5}, 0, \frac{4}{5} \right\rangle$$

Example 14:
Determine the value of c such that the vector **v** = **i** + 2**j** + 3**k** satisfies the relation $\|cv\| = 3$

Solution:

c**v** = c **i** + 2c **j** + 3c **k**

$$\|cv\| = \sqrt{c^2 + 4c^2 + 9c^2} = \sqrt{14}c$$

$$\sqrt{14}\ c = 3$$

$$c = \pm\frac{3\sqrt{14}}{14}$$

Example 15:
Find the vector **v** with a magnitude $\sqrt{5}$ and in the direction of **u** = ⟨-4, 6, 2⟩

Solution:
The unit vector in the direction of **u** is given by

$$n = \frac{u}{\|u\|} = \frac{\langle -4,6,2\rangle}{\sqrt{16+36+4}} = \frac{1}{2\sqrt{14}}\langle -4,6,2\rangle = \frac{1}{\sqrt{14}}\langle -2,3,1\rangle$$

$$v = \sqrt{5}n = \sqrt{\frac{5}{14}}\langle -2,3,1\rangle = \frac{\sqrt{70}}{14}\langle -2,3,1\rangle$$

Example 16:

The lights in an auditorium are 24 N discs of radius 18 cm. Each disc is supported by three equally spaced wires that are L cm long.

(a) Write the tension T in each wire as a function of L.
Determine the domain of the function.

(b) Complete the following table.

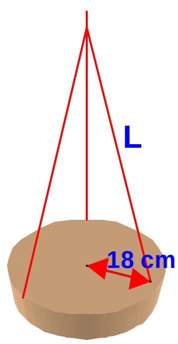
L	20	25	30	35	40	45	50
T							

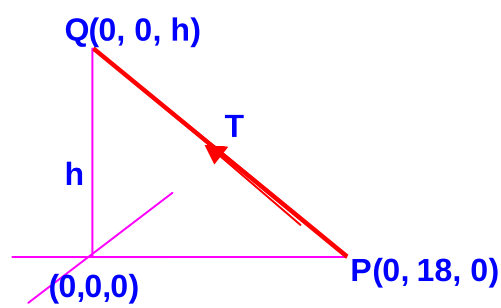
(c) Graph the model in part (a) and determine the asymptotes of the graph.

(d) Confirm the asymptotes analytically.

(e) Determine the minimum length of each cable if a cable is designed to carry a maximum load of 10 N.

Solution:





The tensions on all three cables are the same.
If **T** is the tension on one of the cables as shown in the figure above, it will be a scalar multiple of the vector PQ

$$\vec{PQ} = \langle 0, -18, h\rangle$$

T = c ⟨0, -18, h⟩ where $h = \sqrt{L^2 - 18^2}$

The total tension on the three cables = 3**T** = 3c ⟨0, -18, h⟩ ….(i)

This total tension balances the downward force of 24 N which is the weight of the disc. The vector form of this weight is

W = ⟨0, 0, -24⟩ ….(ii)

Since 3T = W, we have 3ch = -24

Or,

$$c = \frac{-8}{h}$$

$$T = c\langle 0, -18, h\rangle = \frac{-8}{h}\langle 0, -18, h\rangle$$

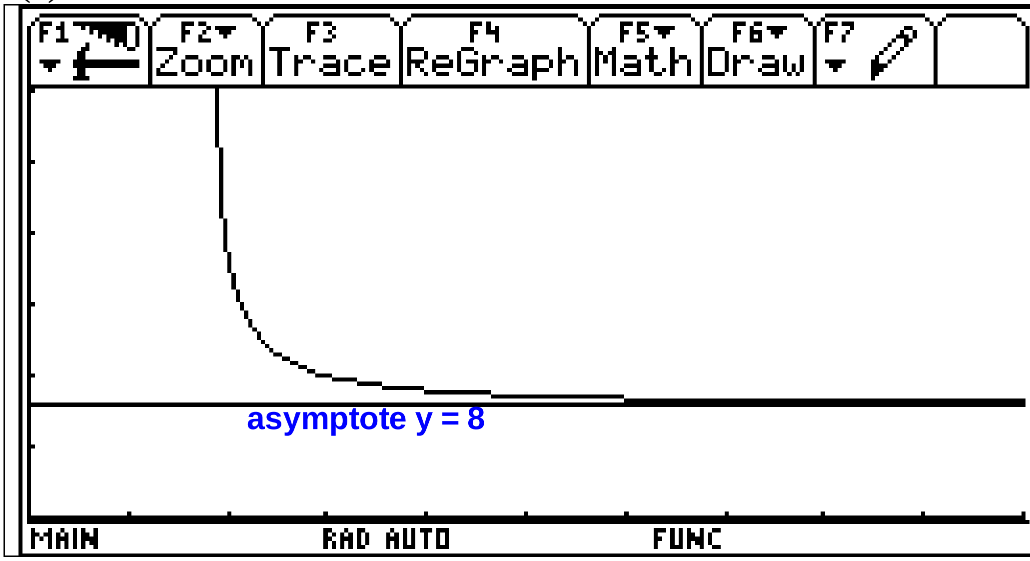
$$\|T\| = \frac{8}{h}\sqrt{18^2 + h^2} = \frac{8}{\sqrt{L^2 - 18^2}}\sqrt{18^2 + L^2 - 18^2}$$

$$= \frac{8L}{\sqrt{L^2 - 18^2}}$$

(b)

L	20	25	30	35	40	45	50
T	18.4	11.5	10	9.3	8.9	8.7	8.5

(c)



(d) $$\lim_{L \to \infty} \frac{8L}{\sqrt{L^2 - 18^2}} = 8$$

(e) From the table in (b) above, when T = 10 N, L = 30 cm

Homework:

1. Find the lengths of the sides of the triangle with vertices (0,0,0), (2,2,1), (2, -4,4) and determine whether is right triangle, isosceles or neither.

2. Find the standard form of the equation of the sphere with end points of its diameter (2,0,0) and (0,6,0)

3. Find the center and radius of the sphere $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$

4. Find the center and radius of the sphere $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

5. Given that $\mathbf{v} = \langle 1,2,2 \rangle$, sketch (a) $2\mathbf{v}$, (b) $-\mathbf{v}$, (c) $(3/2)\mathbf{v}$, (d) $0\mathbf{v}$

6. Given that $\mathbf{u} = \langle 1,2,3 \rangle$, $\mathbf{v} = \langle 2,2,-1 \rangle$, and $\mathbf{w} = \langle 4,0,-4 \rangle$, find the vector $\mathbf{z} = 2\mathbf{u} + 4\mathbf{v} - \mathbf{w}$

7. Determine which of the following vectors is parallel to $\mathbf{z} = \langle 3,2,-5 \rangle$?
   (a) $\langle 6, -4, 10 \rangle$ (b) $\langle 2, 4/3, -10/3 \rangle$
   (c) $\langle 6 ,4 ,10 \rangle$ (d) $\langle 1, -4, 2 \rangle$

8. If vector **z** has the initial point (1, -1, 3) and terminal point (-2, 3, 5), which of the following vectors is parallel to z?
   (a) $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$ (b) $4\mathbf{j} + 2\mathbf{k}$

9. Use vectors to find whether the points (1,2,4), (2,5,0), (0,1,5) are collinear.

10. If $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$, find the magnitude of **v**

11. If $\mathbf{u} = \langle 2,-1,2 \rangle$, find a unit vector (a) in the direction of **u** and (b) in the direction opposite to **u**.

12. Find the vector v which has a magnitude 3/2 and is in the direction of $\mathbf{u} = \langle 2, -2, 1 \rangle$

Lesson 1.3
The Dot Product of Two Vectors

1. Dot Product

The dot product of two vectors **u** and **v** is the product of the magnitude of **u** and the component of **v** in the direction of **u**. If θ is the angle between **u** and **v**, then the component of **v** in the direction of **u** is $\|v\|\cos\theta$

$$u \cdot v = \|u\|\|v\|\cos\theta \qquad \ldots(i)$$

If **u** = $\langle u_1, u_2, u_3\rangle$ and **v** = $\langle v_1, v_2, v_3\rangle$

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$$

The dot product of vectors is a scalar quantity and it has the following properties:
If **u**, **v**, **w** are vectors in a plane or in space, then

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ **(commutative property)**
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ **(Distributive property)**
3. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$
4. $\mathbf{0} \cdot \mathbf{v} = 0$

From equation (i) above, it also follows that:

If vectors u and v are at right angles to each other,

$$\mathbf{u} \cdot \mathbf{v} = 0$$

This is a test we can use to test whether two vectors are orthogonal (perpendicular to each other).

Equation (i) above can also be used to find the angle between two vectors **u** and **v**

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} \qquad \ldots(ii)$$

Example 1: If **u** = 2**i** + **j** – 2**k** and **v** = **i** – 3**j** + 2**k**, find
(a) **u** · **v**, (b) (b) **u** · **u**, (c) (**u** · **v**)**v** and (d) **u** · 2**v**

Solution:

u = 2**i** + **j** -2**k** and **v** = **i** - 3**j** + 2**k**

(a) **u** · **v** = $u_1v_1 + u_2v_2 + u_3v_3$ = 2(1) + 1(-3) + (-2)2 = 2 - 3 - 4 = **-5**
(b) **u** · **u** = 2(2) + 1(1) +(-2)(-2) = 4 + 1 + 4 = **9**
(c) (**u** · **v**)**v** = -5(**i** - 3**j** + 2**k**) = **-5i + 15j - 10k**
(d) 2**v** = 2**i** - 6**j** + 4**k**
u · 2**v** = 2(2) + 1(-6) + (-2)(4) = 4 - 6 -8 = **-10**

Using TI-86:

[2,1,-2]→U [2.000 1.000 -2.000] [1,-3,2]→V [1.000 -3.000 2.000] dot(U,V) -5.000	dot(U,U) 9.0 dot(U,V)*V [-5.0 15.0 -10.0] dot(U,2V) -10.0

Example 2:
$\|u\| = 40, \quad \|v\| = 25$ and the angle between u and v is $5\pi/6$.

Find **u** · **v**.

Solution:

$$u \cdot v = \|u\|\|v\|\cos\theta = 40 \times 25 \times \cos 5\pi/6 = -866$$

Example 3:
Find the angle θ between the vectors **u** and **v** given that:

$$u = \cos\frac{\pi}{6}i + \sin\frac{\pi}{6}j$$
$$v = \cos\frac{3\pi}{4}i + \sin\frac{3\pi}{4}j$$

Solution:

$$u \cdot v = \cos\frac{\pi}{6}\cos\frac{3\pi}{4} + \sin\frac{\pi}{6}\sin\frac{3\pi}{4} = -0.26$$

$$\|u\| = \sqrt{\sin^2(\pi/6) + \sin^2(\pi/6)} = 1$$

$$\|v\| = 1$$

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = -0.26$$

$$\theta = \cos^{-1}(-0.26) = 1.83 = 105^o$$

Using TI-86:

[cos (π/6),sin (π/6)] →U [.87 .50] [cos (3π/4),sin (3π/4)]→V [-.71 .71]	cos^{-1} (dot(U,V)/(norm U*norm V) 1.83 105.00

Example 4:

Find the angle θ between the vectors:

u = 2**i** - 3**j** + **k** and **v** = **i** - 2**j** + **k**

Solution:

u = 2**i** - 3**j** + **k** and **v** = **i** - 2**j** + **k**

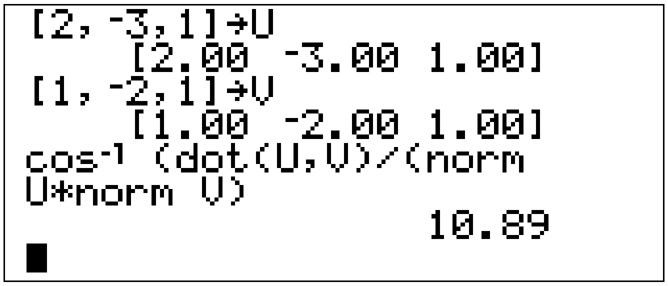
u · **v** = 2(1) + (-3)(-2) + 1(1) = 9

$$\|u\| = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$$

$$\|v\| = \sqrt{1+4+1} = \sqrt{6}$$

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{9}{\sqrt{14\times 6}} = 0.98$$

$$\theta = \cos^{-1}(0.98) = 10.89^o$$



Example 5:
Determine whether **u** = ⟨2, 18 ⟩ and **v** = ⟨3/2, -1/6⟩ are orthogonal.

Solution:

If **u** and **v** are orthogonal, then **u** · **v** = 0.

$$u \cdot v = 2 \times \frac{3}{2} + 18 \times \left(-\frac{1}{6}\right) = 0$$

u and **v** are orthogonal.

Example 6:
Determine whether **u** = -2**i** + 3**j** – **k** and **v** = 2**i** + **j** – **k** are orthogonal.

Solution:

u · **v** = (-2)2 + 3(1) + (-1)(-1) = 0
u and **v** are orthogonal.

2. Direction Cosines of a Vector

Example 7:

If $\vec{u} = \langle 2,1,2 \rangle$ and $\vec{v} = \langle 0,3,4 \rangle$, find

(a) the projection of $\vec{u}$ onto $\vec{v}$,
(b) the vector component of $\vec{u}$ orthogonal to $\vec{v}$,
(c) the scalar component of $\vec{u}$ in the direction of $\vec{v}$.

Solution:

$\vec{u} \cdot \vec{v} = (2)(0) + (1)(3) + (2)(4) = 11,$

$|\vec{v}| = \sqrt{(0)^2 + (3)^2 + (4)^2} = 5,$ and so $|\vec{v}|^2 = 25.$

(a) $proj_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\right)\vec{v} = \frac{11}{25}\langle 0,3,4 \rangle = \langle 0, \frac{33}{25}, \frac{44}{25} \rangle.$

(b) $\vec{u} - proj_{\vec{u}}\vec{v} = \langle 2,1,2 \rangle - \langle 0, \frac{33}{25}, \frac{44}{25} \rangle = \langle 2, \frac{-8}{25}, \frac{6}{25} \rangle.$

(c) $|proj_{\vec{u}}\vec{v}| = \sqrt{(0)^2 + (\frac{33}{25})^2 + (\frac{44}{25})^2} = \frac{11}{5} = 2.2$

Lesson 1.4
The Cross Product of Two Vectors in Space

If **u** = u_1**i**+ u_2**j** + u_3**k** and **v** = v_1**i** + v_2**j** + v_3**k** are two vectors in space, then the cross product of these vectors is a vector given by
u × **v** = $(u_2v_3 - u_3v_2)$**i** $-(u_1v_3 - u_3v_1)$**j** + $(u_1v_2 - u_2v_1)$**k**

If **u** =⟨2, -3, 5⟩ and **v** = ⟨1, 2, 4⟩, we use the following method to find **u** × **v**
Complete the 3 x 3 matrix with **i**, **j**, **k** as the first row; $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ as the second row; $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ as the third row.

$$u\times v=\begin{bmatrix} i & j & k \\ 2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix}=i((-3)4-2\times 5)-j(2\times 4-1\times 5)+k(2\times 2-1(-3)$$

$$=-22i-3j+7k$$

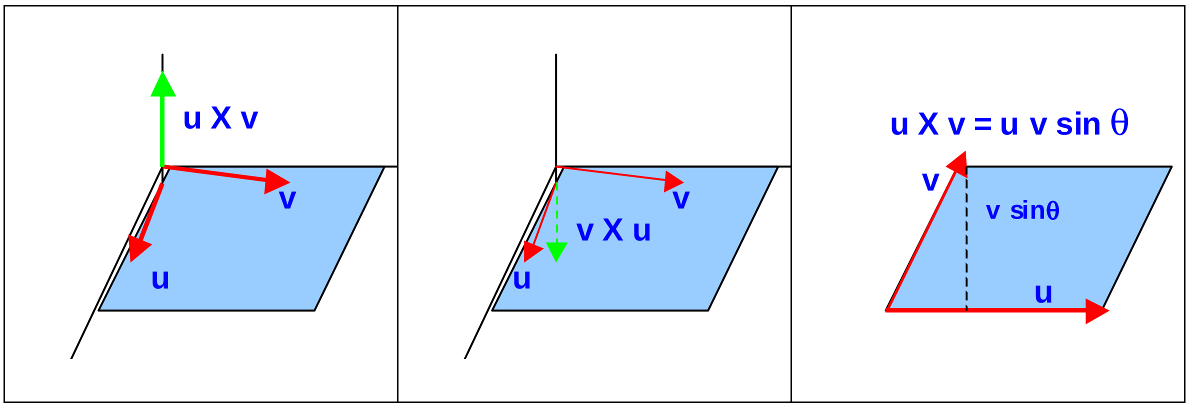
To obtain the x component of the cross product, cross out the row and column containing **i** and multiply the remaining 2 x 2 matrix diagonally and subtract as (-3)4 - 2 x 5 = -22. The x component of the cross product is -22**i**. To find the y component, cross out the row and column containing **j**, then multiply diagonally the remaining 2 x 2 matrix and subtract as 2 x 4 - 1 x 5 = 3. Remember, we take **i** positive, **j** negative and **k** positive. Therefore, the y component of the cross product = -3**j**. To find the z component of the cross product, cross out the row and column containing **k** and multiply and subtract the remaining 2 x 2 matrix. as 2 x 2 - (-3)1 = 7. The z component of the cross product is 7**k**. Thus
u x **v** = -22**i** - 3**j** + 7**k** as shown above.

Unlike the dot product, the cross product of two vectors **u** and **v** is a vector orthogonal to both **u** and **v**. The direction of **u** × **v** can be found using the right hand rule. Hold the four fingers of your right hand at right angles to your thumb. If these four fingers are moved from **u** to **v**, the thumb will point in the direction of the cross product.

The following are some of the properties of cross product of vectors.

If u, v, w are vectors in space and c is a scalar, then
1. **u** × **v** = - **v** × **u**

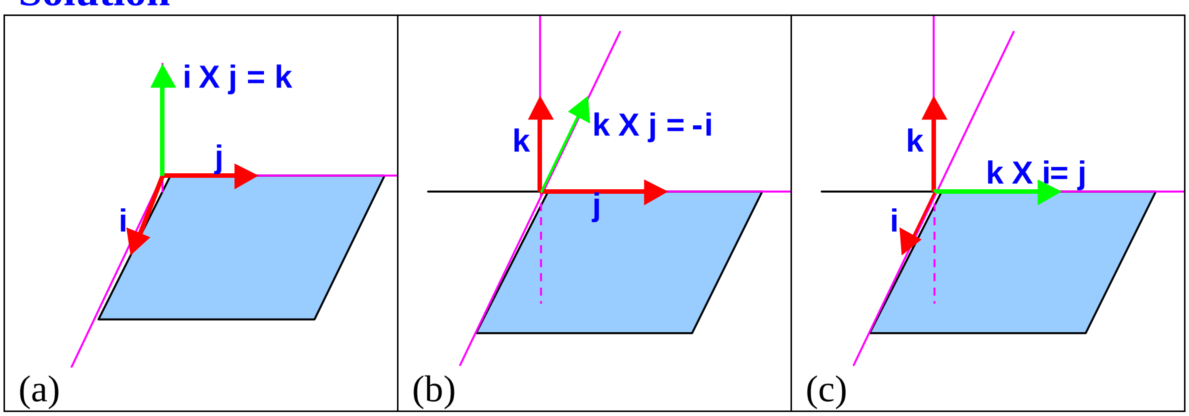
2. **u** $\times$ **u** = **v** $\times$ **v** = 0
3. **u** $\times$ **v** is orthogonal to both **u** and **v**
4. The magnitude of **u** $\times$ **v** is given by
   $\|u\times v\| = \|u\|\|v\|\sin\theta$ where θ is the angle between the vectors
5. **u** $\times$ **v** = 0 if **u** and **v** are parallel
6. **u** $\times$ **v** = area of the parallelogram having **u** and **v** as adjacent sides.



Example 1:

Find the cross product (a) **i** $\times$ **j**, (b) **k** $\times$ **j** and (c) **k** $\times$ **i** and sketch the result.

Solution



Example 2:

Given that **u** = **j** + 6**k** and **v** = **i** – 2**j** + **k**, find **u** $\times$ **v** and show that it is orthogonal to both **u** and **v**.

Solution:

$$u \times v = \begin{bmatrix} i & j & k \\ 0 & 1 & 6 \\ 1 & -2 & 1 \end{bmatrix} = (1-(-12))i - (0-6)j + (0-1)k$$

$$= 13i + 6j - k$$

If **u** and **u** × **v** are orthogonal , then **u** · (**u** × **v**) = 0
u = ⟨0, 1, 6⟩, **v** = ⟨1, -2, 1⟩ and **u** × **v** = ⟨13, 6, -1⟩
u · (**u** × **v**) = 0 + 6 - 6 = 0
v · (**u** × **v**) = 13 - 12 -1 = 0
Therefore, (**u** × **v**) is orthogonal to both **u** and **v**

Example 3:
Given that **u** = ⟨-8, -6, 4⟩ and **v** = ⟨10, -12, -2⟩, use a graphing utility to find **u** × **v** and a unit vector orthogonal to **u** and **v.**

Solution:

```
[-8,-6,4]→U
    [-8.00 -6.00 4.00]
[10,-12,-2]→V
  [10.00 -12.00 -2.00]
cross(U,V)
  [60.00 24.00 156.00]
```

```
[60,24,156]→A
  [60.00 24.00 156.00]
A/norm A
           [.36 .14 .92]
```

The figure above shows the calculator output for **u** × **v**. Since this vector is orthogonal to both **u** and **v**, a unit vector orthogonal to **u** and **v** is a unit vector in the direction of **u** × **v**. In the figure on the right I stored **u** × **v** as A and obtained $\frac{A}{\|A\|}$ which is the required unit vector.

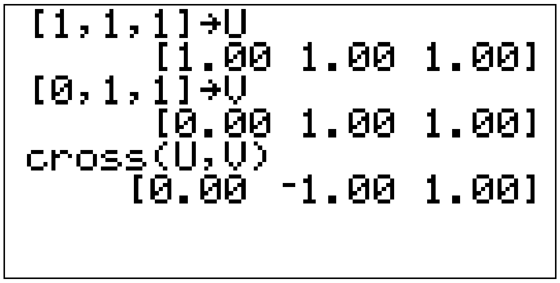
Example 4:
Find the area of the parallelogram with the vectors:
u = **i** + **j** + **k** and **v** = **j** + **k** as adjacent sides. Verify your result using the calculator.

Solution:

The area of the parallelogram is given by the magnitude of **u** × **v**

$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = 0i - j + k = \langle 0, -1, 1 \rangle$$

$$\|u \times v\| = \sqrt{1+1} = \sqrt{2}$$



Example 5:
Find the area of the triangle with vertices P(2, -3, 4), Q(0, 1, 2) and R(-1, 2, 0).

Solution:

$$\vec{PQ} = u = \langle -2, 4, -2 \rangle$$

$$\vec{PR} = v = \langle -3, 5, -4 \rangle$$

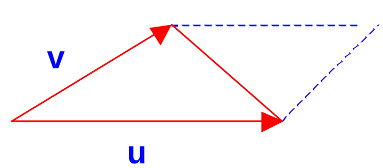
Area of the triangle with **u** and **v** as adjacent sides is given by:

$$A = \frac{1}{2}\|u \times v\|$$

$$u \times v = \begin{bmatrix} i & j & k \\ -2 & 4 & -2 \\ -3 & 5 & -4 \end{bmatrix} = -6i - 2j + 2k$$

$$\|u \times v\| = \sqrt{(-6)^2 + (-2)^2 + 2^2} = \sqrt{44} = 2\sqrt{11}$$

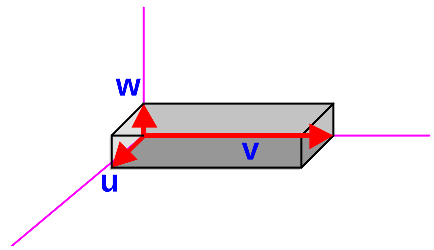
The area of the triangle = $\sqrt{11}$



2. The Triple Scalar Product

If **u** = $\langle u_1, u_2, u_3 \rangle$, **v** = $\langle v_1, v_2, v_3 \rangle$, and **w** = $\langle w_1, w_2, w_3 \rangle$, then the quantity **u** · (**v** × **w**) is called the triple scalar of **u**, **v** and **w**. The magnitude of **u** · (**v** × **w**) gives the volume of a parallelepiped with **u**, **v**, **w** as adjacent sides.

$$u \cdot (v \times w) = \begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$



Example 6:

If **u** = ⟨2, 0, 0⟩, **v** = ⟨1, 1, 1⟩ and **w** = ⟨0, 2, 2⟩, find **u** · (**v** × **w**)

Solution:

$$u \cdot (v \times w) = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} = 2(2-2) - 0(2-0) + 0(2-0) = 0$$

Example 7:

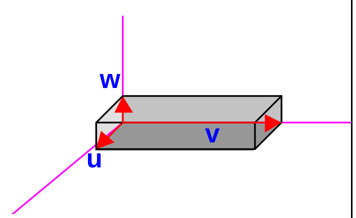
Find the volume of the parallelepiped having adjacent edges given by the vectors **u** = ⟨1, 3, 1⟩, **v** = ⟨0, 5, 5⟩ and **w** = ⟨4, 0, 4⟩.

Solution:

The volume of the parallelepiped is given by the magnitude of the triple scalar product **u** · (**v** × **w**)

u · (**v** × **w**) =

$$\begin{bmatrix} 1 & 3 & 1 \\ 0 & 5 & 5 \\ 4 & 0 & 4 \end{bmatrix} = 1(20-0) - 3(0-20) + 1(0-20) = 60$$



Lesson 1.5
Lines and Planes in Space

1. Lines in Space

In order to write a set of parametric equations for a line in space, we need a point $P(x_1, y_1, z_1)$ through which the line passes and a vector $\mathbf{v} = \langle a, b, c\rangle$ parallel to the line.

Parametric equations for a line passing through $P(x_1, y_1, z_1)$ and parallel to the vector $v = \langle a, b, c\rangle$ are given by

$$\begin{aligned} x &= x_1 + at \\ y &= y_1 + bt \\ z &= z_1 + ct \end{aligned}$$

A symmetric equation for the line is obtained by eliminating the parameter t from the above equations which gives:

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Example 1:
Find a set of parametric equations and a set of symmetric equations for a line passing through P(0, 0, 0) and parallel to the vector $\mathbf{v} = \langle -2, 5/2, 1\rangle$

Solution:

We have $x_1 = 0$, $y_1 = 0$ $z_1 = 0$
$v = \langle -2, 5/2, 1\rangle = \langle -4, 5, 2\rangle$
$a = -4$ $b = 5$ $c = 2$

The parametric equations are:
$x = 0 - 4t$ or, $\mathbf{x = -4t}$
$y = 0 + 5t$ or, $\mathbf{y = 5t}$
$z = 0 + 2t$ or, $\mathbf{z = 2t}$

The symmetric equation is:

$$\frac{x}{-4}=\frac{y}{5}=\frac{z}{2}$$

Example 2:
Find a set of parametric equations and a set of symmetric equations for a line passing through P(-2, 0, 3) and parallel to the vector **v** = 6**i** + 3**j**

Solution:

We have: $x_1 = -2$, $y_1 = 0$, $z_1 = 3$
$a = 6$, $b = 3$, $c = 0$

The parametric equations are:
x = -2 + 6t
y = 3t
z = 3
The symmetric equations are:

$$\frac{x+2}{6}=\frac{y}{3}$$

$$\frac{x+2}{2}=y$$

$$z=3$$

Example 3:
Find a set of parametric equations and a set of symmetric equations for a line passing through P(-3, 5, 4) and parallel to the line:

$$\frac{x-1}{3}=\frac{y+1}{-2}=z-3$$

Solution:

The general form of the symmetric equation of a line that is parallel to v = ⟨a, b, c⟩ is:

$$\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}.$$

Since the line we are looking for is parallel to

$$\frac{x-1}{3}=\frac{y+1}{-2}=z-3,$$

the vector v = ⟨3, -2, 1⟩ is parallel to the line. We are now looking for the equation of a line that passes through P(-3, 5, 4) and parallel to v = ⟨3, -2, 1⟩. So we have:

$x_1 = -3$, $y_1 = 5$ $z_1 = 4$
$a = 3$ $b = -2$ $c = 1$

The parametric equations are:

x = -3 + 3t
y = 5 - 2t
z = 4 + t

The symmetric equations are:

$$\frac{x+3}{3}=\frac{y-5}{-2}=z-4$$

Example 4:
Find a set of parametric equations and a set of symmetric equations for a line that passes through P(1, 0, 1) and Q(1, 3, -2).

