The Hashemite University
Faculty of Engineering
Department of Civil Engineering

# CE 315: Structural Analysis

## Types of Structures and Loads

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## Introduction

A structure refers to a system of connected parts used to support a load. Important examples related to civil engineering include:

- Buildings,
- Bridges and
- Towers;

and in other branches of engineering,

- Ship and aircraft frames,
- Tanks, pressure vessels,
- Mechanical systems, and
- Electrical supporting structures

The design of a structure involves many considerations, among which are four major objectives that must be satisfied:

❖The structure must meet the performance requirement (utility).

❖The structure must carry loads safely (safety).

❖The structure should be economical in material, construction, and cost (economy).

❖The structure should have a good appearance (aesthetics).

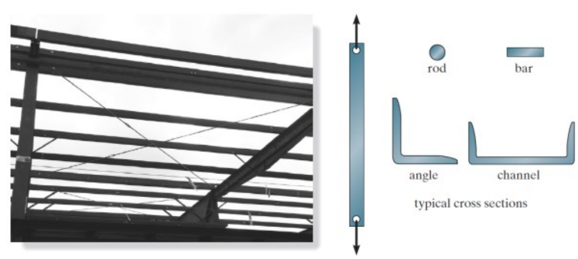
Once a preliminary design of a structure is proposed, the structure must then be analyzed to ensure that it has its required **stiffness** and **strength**. To analyze a structure properly, certain idealizations must be made as to how the members are supported and connected together. The loadings are determined from codes and local specifications, and the forces in the members and their displacements are found using the **theory of structural analysis**.



# Classification of Structures

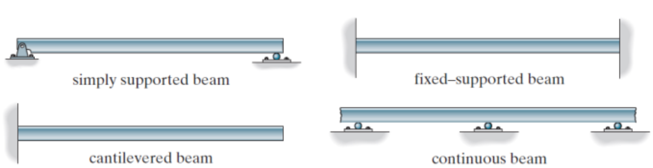
## ❑ Structural elements

➢ **Tie rods**: Structural members subjected to a tensile force are often referred to as tie rods or bracing struts.

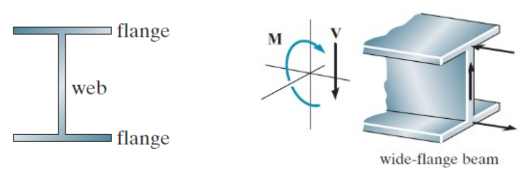


Tie rods used for wind bracing.

➢ **Beams**: usually straight horizontal members used primarily to carry vertical loads. they are classified according to the way they are supported.

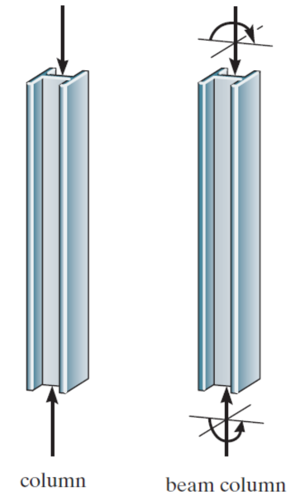



Beams are primarily designed to resist bending moment; however, if they are short and carry large loads, the internal shear force may become quite large and this force may govern their design.



## Classification of Structures (Cont'd)

➢**Columns**: Members that are generally **vertical and resist axial compressive loads** are referred to as columns. Occasionally, columns are subjected to both an axial load and a bending moment as shown in the figure. These members are referred to as **beam columns**.

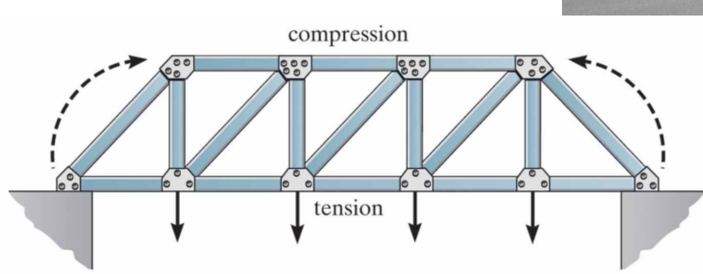





## Classification of Structures (Cont'd)

❑ **Types of structures**

➢ **Trusses**: Used when the large spans are required. spans ranging from 30 ft. (9 m) to 400 ft. (122 m)




Loading causes bending of truss, which develops compression in top members, tension in bottom members

## Classification of Structures (Cont'd)

➢ **Cables & Arches**: Used to span long distances. They are commonly used to support bridges, and building roofs.