Solution:

The vector $\vec{PQ}$ is given by:

$$\vec{PQ}=\langle 0,3,-3\rangle$$

Now we are looking for the equation of the line that passes through P(1, 0, 1) (we choose one of the points), and parallel to the vector ⟨0, 3, -3⟩.
We have:
$x_1 = 1$, $y_1 = 0$, $z_1 = 1$
$a = 0$ $b = 3$ $c = -3$

The parametric equations are:
x = 1
y = 3t
z = 1 - 3t

The symmetric equations are:

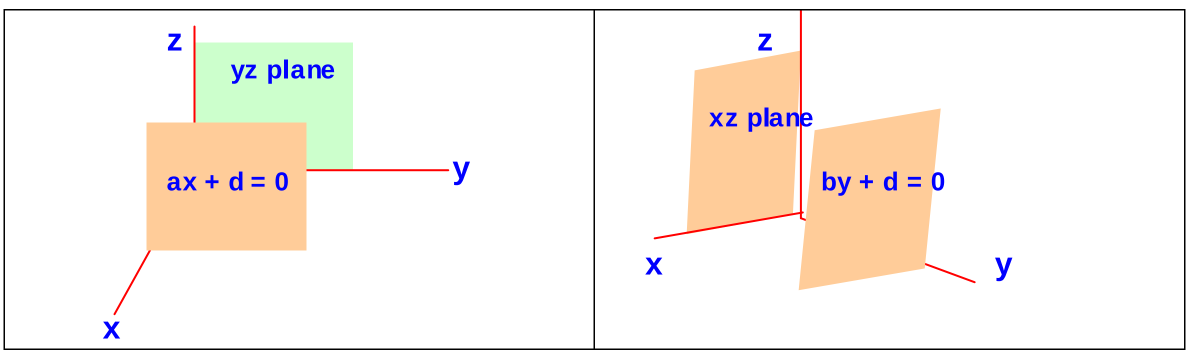
$$x = 1 \text{ and } \frac{y}{3} = \frac{z-1}{-3} \quad or, \quad y = 1 - z$$

2. Planes in space

A plane in space is defined using a point $P(x_1, y_1, z_1)$ in it and a vector $n = \langle a, b, c\rangle$ normal to it.

The standard equation of the plane containing the point $P(x_1, y_1, z_1)$ and a normal vector $n = \langle a, b, c\rangle$ is
$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$
The general equation of the plane is
$ax + by + cz + d = 0$

The plane $ax + d = 0$ is parallel to the yz plane, $by + d = 0$ is parallel to the xz plane and $cz + d = 0$ is parallel to the xy plane.

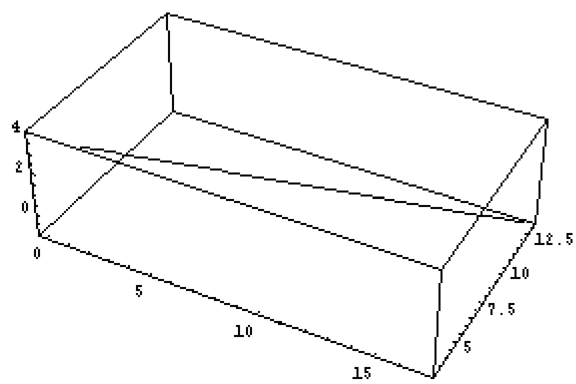


Example 5:
Find a set of parametric equations for the line that passes through the point P(2, 3, 4) and perpendicular to the plane $3x + 2y - z = 6$.

Solution:
The vector normal to the plane $3x + 2y - z = 6$ is $n = \langle 3, 2, -1\rangle$
Since we are looking for the equation of a line that is perpendicular to this plane, the vector $\mathbf{n} = \langle 3, 2, -1\rangle$ is parallel to the line. We need the equation of the line that passes through (2, 3, 4) and parallel to the vector $n = \langle 3, 2, -1\rangle$
We have: $x_1 = 2$, $y_1 = 3$, $z_1 = 4$

a = 3, b = 2 , c = -1

The parametric equations are:

x = 2 + 3t **y = 3 + 2t** **z = 4 - t**	

Example 6:

Two lines in space are given by:

x = -3t + 1, y = 4t + 1, z = 2t + 4

x = 3s + 1, y = 2s + 4, z = -s + 1

Determine whether these two lines intersect. If they do, find the point of intersection and the cosine of the angle of intersection.

Solution:

If the lines intersect, at the point of intersection, we have:

(i) -3t + 1 = 3s + 1,

(ii) 4t + 1 = 2s + 4

(iii) 2t + 4 = -s + 1

From (i) we have s = -t. Using this in (ii), we get 4t + 1 = -2t + 4, or, t = ½. Using s = -t in (iii) we have 2t + 4 = t + 1, or, t = -3. Since t should be the same at the point of intersection, **these lines do not intersect**.

Example 7:

Two lines in space are given by:

$$\frac{x-2}{-3} = \frac{y-2}{6} = z-3$$

$$\frac{x-3}{2} = y+5 = \frac{z+2}{4}$$

Determine whether these two lines intersect. If they do, find the point of intersection and the cosine of the angle of intersection.

Solution:

Writing the equations in the parametric form, we have for line 1

$x = 2 - 3t, \quad y = 2 + 6t, \quad z = 3 + t$

For line 2 we have:

$x = 3 + 2s, \quad y = -5 + s, \quad z = -2 + 4s$

If they intersect, at the point of intersection we have,

(i) $2 - 3t = 3 + 2s$ or, $2s + 3t = -1$

(ii) $2 + 6t = -5 + s$ or, $s - 6t = 7$

(iii) $3 + t = -2 + 4s$ or, $4s - t = 5$

Solving (i) and (ii) above gives $s = 1$ and $t = -1$. These solutions also satisfy the third equation. Therefore, these two lines intersect when $t = -1$ and $s = 1$.

The point of intersection is **P(5, -4, 2)**

To find the cosine of the angle of intersection, we use the dot product of the two vectors **u** and **v** that are parallel to these lines.

The vector **u** parallel to line 1 is $\mathbf{u} = \langle -3, 6, 1\rangle$

The vector **v** parallel to line 2 is $\mathbf{v} = \langle 2, 1, 4\rangle$

$u \cdot v = -6 + 6 + 4 = 4$

$$\|u\| = \sqrt{9+36+1} = \sqrt{46}$$

$$\|v\| = \sqrt{4+1+16} = \sqrt{21}$$

$$\cos\theta = \frac{u \cdot v}{\|u\|\|v\|} = \frac{4}{\sqrt{46}\sqrt{21}} = 0.13$$

$\boldsymbol{\theta = 82.5^{o}}$

Example 8:

Find an equation of the plane containing the point P(1, 0, -3) and perpendicular to the vector **n** = **k**

Solution:

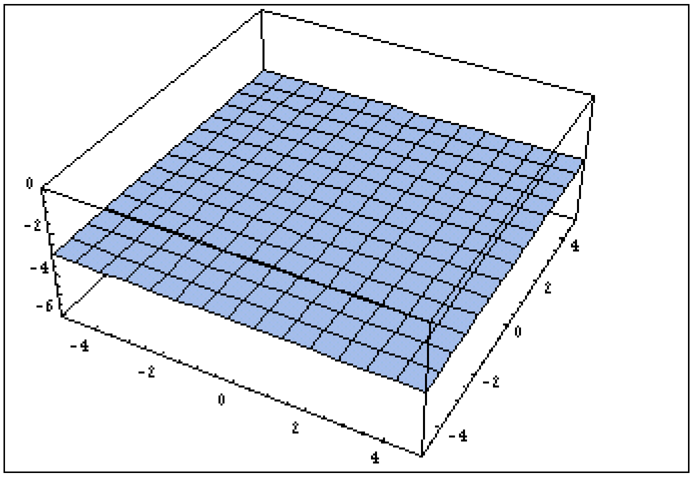
We have $x_1 = 1, \quad y_1 = 0, \quad z_1 = -3$

$a = 0, \quad b = 0 \quad c = 1$

The standard equation is:

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$0(x - 1) + 0(y - 0) + 1(z -(- 3)) = 0$ or, $\boldsymbol{z = -3}$



Example 9:

Find an equation of the plane containing the point P(3, 2, 2) and perpendicular to the line:

$$\frac{x-1}{4} = y+2 = \frac{z+3}{-3}$$

Solution:

The vector perpendicular to the plane is n = ⟨4, 1, -3⟩

We have $x_1 = 3$, $y_1 = 2$, $z_1 = 2$

$a = 4$, $b = 1$, $c = -3$

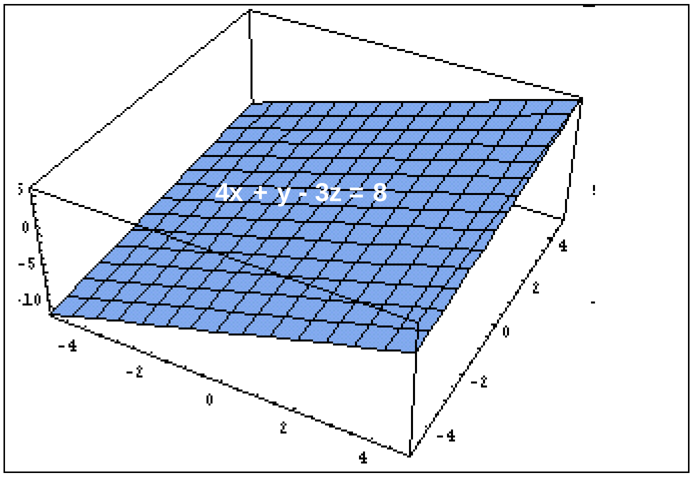
The standard equation of the plane is:

$4(x - 3) + 1(y - 2) + (-3)(z - 2) = 0$

$4x - 12 + y - 2 - 3z + 6 = 0$

$\mathbf{4x + y - 3z = 8}$

The plane $z = \dfrac{4x + y - 8}{3}$ is plotted below using *mathematica*.



Example 10:
Find an equation of the plane that passes through P(1, 2, -3), Q(2, 3, 1) and R(0, -2, -1).

Solution:

Using the points P, Q and R, we can define two vectors, $\vec{PQ}$ *and* $\vec{PR}$ on the plane. The cross product of these two vectors will be normal to the plane.

$$\vec{PQ} = u = \langle 1, 1, 4 \rangle$$

$$\vec{PR} = v = \langle -1, -4, 2 \rangle$$

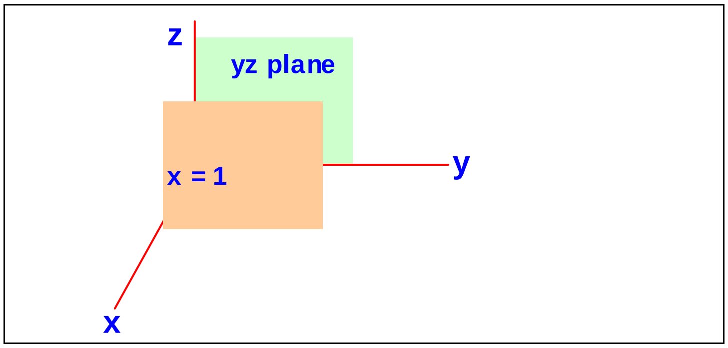
$$u \times v = \begin{bmatrix} i & j & k \\ 1 & 1 & 4 \\ -1 & -4 & 2 \end{bmatrix} = 18i - 6j - 3k = 3(6i - 2j - k)$$

The normal vector n = ⟨6, -2, -1⟩ and the plane contains the point P(1, 2, -3)
The equation of the plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

6(x - 1) + (-2)(y - 2) + (-1)(z - (-3)) = 0
6x - 6 - 2y + 4 - z - 3 = 0
6x - 2y - z = 5

Example 11:
Find an equation of the plane that contains the point P(1, 2, 3) and is parallel to the yz plane.

Solution:



$n = \langle 1, 0, 0 \rangle$

The equation of the plane is $1(x - 1) + 0(y - 2) + 0(z - 3) = 0$

Or, **$x = 1$**

Example 12:

Find an equation for the plane that contains the point (2, 2, 1) and contains the line:

$$\frac{x}{2} = \frac{y-4}{-1} = z$$

Solution:

The direction vector of the line is $u = \langle 2, -1, 1 \rangle$. A point on the line can be obtained by assigning arbitrary values to x and z and solving for y. When $x = z = 0$, $y = 4$. Thus the point (0, 4, 0) is on the line. The vector joining point (2, 2, 1) to (0, 4, 0) is $v = \langle -2, 2, -1 \rangle$. Now we have two vectors on the plane and the cross product of these gives the normal vector to the plane.

$$n = u \times v = \begin{bmatrix} i & j & k \\ 2 & -1 & 1 \\ -2 & 2 & -1 \end{bmatrix} = -i + 0j + 2k$$

We have: $x_1 = 2$, $y_1 = 2$, $z_1 = 1$

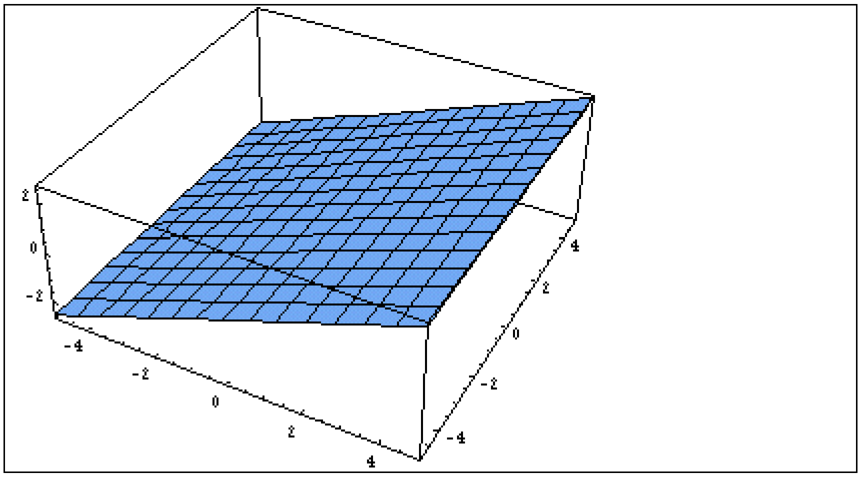
$a = -1$, $b = 0$, $c = 2$

The equation of the plane is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$-1(x - 2) + 0 + 2(z - 1) = 0$

$-x + 2 + 2z - 2 = 0$

$x - 2z = 0$



Example 13:
Find an equation of the plane that passes through P(4, 2, 1) and Q(-3, 5, 7) and is parallel to the z axis.

Solution:
A vector **u** on the plane is given by **u** = ⟨-7, 3, 6⟩. Since the plane is parallel to the z axis, it is parallel to the vector **v** = ⟨0, 0, 1⟩. The normal vector to the plane is given by:

$$n = u \times v = \begin{bmatrix} i & j & k \\ -7 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix} = 3i + 7j$$

We have $x_1 = 4$, $y_1 = 2$, $z_1 = 1$
$a = 3$, $b = 7$, $c = 0$

The equation of the plane is:
3(x - 4) + 7(y - 2) + 0 = 0
3x - 12 + 7y -14 = 0
3x + 7y = 26

Example 14:
Determine whether the planes 3x + 2y – z = 7 and x – 4y + 2z = 0 are parallel, orthogonal or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

Solution:

The normal vector to plane 1 is $\mathbf{n}_1 = \langle 3, 2, -1\rangle$
The normal vector to plane 2 is $\mathbf{n}_2 = \langle 1, -4, 2\rangle$
If the two planes are parallel, then $\mathbf{n}_1$ will be a scalar multiple of $\mathbf{n}_2$. Since in this case, $\mathbf{n}_1$ is not a scalar multiple of $\mathbf{n}_2$, the two planes are not parallel.
If the two planes are orthogonal, then $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$
$\mathbf{n}_1 \cdot \mathbf{n}_2 = 3 - 8 - 2 = -7$. The planes are not orthogonal.
If θ is the angle between the two planes, then

$$\cos\theta = \frac{|n_1 \cdot n_2|}{\|n_1\|\|n_2\|} = \frac{7}{\sqrt{14}\sqrt{21}} = \frac{1}{\sqrt{6}}$$

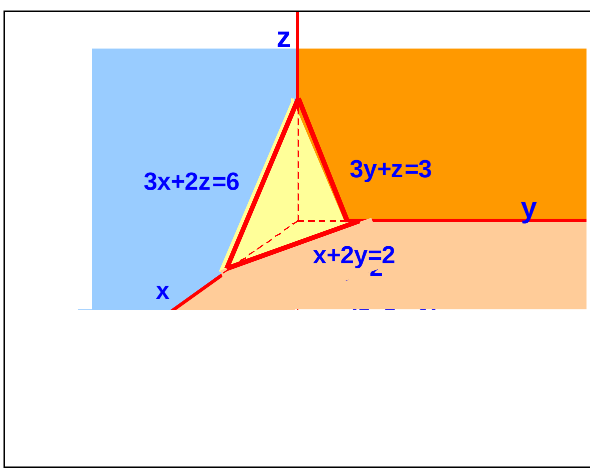
$$\theta = \cos^{-1}(1/\sqrt{6}) = 65.9^o$$

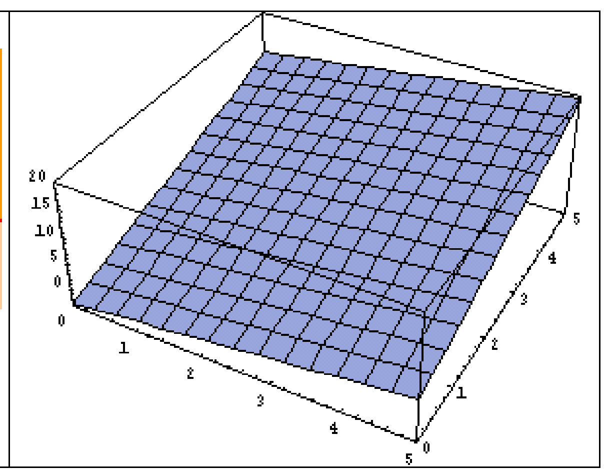
Example 15:

Sketch a graph of the plane 3x + 6y + 2z = 6

Solution:

The intersection of a plane on the xy plane is called the xy trace. The xy trace is obtained by setting z = 0 in the equation of the plane.
The xy trace of this plane is 3x + 6y = 6 , or, x + 2y = 2
The yz trace is 6y + 2z = 6, or, 3y + z = 3
The xz trace is 3x + 2z = 6
The figure on the left below shows how these traces are used to sketch the graph of the plane. The figure on the right is the mathematica generated graph of the plane.





Example 16:

Find a set of parametric equations for the line of intersection of the planes x - 3y + 6z = 4 and 5x + y - z = 4

Solution:

The normal vectors of these planes are:

$n_1 = \langle 1, -3, 6\rangle$ and $n_2 = \langle 5, 1, -1\rangle$. The line of intersection of these planes is orthogonal to both these vectors and is given by $n_1 \times n_2$

$$n_1 \times n_2 = \begin{bmatrix} i & j & k \\ 1 & -3 & 6 \\ 5 & 1 & -1 \end{bmatrix} = -3i + 31j + 16k$$

Now we find a point of intersection of the planes by elimination method.

$$\begin{array}{l} x - 3y + 6z = 4 \\ 15x + 3y - 3z = 12 \\ \hline 16x + 3z = 16 \end{array}$$

If we let z = 0, then x = 1. Using these two values in x - 3y + 6z = 4, y = -1. Therefore, a point of intersection of the planes is (1, -1, 0). We are looking for the parametric equations of the line with the direction vector ⟨-3, 31, 16⟩ and passing through the point (1, -1, 0)

We have $x_1 = 1$, $y_1 = -1$, $z_1 = 0$

a = -3, b = 31 c = 16

x = 1 - 3t

y = -1 + 31t

z = 16t

Example 17:

Find the point of intersection of the plane 2x + 3y = -5 and the line

$\frac{x-1}{4} = \frac{y}{2} = \frac{z-3}{6}$. Does the line lie on the plane?

Solution:

Writing the line in the parametric form, we have

x = 1 + 4t, y = 2t, z = 3 + 6t

Using these values in the equation for the plane, gives

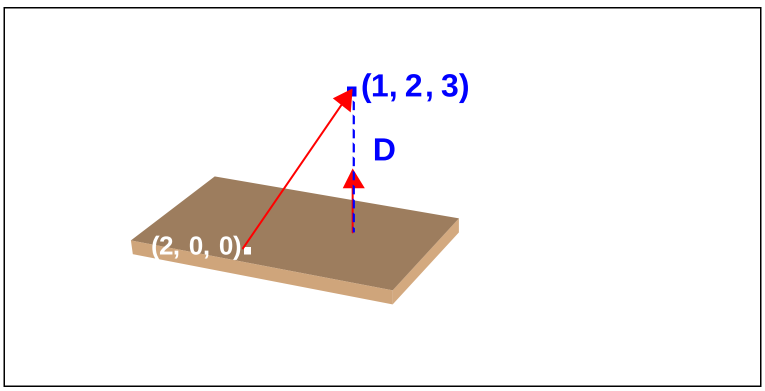
2(1 + 4t) + 3(2t) = -5

2 + 8t + 6t = -5 or, t = -1/2

Using this value of t in the parametric equations give
x = 1 - 2 = -1, y = -1, z = 3 - 3 = 0
The point of intersection is **(-1, -1, 0)**
Since there is only one point of intersection, the line does not lie on the plane.

Example 18:
Find the distance between the point (1, 2, 3) and the plane 2x - y + z = 4.

Solution:
The normal vector to the plane is n = ⟨2, -1, 1⟩
A point on the plane can be obtained by setting y = z = 0 and solving for x.
This gives (2, 0, 0) a point on the plane
P(2, 0, 0) is a point on the plane and Q(1, 2, 3) a point outside it.
Vector **PQ** = ⟨-1, 2, 3⟩



The distance D between the point and the plane is the length of the projection of vector PQ along the normal vector n.

$$D = \frac{\left|\vec{PQ} \cdot n\right|}{\|n\|} = \frac{|-2-2+3|}{\sqrt{6}} = \frac{\sqrt{6}}{6}$$

Homework:

1. Find a set of parametric equations and a set of symmetric equations for a line that passes through (-2, 0, 3) and parallel to the vector v = 2i + 4j – 2k.

2. Find a set of parametric equations and a set of symmetric equations for a line that passes through (1, 0, 1) and parallel to the line given by $x = 3 + 3t$, $y = 5 - 2t$, $z = -7 + t$

3. Find a set of parametric equations and a set of symmetric equations for a line that passes through (5, -3, -2) and (-2/3, 2/3, 1)

4. Determine whether the following lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.

   (i) $x = 4t + 2$, $y = 3$, $z = -t + 1$

   $x = 2s + 2$, $y = 2s + 3$, $z = s + 1$

   (ii) $\dfrac{x}{3} = \dfrac{y-2}{-1} = z+1, \quad \dfrac{x-1}{4} = y+2 = \dfrac{z+3}{-3}$

5. Find an equation of the plane:

   (i) passing through (3, 2, 2) and perpendicular to $n = 2i + 3j - k$

   (ii) passing through (0, 0, 6) and perpendicular to the line given by

   $x = 1 - t$, $y = 2 + t$, $z = 4 - 2t$

6. Find the equation of the plane passing through (0,0,0), (1,2,3) and (-2,3,3)

7. Find an equation of the plane that passes through the point (1,2,3) and parallel to the xy plane.

8. Find an equation of the plane that passes through the points (2,2,1) and (-1,1,-1) and perpendicular to the plane $2x - 3y + z = 3$

9. Determine whether the following planes are orthogonal, parallel or neither. If they are neither parallel nor orthogonal, find the angle of intersection.

   $x - 3y + 6z = 4$

   $5x + y - z = 1$

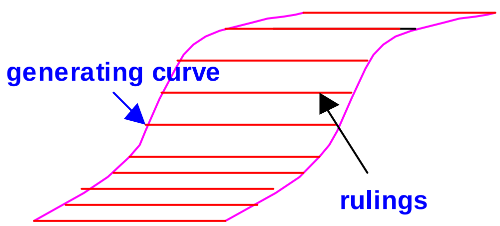
10. Sketch the graph of the plane $4x + 2y + 6z = 12$

11. Find the distance between the point (0,0,0) and the plane $2x + 3y + z = 12$

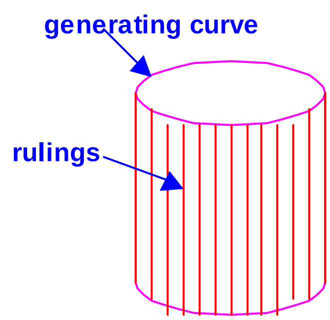
Lesson 1.6
Surfaces in Space

1. Cylindrical Surfaces.

A cylindrical surface in space is constructed using a **generating curve** called the directrix and a set of parallel lines intersecting the generating curve at right angles. These parallel lines are called **rulings**.

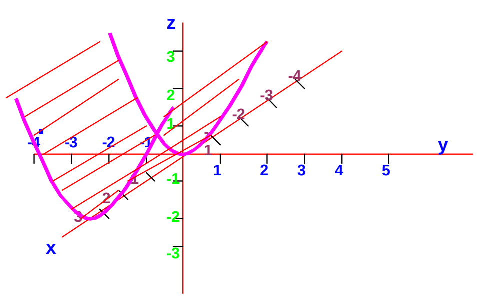


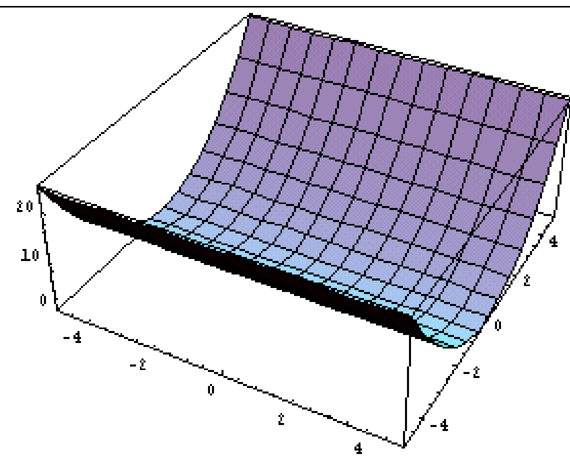
Rulings intersect the generating curve at right angles.



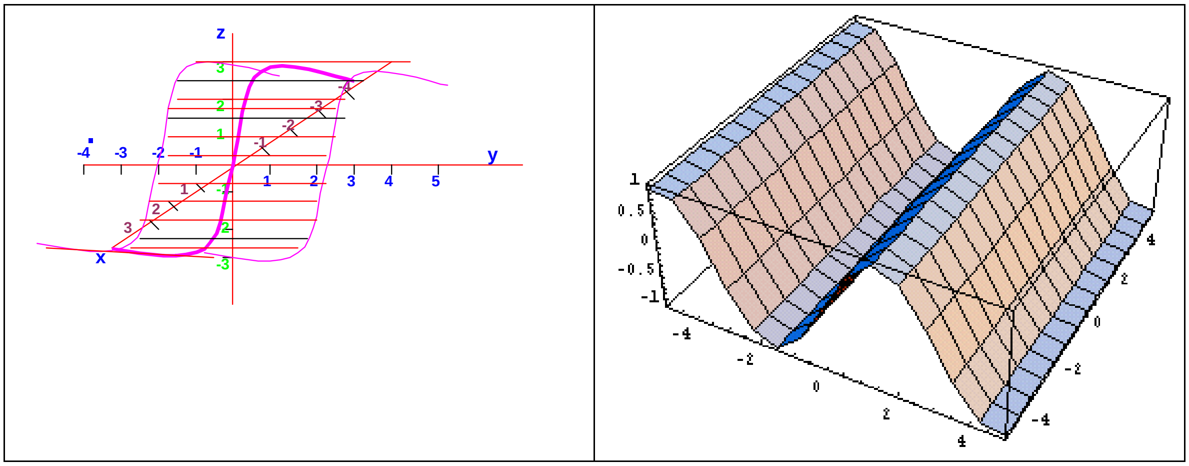
If the rulings of a cylinder are parallel to one of the coordinate axes, the equation of this cylinder will contain only variables containing the other two axes.

For example $z = y^2$ describes a cylinder which has the generating curve $z = y^2$ and rulings that are parallel to the x axis as shown below.



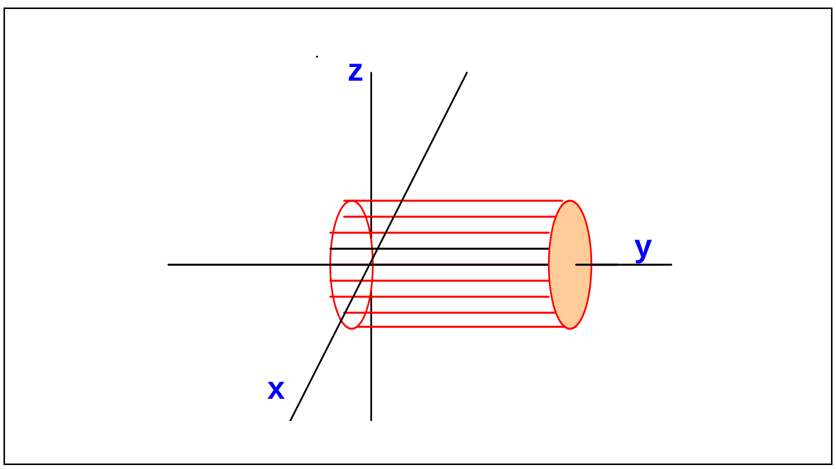


The equation z = sin x describes a cylinder with z = sin x as the generating curve and the rulings parallel to the y axis.



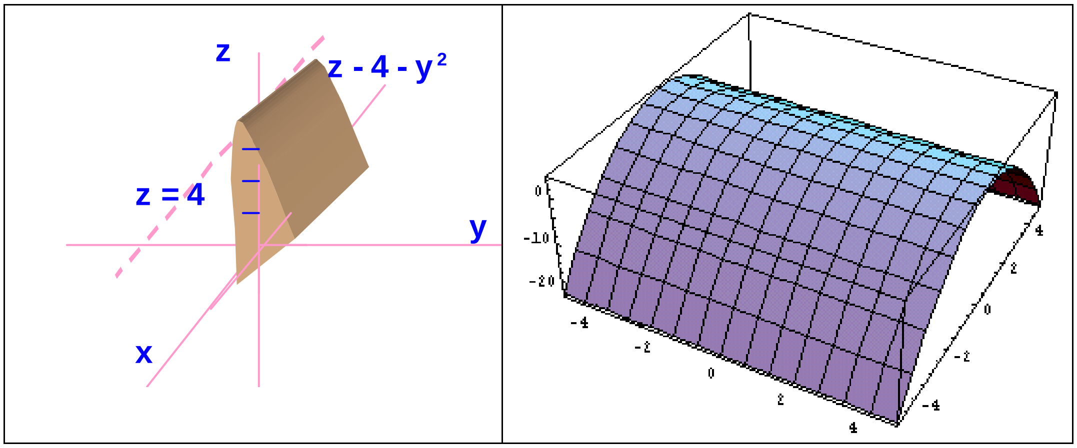
Example 1:
Describe and sketch the surface $x^2 + z^2 = 16$

Solution:
Here the y coordinate is missing. This is a cylinder with rulings parallel to the y axis and the generating curve is the circle given by $x^2 + z^2 = 16$.



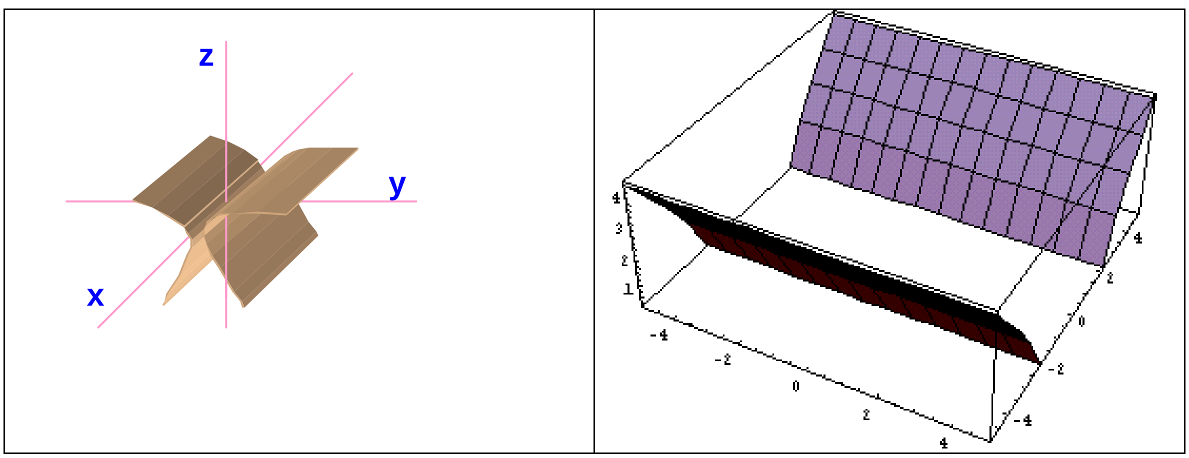
Example 2:
Describe and sketch the surface $y^2 + z = 4$

Solution:
This can be written as $z = 4 - y^2$. This is a cylinder with the generating curve given by the parabola $z = 4 - y^2$ and the rulings parallel to the x axis.



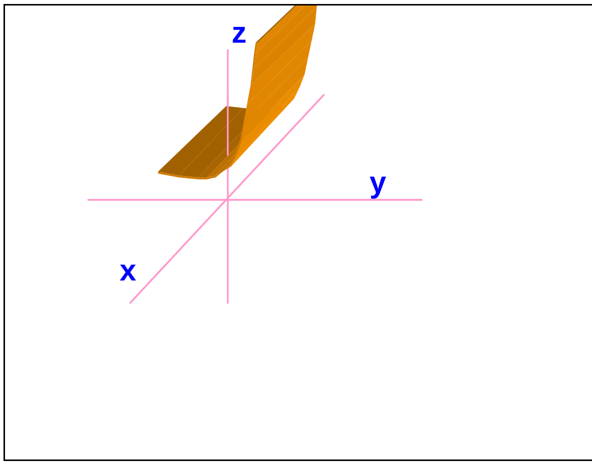
Example 3:
Describe and sketch the surface y^2 - z^2 = 4.

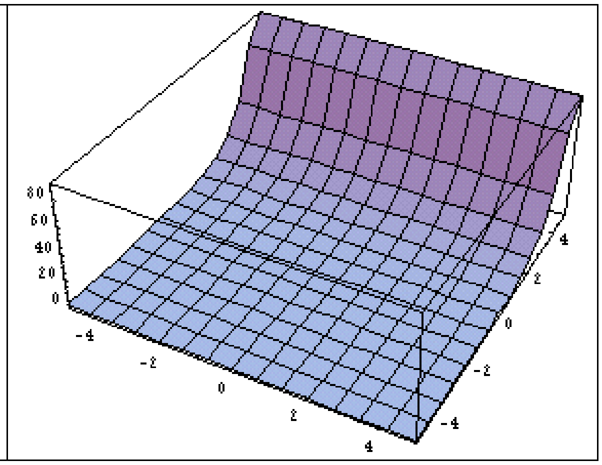
Solution:
This is a cylinder with the generating curve given by the hyperbola
$\frac{y^2}{4} - \frac{z^2}{4} = 1$ and the rulings parallel to the x axis.



Example 4:
Describe and sketch the surface z - e^y = 0

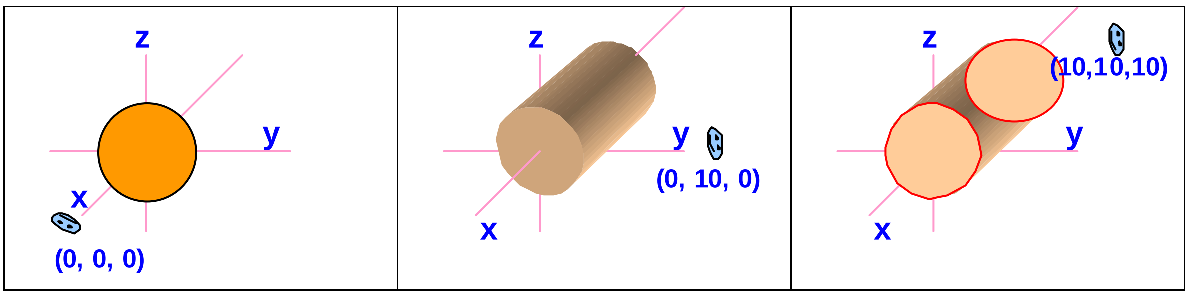
Solution:
This is a cylinder with the generating curve given by the exponential curve z = e^y and the rulings parallel to the x axis.





Example 5:
Sketch a view of the cylinder $y^2 + z^2 = 4$ from each of the following points.
(a) (10, 0, 0) (b) (0, 10, 0) (c) (10, 10, 10)

Solution:



2. Quadric Surfaces

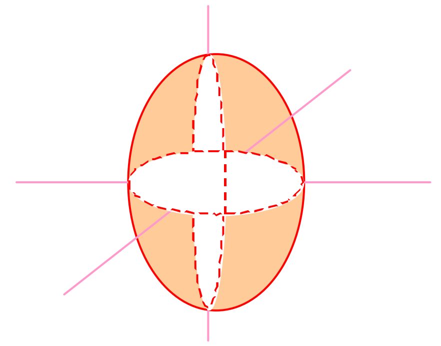
Quadric surfaces are the three dimensional analogues of conic sections. For example, an ellipse is a conic, and revolving it about an axis produces an ellipsoid. The trace of an ellipsoid in each of the three planes is an ellipse. Similarly, revolving a parabola about its axis generates a quadric surface called the paraboloid. There are six basic types of quadric surfaces. We will consider the standard equation of each of these.

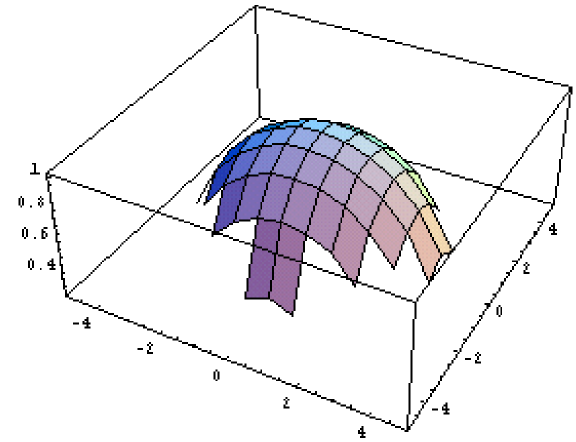
(i) Ellipsoid:

Standard equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Fo a = b = c ≠ 0, the surface will be a sphere.





The xy trace is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is an ellipse

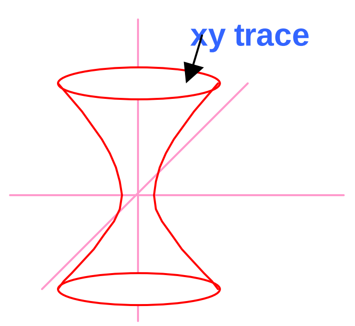
The xz trace is $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ which is an ellipse

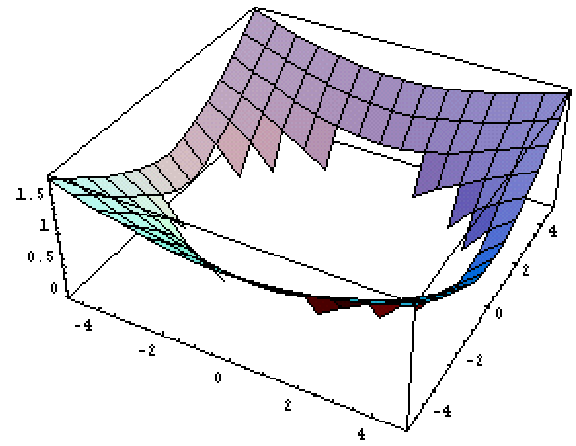
The yz trace is $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which is an ellipse.

(ii) Hyperboloid of One Sheet

The standard equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$





The xy trace is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is an ellipse.

The xz trace is $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1$ which is a hyperbola.

The yz trace is $\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ which is a hyperbola.

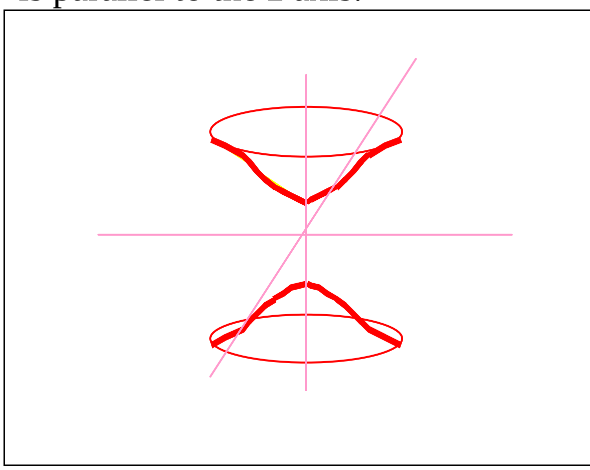
The axis of the hyperboloid corresponds to the variable with negative coefficient. The axis of $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ is the z axis. The hyperboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ has its axis along the y axis.

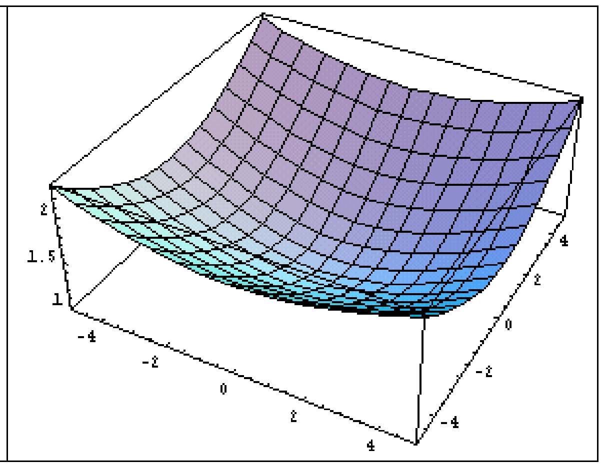
(iii) Hyperboloid of Two Sheets

The standard equation is:

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The axis of the hyperboloid corresponds to the variable that is positive. This is parallel to the z axis.





The xy trace is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which is an ellipse.

The xz trace is $\frac{z^2}{c^2} - \frac{x^2}{a^2} = 1$ which is a hyperbola.

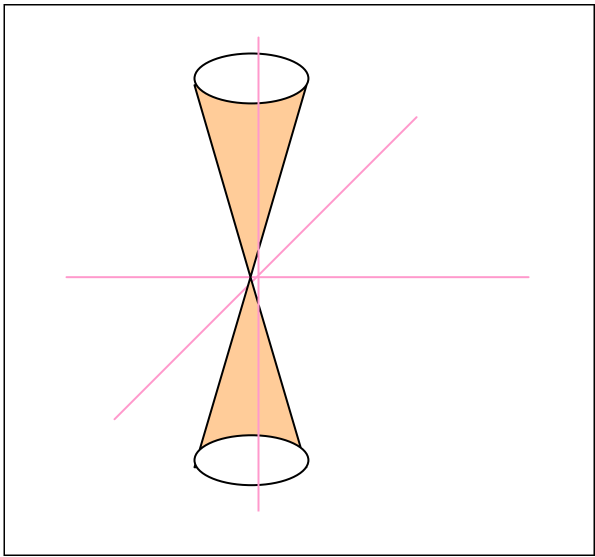
the yz trace is $\frac{z^2}{c^2} - \frac{y^2}{b^2} = 1$ which is a hyperbola.

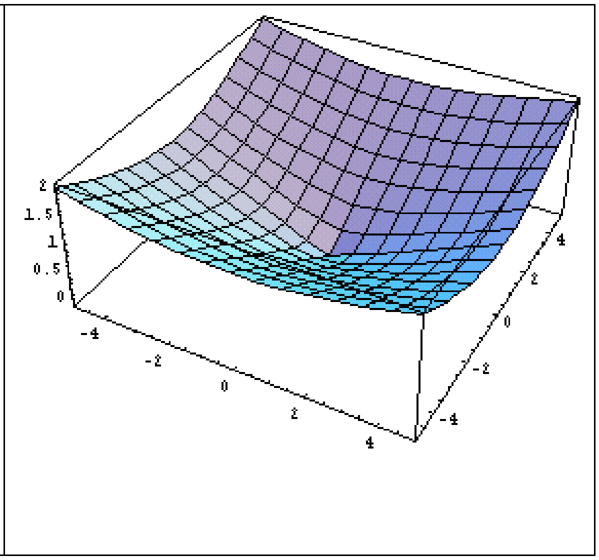
(iv) Elliptic Cone

The standard equation is:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

The axis of the elliptic cone corresponds to the variable with the negative coefficient. In this case the axis is the z axis.





The xy trace is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ which is an ellipse.

The xz trace is $\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$ which is a hyperbola.

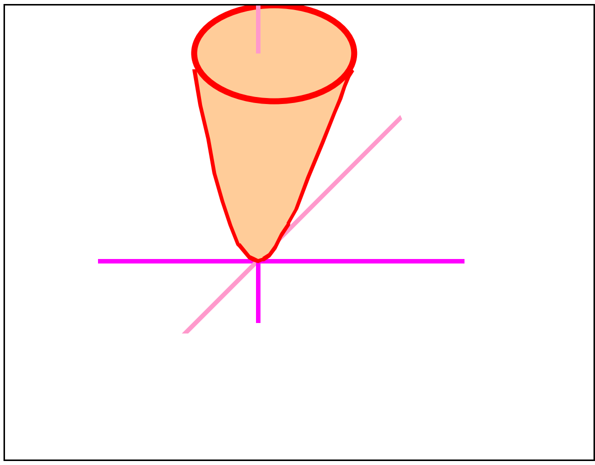
The yz trace is $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ which is a hyperbola.

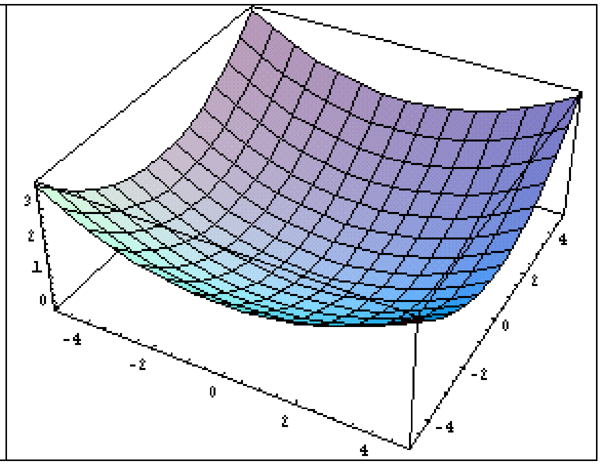
(v) Elliptic paraboloid

The standard equation is:

$$z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

The axis of the paraboloid corresponds to the variable raised to the first power. The axis of this paraboloid is the z axis.



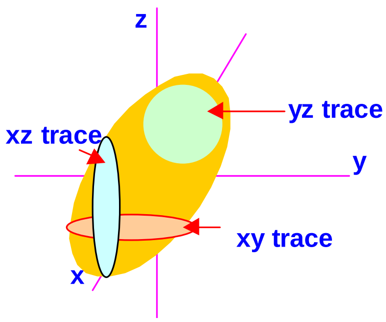


The xy trace is an ellipse
The xz trace is a parabola
The yz trace is a parabola

Example 6:
Identify and sketch the quadric surface
$$\frac{x^2}{9}+\frac{y^2}{16}+\frac{z^2}{16}=1$$

Solution:

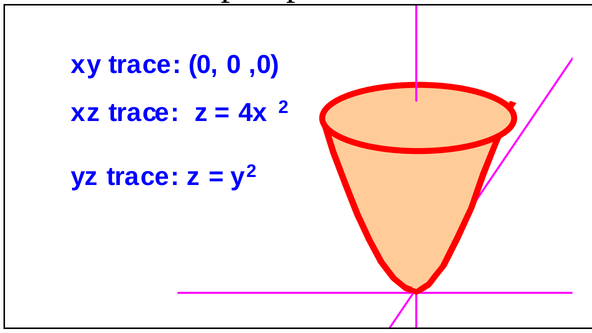
This is an ellipsoid with the ellipse $\frac{x^2}{9}+\frac{y^2}{16}=1$ as the xy trace, the ellipse $\frac{x^2}{9}+\frac{z^2}{16}=1$ as the xz trace, and the circle $y^2 + z^2 = 4$ as the yz trace

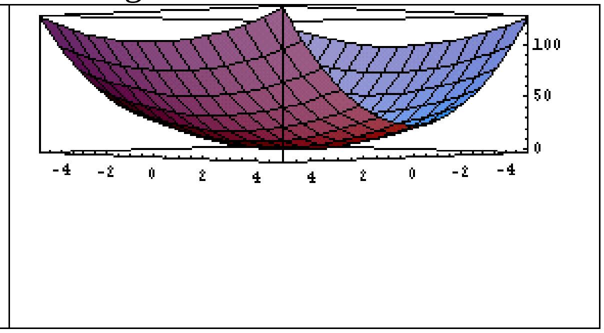


Example 7:
Identify and sketch the quadric surface $z = 4x^2 + y^2$.

Solution:

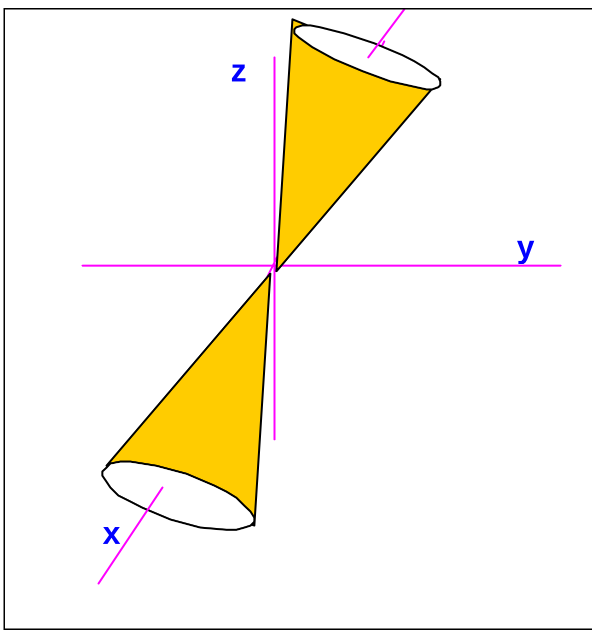
This is an elliptic paraboloid with its axis along the z axis.

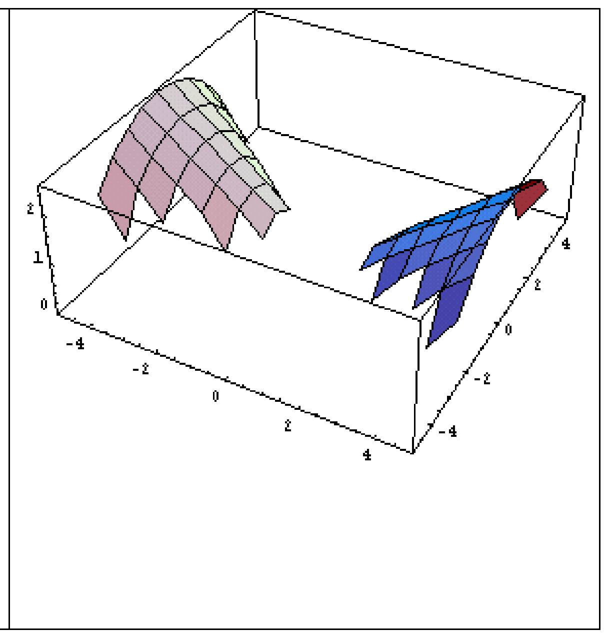




Example 8:
Identify and sketch the graph of $x^2 = 2y^2 + 2z^2$.

Solution:
$y^2 + z^2 - x^2/2 = 0$. This is an elliptic cone with its axis along the x axis.
The xy trace is $x = \pm\sqrt{2}y$
The xz trace is $x = \pm 2\sqrt{z}$
the yz trace is the point (0,0, 0)





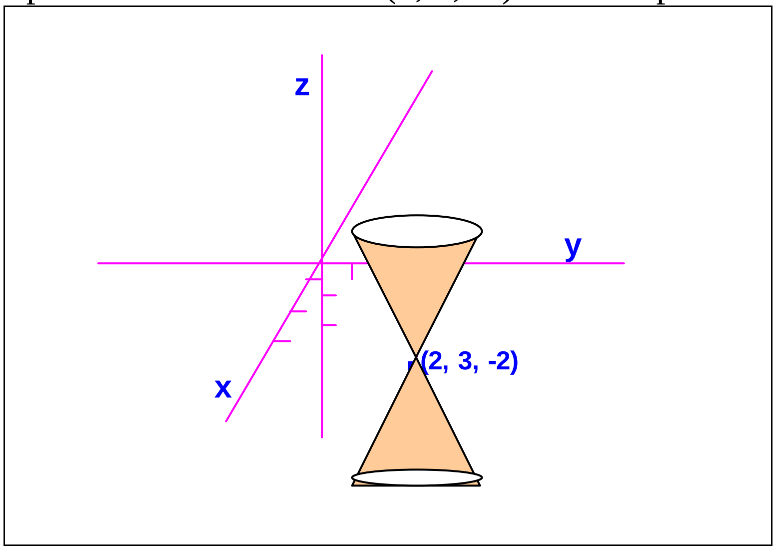
Example 9:
Identify and sketch the quadric surface:
$4x^2 + y^2 - 4z^2 - 16x - 6y - 16z + 9 = 0$

Solution:
$4x^2 - 16x + y^2 - 6y - 4z^2 - 16z = -9$

$4(x^2 - 4x + 4) + y^2 - 6y + 9 - 4(z^2 + 4z + 4) = -9 + 16 + 9 -16$
$4(x - 2)^2 + (y - 3)^2 - 4(z + 2)^2 = 0$

$$(x-2)^2 + \frac{(y-3)^2}{4} - (z+2)^2 = 0$$

This is an elliptic cone with center at (2, 3, -2) and axis parallel to the z axis.



Example 10:

Use a graphing utility to graph the surface $x^2 + y^2 = e^{-z}$

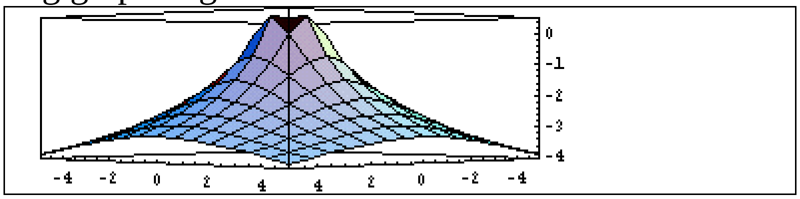
Solution:

First we solve for z before we can use it on a graphing utility.

$$e^z = \frac{1}{x^2 + y^2}$$

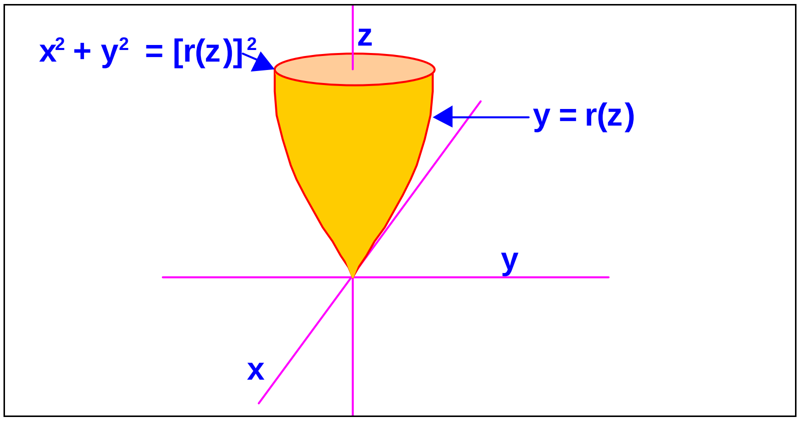
$$z = \ln\left(\frac{1}{x^2 + y^2}\right)$$

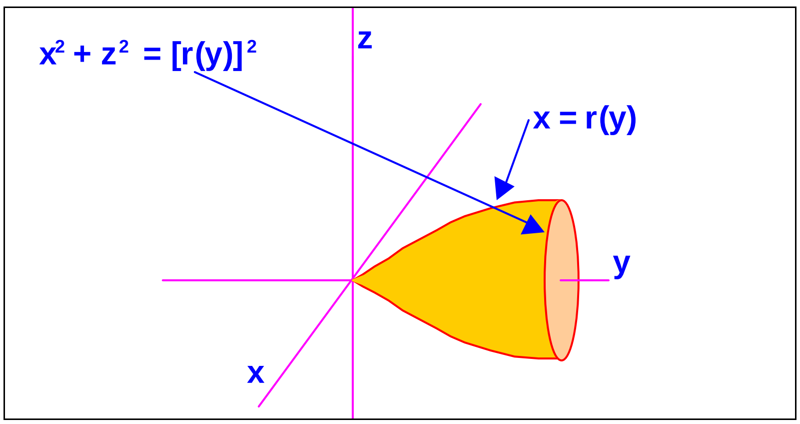
The following graph is generated on mathematica.



3. Surfaces of Revolution:

If a generating curve y = r(z) is revolved about the z axis, the surface produced will be as shown in the figure below. The traces taken parallel to the xy plane will be circles given by $x^2 + y^2 = [r(z)]^2$





Similarly if the generating curve x = r(y) is revolved about the y axis, the surface generated will have the equation $x^2 + z^2 = [r(y)]^2$

The graph of a radius function revolved about one of the coordinate axes, the equation of the resulting surface has one of the following forms:

1. **Revolved about the x axis: $y^2 + z^2 = [r(x)]^2$**
2. **Revolved about the y axis: $x^2 + z^2 = [r(y)]^2$**
3. **Revolved about the z axis: $x^2 + y^2 = [r(z)]^2$**

Example 11:

Find an equation of the surface of revolution by revolving the curve z = 2y in the yz plane about the y axis.

Solution:

We have the generating curve given by z = r(y) = 2y.
The equation of the surface of revolution is given by $x^2 + z^2 = [r(y)]^2$
Or **$x^2 + z^2 = 4y^2$**

Example 12:
Find an equation for the surface of revolution generated by revolving the curve $2z = \sqrt{4 - x^2}$ in the xz plane about the x axis.

Solution:
The radius function must a function of x

$$r(x) = z = \frac{\sqrt{4 - x^2}}{2}.$$

The equation of the surface generated is:

$$y^2 + z^2 = \frac{4 - x^2}{4}$$

$$4y^2 + 4z^2 = 4 - x^2$$

$$x^2 + 4y^2 + 4z^2 = 4$$

Example 13:
Find an equation for the surface of revolution generated by revolving the curve z = ln y in the yz plane about the z axis.

Solution:

Since the axis of revolution is the z axis, the radius function is r(z)
z = ln y. solving for y we get,
$y = e^z$
The equation for the surface is **$x^2 + y^2 = e^{2z}$**

Example 14:
Find an equation of the generating curve which generates the surface given by $x^2 + z^2 = \sin^2 y$

Solution:

x = siny is the equation of the generating curve in the xy plane.

$z = \sin y$ is the equation of the generating curve in the yz plane.

Homework:

1. Describe and sketch the following surfaces:
   (i) $y^2 + z^2 = 9$
   (ii) $4x^2 + y^2 = 4$

2. Identify and sketch the following quadric surfaces.

   (i) $16x^2 - y^2 + 16z^2 = 4$
   (ii) $x^2 - y^2 + z = 0$
   (iii) $16x^2 + 9y^2 + 16z^2 - 32x - 36y + 36 = 0$

3. Find an equation for the surface of revolution generated by the following curve about the given axis.

   (i) $z^2 = 4y$ in the yz plane about the y axis
   (ii) $z = 2y$ in the yz plane about the z axis.