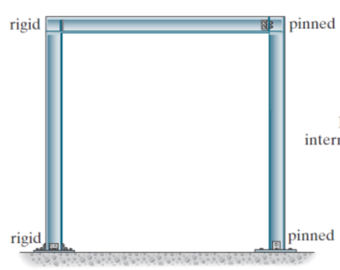
Cables support their loads in tension.

Arches support their loads in compression.



## Classification of Structures (Cont'd)

➢ **Frames**: are composed of beams and columns that are either pin or fixed connected,




Frame members are subjected to internal axial, shear, and moment loadings.

➢ **Surface Structures**: They are made from a material having a very small thickness compared to its other dimensions. (Thin plates or shells)

# Loads

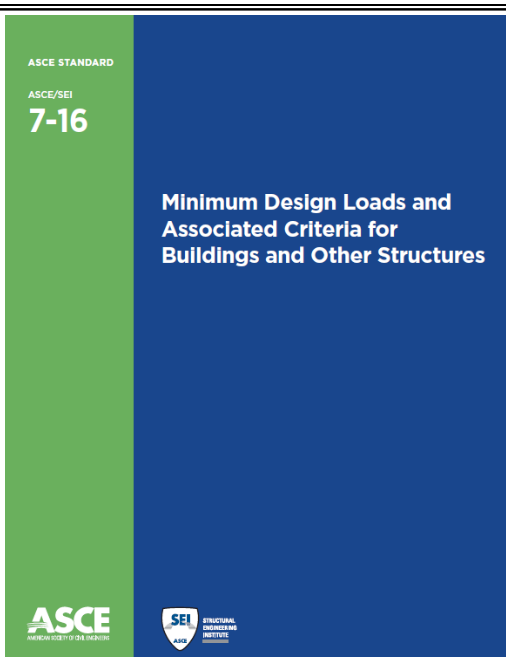
- Once the dimensional requirements and the structural form has been determined for a structure, it is necessary to first specify the **loads** that act on it.
- The design loading for a structure is often specified in **CODES.**
- A code is a set of technical specifications and standards that control major details of analysis, design, and construction of buildings, equipment, and bridges.
- The purpose of codes is to produce safe, economical structures so that the public will be protected from poor or inadequate design and construction.

❑ In general, the structural engineer works with **two types** of codes:

- **General building codes:** They specify the requirements of governmental bodies for minimum design loads on structures and minimum standards for construction

1. Standard Minimum Design Loads for Buildings and Other Structures (ASCE/SEI 7-16) ,published by the American Society of Civil Engineers (ASCE).
2. International Building Code (IBC).