4. Find an equation of a generating curve for the surface of revolution given by $x^2 + y^2 - 2z = 0$

Practice 1, Exam 2
Calculus III

Question 1 **(4 points):** **Consider the function** $f(x,y)=\sqrt{y-x^2}$.

(a) Find and describe the domain of f.

(b) Sketch the level curves k = 1, and k = 2 all on one coordinate grid. What kind of curves are they?

Question 2 **(4 points):** **Evaluate the limit** $\lim_{(x,y)\to(4,3)} \frac{\sqrt{x}-\sqrt{y+1}}{x-y-1}$.

Question 3 **(4 points):** **Let** $f(x,y)=\begin{cases} \frac{xy^2}{x^2+y^4}, & (x,y)\neq(0,0) \\ 0, & (x,y)\neq(0,0). \end{cases}$

Show that $f_y(0,0)$ **exists.**

Question 4 **(4 points):** **Use a chain rule to find the value of** $\left.\frac{\partial z}{\partial\theta}\right|_{r=2,\theta=\pi/6}$ **if**

$$z=xye^{x/y}, \quad x=r\cos\theta, \quad y=r\sin\theta.$$

Question 5 **(4 points):** **Let** $f(x,y)=\frac{x+1}{y+1}$

(a) Find the directional derivative of f **at the point (2, 0) in the direction of the vector** $\vec{v}=\langle -1,\sqrt{3}\rangle$.

(b) Find the equation of the tangent plane of the surface z = f**(x, y) at the point (1, 1, 1).**

Question 6 **(4 points):** **Find the shortest distance from the point (-6, 4, 0) to the cone** $z=\sqrt{x^2+y^2}$.

Question 7 **(6 points):** **Find all the local maxima, local minima, and saddle points of the function** $f(x,y)=x^3-y^3-2xy+6$.

Practice 2, Exam 2
Calculus III

Question 1 (4 points): **Consider the function $f(x,y) = ln(4 - x^2 - y^2)$.**
(a) Determine and sketch the domain of f.
(b) Sketch the level curves k = 0, k = 1, and k = 2 all on one coordinate grid. What kind of curves are they?

Question 2 (4 points): **Determine the set of points at which the function is continuous.**

$$f(x,y) = \frac{1 + x^2 + y^2}{1 - x^2 - y^2}$$

Question 3 (4 points): **By considering different paths of approach, show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2}$ does not exist.**

Question 4 (4 points): **Suppose that**

$$w = \sqrt{x^2 + y^2 + z^2}, \quad x = cos\theta, \quad x = sin\theta, \quad z = tan\theta$$

Use the chain rule to find $\frac{dw}{d\theta}$ when $\theta = \frac{\pi}{4}$.

Question 5 (4 points): **Find an equation for the tangent plane and parametric equations for the normal line to the surface $x^2y - 4z^2 = -7$ at the point $(-3, 1. -2)$.**

Question 6 (4 points): **The temperature (in degrees Celsius) at a point (x, y) on a metal plate in the xy-plane is $T(x,y) = \frac{xy}{1+x^2+y^2}$.**

a) **Find the rate of change of temperature at (1, 1) in the direction of $\vec{v} = 2\vec{\imath} - \vec{\jmath}$.**
b) **An ant at (1, 1) wants to walk in the direction in which the temperature drops most rapidly. Find a unit vector in that direction.**

Question 7 (4 points): **Consider the function $f(x,y) = 2 + x^2 - y^2 - y$.**

(a) Find the critical points of f and classify each one as a local maximum, local minimum, or saddle point.
(b) Find the absolute maximum and minimum values of f on the disk

$$\{(x,y) | x^2 + y^2 \leq 1\}$$

and the points where these extreme values occur.

Calculus III

Study concepts, example questions & explanations

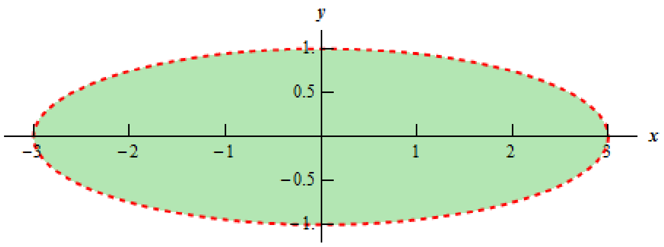
Question #1: Domain of a Function

Let $f(x, y) = ln(9 - x^2 - 9y^2)$. Evaluate $f(2, 0)$ and find and sketch the domain of *f*.

Answer:

$f(2, 0) = ln(9 - 2^2 - 9 \times 0^2) = ln(5)$.

$ln(9 - x^2 - 9y^2)$ is defined whenever $9 - x^2 - 9y^2 > 0$ or $\frac{x^2}{9} + y^2 < 1$. Thus the domain of *f* is $D = \left\{(x, y) \mid \frac{x^2}{9} + y^2 < 1\right\}$, the points inside the ellipse $\frac{x^2}{9} + y^2 = 1$.



Question #2: Domain and Range of a Function

Let $f(x, y) = \sqrt{36 - 9x^2 - 4y^2}$.

- **(a) Evaluate $f(2, 0)$.**
- **(b) Find and sketch the domain of *f*.**
- **(c) Find the range of *f*.**

Answer:

$\boldsymbol{f(1,2) = \sqrt{36 - 9(1)^2 - 4(2)^2} = \sqrt{11}.}$

For the square root to be defined, we need $\mathbf{36 - 9x^2 - 4y^2 \geq 0}$ or $\frac{x^2}{4} + \frac{y^2}{9} \leq 1$. Thus the domain of *f* is $\boldsymbol{D} = \left\{(\boldsymbol{x}, \boldsymbol{y}) \mid \frac{x^2}{4} + \frac{y^2}{9} \leq 1\right\}$, the points on or inside the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Since $\mathbf{0} \leq \sqrt{\mathbf{36 - 9x^2 - 4y^2}} \leq \mathbf{6}$, the range is $\{z \mid 0 \leq z \leq 6\} = [0,6]$.

Question #3: Domain and Range of a Function

Find the domain and range of the function $\boldsymbol{f(x,y) = \frac{1}{\sqrt{x^2+y^2-1}}}$.

Answer:

$\sqrt{\boldsymbol{x^2 + y^2 - 1}}$ is defined whenever $\boldsymbol{x^2 + y^2 - 1} \geq 0$ and since it is in the denominator we consider only $\boldsymbol{x^2 + y^2 - 1} > 0$ or $\boldsymbol{x^2 + y^2 > 1}$. . Thus the domain of *f* is $\boldsymbol{D} = \{(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x^2 + y^2 > 1}\}$, the points outside the unit circle $\boldsymbol{x^2 + y^2 = 1}$.

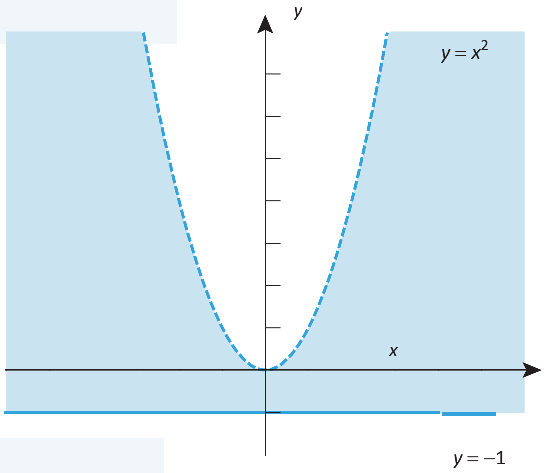
Since $\sqrt{\boldsymbol{x^2 + y^2 - 1}} > 0$, the range of *f* is $(\mathbf{0}, \infty)$.

Question #4: Domain of a Function

Find and sketch the domain of the function $\boldsymbol{f(x,y) = \sqrt{y+1} + ln(x^2 - y)}$.

Answer:

We note that $\sqrt{\boldsymbol{y}+\mathbf{1}}$ is defined only when $\boldsymbol{y}+\mathbf{1} \geq 0$ or $\boldsymbol{y} \geq -1$, while $\boldsymbol{ln}(\boldsymbol{x}^2-\boldsymbol{y})$ is defined only when $\boldsymbol{x}^2-\boldsymbol{y} > 0$ or $\boldsymbol{y} < \boldsymbol{x}^2$. Thus the domain of *f* is $\boldsymbol{D} = \{(\boldsymbol{x},\boldsymbol{y})\ |{-\mathbf{1}} \leq \boldsymbol{y} < \boldsymbol{x}^2\}$. The natural domain of *f* is then the region lying above or on the line *y* = −1 and below the parabola $\boldsymbol{y} = \boldsymbol{x}^2$.



Question #5: Domain of a Function

Find the domain of the function $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \frac{\sqrt{9-x^2-y^2}}{x+2y}$.

Answer:

$\sqrt{\mathbf{9}-\boldsymbol{x}^2-\boldsymbol{y}^2}$ is defined whenever $\mathbf{9}-\boldsymbol{x}^2-\boldsymbol{y}^2 \geq 0$ or $\boldsymbol{x}^2+\boldsymbol{y}^2 \leq \mathbf{9}$ and the denominator $\boldsymbol{x}+\mathbf{2}\boldsymbol{y} \neq 0$. So, the domain of *f* is $\boldsymbol{D} = \{(\boldsymbol{x},\boldsymbol{y})\ |\boldsymbol{x}^2+\boldsymbol{y}^2 \leq \mathbf{9}\ \boldsymbol{and}\ \boldsymbol{x} \neq -\mathbf{2}\boldsymbol{y}\}$.

Question #6: Graph of a Function

In each part, describe the graph of the function in the xyz-coordinate system.

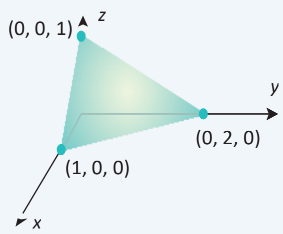
(a) $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \mathbf{1}-\boldsymbol{x}-\frac{1}{2}\boldsymbol{y}$.

(b) $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \sqrt{\mathbf{1}-\boldsymbol{x}^2-\boldsymbol{y}^2}$

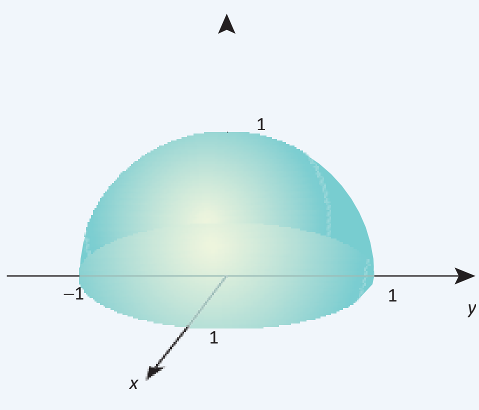
(c) $f(x, y) = -\sqrt{x^2 + y^2}$

Answer:

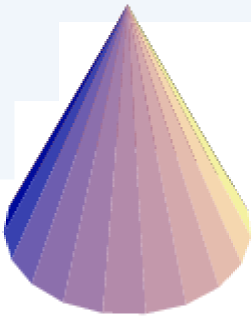
(a) The graph of the given function is the graph of the equation $z = 1 - x - \frac{1}{2}y$ which is a plane. A triangular portion of the plane can be sketched by plotting the intersections with the coordinate axes and joining them with line segments.



(b) The graph of the given function is the graph of the equation $z = \sqrt{1 - x^2 - y^2}$. After squaring both sides, this can be rewritten as $x^2 + y^2 + z^2 = 1$. which represents a sphere of radius 1, centered at the origin. Since $z = \sqrt{1 - x^2 - y^2}$ imposes the extra condition that z ≥ 0, the graph is just the upper hemisphere.



(c) The graph of the given function is the graph of the equation $z = -\sqrt{x^2 + y^2}$. After squaring both sides, this can be rewritten as $z^2 = x^2 + y^2$, which is the equation of a circular cone. Since $z = -\sqrt{x^2 + y^2}$ imposes the extra condition that $z \leq 0$, the graph is just the lower nappe of the cone.



Question #7: Level curves

Let $f(x, y) = \sqrt{x^2 + y^2}$.

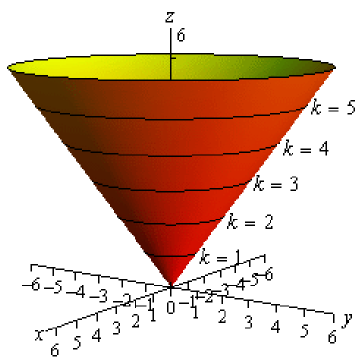
(a) Sketch the graph of the function.

(b) Identify the level curves of *f*.

(c) Sketch the level curves for $k = 1, 2, 3, 4, 5$.

Answer:

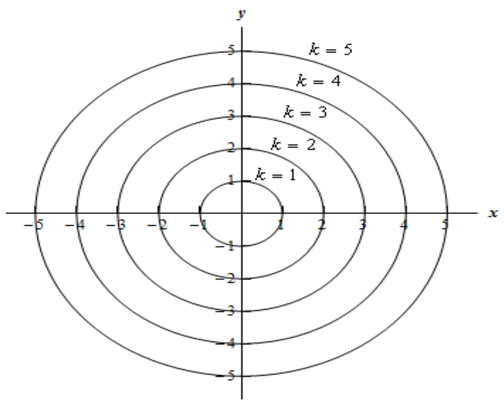
$z = f(x, y) = \sqrt{x^2 + y^2}$. Recall from the Quadric Surfaces section that this the upper portion of the cone $z^2 = x^2 + y^2$.



The level curves are

$$\sqrt{x^2+y^2}=k \text{ or } x^2+y^2=k^2$$

This is a family of circles with center (0, 0) and radius *k*. The cases $k=1,2,3,4,5$ are shown in the figure below.



Question #8: Level Curves

Identify and sketch the level curves (or contours) for the following function $y^2=2x^2+z$.

Answer:

We know that level curves or contours are given by setting *z* = *k*. Doing this in our equation gives,

$$y^2=2x^2+k$$

If k = 0, the equation will be $y^2=2x^2$ or $y=\pm\sqrt{2}x$. So, in this case the level curve(s) will be two lines through the origin.

Next, if *k* > 0, the level curves will be

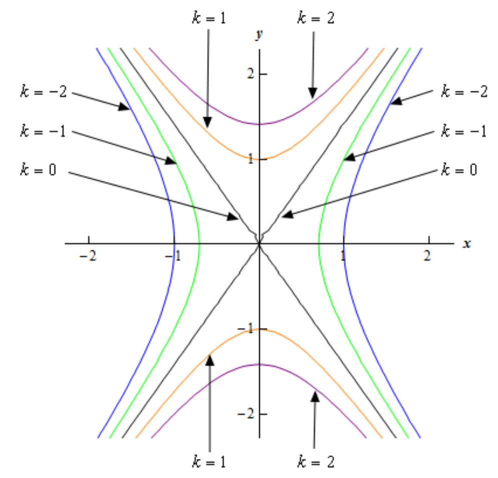
$$\frac{y^2}{k}-\frac{x^2}{k/2}=1$$

which are hyperbolas symmetric about the *y*-axis and open up and down.

Finally, if k < 0 the level curves are in the form

$$\frac{x^2}{(-\mathrm{k}/2)} - \frac{y^2}{(-k)} = 1.$$

This is a family of hyperbolas that are symmetric about the *x*-axis and open right and left.



Question #9: Level Surfaces

Let $f(x,y) = \frac{1}{\sqrt{x^2+y^2+z^2-4}}$.

(a) Find the domain of *f*.
(b) Find the level surfaces of *f*.

Answer:

$\sqrt{x^2+y^2+z^2-4}$ is defined whenever $x^2+y^2+z^2-4 \geq 0$ or $x^2 + y^2 + z^2 \geq 4$ and to avoid division by zero we need that $x^2+y^2+z^2 >$

4. Thus the domain of f is $\boldsymbol{D} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})\ |\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2 > 4\}$, the points outside the sphere of center (0, 0, 0) and radius 2.

The level surfaces are $\frac{\mathbf{1}}{\sqrt{\boldsymbol{x}^2+\boldsymbol{y}^2+\boldsymbol{z}^2-\mathbf{4}}} = \boldsymbol{k}$, where $k \geq 0$. These form a family of concentric spheres with center (0, 0, 0) and radius $\sqrt{\mathbf{4} + \frac{\mathbf{1}}{\boldsymbol{k}^2}}$

$$\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2 = \mathbf{4} + \frac{\mathbf{1}}{\boldsymbol{k}^2}.$$

Calculus III

Study concepts, example questions & explanations

Question #1: Limits

Find the limit in the following:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$

(b) $\lim_{(x,y)\to(0,0)} \frac{e^y sinx}{x}$

(c) $\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y}$

(d) $\lim_{(x,y)\to(0,0)} \frac{x-y}{x^4-y^4}$

(e) $\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y-xy+4x^2-4x}$

(f) $\lim_{(x,y)\to(0,0)} \frac{sin(x^2+y^2)}{x^2+y^2}$

Answer:

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\to(0,0)} \frac{x(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})} = \lim_{(x,y)\to(0,0)} x(\sqrt{x}+\sqrt{y}) = 0.$

(b) $\lim_{(x,y)\to(0,0)} \frac{e^y sinx}{x} = \lim_{(x,y)\to(0,0)} e^y \frac{sinx}{x} = 1 \times 0 = 0.$

(c) $\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,1)} \frac{(x-y)(x+y)}{(x-y)} = \lim_{(x,y)\to(1,1)} (x+y) = 2.$

(d) $\lim_{(x,y)\to(2,2)} \frac{x-y}{x^4-y^4} = \lim_{(x,y)\to(2,2)} \frac{(x-y)}{(x^2-y^2)(x^2+y^2)} = \lim_{(x,y)\to(2,2)} \frac{(x-y)}{(x-y)(x+y)(x^2+y^2)} =$
$\lim_{(x,y)\to(2,2)} \frac{1}{(x+y)(x^2+y^2)} = \frac{1}{32}.$

(e) $\lim_{(x,y)\to(2,-4)} \frac{y+4}{x^2y-xy+4x^2-4x} = \lim_{(x,y)\to(2,-4)} \frac{y+4}{yx(x-1)+4x(x-1)} = \lim_{(x,y)\to(2,-4)} \frac{y+4}{x(x-1)(y+4)} =$
$\lim_{(x,y)\to(2,-4)} \frac{1}{x(x-1)} = \frac{1}{2}.$

(f) Let $w = x^2 + y^2$, then $\lim_{(x,y)\to(0,0)} \frac{sin(x^2+y^2)}{x^2+y^2} = \lim_{w\to0^+} \frac{sin(w)}{w} = 1.$

Question #2: Limits

By considering different paths of approach, show that the limits do not exist.

(a) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4-y^2}{x^4+y^2}$

(b) $\lim\limits_{(x,y)\to(1,1)} \frac{xy^2-1}{y-1}$

(c) $\lim\limits_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$

(d) $\lim\limits_{(x,y)\to(1,-1)} \frac{xy+1}{x^2-y^2}$

Answer:

(a) We examine the values of *f* along curves that end at (0, 0).
Along the *x*-axis, *y* = 0, the limit is

$$\lim_{\substack{(x,y)\to(0,0)\\ Along\ x-axis}} \frac{x^4-y^2}{x^4+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^4}{x^4} = 1.$$

Along the *y*-axis, *x* = 0, the limit is

$$\lim_{\substack{(x,y)\to(0,0)\\ Along\ y-axis}} \frac{x^4-y^2}{x^4+y^2} = \lim_{(x,y)\to(0,y)} \frac{-y^2}{y^2} = -1.$$

Since *f* has two different limits along two different curves, the given limit does not exist.

(b) First let's approach (1, 1) along the line x = 1. Then

$$\lim_{\substack{(x,y)\to(1,1)\\ Along\ x=1}} \frac{xy^2-1}{y-1} = \lim_{y\to1} \frac{y^2-1}{y-1} = \lim_{y\to1} \frac{(y-1)(y+1)}{y-1} = \lim_{y\to1} y+1 = 2.$$

We now approach along the line *y* = x, Then

$$\lim_{\substack{(x,y)\to(1,1)\\ Along\ y=x}} \frac{x^4-y^2}{x^4+y^2} = \lim_{x\to1} \frac{x^3-1}{x-1} = \lim_{x\to1} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to1}(x^2+x+1) = 3.$$

Since *f* has two different limits along two different curves, the given limit does not exist.

(c) First let's approach (0, 0) along the x-axis, y = 0. Then

$$\lim_{\substack{(x,y)\to(0,0) \\ Along\ x-axis}} \frac{x^2y}{x^4+y^2} = \lim_{y\to 0}\frac{0}{y^2} = 0.$$

Next, we approach (1, 1) along the curve $y = x^2$, Then

$$\lim_{\substack{(x,y)\to(0,0) \\ Along\ y=x^2}} \frac{x^2y}{x^4+y^2} = \lim_{x\to 0}\frac{x^4}{x^4+x^4} = \lim_{x\to 0}\frac{x^4}{2x^4} = \frac{1}{2}.$$

Since we have obtained different limits along different paths, the given limit does not exist.

(d) We examine the values of *f* along curves that end at (1, -1). Along the vertical line $x = 1$, the limit is

$$\lim_{\substack{(x,y)\to(1,-1) \\ Along\ x=1}} \frac{xy+1}{x^2-y^2} = \lim_{y\to -1}\frac{y+1}{1-y^2} = \lim_{y\to -1}\frac{y+1}{(1-y)(1+y)} = \lim_{y\to -1}\frac{1}{(1-y)} = \frac{1}{2}.$$

Along the line $y = -1$, the limit is

$$\lim_{\substack{(x,y)\to(1,-1) \\ Along\ y=-1}} \frac{xy+1}{x^2-y^2} = \lim_{x\to 1}\frac{-x+1}{x^2-1} = \lim_{x\to 1}\frac{-(x-1)}{(x-1)(x+1)} = \lim_{x\to 1}\frac{-1}{(x+1)} = -\frac{1}{2}.$$

Since *f* has two different limits along two different curves, the given limit does not exist.

Question #3: Continuity

Where is the function $f(x,y) = \frac{xy}{x^2-y^2}$ continuous?

Answer:

The function *f* is not defined when $x^2 - y^2 = 0$ and so when $y = x$ or $y = -x$. Since *f* is a rational function, it is continuous on its domain, which is the set $D = \{(x,y)\ |y \neq x\ \ and\ \ y \neq -x\}$.

Question #4: Continuity

Discuss the continuity of the function

$$\boldsymbol{f(x,y)} = \begin{cases} \frac{xy}{x^2+xy+y^2} & \boldsymbol{if\ (x,y) \neq (0,0)} \\ \boldsymbol{0} & \boldsymbol{if\ (x,y) = (0,0)} \end{cases}$$

Answer:

The function *f* is defined everywhere. For $(\boldsymbol{x}, \boldsymbol{y}) \neq (\boldsymbol{0}, \boldsymbol{0})$, *f* is a rational function and $\boldsymbol{x^2} + \boldsymbol{xy} + \boldsymbol{y^2} \neq 0$. Thus f is continuous throughout $(\boldsymbol{x}, \boldsymbol{y}) \neq (\boldsymbol{0}, \boldsymbol{0})$. To examine the continuity of *f* at (0, 0) we need to find $\lim_{(x,y)\to(0,0)} f(x,y)$. By considering different paths of approach, we show that the limit does not exist. First, we approach (0, 0) along the *x*-axis, *y* = 0, to get

$$\lim_{\substack{(x,y)\to(0,0) \\ Along\ x-axis}} \frac{xy}{x^2+xy+y^2} = \lim_{y\to 0} \frac{0}{x^2} = 0.$$

If we approach (0, 0) along the line *y* = *x*, we get

$$\lim_{\substack{(x,y)\to(0,0) \\ Along\ y=x}} \frac{xy}{x^2+xy+y^2} = \lim_{x\to 0} \frac{x^2}{x^2+x^2+x^2} = \lim_{x\to 0} \frac{x^2}{3x^2} = \frac{1}{3}.$$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so *f* is not continuous at (0, 0) and the largest set on which *f* is continuous is $\{(\boldsymbol{x}, \boldsymbol{y}) \mid (\boldsymbol{x}, \boldsymbol{y}) \neq (\boldsymbol{0}, \boldsymbol{0})\}$.

Question #5: Continuity

Discuss the continuity of the function $\boldsymbol{f(x,y,z)} = \frac{xyz}{x^2+y^2-z^2}$

Answer:

The function *f* is a rational function and thus is continuous on its domain $\{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \mid \boldsymbol{x^2} + \boldsymbol{y^2} - \boldsymbol{z^2} \neq \boldsymbol{0}\} = \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \mid \boldsymbol{z^2} \neq \boldsymbol{x^2} + \boldsymbol{y^2}\}$. So *f* is continuous on $\mathbb{R}^3$ except on the cone $\boldsymbol{z^2} = \boldsymbol{x^2} + \boldsymbol{y^2}$.

Question #6: Differentiability

Show that *f* is not differentiable at (0, 0).

$$\boldsymbol{f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4} & if\ \ (x,y) \neq (0,0) \\ 0 & if\ \ (x,y) = (0,0) \end{cases}}$$

Answer:

Recall that If a function *f*(x, y) is differentiable at (x_0,y_0), then *f* is continuous at (x_0,y_0). Thus, if *f* is not continuous at (x_0,y_0) then it is not differentiable at (x_0,y_0).

We will show that *f* is not continuous at (0, 0) and so is differentiable at (0, 0). To do that, we first find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

By considering different paths of approach, we show that the limit does not exist. If we approach (0, 0) along the *x*-axis, *y* = 0, to get

$$\boldsymbol{\lim_{\substack{(x,y)\to(0,0) \\ Along\ x-axis}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{0}{x^2} = 0}.$$

If we approach (0, 0) along the curve $x = y^2$, we get

$$\boldsymbol{\lim_{\substack{(x,y)\to(0,0) \\ Along\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}}.$$

Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so *f* is not continuous at (0, 0) and as a result *f* is not differentiable at (0, 0).

Calculus III

Study concepts, example questions & explanations

Question #1: Partial Derivatives

Find the first partial derivatives of the function $\boldsymbol{f(x,y) = y\ln(x^3 + y^2)}$**.**

Answer:

Recall that $\frac{d}{dx} lnu = \frac{u\prime}{u}$ and $\frac{d}{dx}(u \cdot v) = u \cdot v' + u' \cdot v$

$$f_x = \frac{\partial f}{\partial x} = y\frac{3x^2}{\boldsymbol{x^3+y^2}} = \frac{3yx^2}{\boldsymbol{x^3+y^2}}$$

$$f_y = \frac{\partial f}{\partial y} = y\frac{2y}{\boldsymbol{x^3+y^2}} + (1)\,\mathbf{ln}(\boldsymbol{x^3+y^2}) = \frac{2y^2}{\boldsymbol{x^3+y^2}} + \mathbf{ln}(\boldsymbol{x^3+y^2})$$

Question #2: Partial Derivatives

Find the first partial derivatives of the function $\boldsymbol{f(x,y) = x^2 sin\left(\frac{x}{x+y}\right)}$**.**

Answer:

Recall that $\frac{d}{dx}\sin(u) = \cos(u) \cdot u'$ and $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \cdot u' - u \cdot v\prime}{v^2}$

$$f_x = \frac{\partial f}{\partial x} = x^2 \boldsymbol{cos}\left(\frac{x}{x+y}\right)\left(\frac{(x+y)\cdot 1 - x\cdot 1}{(x+y)^2}\right) + 2x\boldsymbol{sin}\left(\frac{x}{x+y}\right) = \frac{yx^2}{(x+y)^2}\boldsymbol{cos}\left(\frac{x}{x+y}\right) + 2x\boldsymbol{sin}\left(\frac{x}{x+y}\right)$$

$$f_y = \frac{\partial f}{\partial y} = x^2 \boldsymbol{cos}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}+\boldsymbol{y}}\right)\left(\frac{(\boldsymbol{x}+\boldsymbol{y})\cdot \mathbf{0} - \boldsymbol{x}\cdot \mathbf{1}}{(\boldsymbol{x}+\boldsymbol{y})^2}\right) = \frac{-\boldsymbol{x}^3}{(\boldsymbol{x}+\boldsymbol{y})^2}\boldsymbol{cos}\left(\frac{\boldsymbol{x}}{\boldsymbol{x}+\boldsymbol{y}}\right)$$

Question #3: Partial Derivatives

Find the first partial derivatives of the function

$$\boldsymbol{f(x,y) = e^{xy} sin(3x) cos(2y).}$$

Answer:

Recall that $\frac{d}{dx}\mathrm{e}^u = \mathrm{e}^u \cdot u'$ and $\frac{d}{dx}\cos(u) = -\sin(u)\cdot u'$.

$$f_x = \frac{\partial f}{\partial x} = \boldsymbol{e^{xy}\big(3cos(3x)\big)cos(2y) + (ye^{xy})sin(3x)cos(2y)}$$

$$= \boldsymbol{e^{xy}cos(2y)[3cos(3x) + ysin(3x)]}$$

$$f_y = \frac{\partial f}{\partial y} = \boldsymbol{e^{xy}sin(3x)(-2sin(2y)) + (xe^{xy})sin(3x)cos(2y)}$$

$$= \boldsymbol{e^{xy}sin(3x)[xcos(2y) - 2sin(2y)]}$$

Question #4: Partial Derivatives

Find the first partial derivatives of the function

$$\boldsymbol{f(x,y,z) = e^{x} sin(xyz^2) cos(y).}$$

Answer:

$$f_x = \frac{\partial f}{\partial x} = \boldsymbol{e^{x}\left(yz^2cos\left(xyz^2\right)\right)cos(y) + (e^{x})sin\left(xyz^2\right)cos(y)}$$

$$= \boldsymbol{e^{x}cos(y)\left[yz^2cos\left(xyz^2\right) + sin\left(xyz^2\right)\right]}$$

$$f_y = \frac{\partial f}{\partial y} = e^x sin(xyz^2)(-sin(y)) + e^x \left(xz^2 cos(xyz^2)\right) cos(y)$$
$$= e^x[-sin(xyz^2)sin(y) + xz^2 cos(xyz^2)cos(y)]$$

$$f_z = \frac{\partial f}{\partial z} = e^x \left(2zxycos(xyz^2)\right) cos(y) = 2xyze^x cos(xyz^2)cos(y)$$

Question #5: Partial Derivatives

Use the limit definition of partial derivative to compute the partial derivatives $f_x(0,0)$ and $f_y(0,0)$ of the function

$$f(x,y) = \begin{cases} \frac{sin(x^3+y^4)}{x^2+y^2} & if \ \ (x,y) \neq (0,0) \\ 0 & if \ \ (x,y) = (0,0) \end{cases}$$

Answer:
Using the limit definition of partial derivative we get

$$f_x(0,0) = \lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{f(h,0)-0}{h} = \lim_{h\to 0}\frac{\frac{\sin(h^3)}{h^2}}{h} = \lim_{h\to 0}\frac{\sin(h^3)}{h^3} = 1$$

$$f_y(0,0) = \lim_{h\to 0}\frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0}\frac{f(0,h)-0}{h} = \lim_{h\to 0}\frac{\frac{\sin(h^4)}{h^2}}{h} = \lim_{h\to 0}\frac{\sin(h^4)}{h^3}$$
$$= \lim_{h\to 0} h\frac{\sin(h^4)}{h^4} = 0\cdot 1 = 0$$

Question #6: Implicit Differentiation

Find $\frac{\partial z}{\partial x}$ if *z* is defined implicitly as a function of x and y by the equation

$$xyz + cos(x+2y+5z) = 9.$$

Answer:
To find $\frac{\partial z}{\partial x}$, we differentiate implicitly with respect to *x*, being careful to treat *y* as a constant

$$\frac{\partial}{\partial x}[xyz + cos(x + 2y + 5z)] = \frac{\partial}{\partial x}[9]$$

$$(xy)\frac{\partial z}{\partial x} + (1)yz - sin(x + 2y + 5z)\left(1 + 0 + 5\frac{\partial z}{\partial x}\right) = 0$$

$$(xy)\frac{\partial z}{\partial x} + yz - sin(x + 2y + 5z) - 5sin(x + 2y + 5z)\frac{\partial z}{\partial x} = 0$$

$$\left(xy - 5sin(x + 2y + 5z)\right)\frac{\partial z}{\partial x} = sin(x + 2y + 5z) - yz$$

$$\frac{\partial z}{\partial x} = \frac{sin(x + 2y + 5z) - yz}{\left(xy - 5sin(x + 2y + 5z)\right)}$$

Question #7: Implicit Differentiation

Find $\frac{\partial z}{\partial y}$ at the point (2, 0, 1) if *z* is defined implicitly as a function of *x* and *y* by the equation

$$xz + ln(z) = x + y.$$

Answer:
We differentiate both sides of the equation with respect to *y*, holding *x* constant and treating *z* as a differentiable function of *y*:

$$\frac{\partial}{\partial y}[xz + ln(z)] = \frac{\partial}{\partial y}[x + y]$$

$$x\frac{\partial z}{\partial y} + \frac{\frac{\partial z}{\partial y}}{z} = 0 + 1$$

$$\left(x + \frac{1}{z}\right)\frac{\partial z}{\partial y} = 1$$

$$\frac{\partial z}{\partial y} = \frac{1}{\left(x + \frac{1}{z}\right)} = \frac{z}{zx + 1}$$

So, $\frac{\partial z}{\partial y}|_{(2,0,1)} = \frac{1}{(1)(2)+1} = \frac{1}{3}$.

Question #8: Implicit Differentiation

If resistors of R_l , R_2, and R_3 ohms are connected in parallel to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$

Find the value of $\frac{\partial R}{\partial R_2}$ when R_l = 30, R_2 = 45, and R_3 = 90 ohms

Answer:

To find $\frac{\partial R}{\partial R_2}$, we treat R_l and R_3 as constants and, using implicit differentiation, differentiate both sides of the equation with respect to R_2:

$$\frac{\partial}{\partial R_2}\left(\frac{1}{R}\right) = \frac{\partial}{\partial R_2}\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)$$

$$\frac{-1}{R^2}\frac{\partial R}{\partial R_2} = 0 - \frac{1}{{R_2}^2} + 0$$

$$\frac{\partial R}{\partial R_2} = \frac{R^2}{{R_2}^2} = \left(\frac{R}{R_2}\right)^2$$

When R_l = 30, R_2 = 45, and R_3 = 90 ohms

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{30} + \frac{1}{45} + \frac{1}{90} = \frac{1}{15}.$$

So, R = 15 and

$$\frac{\partial R}{\partial R_2} = \left(\frac{15}{45}\right)^2 = \frac{1}{9}.$$

Question #9: Higher Derivatives

Find the second partial derivatives for $\boldsymbol{f(x,y) = x^2y^4 + y\sqrt{x} + 4y}$.

Answer:

$$f_x = \frac{\partial f}{\partial x} = 2xy^4 + y\frac{1}{2\sqrt{x}} + 0 = 2xy^4 + \frac{1}{2}yx^{-1/2}$$

$$f_y = \frac{\partial f}{\partial y} = 4x^2y^3 + \sqrt{x} + 4$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left[2xy^4 + \frac{1}{2}yx^{-1/2}\right] = 2y^4 + \frac{1}{2}y\left(-\frac{1}{2}x^{-3/2}\right) = 2y^4 - \frac{1}{4}yx^{-3/2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left[4x^2y^3 + \sqrt{x} + 4\right] = 12x^2y^2 + \frac{1}{2}x^{-1/2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left[2xy^4 + \frac{1}{2}yx^{-1/2}\right] = 8xy^3 + \frac{1}{2}x^{-1/2}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[4x^2y^3 + \sqrt{x} + 4\right] = 8xy^3 + \frac{1}{2}x^{-1/2}$$

Calculus III

Study concepts, example questions & explanations

Question #1: Differentiability

Show that $f(x,y) = ye^{xy}$ is differentiable at (0, 1).

Answer:

Recall that if the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then *f* is differentiable at (a, b).

The partial derivatives are

$$f_x = y^2 e^{xy}, \qquad f_x(0,1) = 1$$

$$f_y = yxe^{xy} + e^{xy}, \qquad f_y(0,1) = 1$$

Both f_x and f_y are continuous functions, so is *f* differentiable at (0, 1).

Question #2: Differentiability

Show that *f* is not differentiable at (0, 0).

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Answer:

Recall that If a function $f(x, y)$ is differentiable at (x_0,y_0), then f is continuous at (x_0,y_0). Thus, if f is not continuous at (x_0,y_0) then it is not differentiable at (x_0,y_0).

We will show that f is not continuous at (0, 0) and so is differentiable at (0, 0). To do that, we first find $\lim_{(x,y)\to(0,0)} f(x,y)$.

By considering different paths of approach, we show that the limit does not exist. If we approach (0, 0) along the x-axis, $y = 0$, to get

$$\lim_{\substack{(x,y)\to(0,0) \\ Along\ x-axis}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{0}{x^2} = 0.$$

If we approach (0, 0) along the curve $x = y^2$, we get

$$\lim_{\substack{(x,y)\to(0,0) \\ Along\ x=y^2}} \frac{xy^2}{x^2+y^4} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \lim_{y\to 0} \frac{y^4}{2y^4} = \frac{1}{2}.$$

Thus $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at (0, 0) and as a result f is not differentiable at (0, 0).

Question #3: Tangent Plane

Find an equation of the tangent plane to the given surface at the specified point.

$$\boldsymbol{f(x,y) = x\sin(x+y), \qquad (-1, 1, 0).}$$

Answer:

Recall that if f has continuous partial derivatives, then nn equation of the tangent plane to the surface at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0)$$

The partial derivatives are

$$f_x = x\cos(x+y) + \sin(x+y), \qquad f_x(-1,1) = -1,$$

$$f_y = x cos(x+y), \qquad f_y(-1,1) = -1.$$

Thus an equation of the tangent plane at $(\mathbf{-1}, \mathbf{1}, \mathbf{0})$ is

$$z - z_0 = f_x(x_0, y_0)(\mathrm{x} - x_0) + f_y(x_0, y_0)(\mathrm{y} - y_0)$$
$$z - 0 = f_x(-1,1)(\mathrm{x} - (-1)) + f_y(-1,1)(\mathrm{y} - 1)$$
$$\mathrm{z} = (-1)(\mathrm{x} + 1) + (-1)(\mathrm{y} - 1)$$
$$z = -x - y$$

Question #4: Tangent Plane

Find an equation of the tangent plane to the given surface at the specified point.

$$\boldsymbol{f(x,y) = \ln(\mathrm{x} - 2\mathrm{y}), \qquad (3, 1, 0).}$$

Answer:
Recall that if *f* has continuous partial derivatives, then nn equation of the tangent plane to the surface at the point (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(\mathrm{x} - x_0) + f_y(x_0, y_0)(\mathrm{y} - y_0)$$

The partial derivatives are

$$f_x = \frac{1}{x-2y}, \qquad f_x(3,1) = 1,$$

$$f_y = \frac{-2}{x-2y}, \qquad f_y(3,1) = -2.$$

Thus an equation of the tangent plane at $(3,1,0)$ is

$$z - z_0 = f_x(x_0, y_0)(\mathrm{x} - x_0) + f_y(x_0, y_0)(\mathrm{y} - y_0)$$
$$z - 0 = f_x(3,1)(\mathrm{x} - 3) + f_y(3,1)(\mathrm{y} - 1)$$
$$\mathrm{z} = (1)(\mathrm{x} + 1) + (-2)(\mathrm{y} - 1)$$
$$z = x - 2y + 3$$

Calculus III

Study concepts, example questions & explanations

Question #1: Chain Rule

Use the Chain Rule to find $\frac{\partial z}{\partial t}$ **where** $\boldsymbol{z = cos(x + 4y), \;\; x = 5t^4, \;\; y = 1/t.}$

Answer:
To remember the Chain Rule, it's helpful to draw the tree diagram.
With the help of the tree diagram, we have the Chain Rule

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Applying the above Chain Rule, we get

$$\frac{\partial z}{\partial t} = (-\sin(x + 4y))(20t^3) + (-4\sin(x + 4y))\left(-\frac{1}{t^2}\right)$$

$$\frac{\partial z}{\partial t} = -20t^3 \sin(5t^4 + 4/t) + \frac{16}{t^2}\sin(5t^4 + 4/t)$$

$$\frac{\partial z}{\partial t} = \left(-20t^3 + \frac{16}{t^2}\right)\sin(5t^4 + 4/t)$$

Question #2: Chain Rule

Use the Chain Rule to find $\frac{\partial z}{\partial s}$ **where**

$$\boldsymbol{z = \sqrt{x - 3y}, \qquad x = s^2 + t^3, \qquad y = 1 - 2st.}$$

Answer:
To remember the Chain Rule, it's helpful to draw the tree diagram. With the help of the tree diagram, we have the Chain Rule

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

Applying the above Chain Rule, we get

$$\frac{\partial z}{\partial s} = \left(\frac{1}{2\sqrt{x-3y}}\right)(2s) + \left(\frac{-3}{2\sqrt{x-3y}}\right)(-2t)$$

$$\frac{\partial z}{\partial s} = \frac{s+3t}{\sqrt{x-3y}}$$

Question #3: Chain Rule

Use the Chain Rule to find $\frac{\partial w}{\partial \theta}$ when r = 2 and $\theta = \pi/2$.

$$\boldsymbol{w = xy + xz + yz, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = r\theta.}$$

Answer:
To remember the Chain Rule, it's helpful to draw the tree diagram. With the help of the tree diagram, we have the Chain Rule

$$\frac{\partial z}{\partial \theta} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial \theta}$$

Applying the above Chain Rule, we get

$$\frac{\partial z}{\partial \theta} = (y+z)(-rsin\theta) + (x+z)(rcos\theta) + (x+y)(r)$$

$$\frac{\partial z}{\partial \theta} = (rsin\theta + r\theta)(-rsin\theta) + (rcos\theta + r\theta)(rcos\theta) + (rcos\theta + rsin\theta)(r)$$

$$\frac{\partial z}{\partial \theta}|_{r=2,\theta=\pi/2} = (2+\pi)(-2) + (0+\pi)(0) + (0+2)(2) = -2\pi$$

Question #4: Chain Rule

Use the Chain Rule to find $\boldsymbol{\frac{\partial T}{\partial p}}$ when $\boldsymbol{p}$ = 4, $\boldsymbol{q}$ = 1 and $\boldsymbol{r = 9}$.

$$\boldsymbol{T = 3u^3v^2 + uv + v^4, \qquad u = pq\sqrt{r}, \qquad v = q\sqrt{pr}.}$$

Answer:
To remember the Chain Rule, it's helpful to draw the tree diagram. With the help of the tree diagram, we have the Chain Rule

$$\frac{\partial T}{\partial p} = \frac{\partial T}{\partial u}\frac{\partial u}{\partial p} + \frac{\partial T}{\partial v}\frac{\partial v}{\partial p}$$

Applying the above Chain Rule, we get

$$\frac{\partial T}{\partial p} = (9u^2v^2 + v + 4v^3)\left(q\sqrt{r}\right) + (6u^3v)\left(\frac{qr}{2\sqrt{p}}\right)$$

When p = 4, q = 1 and $\boldsymbol{r = 9}$, we have that

$$\boldsymbol{u = pq\sqrt{r} = 12} \qquad \text{and} \qquad \boldsymbol{v = q\sqrt{pr} = 18}.$$

Thus

$$\frac{\partial T}{\partial p}|_{\mathrm{p}\,=\,4,\mathrm{q}\,=\,1,\boldsymbol{r=9}} = 1749654$$

Question #5: Chain Rule

If $z = f(x, y)$ has continuous second-order partial derivatives and $x = r^2 + s^2$ and $y = 2rs$. Find $\frac{\partial^2 z}{\partial r^2}$.

Answer:
The Chain Rule gives

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}$$

Applying the Product Rule to the last expression, we

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y}\right)$$

$$= 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2\frac{\partial z}{\partial x} + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

But, using the Chain Rule again, we have

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r} = 2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}$$

$$\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial r} = 2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}$$

Putting these expressions into the equation of $\frac{\partial^2 z}{\partial r^2}$ and using the equality of the mixed second-order derivatives, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, we obtain

$$\frac{\partial^2 z}{\partial r^2} = 2r\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}\right) + 2\frac{\partial z}{\partial x} + 2s\frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}\right)$$

$$= 2r\left(2r\frac{\partial^2 z}{\partial x^2} + 2s\frac{\partial^2 z}{\partial y \partial x}\right) + 2\frac{\partial z}{\partial x} + 2s\left(2r\frac{\partial^2 z}{\partial x \partial y} + 2s\frac{\partial^2 z}{\partial y^2}\right)$$

$$= 2\frac{\partial z}{\partial x} + 4r^2\frac{\partial^2 z}{\partial x^2} + 8rs\frac{\partial^2 z}{\partial x \partial y} + 4s^2\frac{\partial^2 z}{\partial y^2}$$

Question #6: Chain Rule

Let $W(s,t) = F(u(s,t), v(s,t))$ where are differentiable, and

$$u(1.0) = 2,\ \ u_s(1,0) = -2,\ \ u_t(1,0) = 6,$$
$$v(1.0) = 3,\ \ v_s(1,0) = 5,\ \ v_t(1,0) = 4,$$
$$F_u(2,3) = -1,\ \ F_v(2,3) = 10$$

Find $W_s(1,0)$.

Answer:

With the help of the tree diagram, we have the Chain Rule

$$\frac{\partial W}{\partial s} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial s}$$

Applying the above Chain Rule, we get

$$\frac{\partial W}{\partial s}|_{(1,0)} = F_u(u(1,0), v(1,0))u_s(1,0) + F_v(u(1,0), v(1,0))v_s(1,0)$$

$$\frac{\partial W}{\partial s}|_{(1,0)} = F_u(2,3)(-2) + F_v(2,3)(5)$$

$$\frac{\partial W}{\partial s}|_{(1,0)} = (-1)(-2) + (10)(5) = 52$$

Question #7: Chain Rule

Use the Chain Rule to find $\frac{\partial u}{\partial z}$ when $x = \sqrt{3},\ y = 2,\ z = 1.$

$$u = \frac{p-q}{q-r}, \qquad p = x + y + z, \qquad q = x - y + z, \qquad r = x + y - z.$$

Answer:

With the help of the tree diagram, we have the Chain Rule

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial u}{\partial r}\frac{\partial r}{\partial z}$$

Applying the above Chain Rule, we get

$$\frac{\partial u}{\partial z} = \left(\frac{1}{q-r}\right)(1) + \left(\frac{(q-r)(-1)-(p-q)(1)}{(q-r)^2}\right)(1) + \left(\frac{p-q}{(q-r)^2}\right)(-1)$$

$$\frac{\partial u}{\partial z} = \frac{1}{q-r} + \frac{r-p}{(q-r)^2} + \frac{q-p}{(q-r)^2} = \frac{1}{q-r} + \frac{r+q-2p}{(q-r)^2}$$

When $x = \sqrt{3},\ y = 2,\ z = 1$, we have

$$p = x + y + z = \sqrt{3} + 3, \quad q = \sqrt{3} - 1, \quad r = \sqrt{3} + 1.$$

Thus,

$$\frac{\partial u}{\partial z}\Big|_{x=\sqrt{3},y=2,z=1} = \frac{1}{-2} + \frac{-6}{(-2)^2} = -2$$

Calculus III

Study concepts, example questions & explanations

Question #1: Directional Derivative

Find the directional derivative of the function $\boldsymbol{f(x,y) = ysin(xy)}$ at the point $(\frac{\pi}{3}, \mathbf{1})$ in the direction of the vector $\vec{\boldsymbol{v}} = \langle \mathbf{1}, \mathbf{3} \rangle$.

Answer:
The directional derivative of *f* at (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

We first compute the gradient vector at $(\frac{\pi}{3}, \mathbf{1})$:

$$f_x = y^2 \cos(xy), \quad f_y = xy \cos(xy),$$

$$f_x(\frac{\boldsymbol{\pi}}{\mathbf{3}}, \mathbf{1}) = 1/2, \quad f_y(\frac{\boldsymbol{\pi}}{\mathbf{3}}, \mathbf{1}) = \pi/6$$

$$\nabla f = \langle f_x, f_y \rangle$$

$$\nabla f\left(\frac{\boldsymbol{\pi}}{\mathbf{3}}, \mathbf{1}\right) = \langle f_x\left(\frac{\boldsymbol{\pi}}{\mathbf{3}}, \mathbf{1}\right), f_y\left(\frac{\boldsymbol{\pi}}{\mathbf{3}}, \mathbf{1}\right)\rangle = \langle 1/2, \pi/6 \rangle$$

Note that $\vec{\boldsymbol{v}}$ is not a unit vector, but since $|\vec{\boldsymbol{v}}| = \sqrt{10}$, the unit vector in the direction $\vec{\boldsymbol{v}}$ of is $\vec{u} = \frac{1}{|\vec{v}|}\vec{\boldsymbol{v}} = \langle \frac{\mathbf{1}}{\sqrt{\mathbf{10}}}, \frac{\mathbf{3}}{\sqrt{\mathbf{10}}} \rangle$.
Therefore, we have

$$D_{\vec{u}} f\left(\frac{\boldsymbol{\pi}}{\mathbf{3}}, \mathbf{1}\right) = \nabla f\left(\frac{\boldsymbol{\pi}}{\mathbf{3}}, \mathbf{1}\right) \cdot \vec{u} = \langle \frac{1}{2}, \frac{\pi}{6} \rangle \cdot \langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \rangle = \frac{1}{2\sqrt{10}} + \frac{\pi}{2\sqrt{10}} = \frac{1+\pi}{2\sqrt{10}}$$

Question #2: Directional Derivative

If $\boldsymbol{f(x,y) = ln(x^2 - 3xy + y)}$, (a) find the gradient of *f* and (b) find the directional derivative of *f* at (5, 1) in the direction of $\vec{\boldsymbol{\imath}} + \boldsymbol{2}\vec{\boldsymbol{\jmath}}$.

Answer:

The gradient vector of *f* is at $(\boldsymbol{\frac{\pi}{3}}, \boldsymbol{1})$:

$$f_x = \frac{2x - 3y}{\boldsymbol{x^2 - 3xy + y}}, \quad f_y = \frac{-3x + 1}{\boldsymbol{x^2 - 3xy + y}},$$

$$\nabla f = \langle f_x, f_y \rangle = \langle \frac{2x - 3y}{\boldsymbol{x^2 - 3xy + y}}, \frac{-3x + 1}{\boldsymbol{x^2 - 3xy + y}} \rangle$$

The directional derivative of *f* at (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

At (5,1) we have

$$\nabla f(\boldsymbol{5}, \boldsymbol{1}) = \langle f_x(\boldsymbol{5}, \boldsymbol{1}), f_y(\boldsymbol{5}, \boldsymbol{1}) \rangle = \langle \frac{7}{11}, -14/11 \rangle$$

Note that $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\imath}} + \boldsymbol{2}\vec{\boldsymbol{\jmath}}$ is not a unit vector, but since $|\vec{\boldsymbol{v}}| = \sqrt{5}$, the unit vector in the direction $\vec{\boldsymbol{v}} =$ of is $\vec{u} = \frac{\boldsymbol{1}}{|\vec{\boldsymbol{v}}|}\vec{\boldsymbol{v}} = \langle \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{5}}}, \frac{\boldsymbol{2}}{\sqrt{\boldsymbol{5}}} \rangle$.

Therefore, we have

$$D_{\vec{u}} f(\boldsymbol{5}, \boldsymbol{1}) = \nabla f(\boldsymbol{5}, \boldsymbol{1}) \cdot \vec{u} = \langle \frac{7}{11}, -\frac{14}{11} \rangle \cdot \langle \frac{\boldsymbol{1}}{\sqrt{\boldsymbol{5}}}, \frac{\boldsymbol{2}}{\sqrt{\boldsymbol{5}}} \rangle = \frac{7}{11\sqrt{5}} - \frac{28}{11\sqrt{5}} = \frac{-21}{11\sqrt{5}}$$

Question #3: Directional Derivative

If $\boldsymbol{f(x,y) = xe^y + cos(xy)}$, find the rate of change of *f* at the point *P*(2, 0) in the direction from *P* to *Q*(5,-4).

Answer:

We first compute the gradient vector:

$$\nabla f = \langle f_x, f_y \rangle = \langle \boldsymbol{e^y - ysin(xy), xe^y - xsin(xy)} \rangle$$

The gradient of *f* at (2, 0) is

$$\nabla f(2,0) = \langle f_x(2,0), f_y(2,0) \rangle = \langle \mathbf{1}, \mathbf{2} \rangle$$

The directional derivative of *f* at (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

The unit vector in the direction of $\overrightarrow{PQ} = \langle 3, -4 \rangle$ is $\vec{u} = \frac{1}{|\overrightarrow{PQ}|} \overrightarrow{PQ} = \langle \frac{3}{5}, \frac{-4}{5} \rangle$.

Therefore, we have

$$D_{\vec{u}} f(2,0) = \nabla f(2,0) \cdot \vec{u} = \langle 1,2 \rangle \cdot \langle \frac{3}{5}, \frac{-4}{5} \rangle = \frac{3}{5} - \frac{8}{5} = -1.$$

Question #4: Directional Derivative

Find the derivative of $f(x,y) = x^3 - xy^2 - z$ at P_0(1, 1, 0) in the direction of $\vec{v} = 2\vec{i} - 3\vec{j} + 6\vec{k}$.

Answer:

We first compute the gradient vector:

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle \mathbf{3x^2 - y^2, -2xy, -1} \rangle$$

The gradient of *f* at (1, 1, 0) is

$$\nabla f(1,1,0) = \langle f_x(1,1,0), f_y(1,1,0), f_z(1,1,0) \rangle = \langle 2, -2, -1 \rangle$$

The directional derivative of *f* at (x_0, y_0, z_0) in the direction of the unit vector $\vec{u} = \langle a, b, c \rangle$ is

$$D_{\vec{u}} f = \nabla f \cdot \vec{u}$$

The unit vector in the direction of $\vec{\boldsymbol{v}} = \mathbf{2}\vec{\boldsymbol{i}} - \mathbf{3}\vec{\boldsymbol{j}} + \mathbf{6}\vec{\boldsymbol{k}}$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \langle \frac{\mathbf{2}}{\mathbf{7}}, \frac{\mathbf{-3}}{\mathbf{7}}, \frac{\mathbf{6}}{\mathbf{7}} \rangle.$$

Therefore, we have

$$D_{\vec{u}} f(1,1,0) = \nabla f(1,1,0) \cdot \vec{u} = \langle 2, -2, -1 \rangle \cdot \langle \frac{2}{7}, \frac{-3}{7}, \frac{6}{7} \rangle = \frac{4}{7}.$$

Question #5: Directional Derivative

Find the maximum rate of change of $\boldsymbol{f(x,y)=\frac{x^2}{2}+\frac{y^2}{2}}$ at the point (1, 1) and the direction in which it occurs.

Answer:

The maximum value of the directional derivative $D_{\vec{u}}f(\boldsymbol{x})$ is $|\nabla f(\boldsymbol{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\boldsymbol{x})$.

We first compute the gradient vector:

$$\nabla f=\langle f_x,f_y\rangle=\langle \boldsymbol{x},\boldsymbol{y}\rangle$$

and so

$$\nabla f(1,1)=\langle \mathbf{1},\mathbf{1}\rangle$$

The maximum rate of change of $\boldsymbol{f}$ at the point (1, 1) is

$$|\nabla f(1,1)|=|\langle \mathbf{1},\mathbf{1}\rangle|=\sqrt{1^2+1^2}=\sqrt{2}$$

and it occurs in the direction of the gradient vector

$$\nabla f(1,1)=\langle \mathbf{1},\mathbf{1}\rangle.$$

Question #6: Directional Derivative

Suppose that the temperature at a point (x, y, z) in space is given by $T(x, y, z) = \frac{80}{1+x^2+2y^2+3z^2}$, where *T* is measured in degrees Celsius and *x*, *y*, *z* in meters. In which direction does the temperature increase fastest at the point $(1, 1, -2)$? What is the maximum rate of increase?

Answer:

The maximum value of the directional derivative $D_{\vec{u}}f(\boldsymbol{x})$ is $|\nabla f(\boldsymbol{x})|$ and it occurs when $\vec{u}$ has the same direction as the gradient vector $\nabla f(\boldsymbol{x})$.

The gradient of *T* is:

$$\nabla T = \langle T_x, T_y, T_z \rangle$$
$$= \langle \frac{-160x}{(1 + x^2 + 2y^2 + 3z^2)^2}, \frac{-320y}{(1 + x^2 + 2y^2 + 3z^2)^2}, \frac{-480z}{(1 + x^2 + 2y^2 + 3z^2)^2} \rangle$$

At the point $(1, 1, -2)$ the gradient is

$$\nabla T(1, 1, -2) = \langle -\frac{5}{8}, -\frac{10}{8}, \frac{30}{8} \rangle$$

The temperature increases fastest in the direction of the gradient vector $\nabla T(1, 1, -2)$. The maximum rate of increase is the length of the gradient vector:

$$|\nabla T(1, 1, -2)| = |\langle 1, 1 \rangle| = \frac{5}{8}\sqrt{41} \approx 4^{\circ}C/m.$$

Question #7: Directional Derivative

Find the directions in which the function increase and decrease most rapidly at P_0. Then find the derivatives of the function in these directions.

$$\boldsymbol{f(x,y) = \mathrm{x}^2 y + e^{xy} siny, \qquad P_0(1,0).}$$

Answer:

The gradient of *f* is:

$$\nabla f = \langle f_x, f_y\rangle = \langle \boldsymbol{2xy + ye^{xy}siny, x^2 + e^{xy}cosy + xe^{xy}siny}\rangle.$$

At the point $(\mathbf{1},\mathbf{0})$ the gradient is

$$\nabla f(\mathbf{1},\mathbf{0}) = \langle \mathbf{0}, 2\rangle.$$

The function increases most rapidly in the direction of $\nabla f(\mathbf{1},\mathbf{0}) = \langle \mathbf{0}, 2\rangle$ and the rate of change in this direction is

$$|\nabla f(\mathbf{1},\mathbf{0})| = |\langle \mathbf{0}, \mathbf{2}\rangle| = 2.$$

The function decreases most rapidly in the direction of $-\nabla f(\mathbf{1},\mathbf{0}) = \langle \mathbf{0}, -2\rangle$ and the rate of change in this direction is

$$-|\nabla f(\mathbf{1},\mathbf{0})| = -|\langle \mathbf{0}, \mathbf{2}\rangle| = -2.$$

Question #8: Directional Derivative

Find the directions in which the directional derivative of $f(x,y) = ye^{-xy}$ at the point (0, 2) has the value 1.

Answer:
The directional derivative of *f* at (x_0, y_0) in the direction of the unit vector $\vec{u} = \langle a, b\rangle$ is

$$D_{\vec{u}}f = \nabla f \cdot \vec{u}$$

The gradient of *f* is:

$$\nabla f = \langle f_x, f_y\rangle = \langle -\boldsymbol{y^2 e^{-xy}}, -\boldsymbol{xye^{-xy}} + \boldsymbol{e^{-xy}}\rangle.$$

At the point $(\mathbf{0}, \mathbf{2})$ the gradient is

$$\nabla f(\mathbf{0}, \mathbf{2}) = \langle -4, 1\rangle.$$

The directional derivative of *f* at $(0,2)$ in the direction of the unit vector $\vec{u} = \langle a, b\rangle$, is

$$D_{\vec{u}}f = \nabla f \cdot \vec{u} = \langle -4,1\rangle \cdot \langle a, b\rangle = -4a + b.$$

Thus $D_{\vec{u}}f$ has the value 1 in the direction of

$$D_{\vec{u}}f = -4a + b = 1 \Rightarrow b = 1 + 4a.$$

Since $\vec{u} = \langle a, b\rangle$ is a unit vector we have that $\sqrt{a^2 + b^2} = 1$ or $a^2 + b^2 = 1$.

In this case we have

$$a^2 + b^2 = 1$$

$$a^2 + (1 + 4a)^2 = 1$$

$$a^2 + 1 + 8a + 16a^2 = 1$$

$$17a^2 + 8a = 0$$

$$a(17a + 8) = 0$$

$$a = 0 \quad or \quad a = -8/17.$$

Then

$$a = 0 \ \Rrightarrow b = 1 \ \text{ and } \ a = -\frac{8}{17} \ \Rrightarrow b = 1 + 4\left(-\frac{8}{17}\right) = -\frac{15}{17},$$

The directions in which $D_{\vec{u}}f = 1$ are

$$\langle 0,1\rangle, \qquad \langle -\frac{8}{17}, -\frac{15}{17}\rangle.$$

Question #9: Tangent Plane and Normal Line

Find the equations of the tangent plane and normal line at the point (0, 1, 1) to the surface

$$\boldsymbol{x + y + z = 2e^{xyz}}.$$

Answer:

The tangent plane at the point $P(x_0, y_0, z_0)$ to the surface F(x, y, z) = k has normal vector $\vec{n} = \nabla F(x_0, y_0, z_0)$.

The normal line at the point $P(x_0, y_0, z_0)$ to the surface F(x, y, z) = k is in the direction of the vector $\vec{v} = \nabla F(x_0, y_0, z_0)$.

First we rewrite the equation of the surface in the form F(x, y, z) = k as:

$$\boldsymbol{x + y + z - 2e^{xyz} = 0}$$

and so

$$\boldsymbol{F(x, y, z) = x + y + z - e^{xyz}}.$$

Therefore we have

$$F_x = 1 - 2yz\boldsymbol{e^{xyz}}, \qquad F_y = 1 - 2xz\boldsymbol{e^{xyz}}, \qquad F_z = 1 - 2xy\boldsymbol{e^{xyz}},$$

$$F_x(0,1,1) = -1, \qquad F_y(0,1,1) = 1, \qquad F_z(0,1,1) = 1,$$

$$\nabla F(0,1,1) = \langle F_x(0,1,1), F_y(0,1,1), F_z(0,1,1)\rangle = \langle -1,1,1\rangle$$

The tangent plane at the point $(0,1,1)$ to the surface has normal vector $\vec{n} = \nabla F(0,1,1) = \langle -1,1,1\rangle$. Therefore the equation of the tangent plane is

$$(-1)(x - 0) + (1)(y - 1) + (1)(z - 1) = 0$$

which simplifies to $-x + y + z = 2$.

The normal line at the point $(0,1,1)$ to the surface is in the direction of the vector $\vec{v} = \nabla F(0,1,1) = \langle -1,1,1\rangle$. Parametric equations of the normal line are

$$x = 0 - t, \quad y = 1 + t, \quad z = 1 + t.$$

Question #10: Tangent Plane and Normal Line

Find the equations of the tangent plane and normal line at the point (1, 2, 4) to the surface

$$\boldsymbol{x^2 + y^2 + z = 9.}$$

Answer:

The tangent plane at the point $P(x_0, y_0, z_0)$ to the surface F(x, y, z) = k has normal vector $\vec{n} = \nabla F(x_0, y_0, z_0)$.

The normal line at the point $P(x_0, y_0, z_0)$ to the surface F(x, y, z) = k is in the direction of the vector $\vec{v} = \nabla F(x_0, y_0, z_0)$.

The equation of the surface is already written in the form F(x, y, z) = k, and so

$$\boldsymbol{F(x, y, z) = x^2 + y^2 + z.}$$

The gradient is

$$F_x = 2x, \quad F_y = 2y, \quad F_z = 1,$$

$$\nabla F(1,2,4) = \langle F_x(1,2,4), F_y(1,2,4), F_z(1,2,4) \rangle = \langle 2,4,1 \rangle$$

The tangent plane at the point $(1,2,4)$ to the surface has normal vector $\vec{n} = \nabla F(1,2,4) = \langle 2,4,1 \rangle$. The tangent plane is therefore the plane

$$2(x-1) + 4(y-2) + 1(z-4) = 0$$

which simplifies to $2x + 4y + z = 14$.

The normal line at the point $(1,2,4)$ to the surface is in the direction of the vector $\vec{v} = \nabla F(1,2,4) = \langle 2,4,1 \rangle$. Parametric equations of the normal line are

$$x = 1 + 2t, \quad y = 2 + 4t, \quad z = 4 + t.$$

Question #11: Tangent Plane and Normal Line

Find the tangent plane to the surface $z = xcos(y) - ye^{x}$ at (0, 0, 0).

Answer:

The tangent plane at the point $P(x_0, y_0, z_0)$ to the surface F(x, y, z) = k has normal vector $\vec{n} = \nabla F(x_0, y_0, z_0)$.

First we rewrite the equation of the surface in the form F(x, y, z) = k as:

$$\boldsymbol{xcos(y) - ye^{x}} - z = 0$$

and so

$$\boldsymbol{F(x, y, z) = xcos(y) - ye^{x}} - z.$$

We first calculate the partial derivatives of *F*

$$F_x = \cos(y) - y\boldsymbol{e}^{x}, \qquad F_y = -xsin(y) - \boldsymbol{e}^{x}, \qquad F_z = -1,$$

$$F_x(0,0,0) = 1, \qquad F_y(0,0,0) = -1, \qquad F_z(0,1,1) = -1,$$

Then,

$$\nabla F(0,0,0) = \langle F_x(0,0,0), F_y(0,0,0), F_z(0,0,0)\rangle = \langle 1, -1, -1\rangle.$$

The tangent plane at the point $(0,0,0)$ to the surface has normal vector $\vec{n} = \nabla F(0,0,0) = \langle 1, -1, -1\rangle$. Therefore the equation of the tangent plane is

$$(1)(x - 0) + (-1)(y - 0) + (-1)(z - 0) = 0$$

which simplifies to $x - y - z = 0$.

Question #12: Tangent Plane

At what point on the paraboloid $y = x^2 + z^2$ is the tangent plane parallel to the plane $x + 2y + 3z = 1$?

Answer:

The tangent plane at the point $P(x_0, y_0, z_0)$ to the surface F(x, y, z) = k has normal vector $\vec{n} = \nabla F(x_0, y_0, z_0)$.

The normal line at the point $P(x_0, y_0, z_0)$ to the surface F(x, y, z) = k is in the direction of the vector $\vec{v} = \nabla F(x_0, y_0, z_0)$.

First we rewrite the equation of the paraboloid in the form F(x, y, z) = k as:

$$\boldsymbol{x^2 + z^2 - y = 0}$$

and so

$$\boldsymbol{F(x, y, z) = x^2 + z^2 - y.}$$

Therefore we have

$$F_x = 2x, \qquad F_y = -1, \qquad F_z = 2z,$$

$$\nabla F(x_0, y_0, z_0) = \langle 2x_0, -1, 2z_0 \rangle.$$

The tangent plane at the point (x_0, y_0, z_0) to the paraboloid has normal vector $\overrightarrow{n_1} = \nabla F(x_0, y_0, z_0) = \langle 2x_0, -1, 2z_0 \rangle$. The given plane $x + 2y + 3z = 1$ has normal vector $\overrightarrow{n_2} = \langle 1,2,3 \rangle$. The two planes are parallel if their normal vectors are parallel, $\overrightarrow{n_1} // \overrightarrow{n_2}$. Thus,

$$\overrightarrow{n_1} = k\overrightarrow{n_2} \Rightarrow \langle 2x_0, -1, 2z_0 \rangle = k\langle 1,2,3 \rangle \Rightarrow 2x_0 = k, 2k = -1, 2z_0 = 3k$$

So, $k = -\frac{1}{2}$, $x_0 = \frac{k}{2} = -\frac{1}{4}$, $z_0 = \frac{3k}{2} = -\frac{3}{4}$.

The point (x_0, y_0, z_0) is a point on the paraboloid if $y_0 = {x_0}^2 + {z_0}^2 = \frac{5}{8}$. Therefore, the point on the paraboloid $\boldsymbol{y = x^2 + z^2}$ where the tangent plane is parallel to the plane $x + 2y + 3z = 1$ is

$$(x_0, y_0, z_0) = \left(-\frac{1}{4}, \frac{5}{8}, -\frac{3}{4}\right).$$

Question #13: The Gradient

The cylinder $x^2 + y^2 = 2$ and the plane $x + z = 4$ meet in an ellipse E. Find parametric equations for the line tangent to E at the point *P*$_0$(l, 1, 3).

Answer:

First we rewrite the equations of the two surfaces in the form F(x, y, z) = k as:

$$\boldsymbol{F(x,y,z) = x^2 + y^2}, \qquad \boldsymbol{G(x,y,z) = x + z}$$

The tangent line is orthogonal to both ∇F and ∇G at P_0, and therefore parallel to $\vec{v} = \nabla F \times \nabla G$. We have

$$\nabla F(1,1,3) = \langle 2,2,0 \rangle, \quad \nabla G(1,1,3) = \langle 1,0,1 \rangle,$$

$$\vec{v} = \nabla F \times \nabla G = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\vec{i} - 2\vec{j} - 2\vec{k}.$$

The tangent line is

$$x = 1 + 2t, \quad y = 1 - 2t, \quad z = 3 - 2t.$$

Calculus III

14.7: Maximum and Minimum Values

Study concepts, example questions & explanations

In almost all the problems in this sheet, we will use the Second Derivative Test for functions of two variable to maximize and minimize the given function.

$\boxed{3}$ **Second Derivatives Test** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b) f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1: In case (c) the point (a, b) is called a saddle point of *f* and the graph of *f* crosses its tangent plane at (a, b).

NOTE 2: If *D* = 0, the test gives no information (The Test Fails): could have a local maximum or local minimum at (a, b), or could be a saddle point of (a, b). <u>In this case, we must find some other way to determine the behavior of *f* at (a, b).</u>

NOTE 3: We cannot apply this test if the partial derivatives of *f* are not exist.

Question #1: Maximum and Minimum Values

Locate and classify the critical points of

$$\boldsymbol{f(x, y) = (x-5)^2 + (y+8)^2.}$$

Answer:

First compute the partial derivatives of *f*:

$$f_x = 2(x-5), \ \ f_y = 2(y+8).$$

The critical points satisfy the equations $f_x = 0$ and $f_y = 0$ simultaneously. Hence

$$2(x-5) = 0 \quad \Rightarrow \quad x = 5$$

$$2(y+8) = 0 \quad \Rightarrow \quad y = -8$$

So, $(5, -8)$ is the only critical point for *f*.

We now need to find *D* defined as

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2.$$

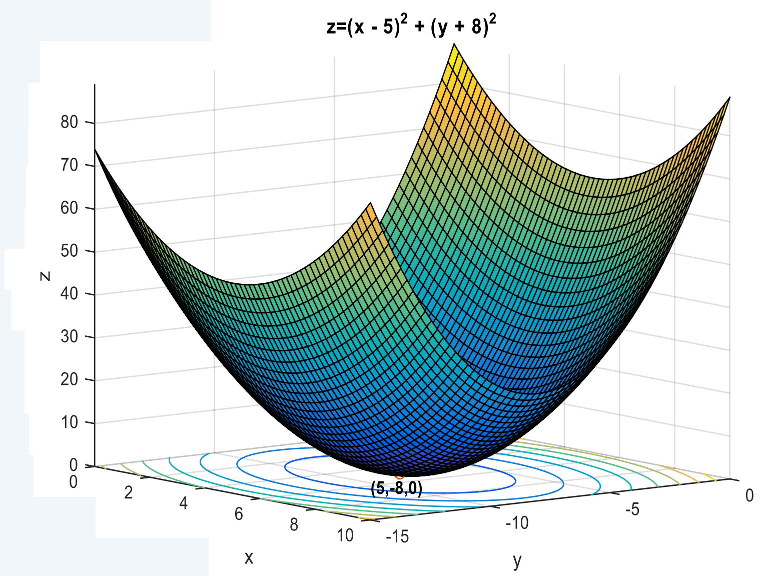
To do so, we find the second partial derivatives

$$f_{xx} = 2, \quad f_{yy} = 2, \ \ f_{xy} = 0.$$

Thus,

$$D(5, -8) = f_{xx}(5, -8) f_{yy}(5, -8) - \left(f_{xy}(5, -8)\right)^2 = (2)(2) - (0)^2 = 4.$$

Since $D > 0$ and $f_{xx}(2, -1) = 4 > 0$, *f* has a local minimum at $(5, -8)$ and so $f(5, -8) = 0$ is a local minimum.



Question #2: Maximum and Minimum Values

Find the local maximum and minimum values and saddle point(s) of the function

$$\boldsymbol{f(x, y) = 2x^2 + 2xy + 2y^2 - 6x.}$$

Answer:

We first compute the partial derivatives of *f*:

$$f_x = 4x + 2y - 6, \;\; f_y = 2x + 4y.$$

The critical points satisfy the equations $f_x = 0$ and $f_y = 0$ simultaneously. Hence

$$4x + 2y - 6 = 0 \;\; \Longrightarrow \;\; 4x + 2y = 6$$

$$2x + 4y = 0$$

The above system of equations has one solution at the point (2, -1). So, (2, -1) is the only critical point for *f*.

We now need to find D defined as

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2.$$

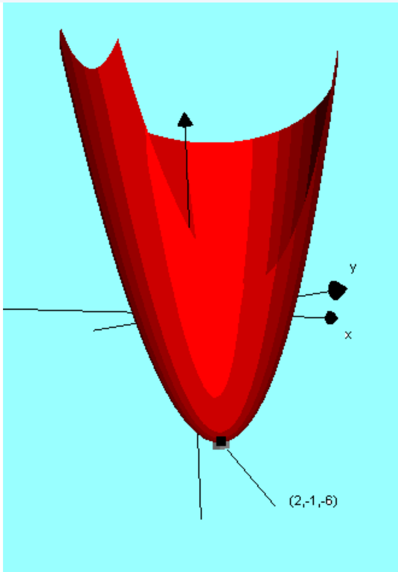
To do so, we find the second partial derivatives

$$f_{xx} = 4, \quad f_{yy} = 4, \quad f_{xy} = 2.$$

Thus,

$$D(2,-1) = f_{xx}(2,-1) f_{yy}(2,-1) - \left(f_{xy}(2,-1)\right)^2 = (4)(4) - (2)^2 = 12.$$

Since $D > 0$ and $f_{xx}(2,-1) = 4 > 0$, f has a local minimum at $(2,-1)$ and so $f(2,-1) = -6$ is a local minimum.



Question #3: Maximum and Minimum Values

Determine the critical points and locate any local minima, maxima and saddle points of the function

$$\boldsymbol{f(x,y) = 2x^2 - 4xy + y^4 + 2}$$

Answer:

We first compute the partial derivatives of *f*:

$$f_x = 4x - 4y, \qquad f_y = -4x + 4y^3.$$

The critical points satisfy the equations $f_x = 0$ and $f_y = 0$ simultaneously. Hence

$$4x - 4y = 0 \Longrightarrow \quad x = y$$

$$-4x + 4y^3 = 0$$

Substituting the first equation in the second equation gives

$$-4y + 4y^3 = 0$$
$$4y(y^2 - 1) = 0$$
$$4y(y-1)(y+1) = 0$$
$$y = 0, \qquad y = 1, \qquad y = -1.$$

We now use the equation x = y to find the critical points:

$$(0,0), \qquad (1,1), \qquad (-1,-1).$$

We now need to find *D* defined as

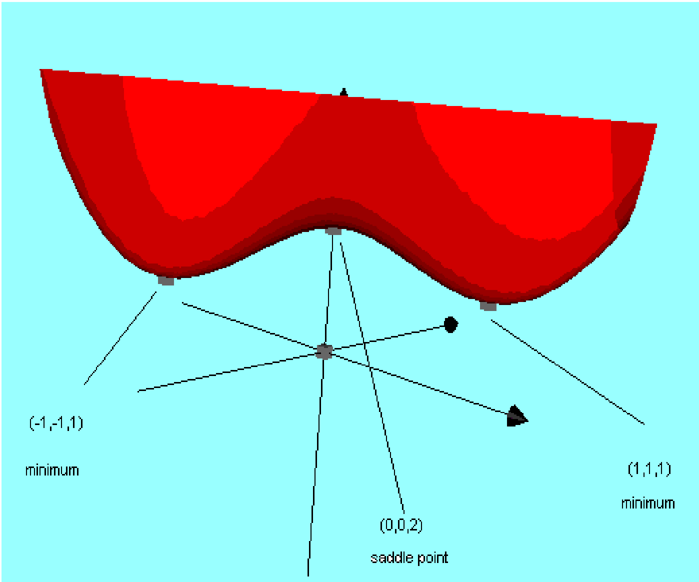
$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2$$

To do so, we find the second partial derivatives

$$f_{xx} = 4, \quad f_{yy} = 12y^2, \ f_{xy} = -4.$$

Thus,

	$(0,0)$	$(1,1)$	$(-1,-1)$
$f_{xx}(a,b)$	4	4	4
$f_{yy}(a,b)$	0	0	0
$f_{xy}(a,b)$	-4	-4	-4
D	-16	32	32
	Saddle point	local minimum	local minimum



Question #4: Maximum and Minimum Values

Find and classify the critical points of the function

$$\boldsymbol{f(x,y) = e^{\left(-\frac{1}{3}x^3+x-y^2\right)}}$$

Answer:

The partial derivatives of *f* are

$$f_x = (1-x^2)e^{\left(-\frac{1}{3}x^3+x-y^2\right)}, \qquad f_y = -2ye^{\left(-\frac{1}{3}x^3+x-y^2\right)}.$$

The critical points satisfy the equations $f_x = 0$ and $f_y = 0$ simultaneously. Hence

$$\begin{aligned} (1-x^2)e^{\left(-\frac{1}{3}x^3+x-y^2\right)} &= 0 \quad \Longrightarrow \quad (1-x^2) = 0 \\ -2ye^{\left(-\frac{1}{3}x^3+x-y^2\right)} &= 0 \quad \Longrightarrow \quad y = 0 \end{aligned}$$

The first equation gives $x = 1 \ or \ x = -1$ and the second equation gives $y = 0$. Thus, the critical points are

$$(1,0), \qquad (-1,0).$$

We now need to find *D* defined as

$$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2.$$

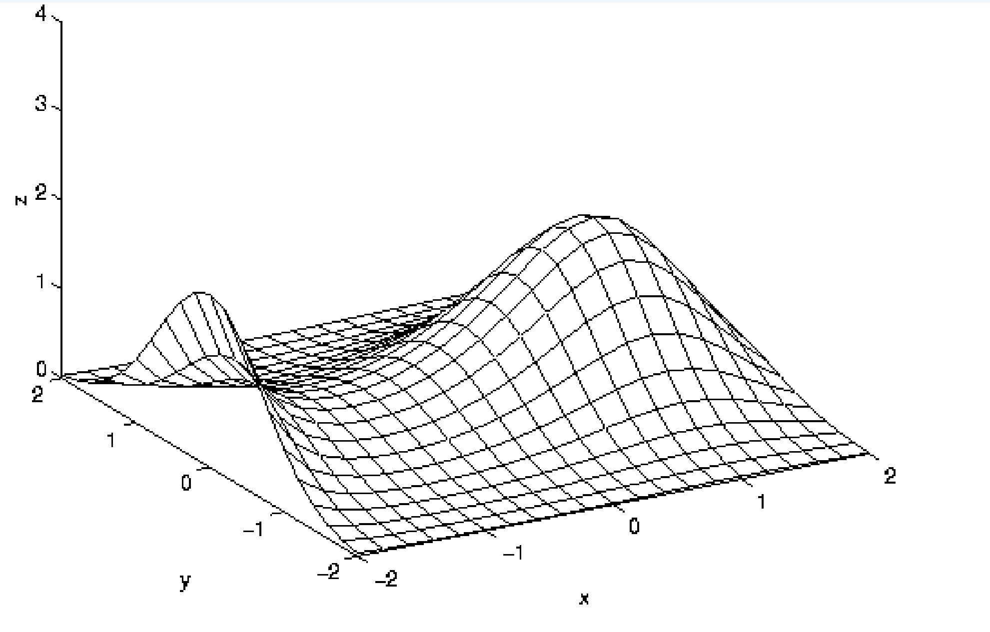
To do so, we find the second partial derivatives

$$f_{xx} = (-2x + (1 - x^2)^2)e^{\left(-\frac{1}{3}x^3+x-y^2\right)}, \quad f_{yy} = (-2 + 4y^2)e^{\left(-\frac{1}{3}x^3+x-y^2\right)},$$

$$f_{xy} = -2y(1 - x^2)e^{\left(-\frac{1}{3}x^3+x-y^2\right)}.$$

Thus,

	$(1,0)$	$(-1,0)$
$f_{xx}(a,b)$	$-2e^{2/3}$	$2e^{-2/3}$
$f_{yy}(a,b)$	$-2e^{2/3}$	$-2e^{-2/3}$
$f_{xy}(a,b)$	0	0
D	$4e^{4/3}$	$-4e^{-4/3}$
	local maximum	Saddle point



Question #5: Maximum and Minimum Values

Find and classify the critical points for

$$\boldsymbol{f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2}$$

Answer:

We will first need to get all the first partial derivatives

$$f_x = 6xy - 6x, \qquad f_y = 3x^2 + 3y^2 - 6y.$$

The critical points satisfy the equations $f_x = 0$ and $f_y = 0$ simultaneously. Hence

$$6xy - 6x = 0 \quad \Longrightarrow \quad 6x(y-1) = 0$$
$$3x^2 + 3y^2 - 6y = 0$$

The first equation gives $x = 0$ or $y = 1$. We plug these values in the second equation to get

$$x = 0 \quad \Longrightarrow \quad 3y^2 - 6y = 3y(y-2) = 0 \quad \Longrightarrow \quad y = 0, \qquad y = 2$$

$$y = 1 \quad \Longrightarrow \quad 3x^2 - 3 = 3(x-1)(x+1) = 0 \quad \Longrightarrow \quad x = 1, \qquad x = -1$$

Thus, the critical points are

$$(0,0), \quad (0,2), \quad (1,1), \quad (-1,1).$$

We now need to find *D* defined as

$$D = f_{xx}f_{yy} - \left(f_{xy}\right)^2.$$

To do so, we find the second partial derivatives

$$f_{xx} = 6y - 6, \quad f_{yy} = 6y - 6, \quad f_{xy} = 6x.$$

Thus,

	$(0,0)$	$(0,2)$	$(1,1)$	$(-1,1)$
$f_{xx}(a,b)$	-6	6	0	0
$f_{yy}(a,b)$	-6	6	0	0
$f_{xy}(a,b)$	0	0	6	-6
D	36	36	-36	-36
	local maximum	local minimum	Saddle point	Saddle point

Question #6: Maximum and Minimum Values

Locate and classify the critical points of

$$\boldsymbol{f(x,y) = 3xy - x^3 - y^3.}$$

Answer:

First compute the partial derivatives of *f*:

$$f_x = 3y - 3x^2, \quad f_y = 3x - 3y^2.$$

The critical points satisfy the equations $f_x = 0$ and $f_y = 0$ simultaneously. Hence

$$3y - 3x^2 = 0 \quad \Longrightarrow \quad y = x^2$$

$$3x - 3y^2 = 0 \quad \Longrightarrow \quad x = y^2$$

The first equation gives $y = x^2$. We plug these values in the second equation to get

$$x = y^2 = (x^2)^2 = x^4 \quad \Longrightarrow \; x^4 - x = x(x^3 - 1) = 0 \quad \Longrightarrow \quad x = 0 \;\; or \;\; 1$$

Now, $x = 0 \quad \Longrightarrow \quad y = 0^2 = 0$ and $x = 1 \quad \Longrightarrow \quad y = 1^2 = 1$.
The critical points for *f* are

$$(0,0), \quad (1,1).$$

We now need to find *D* defined as

$$D = f_{xx} f_{yy} - \left(f_{xy}\right)^2.$$

To do so, we find the second partial derivatives

$$f_{xx} = -6x, \quad f_{yy} = -6y, \;\; f_{xy} = 3.$$

Thus,

$$D(0,0) = f_{xx}(0,0) f_{yy}(0,0) - \left(f_{xy}(0,0)\right)^2 = (0)(0) - (3)^2 = -9,$$

$$D(1,1) = f_{xx}(1,1) f_{yy}(1,1) - \left(f_{xy}(1,1)\right)^2 = (-6)(-6) - (3)^2 = 27.$$

We can obtain that:
$D(0,0) < 0$, and so (0,0) is a saddle point.
$D(1,1) > 0$ with $f_{xx}(2,-1) < 0$ gives that *f* has a local maximum at (1, 1).

Question #7: Applications on Extreme Values

Find the shortest distance from the point (1, 0, -2) to the plane $\boldsymbol{x+2y+z=4}$**.**

Answer:

The distance from any point (x, y, z) to the point (1, 0, -2) is

$$d=\sqrt{(x-1)^2+(y-0)^2+(z+2)^2}$$

but if (x, y, z) lies on the plane $x+2y+z=4$, then $\mathrm{z}=4-x-2y$ and so we have

$$d=\sqrt{(x-1)^2+(y-0)^2+(4-x-2y+2)^2}.$$

We can minimize *d* by minimizing the simpler expression

$$d^2=f(x,y)=(x-1)^2+y^2+(6-x-2y)^2.$$

By solving the equations

$$f_x=2(x-1)-2(6-x-2y)=4x+4y-14=0,$$

$$f_x=2y-4(6-x-2y)=4x+10y-24=0,$$

we find that the only critical point is $\left(\frac{11}{6},\frac{5}{3}\right)$.

Since $f_{xx}=4$, $f_{xy}=4$, $f_{yy}=10$, we have

$$D=f_{xx}f_{yy}-\left(f_{xy}\right)^2=24>0 \text{ and } f_{xx}>0,$$

so by the Second Derivatives Test *f* has a local minimum at $\left(\frac{11}{6},\frac{5}{3}\right)$. Intuitively, we can see that this local minimum is actually an absolute minimum because there must be a point on the given plane that is closest to (1, 0, -2). At this point we get $d=\frac{5}{6}\sqrt{6}$.

So, the shortest distance from the point (1, 0, -2) to the plane $x+2y+z=4$ is $\frac{5}{6}\sqrt{6}$.

Question #8: Finding Absolute Extrema

Find the absolute maximum and minimum values of the function

$$f(x, y) = x^2 - 2xy + 2y$$

on the rectangle $D = \{(x, y) | 0 \leq x \leq 3,\ 0 \leq y \leq 2\}$.

Answer:

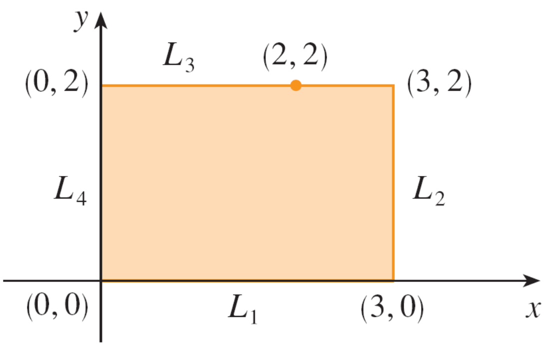
Since f is a polynomial, it is continuous on the closed, bounded rectangle D, so Theorem 8 tells us there is both an absolute maximum and an absolute minimum.

We first find the critical points. These occur when

$$f_x = 2x - 2y = 0, \qquad f_y = -2x + 2 = 0,$$

so the only critical point is (1, 1), and the value of f there is $f(1,1) = 1$.

Next, we look at the values of f on the boundary of D, which consists of the four line segments L_1, L_2, L_3, L_4 shown in the Figure below.



On L_1 we have $y = 0$ and

$$f(x, 0) = x^2 \qquad 0 \leq x \leq 3.$$

This is an increasing function of x, so its minimum value is $f(0,0) = 0$ and its maximum value is $f(3,0) = 9$.

On L_2 we have $x = 3$ and

$$f(3, y) = 9 - 4y \qquad\qquad 0 \le y \le 2$$

This is a decreasing function of y, so its maximum value is $f(3,0) = 9$ and its minimum value is $f(3,2) = 1$.

On L_3 we have $y = 2$ and

$$f(x, 2) = x^2 - 4x + 4 \qquad\qquad 0 \le x \le 3.$$

Simply by observing that $f(x, 2) = (x - 2)^2$, we see that the minimum value of this function is $f(2,2) = 0$ and the maximum value is $f(0,2) = 4$.

Finally, on L_4 we have $x = 0$ and

$$f(0, y) = 2y \qquad\qquad 0 \le y \le 2$$

with maximum value $f(0,2) = 4$ and minimum value $f(0,0) = 0$.

Thus, on the boundary, the minimum value of f is 0 and the maximum is 9.

Finally, we compare these values with the value $f(1,1) = 1$ at the critical point and conclude that the absolute maximum value of f on D is $f(3, 0) = 9$ and the absolute minimum value is $f(0,0) = f(2, 2) = 0$.

	(a,b)	$f(a,b)$	Classification
Interior point(s)	$(1,1)$	1	
Boundary point(s)	$(0,0)$	0	absolute minimum
	$(3,0)$	9	absolute maximum
	$(3,2)$	1	
	$(0,2)$	4	
	$(2,2)$	0	absolute minimum

Question #9: : Finding Absolute Extrema

Find the absolute maximum and minimum values of

$$\boldsymbol{f(x,y) = 2 + 2x + 2y - x^2 - y^2}$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.

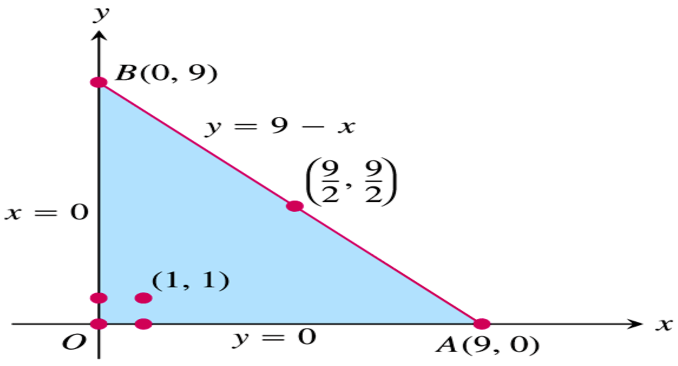


FIGURE 14.46 This triangular region is the domain of the function in Example 5.

Answer:

Since *f* is differentiable, the only places where *f* can assume these values are points inside the triangle where $f_x = f_y = 0$ and points on the boundary.

(a) Interior points. For these we have

$$f_x = 2 - 2x = 0,\ f_y = 2 - 2y = 0,$$

yielding the single point (x, y) = (1, 1). The value of *f* there is *f*(1,1) = 4.

(b) Boundary points. We take the triangle one side at a time:

i) On the segment OA, y = 0. The function

$$f(x, y) = f(x, 0) = 2 + 2x - x^2$$

may now be regarded as a function of *x* defined on the closed interval [0, 9]. Its extreme values may occur at the endpoints

$$x = 0 \qquad where\ \ f(0,0) = 2$$

$$x = 9 \qquad where\ \ f(9,0) = -61$$

and at the interior points where $f'(x, 0) = 2 - 2x = 0$. The only interior point where $f'(x, 0) = 0$ is *x* = 1, where *f(1,0)* = 3.

ii) On the segment *OB*, $x = 0$ and $f(x,y) = f(0,y) = 2 + 2y - y^2$. We know from the symmetry of *f* in *x* and *y* and from the analysis we just carried out that the candidates on this segment are: $f(0,0) = 2$, $f(0,9) = -61$, $f(0,1) = 3$.

iii) We have already accounted for the values of *f* at the endpoints of *AB,* so we need only look at the interior points of *AB.* With $\mathrm{y} = 9 - x$, we have

$$f(x, 9 - x) = 2 + 2x + 2(9 - x) - x^2 - (9 - x)^2 = -61 + 18x - 2x^2.$$

Setting $f'(x, 9 - x) = 18 - 4x = 0$ gives $x = 9/2$.

At this value of *x*, $y = 9 - x = 9 - \frac{9}{2} = 9/2$ and

$$f\left(\frac{9}{2}, \frac{9}{2}\right) = -41/2.$$

We list all the candidates: 4, 2, – 61, 3, – 41/2. The absolute maximum is 4, which *f* assumes at (1, 1). The absolute minimum is – 61, which *f* assumes at (0, 9) and (9, 0).

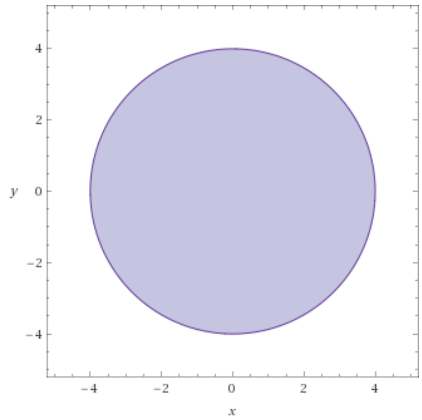
	(a, b)	$f(a, b)$	Classification
Interior point(s)	$(1,1)$	4	absolute maximum
Boundary point(s)	$(1,0)$	3	
	$(0,1)$	3	
	$(0,0)$	2	
	$(9,0)$	-61	absolute minimum
	$(0,9)$	-61	absolute minimum
	$\left(\frac{9}{2}, \frac{9}{2}\right)$	$-\frac{41}{2}$	

Question #10: Finding Absolute Extrema

Find the absolute minimum and absolute maximum of

$$\boldsymbol{f(x,y) = 2x^2 - y^2 + 6y}$$

on the disk $\boldsymbol{x^2 + y^2 \leq 16}$.



Answer:

Since *f* is differentiable, the only places where *f* can assume these values are points inside the disk and points on the boundary of the disk.

Let's first find the critical points of the function that lies inside the disk. To do so, we solve

$$f_x = 4x = 0, \quad f_y = -2y + 6 = 0,$$

yielding the single point (0, 3). The value of f there is f(0, 3) = 9.

Now we need to look at the boundary. On the boundary we have $\boldsymbol{x^2 + y^2 = 16}$ and so $\boldsymbol{x^2 = 16 - y^2}$. If we plug this in the rule of $\boldsymbol{f(x,y)}$ we get that

$$f(x,y) = 2x^2 - y^2 + 6y = 2(16 - y^2) - y^2 + 6y$$

This is a function of *y*,

$$g(\mathrm{y}) = 32 - 3y^2 + 6y$$

We will need to find the absolute extrema of this function on the range $-4 \leq y \leq 4$ (this is the range of y on the disk).

Note that

$$g'(y) = -6\mathrm{y} + 6 = 0 \quad \Rightarrow \quad y = 1.$$

The value of this function at the critical point and the endpoints are,

$$g(-4) = -40, \quad g(1) = 35, \quad g(-4) = 8.$$

To find the values of x that correspond to these values of y, we use that $\boldsymbol{x^2} = \mathbf{16} - \boldsymbol{y^2}$. Thus,

$$y = -4 \Rightarrow x = 0, \quad y = 1 \Rightarrow x = \pm\sqrt{15}, \quad y = 4 \Rightarrow x = 0.$$

We then can find that

$$f(0, -4) = -40, \quad f(0,4) = 8, \qquad f\left(\sqrt{15}, 1\right) = 35, \quad f\left(-\sqrt{15}, 1\right) = 35.$$

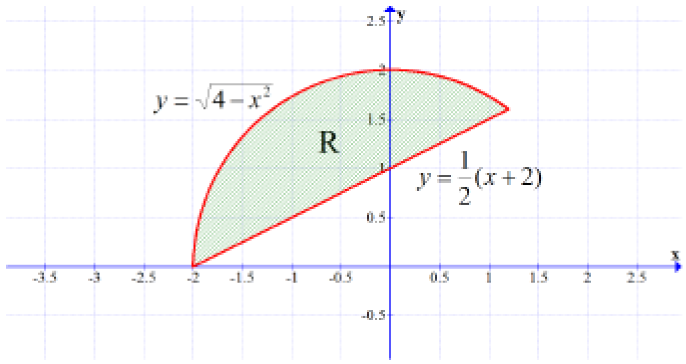
So, comparing these values to the value of the function at the critical point, f(0, 3) = 9, we can see that the absolute minimum occurs at (0,−4) while the absolute maximum occurs twice at $\left(\sqrt{15}, 1\right)$ and $\left(-\sqrt{15}, 1\right)$.

	(a, b)	$f(a, b)$	Classification
Interior point(s)	$(0,3)$	9	
Boundary point(s)	$(0, -4)$	-40	absolute minimum
	$(0,4)$	8	
	$\left(\sqrt{15}, 1\right)$	35	absolute maximum
	$\left(-\sqrt{15}, 1\right)$	35	absolute maximum

Practice Problems, Chapter 15
Calculus III

Question 1: Evaluate $\iint_D (x+y)\, dA$, where D is the region enclosed by $y = x^3,\ y = 0,\ x = 1.$

Question 2: Set up $\iint_R (xy)\, dA$, where R is the region enclosed by $y = \sqrt{4-x^2}$ and $y = \frac{1}{2}x = 1.$

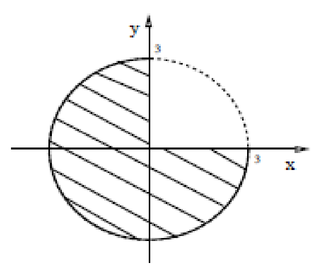


Question 3: Use a double integral to find the volume of a parallelepiped whose base is a rectangle in the xy- plane given by $D = \{(x,y)|0 \le x \le 1, 0 \le y \le 2\}$, while the top side lies in the plane $x + y + z = 3$.

Question 4: Transform to polar coordinates and then evaluate the integral

$$I = \int_0^{\sqrt{2}} \int_x^{\sqrt{4-x^2}} (x^2 + y^2)\, dydx$$

Question 5: Evaluate $\iint_D y\, dA$, where D is the region given by the disk $x^2 + y^2 \le 9$ minus the first quadrant.



Question 6: Evaluate $\int_0^1 \int_0^1 x\, max(x,y)\ dydx$

Question 7: **Use spherical coordinates to find the volume of the region outside the sphere $x^2+y^2+(z-1)^2=1$ and inside the upper half of the sphere $x^2+y^2+z^2=4$.**

Question 8: **Use cylindrical coordinates to find the volume of a curved wedge cut out from a cylinder $(x-2)^2+y^2=4$ by the planes $z=0$ and $z=-y$.**

Question 9: **Use a triple integral to find the volume of the solid enclosed by the paraboloid $z=x^2+y^2$ and the planes $z=1$ and $z=2$.**

Question 10: **Let *E* be the solid enclosed by the paraboloids $z=x^2+y^2$ and $z=8-x^2-y^2$. Write $\iiint_E xyz\,dV$ as an iterated integral in cylindrical coordinates.**

Question 11: **Let *E* be the be the "ice cream cone" bounded below by the cone $z=\sqrt{3(x^2+y^2)}$ and above by the sphere $x^2+y^2+z^2=1$. Write an iterated integral in spherical coordinates which gives the volume of *E*.**

Question 12: **Evaluate**

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}}\left(\frac{1}{x^2+y^2+z^2}\right)dzdydx$$

Calculus III

15.2: Iterated Integrals

Study concepts, example questions & explanations

Question #1: Iterated Integrals

Calculate the iterated integral $\int_0^4 \int_0^2 x\sqrt{y}dxdy$.

Answer:

$$\int_0^4 \int_0^2 x\sqrt{y}dxdy = \int_0^4 \left(\int_0^2 x\sqrt{y}dx\right) dy$$

$$= \int_0^4 \left(\sqrt{y}\frac{x^2}{2}\right)\Big|_0^2 dy$$

$$= \int_0^4 2\sqrt{y}\, dy$$

$$= \left(\frac{4}{3}y^{3/2}\right)\Big|_0^4 = \frac{32}{3}$$

Question #2: Iterated Integrals

Calculate the double integral $\iint_R \left(xy^2 + \frac{y}{x}\right) dA$**,**
$R = \{(x, y)|2 \le x \le 3, -1 \le y \le 0\}$.

Answer:

$$\iint_R \left(xy^2 + \frac{y}{x}\right) dA = \int_2^3 \int_{-1}^0 \left(xy^2 + \frac{y}{x}\right) dydx$$
$$= \int_2^3 \left(\int_{-1}^0 \left(xy^2 + \frac{y}{x}\right) dy\right) dx$$
$$= \int_2^3 \left(\left(x\frac{y^3}{3} + \frac{y^2}{2x}\right)\Bigg|_{-1}^0\right) dx$$
$$= \int_2^3 \left(\frac{1}{3}x - \frac{1}{2x}\right) dx = \left(\frac{x^2}{6} - \frac{1}{2}lnx\right)\Bigg|_2^3 = \frac{5}{6} - \frac{1}{2}(ln3 - ln2).$$

Question #3: Volume

Find the volume of the solid lying under the plane $z = 2x + 5y + 1$ **and above the rectangular region** $R = \{(x, y)| -1 \le x \le 0, 1 \le y \le 4\}$

Answer:

Volume = $\iint_R (2x + 5y + 1)dA = \int_1^4 \int_{-1}^0 (2x + 5y + 1)dxdy$

$$= \int_1^4 \left(\int_{-1}^0 (2x + 5y + 1)\, dx\right) dy$$
$$= \int_1^4 (x^2 + 5yx + x)|_{-1}^0\, dy$$
$$= \int_1^4 5y\, dy = \left(\frac{5}{2}y^2\right)\Bigg|_1^4 = \frac{75}{2}$$

Question #4: Volume

Evaluate the double integral $\iint_R (ye^{-xy})dA, \quad R = [0,2] \times [0,3]$.

Answer:

It's easier if we first integrate with respect to *x*,

$$\iint_R (ye^{-xy})dA = \int_0^3 \int_0^2 (ye^{-xy})dxdy$$

$$= \int_0^3 \left(\int_0^2 (ye^{-xy})\, dx \right) dy$$

$$= \int_0^3 (-e^{-xy})|_0^2 \, dy$$

$$= \int_0^3 (1 - e^{-2y})\, dy = \left(y - \frac{e^{-2y}}{-2} \right)\Bigg|_0^3 = \frac{5}{2} + \frac{1}{2} e^{-6}$$

Question #5: Volume

Find the volume of the solid enclosed by the paraboloid $z = 2 + x^2 + (y-2)^2$ and the planes z = 1, x = 1, x = -1, y = 0, y = 4.

Answer:

Volume = $\int_{-1}^{1}\int_{0}^{4}(2 + x^2 + (y-2)^2)dydx - \int_{-1}^{1}\int_{0}^{4}(1)dydx$

$$= \int_{-1}^{1}\int_{0}^{4}(1 + x^2 + (y-2)^2)dydx$$

$$= \int_{-1}^{1}\left(\int_{0}^{4}(1 + x^2 + (y-2)^2)\,dy\right)dx$$

$$= \int_{-1}^{1}\left(y + yx^2 + \frac{(y-2)^3}{3}\right)\Bigg|_{0}^{4} dx$$

$$= \int_{-1}^{1}\left(\frac{28}{3} + 4x^2\right)dxy = \left(\frac{28}{3}x + \frac{4}{3}x^3\right)\Bigg|_{-1}^{1} = \frac{64}{3}$$

Calculus III

15.3: Double Integrals over General Regions

Study concepts, example questions & explanations

Question #1: Iterated Integral

Evaluate $\int_0^2 \int_{\sqrt{x}}^{x^2} xy\, dydx.$

Answer:

$$\int_0^2 \int_{\sqrt{x}}^{x^2} xy\ dydx = \int_0^2 \left(\int_{\sqrt{x}}^{x^2} xydy \right) dx$$

$$= \int_0^2 \left[x \frac{y^2}{2} \right]_{\sqrt{x}}^{x^2} dx$$

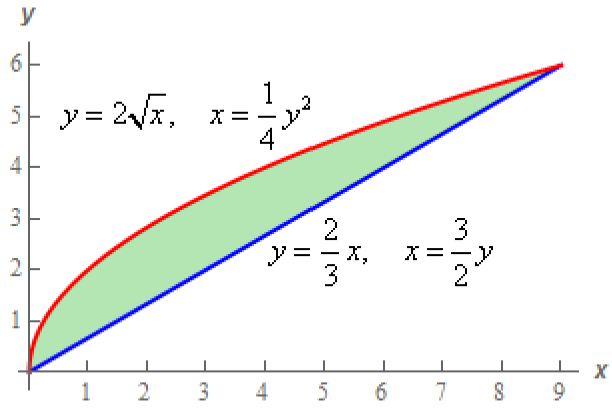
$$= \frac{1}{2} \int_0^2 (x^5 - x^3)\, dx$$

$$= \frac{1}{2} \left[\frac{x^6}{6} - \frac{x^3}{3} \right]_0^2 = 4$$

Question #2: Double Integral

Evaluate $\iint_D (2yx^2 + 9y^3)dA$, where *D* is the region bounded by $y = \frac{2}{3}x$ and $y = 2\sqrt{x}$.

Answer:
Let's first sketch the region *D*



To find the points of intercession of the two curves, we solve

$$\frac{2}{3}x = 2\sqrt{x} \Rightarrow \frac{x^2}{9} = x \Rightarrow x^2 - 9x = 0 \Rightarrow x = 0 \;\; or \;\; x = 9.$$

If we consider *D* as a Type I region, then

$$\iint_D (2yx^2 + 9y^3)dA = \int_0^9 \int_{\frac{2}{3}x}^{2\sqrt{x}} (2yx^2 + 9y^3)dydx$$

$$= \int_0^9 \left(\int_{\frac{2}{3}x}^{2\sqrt{x}} (2yx^2 + 9y^3)\, dy \right) dx$$

$$= \int_0^9 \left[y^2x^2 + \frac{9}{4}y^4 \right]_{\frac{2}{3}x}^{2\sqrt{x}} dx$$

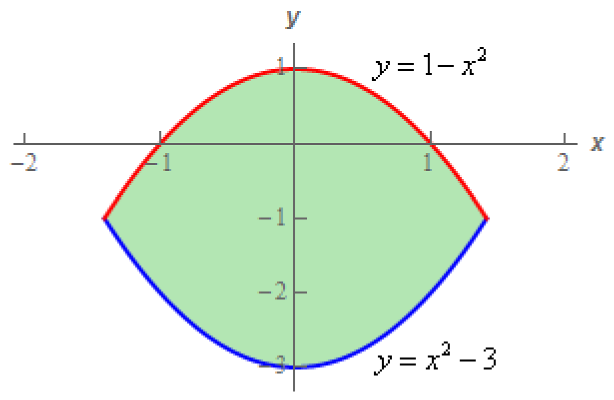
$$= \int_0^9 \left(36x^2 + 4x^3 - \frac{8}{9}x^4 \right) dx$$

$$= \left[12x^3 + x^4 - \frac{8}{45}x^5 \right]_0^9 = \frac{24057}{6}.$$

Question #3: Double Integral

Evaluate $\iint_D x(y-1)\ dA$, where *D* is the region enclosed by $y = 1 - x^2$ and $y = x^2 - 3$.

Answer:
Let's first sketch the region *D*



To find the points of intercession of the two curves, we solve

$$x^2 - 3 = 1 - x^2 \Rrightarrow 2x^2 = 4 \Rrightarrow x^2 = 2 \Rrightarrow x = \pm\sqrt{2}.$$

If we consider *D* as a Type I region, then

$$\iint_D x(y-1)\,dA = \int_{-\sqrt{2}}^{\sqrt{2}} \int_{x^2-3}^{1-x^2} x(y-1)\,dydx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{x^2-3}^{1-x^2} x(y-1)\,dy \right) dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left[x\left(\frac{y^2}{2} - y\right) \right]_{x^2-3}^{1-x^2} dx$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} (4x^3 - 8x)\,dx$$

$$= \left[x^4 - 4x^2\right]_{-\sqrt{2}}^{\sqrt{2}} = 0.$$

Question #4: Double Integral

Evaluate $\iint_D (7x^2 + 14y)\, dA$, where *D* is the region enclosed by $x = 2y^2$ and $x = 8$ by considering *D* as

(a) a Type I region **(b) a Type II region**

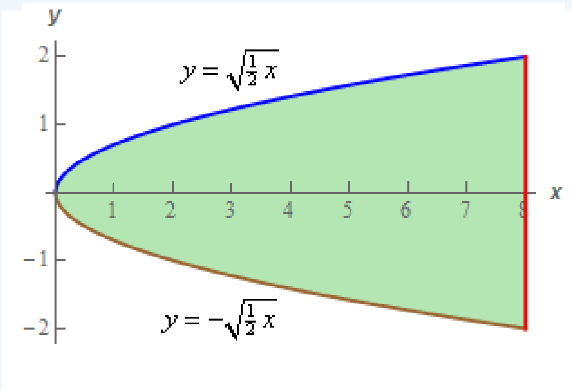
Answer:

(a) If we consider *D* as a Type I region, then

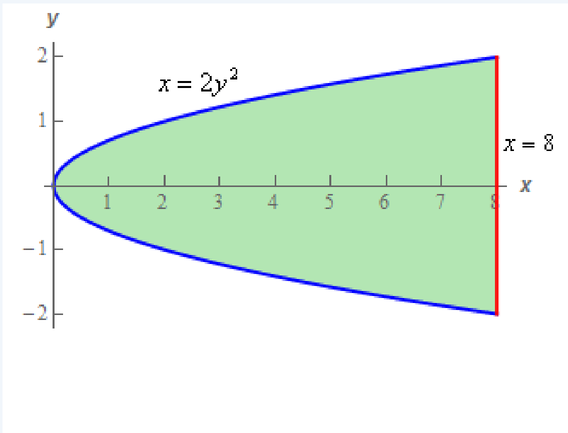
$$\iint_D (7x^2 + 14y)\, dA$$

$$= \int_0^8 \int_{-\sqrt{\frac{x}{2}}}^{\sqrt{\frac{x}{2}}} (7x^2 + 14y)\, dydx$$

$$= 4096$$



(b) If we consider *D* as a Type II region, then



$$\iint_D (7x^2 + 14y)\, dA$$

$$= \int_{-2}^{2} \int_{2y^2}^{8} (7x^2 + 14y)\, dxdy$$

$$= 4096$$

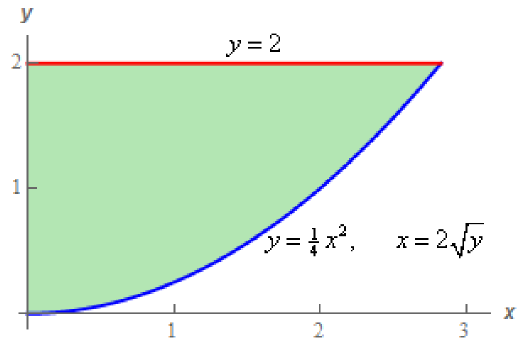
Question #5: Reverse the Order of Integration

Evaluate $\int_0^{2\sqrt{2}} \int_{x^2/4}^{2} x^3 cos(y^3)\ dydx$**.**

Answer:
If we try to evaluate the integral as it stands, we face the impossible task of first evaluating $\int_{x^2/4}^{2} cos(y^3)\ dy$. So we must change the order of integration. To do so we express the given iterated integral as a double integral.

$$\int_0^{2\sqrt{2}} \int_{x^2/4}^{2} x^3 cos(y^3)\ dydx = \iint_D x^3 cos(y^3)\ dA$$

where D is sketched below



If we consider *D* as a Type II region, then

$$\iint_D x^3 cos(y^3)\ dA = \int_0^2 \int_0^{2\sqrt{y}} x^3 cos(y^3)\ dxdy$$

$$= \int_0^2 \left(\int_0^{2\sqrt{y}} x^3 cos(y^3)\ dx \right) dy$$

$$= \int_0^2 \left[\frac{x^4}{4} cos(y^3) \right]_0^{2\sqrt{y}} dy$$

$$= \int_0^2 4y^2 cos(y^3)\, dy$$

$$= \frac{4}{3} \int_0^8 cos(u)\, du$$

$$= \frac{4}{3} \sin(8)$$

Let $u = y^3$

$du = 3y^2\, dy$

$dy = du/3y^2$

$y = 0 \rightarrow u = 0$

$y = 2 \rightarrow u = 8$

Question #6: Volume

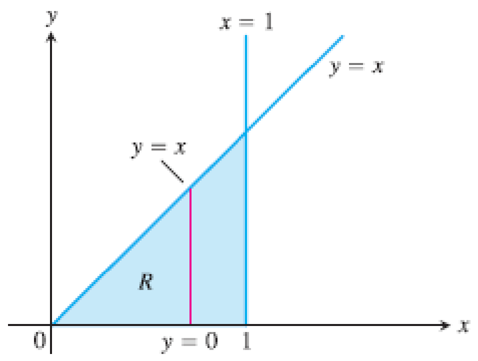
Find the volume of the prism whose base is the triangle in the *xy*-plane bounded by the *x*-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = 3 - x - y$.

Answer:

Recall that if *f*(*x*, *y*) is positive and continuous over *D*, then the volume of the solid region between *D* and the surface z = *f*(*x*,*y*) is

$$Volume = \iint_D f(x,y)\, dA$$

We first sketch the projection *D* of the prism onto the xy-plane



Hence,

$$Volume = \iint_D (3 - x - y)\, dA$$

If we consider *D* as a Type I region, then

$$Volume = \int_0^1 \int_0^x (3 - x - y)\, dydx = \int_0^1 \left(\int_0^x (3 - x - y)\, dy \right) dx$$

$$= \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_0^x dx$$

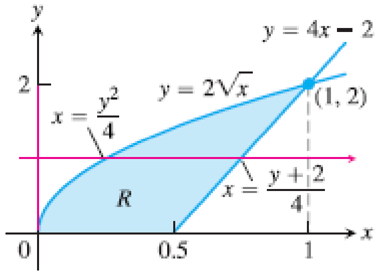
$$= \int_0^1 \left(3x - \frac{3}{2}x^2 \right) dx = \left[\frac{3}{2}x^2 - \frac{x^3}{2} \right]_0^1 = 1.$$

Question #7: Volume

Find the volume of the solid that lies under the surface $z = 16 - x^2 - y^2$ and above the region *D* bounded by the curve $y = 2\sqrt{x}$, the line $y = 4x - 2$, and the *x*-axis.

Answer:

We first sketch the projection *D* of the solid onto the xy-plane



Hence,

$$Volume = \iint_D (16 - x^2 - y^2)\, dA$$

It's easier to consider *D* as a Type II region, then

$$Volume = \int_0^2 \int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dxdy$$

$$= \int_0^2 \left(\int_{y^2/4}^{(y+2)/4} (16 - x^2 - y^2)\, dx \right) dy$$

$$= 12.4$$

Calculus III

15.4: Double Integrals in Polar Coordinates

Study concepts, example questions & explanations

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta$$

3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

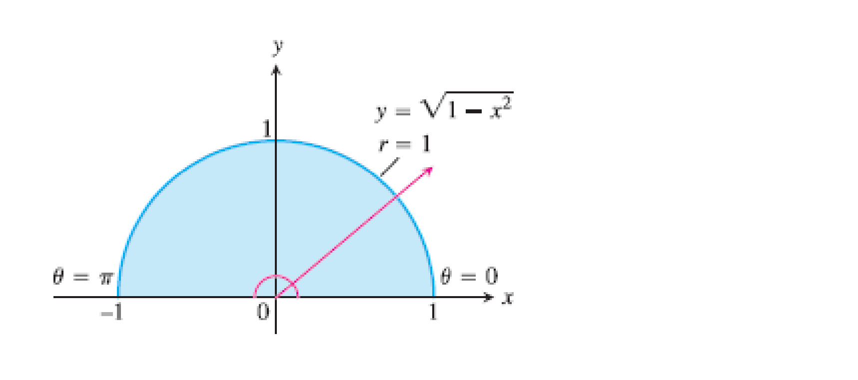
then $$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Question #1: Double Integral in Polar

Evaluate $\iint_D e^{x^2+y^2} dA$**, where** ***D*** **is the semicircular region bounded by the** ***x*****-axis and the curve** $y = \sqrt{1-x^2}$**.**

Answer:

We sketch the region D



The region *D* in polar coordinates can be written as

$$D = \{(r, \theta) | 0 \le \theta \le \pi, 0 \le r \le 1\}$$

We can evaluate the integral in polar coordinates as

$$\iint_D e^{x^2+y^2} dA = \int_0^{\pi} \int_0^{1} e^{r^2} r dr d\theta$$

$$= \int_0^{\pi} \left(\int_0^{1} e^{r^2} r dr \right) d\theta$$

$$= \int_0^{\pi} \left(\int_0^{1} \frac{1}{2} e^u du \right) d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{2} e^u \right]_0^1 dx$$

$$= \int_0^{\pi} \frac{1}{2}(e-1)\, dx$$

$$= \frac{\pi}{2}(e-1)$$

Let $u = r^2$

$du = 2rdr$

$dr = du/2r$

$r = 0 \rightarrow u = 0$

$r = 1 \rightarrow u = 1$

Question #2: Double Integral

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$.

Answer:

Integration with respect to *y* gives

$$\int_0^1 \left(x^2\sqrt{1-x^2} + \frac{(1-x^2)^{3/2}}{3} \right) dx,$$

which is a difficult integral. So we convert the double integral into polar coordinates.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx = \iint_D (x^2 + y^2)\, dA$$

where *D* is the interior of the unit quarter circle $x^2 + y^2 = 1$ in the first quadrant

$$D = \{(r, \theta) | 0 \le \theta \le \pi/2, 0 \le r \le 1\}$$

Thus,

$$\iint_D (x^2 + y^2) dA = \int_0^{\pi/2} \int_0^1 (r^2) r dr d\theta = \int_0^{\pi/2} \left(\int_0^1 r^3\, dr \right) d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{4} r^4 \right]_0^1 d\theta$$

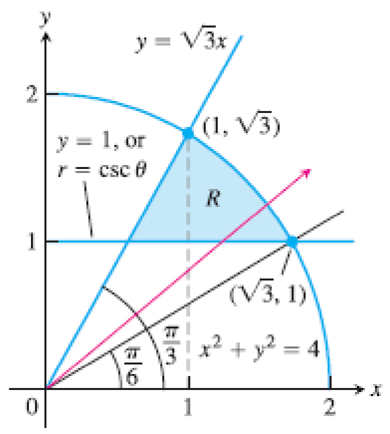
$$= \int_0^{\pi/2} \frac{1}{4} d\theta = \frac{\pi}{8}.$$

Question #3: Double Integral

Evaluate $\iint_D 1\ dA$, where *D* is the region enclosed by the circle $x^2 + y^2 = 4$, above the line y =1, and below the line $y = \sqrt{3}x$.

Answer:

Let's first sketch the region *D*



Then we convert the given curves to polar coordinates

$$x^2 + y^2 = 4 \Rightarrow r = 4$$

$$y = 1 \Rightarrow rsin\theta = 1 \Rightarrow r = csc\theta$$

$$y = \sqrt{3}x \Rightarrow rsin\theta = \sqrt{3}rcos\theta \Rightarrow tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$$

Moreover, the radial line from the origin through the point ($\sqrt{3}$,1) has equation $y = \frac{1}{\sqrt{3}}x$ which is in polar $\theta = \pi/6$.

Thus the region *D* in polar coordinates is

$$D = \{(r, \theta) | \pi/6 \leq \theta \leq \pi/3, csc\theta \leq r \leq 2\}$$

$$\iint_D 1\ dA = \int_{\pi/6}^{\pi/3} \int_{csc\theta}^{2} r\ drd\theta$$

$$= \frac{\pi - \sqrt{3}}{3}$$

Question #4: Double Integral

Find the volume of the solid region bounded above by the paraboloid $z = 9 - x^2 - y^2$ and below by the unit circle in the *xy*-plane.

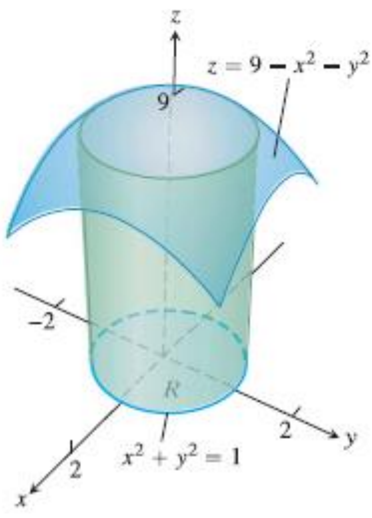
Answer:

The volume of the solid is

$$Volume = \iint_D (9 - x^2 - y^2)\, dA$$

Where D is the unit disk $x^2 + y^2 \leq 1$. It's easier to represent the region D in polar coordinates as

$$D = \{(r, \theta) | 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}$$



Thus,

$$Volume = \iint_D (9 - x^2 - y^2)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (9 - r^2) r \, dr d\theta = \frac{17\pi}{2}$$

Question #5: Reverse the Order of Integration

Find the volume of the solid that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the disk $(x-1)^2 + y^2 = 1$ on the xy-plane.

Answer:

The volume of the solid is

$$Volume = \iint_D (4 - x^2 - y^2)\, dA$$

where *D* is the disk $(x-1)^2 + y^2 \leq 1$. To convert the circle $(x-1)^2 + y^2 = 1$ to polar coordinates,

$$(x-1)^2 + y^2 = 1 \Rightarrow x^2 + y^2 - 2x = 0 \Rightarrow r^2 = 2rcos\theta \Rightarrow r = 2cos\theta$$

Thus,

$$D = \{(r,\theta) | 0 \leq \theta \leq \pi, 0 \leq r \leq 2cos\theta\}.$$

Hence the volume is

$$Volume = \iint_D (4 - x^2 - y^2)\, dA$$

$$= \int_0^{\pi} \int_0^{2cos\theta} (4 - r^2)\ rdrd\theta = \frac{5}{2}\pi$$

Calculus III

15.7: Triple Integrals

Study concepts, example questions & explanations

Question #1: Triple Integral

Evaluate $\int_0^1 \int_0^{x^2} \int_y^{\sqrt{y}} xyz \, dzdydx$ **.**

Answer:

$$\int_0^1 \int_0^{x^2} \int_y^{\sqrt{y}} xyz \, dzdydx = \int_0^1 \int_0^{x^2} \left(\int_y^{\sqrt{y}} xyz \, dz \right) dydx$$

$$= \int_0^1 \int_0^{x^2} \left[xy \frac{z^2}{2} \right]_y^{\sqrt{y}} dydx$$

$$= \frac{1}{2} \int_0^1 \left(\int_0^{x^2} xy(y - y^2) dy \right) dx$$

$$= \frac{1}{2} \int_0^1 \left[x \left(\frac{y^3}{3} - \frac{y^4}{4} \right) \right]_0^{x^2} dx$$

$$= \frac{1}{2} \int_0^1 \left(\frac{x^7}{3} - \frac{x^9}{4} \right) dx$$

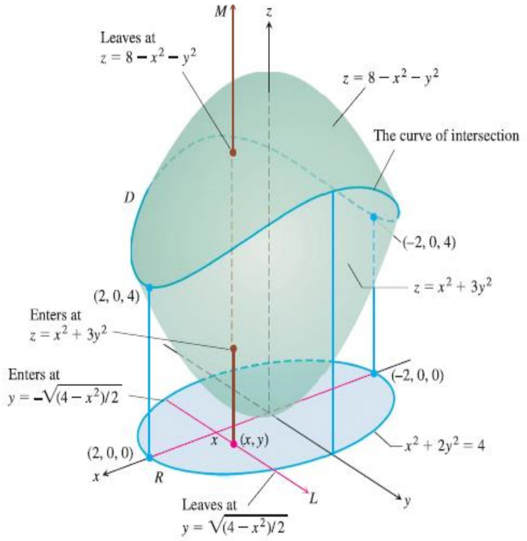
$$= \left[\left(\frac{x^8}{24} - \frac{x^{10}}{40} \right) \right]_0^1$$

$$= \frac{1}{60}$$

Question #2: Triple Integral

Set up, but do not evaluate, the limits of integration for evaluating the triple integral $\iiint_E (\mathrm{x} + 9y^3)dV$, where *E* is the region enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

Answer:
To find the limits of integration for evaluating the integral, we first sketch the region.



The two surfaces intersect on

$$x^2 + 3y^2 = 8 - x^2 - y^2 \Rightarrow x^2 + 2y^2 = 4.$$

The projection *D* of *E* onto the *xy*-plane is the region inside the ellipse $x^2 + 2y^2 = 4$.

The upper boundary of *E* is the surface $z = 8 - x^2 - y^2$ and the lower boundary is the surface $z = x^2 + 3y^2$.

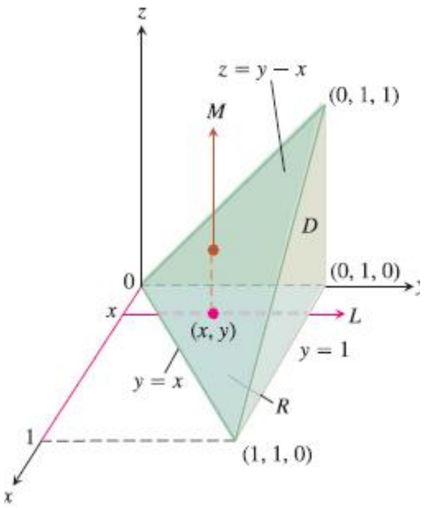
Thus,

$$\iiint_E (x + 9y^3)dV = \iint_D \int_{x^2+3y^2}^{8-x^2-y^2} (2yx^2 + 9y^3)dz\, dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} (2yx^2 + 9y^3)dz\, dydx$$

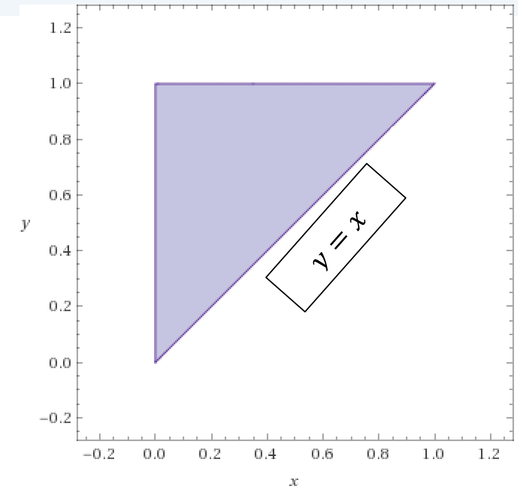
Question #3: Triple Integral

Set up, but do not evaluate, the limits of integration for evaluating the triple integral $\iiint_E (xyz^2)\, dV$, where *E* is the tetrahedron shown below.



Answer:
The z-limts of integration are $0 \le z \le y - x$.

The projection of the tetrahedron *E* onto the xy-plane is the triangular region



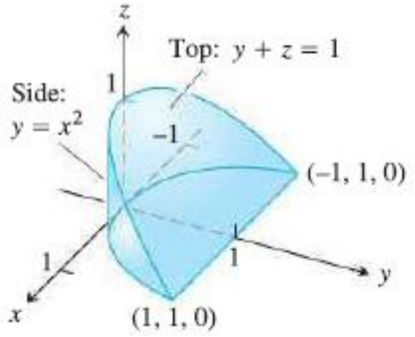
If we consider *D* as a Type I region, then

$$\iiint_E (xyz^2)\, dV = \iint_D \int_0^{y-x} (xyz^2)dz\, dA$$

$$= \int_0^1 \int_x^1 \int_0^{y-x} (xyz^2)dz\, dydx$$

Question #4: Triple Integral

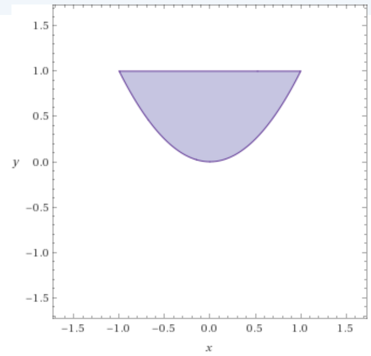
Set up, but do not evaluate, the limits of integration for evaluating the triple integral $\iiint_E (\mathbf{x}+\mathbf{y})\, dV$, where *E* is the region shown below.



Answer:

The *z*-limits of integration are $0 \le z \le 1-y$. The projection of E onto the *xy*-plane is shown below.

So, the region *D* in the *xy*-plane is



$$-1 \le x \le -1$$

$$x^2 \le y \le 1$$

Thus,

$$\iiint_E (x+y)\, dV = \iint_D \int_0^{1-y} (x+y) dz\, dA$$

$$= \int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} (xyz^2) dz\, dydx$$

Question #5: Volume

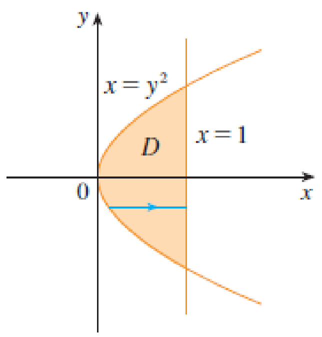
Find the volume of the solid bounded by the parabolic cylinder $x = y^2$ and the planes $z = x$, $x = 1$, $z = 0$.

Answer:

$$Volume\ of\ E = V = \iiint_E 1\, dV$$

The lower and upper surfaces of *E* are $0 \le z \le x$.

To find the projection *D* of *E* onto the *xy*-plane we set *z* = 0.
The cylinder $y = x^2$ intersects the *xy*-plane at the parabola $x = y^2$, the plane $z = x$ intersects the *xy*-plane at the line $x = 0$, and the plane $x = 1$ intersects the *xy*-plane at the line $x = 1$. So *D* is the parabolic region



If we consider *D* as a Type II region, then

$$V = \iiint_E 1\, dV = \iint_D \int_0^x 1\, dz\, dA$$

$$= \int_{-1}^{1} \int_{y^2}^{1} \int_0^x 1\, dz\, dxdy = \frac{4}{5}$$

Question #6: Volume

Find the volume of the solid *E* bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0,\ y = 3x,\ z = 0$ in the first octant.

Answer:

The *x*-limits of integration are $0 \le x \le y/3$.

The projection of *E* onto the *yz*-plane, when *x* = 0, is

$$0 \le z \le 3$$

$$0 \le y \le \sqrt{9 - z^2}$$

Hence,

$$V = \iiint_E 1\, dV = \iint_D \int_0^{y/3} 1\, dx\, dA$$

$$= \int_0^3 \int_0^{\sqrt{9-z^2}} \int_0^{y/3} 1\, dx\, dydz$$

$$= \int_0^3 \int_0^{\sqrt{9-z^2}} \frac{y}{3}\, dydz$$

$$= \int_0^3 \left[\frac{y^2}{6}\right]_0^{\sqrt{9-z^2}} dz$$

$$= \frac{1}{6}\int_0^3 (9 - z^2)\, dz$$

$$= \frac{1}{6}\left[9z - \frac{z^3}{3}\right]_0^3$$

$$= 3$$

Question #7: Volume

Use a triple integral to find the volume of the solid enclosed by the paraboloids $z = 18 - x^2 - y^2$ and $z = x^2 + y^2$.

Answer:

The paraboloids $z = 18 - x^2 - y^2$ and $z = x^2 + y^2$ intersect when

$$18 - x^2 - y^2 = x^2 + y^2 \Rrightarrow x^2 + y^2 = 9.$$

So, the required solid *E* is bounded above by $z = 18 - x^2 - y^2$ and below by $z = x^2 + y^2$. The projection *D* of the solid onto the *xy*-plane is the disk $x^2 + y^2 \leq 9$. Then,

$$V = \iiint_E 1\,dV = \iint_D \int_{x^2+y^2}^{18-x^2-y^2} 1\,dz\,dA$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} \int_{x^2+y^2}^{18-x^2-y^2} 1\,dz\,dydz$$

$$= \int_{-3}^{3} \int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} (18 - x^2 - y^2) dydz$$

It's much simpler to use polar coordinates here. Thus,

$$V = \int_0^{2\pi} \int_0^3 (18 - r^2)\, rdrd\theta$$

$$= \int_0^{2\pi} \int_0^3 (18r - 2r^3) drd\theta$$

$$= \int_0^{2\pi} \frac{1}{6} \left[9r^2 - \frac{r^4}{2}\right]_0^3 d\theta$$

$$= \int_0^{2\pi} \frac{81}{2} d\theta$$

$$= 81\pi$$

Calculus III

15.8: Triple Integrals in Cylindrical Coordinates

Study concepts, example questions & explanations

DEFINITION **Cylindrical coordinates** represent a point P in space by ordered triples (r, θ, z) in which

1. r and θ are polar coordinates for the vertical projection of P on the xy-plane
2. z is the rectangular vertical coordinate.

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$

Evaluating Triple Integrals with Cylindrical Coordinates

4
$$\iiint_E f(x, y, z)\, dV = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

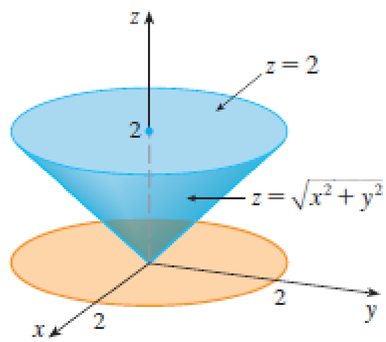
Question #1: Triple Integrals in Cylindrical

Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dzdydx$ **.**

Answer:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\, dzdydx = \iiint_E (x^2+y^2)\, dV$$

The lower surface of E is the cone $z = \sqrt{x^2+y^2}$ and its upper surface is the plane $z = 2$. The projection of E onto the xy-plane is the disk $x^2+y^2 \le 4$.



This region has a much simpler description in cylindrical coordinates:

$$E = \{(r,\theta,z) | 0 \le \theta \le 2\pi, 0 \le r \le 2, r \le z \le 2\}$$

Therefore we have

$$\iiint_E (x^2+y^2)\, dV = \int_0^{2\pi}\int_0^2\int_r^2 r^2\, r dzdrd\theta$$

$$= \int_0^{2\pi}\int_0^2 \left(\int_r^2 r^3\, dz\right) drd\theta$$

$$= \int_0^{2\pi}\int_0^2 [r^3 z]_r^2 drd\theta$$

$$= \int_0^{2\pi}\left(\int_0^2 (2r^3 - r^4)dr\right) d\theta$$

$$= \int_0^{2\pi}\left[\left(\frac{r^4}{2} - \frac{r^5}{5}\right)\right]_0^2 d\theta$$

$$= \int_0^{2\pi} \frac{8}{5} d\theta = \frac{16\pi}{5}$$

Question #2: Triple Integrals in Cylindrical

Evaluate $\iiint_E \sqrt{x^2+y^2}\, dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

Answer:

The solid *E* is bounded above by the plane $z=4$ and below by the plane $z=-5$. The projection *D* of *E* onto the *xy*-plane is the disk $x^2+y^2 \le 4$. Thus,

$$\iiint_E \sqrt{x^2+y^2}\, dV = \iint_D \left(\int_{-5}^{4} \sqrt{x^2+y^2}\, dz\right) dA$$

$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} \int_{-5}^{4} \sqrt{x^2+y^2}\, dz\, dydx$$

This region has a much simpler description in cylindrical coordinates:

$$E = \{(r,\theta,z) | 0 \le \theta \le 2\pi, 0 \le r \le 2, -5 \le z \le 4\}$$

Therefore we have

$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_0^{2\pi} \int_0^2 \int_{-5}^{4} (r)\, rdzdrd\theta$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^2 r^2\, dr\right)\left(\int_{-5}^{4} 1\, dz\right)$$

$$= 48\pi$$

Question #3: Triple Integrals in Cylindrical

Evaluate the integral by changing to cylindrical coordinates

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-\mathrm{x}^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dydx$$

Answer:

First we write

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-\mathrm{x}^2}}\int_{0}^{9-x^2-y^2}\sqrt{x^2+y^2}\,dz\,dydx = \iiint_E \sqrt{x^2+y^2}\,dV$$

The z-limits of integration are $0 \le z \le 9 - x^2 - y^2 = 9 - r^2$.

The projection of *E* onto the xy-plane is the upper semi-disk:

$$-3 \le x \le 3, \quad 0 \le y \le \sqrt{9-\mathrm{x}^2}$$

This solid *E* has the description in cylindrical coordinates:

$$E = \{(r,\theta,z)|0 \le \theta \le \pi, 0 \le r \le 3, 0 \le z \le 9 - r^2\}$$

Therefore we have

$$\iiint_E \sqrt{x^2+y^2}\,dV = \int_0^{\pi}\int_0^{3}\int_0^{9-r^2}(r)\,rdzdrd\theta$$

$$= \int_0^{\pi}\int_0^{3}(9r^2 - r^4)\,drd\theta$$

$$= \int_0^{\pi}\left[\left(3r^3 - \frac{r^5}{5}\right)\right]_0^3 d\theta$$

$$= \int_0^{\pi}\frac{162}{5}d\theta$$

$$= \frac{162}{5}\pi$$

Question #4: Triple Integrals in Cylindrical

Evaluate $\iiint_E x\, dV$, where E is enclosed by the planes $z = 0$ and $z = x + y + 5$ and by the cylinders $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Answer:
The z-limits of integration are

$$0 \le z \le x + y + 5 = rcos\theta + rsin\theta + 5.$$

The projection of *E* onto the xy-plane is the region enclosed between the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

This solid *E* in cylindrical coordinate is

$$E = \{(r, \theta, z) | 0 \le \theta \le 2\pi, 2 \le r \le 3, 0 \le z \le rcos\theta + rsin\theta + 5\}$$

Therefore we have

$$\iiint_E x\, dV = \int_0^{2\pi} \int_2^3 \int_0^{rcos\theta + rsin\theta + 5} (rcos\theta)\, rdzdrd\theta$$

$$= \int_0^{2\pi} \int_2^3 (r^3cos^2\theta + r^3cos\theta sin\theta + 5r^2cos\theta)\, drd\theta$$

$$= \int_0^{\pi} \left[\left(\frac{r^4}{4}cos^2\theta + \frac{r^4}{4}cos\theta sin\theta + \frac{5}{3}r^3cos\theta\right)\right]_2^3 d\theta$$

$$= \int_0^{\pi} \left(\frac{65}{4}cos^2\theta + \frac{65}{4}cos\theta sin\theta + \frac{95}{3}cos\theta\right) d\theta$$

$$= \int_0^{\pi} \left(\frac{65}{8}(1 + cos2\theta) + \frac{65}{8}sin2\theta + \frac{95}{3}cos\theta\right) d\theta$$

$$= \left[\left(\frac{65}{8}\left(\theta + \frac{sin2\theta}{2}\right) - \frac{65}{16}cos2\theta + \frac{95}{3}sin\theta\right)\right]_0^{\pi}$$

$$= \frac{65}{8}\pi$$

Question #5: Volume

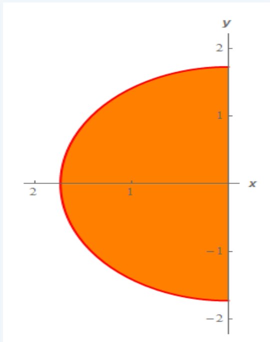
Use a triple integral to find the volume of the solid below the plane $z = 6 - x$, above $z = -\sqrt{4x^2 + 4y^2}$ inside the cylinder $x^2 + y^2 = 3$ with $x \le 0$.

Answer:

$$Volume\ of\ E = V = \iiint_E 1\, dV$$

The *z*-limits are $-\sqrt{4x^2 + 4y^2} \le z \le 6 - x$ or $-2r \le z \le 6 - r\cos\theta$.

To find the projection *D* of *E* onto the *xy*-plane we set *z* = 0. So, *D* is the portion of the disk $x^2 + y^2 \le 3$ ($r \le \sqrt{3}$) with $x \le 0$.



This solid *E* in cylindrical coordinate is

$$E = \{(r, \theta, z) | \pi/2 \le \theta \le 3\pi/2, 0 \le r \le \sqrt{3}, -2r \le z \le 6 - r\cos\theta\}$$

Therefore we have

$$V = \iiint_E 1\, dV = \int_{\pi/2}^{3\pi/2} \int_0^{\sqrt{3}} \int_{-2r}^{6 - r\cos\theta} (1)\, r\,dz\,dr\,d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \int_0^{\sqrt{3}} (6r - r^2\cos\theta + 2r^2)\, dr\,d\theta$$

$$= \int_{\pi/2}^{3\pi/2} \left[\left(3r^2 - \frac{r^3}{3}\cos\theta + \frac{2}{3}r^3\right)\right]_0^{\sqrt{3}} d\theta$$

$$= \int_{\pi/2}^{3\pi/2} (9 + 2\sqrt{3} - \sqrt{3}\cos\theta)\, d\theta = 2\sqrt{3} + (9 + 2\sqrt{3})\pi \approx 42.6$$

Calculus III

15.9: Triple Integrals in Spherical Coordinates

Study concepts, example questions & explanations

We use the following relations in converting from rectangular to spherical coordinates:

1 $$x = \rho \sin \phi \cos \theta \qquad y = \rho \sin \phi \sin \theta \qquad z = \rho \cos \phi$$

Also, the distance formula shows that

2 $$\rho^2 = x^2 + y^2 + z^2$$

Evaluating Triple Integrals with Spherical Coordinates

3 $$\iiint\limits_E f(x, y, z)\, dV$$

$$= \int_c^d \int_\alpha^\beta \int_a^b f(\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)\, \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi$$

where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$$

Question #1: Spherical Coordinates

Describe the surface whose equation in spherical coordinates is

a) $\boldsymbol{\phi = \frac{\pi}{6}}$

b) $\boldsymbol{\rho = 2cos\phi}$

Answer:

a) We convert the equation to rectangular coordinates:

$$\phi = \frac{\pi}{6}$$

$$cos\phi = cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

$$\rho cos\phi = \rho\frac{\sqrt{3}}{2}$$

$$z = \sqrt{x^2 + y^2 + z^2}\frac{\sqrt{3}}{2}$$

$$\frac{3}{4}z^2 = \frac{3}{4}(x^2 + y^2)$$

$$z = \sqrt{3(x^2 + y^2)}$$

This is a cone centered on the *z*-axis.

b) To convert the equation to rectangular coordinates we use that:

$$\rho = 2cos\phi$$
$$\rho^2 = 2\rho cos\phi$$
$$x^2 + y^2 + z^2 = 2z$$
$$x^2 + y^2 + z^2 - 2z = 0$$
$$x^2 + y^2 + z^2 - 2z + 1 - 1 = 0$$
$$x^2 + y^2 + (z - 1)^2 = 1$$

This is a sphere with center (0,0,1) and radius 1.

Question #2: Triple Integrals in Spherical

Evaluate $\iiint_E \sqrt{x^2+y^2+z^2}\, dV$, where E is the ball given by the equation $x^2+y^2+z^2 \le 25.$

Answer:

Since the boundary of *E* is a sphere, we use spherical coordinates:

$$E = \{(\rho,\theta,\phi) | 0 \le \rho \le 5, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$$

Therefore we have

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{5} (\rho)\, \rho^2 sin\phi \;\; d\rho d\phi d\theta$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^{5} \rho^3\, d\rho\right)\left(\int_0^{\pi} sin\phi\, d\phi\right)$$

$$= 625\pi$$

Question #3: Triple Integrals in Spherical

Evaluate the integral by changing to spherical coordinates

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^2 \, dz\, dydx$$

Answer:

First we write

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} (x^2+y^2+z^2)^2 \, dz\, dydx$$

$$= \iiint_E \, (x^2+y^2+z^2)^2 \, dV$$

The region *E* is a portion of the unit ball lying in the first octant and hence it is bounded by the inequalities

$$E = \{(\rho, \theta, \phi) | 0 \le \rho \le 1, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/2\}$$

Therefore we have

$$\iiint_E \, (x^2+y^2+z^2)^2 \, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho^4)\, \rho^2 sin\phi \, d\rho d\phi d\theta$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 (\rho^6 sin\phi) \, d\rho d\phi d\theta$$

$$= \left(\int_0^{\pi/2} 1 \, d\theta\right)\left(\int_0^1 \rho^6 \, d\rho\right)\left(\int_0^{\pi} sin\phi \, d\phi\right)$$

$$= \frac{\pi}{14}.$$

Question #4: Triple Integrals in Spherical

Evaluate $\iiint_E xyz\, dV$, where the region E is a portion of the ball $x^2+y^2+z^2 \le 4$, lying in the first octant $x \ge 0,\ \ y \ge 0,\ \ z \ge 0$.

Answer:

We convert the integral to spherical coordinates

$$E = \{(\rho,\theta,\phi)|0 \le \rho \le 2, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/2\}$$

Note that the integrand in spherical coordinates is

$$xyz = (\rho sin\phi cos\theta)(\rho sin\phi sin\theta)(\rho cos\phi)$$

$$= \rho^3 sin^2\phi cos\phi sin\theta cos\theta$$

Therefore we have

$$\iiint_E xyz\, dV$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 (\rho^3 sin^2\phi cos\phi sin\theta cos\theta)\, \rho^2 sin\phi\ \ d\rho d\phi d\theta$$

$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 (\rho^5 sin^3\phi cos\phi sin\theta cos\theta)\ \ d\rho d\phi d\theta$$

$$= \left(\int_0^{\pi/2} sin\theta cos\theta\, d\theta\right)\left(\int_0^2 \rho^5\, d\rho\right)\left(\int_0^{\pi/2} sin^3\phi cos\phi\, d\phi\right)$$

$$= \frac{4}{3}$$

Question #5: Volume

Use a triple integral to find the volume of the solid bounded by the sphere $x^2 + y^2 + z^2 = 25$, the cone $z = \sqrt{x^2 + y^2}$ and the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$.

Answer:

$$Volume\ of\ E = V = \iiint_E 1\, dV$$

Since we are dealing with spherical regions, we will use spherical coordinates.

Let's first convert the boundary surfaces of *E*:

$x^2 + y^2 + z^2 = 25 \Rightarrow \rho^2 = 25 \Rightarrow \rho = 5,$

$z = \sqrt{x^2 + y^2} \Rightarrow z^2 = x^2 + y^2 \Rightarrow$
$(\rho cos\phi)^2 = (\rho sin\phi cos\theta)^2 + (\rho sin\phi sin\theta)^2$
$\Rightarrow \rho^2 cos^2\phi = \rho^2 sin^2\phi cos^2\theta + \rho^2 sin^2\phi sin^2\theta$
$\Rightarrow \rho^2 cos^2\phi = \rho^2 sin^2\phi(sin^2\theta + cos^2\theta) \Rightarrow \rho^2 cos^2\phi = \rho^2 sin^2\phi$
$\Rightarrow cos\phi = sin\phi \Rightarrow tan\phi = 1 \Rightarrow \phi = \dfrac{\pi}{4}$

In the same way, $z = \sqrt{3}\sqrt{x^2 + y^2} \Rightarrow tan\phi = \frac{1}{\sqrt{3}} \Rightarrow \phi = \frac{\pi}{6}$.

This solid *E* in spherical coordinates is

$$E = \{(\rho, \theta, \phi) | 0 \le \rho \le 5, 0 \le \theta \le 2\pi, \pi/6 \le \phi \le \pi/4\}$$

Therefore we have

$$V = \iiint_E 1\, dV = \int_0^{2\pi} \int_{\pi/6}^{\pi/4} \int_0^5 (1)\, \rho^2 sin\phi\ \ d\rho d\phi d\theta$$

$$= \left(\int_0^{2\pi} 1\, d\theta\right)\left(\int_0^5 \rho^2\, d\rho\right)\left(\int_{\pi/6}^{\pi/4} sin\phi\, d\phi\right)$$

$$= (2\pi)\left(\frac{125}{3}\right)\left(\frac{\sqrt{3} - \sqrt{2}}{2}\right) \approx 41.6$$