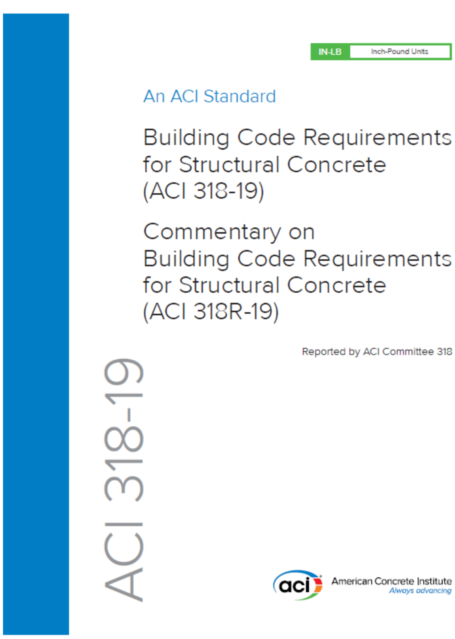


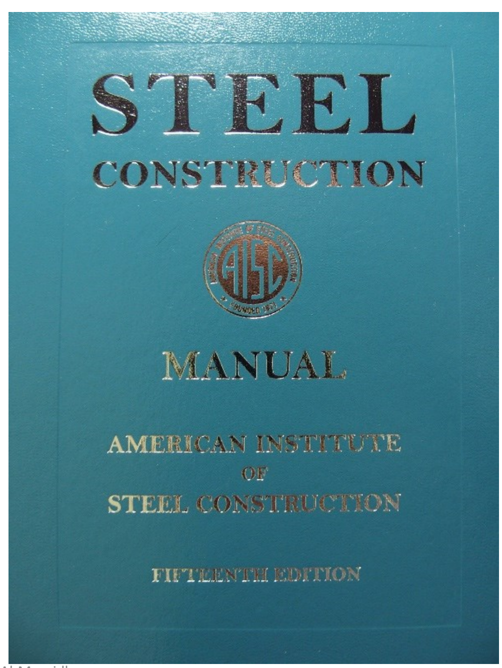



- **Design Codes (Structural codes):** They provide detailed technical standards and are used to establish the requirements for the actual structural design

1. Standard Specifications for Highway Bridges by the American Association of State Highway and Transportation Officials (**AASHTO**) covers the design and analysis of highway bridges.
2. Manual for Railway Engineering by the American Railway Engineering and Maintenance of Way Association (**AREMA**) covers the design and analysis of railroad bridges.
3. Building Code Requirements for Reinforced Concrete (ACI 318) by the American Concrete Institute (**ACI**) covers the analysis and design of concrete structures.
4. Manual of Steel Construction by the American Institute of Steel Construction (**AISC**) covers the analysis and design of steel structures.
5. National Design Specifications for Wood Construction by the American Forest & Paper Association (**AFPA**) covers the analysis and design of wood structures.





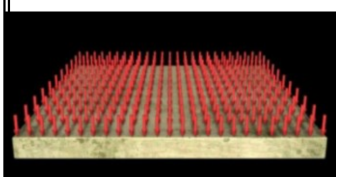
# Types of loads

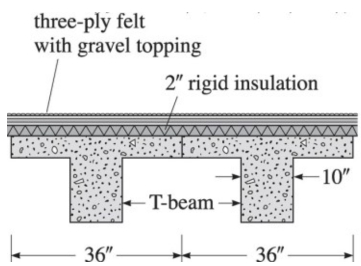
❖ **Dead Loads**

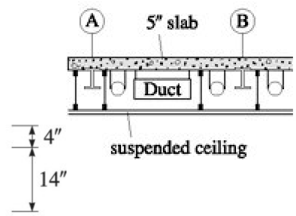
➢ Loads that are constant in magnitude and fixed in location throughout the lifetime of the structure. They Include the weights of the columns, beams, and girders, the floor slab, roofing, walls, windows, plumbing, electrical fixtures….etc.

➢ Dead loads can be calculated with good accuracy from the design configuration, dimensions of the structures, and density of the materials.

➢ Code assumes most dead loads can be simplified as uniformly distributed area load





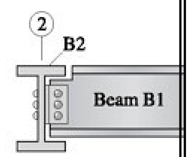






**Table C3.1-2 Minimum Densities for Design Loads from Materials**

| Material | Density (lb/ft³) | Density (kN/m³) |
|---|---|---|
| Aluminum | 170 | 27 |
| Bituminous products | | |
| Asphaltum | 81 | 12.7 |
| Graphite | 135 | 21.2 |
| Paraffin | 56 | 8.8 |
| Petroleum, crude | 55 | 8.6 |
| Petroleum, refined | 50 | 7.9 |
| Petroleum, benzine | 46 | 7.2 |
| Petroleum, gasoline | 42 | 6.6 |
| Pitch | 69 | 10.8 |
| Tar | 75 | 11.8 |
| Brass | 526 | 82.6 |
| Bronze | 552 | 86.7 |
| Cast-stone masonry (cement, stone, sand) | 144 | 22.6 |
| Cement, Portland, loose | 90 | 14.1 |
| Ceramic tile | 150 | 23.6 |
| Charcoal | 12 | 1.9 |
| Cinder fill | 57 | 9.0 |
| Cinders, dry, in bulk | 45 | 7.1 |
| Coal | | |
| Anthracite, piled | 52 | 8.2 |
| Bituminous, piled | 47 | 7.4 |
| Lignite, piled | 47 | 7.4 |
| Peat, dry, piled | 23 | 3.6 |
| Concrete, plain | | |
| Cinder | 108 | 17.0 |
| Expanded-slag aggregate | 100 | 15.7 |
| Haydite (burned-clay aggregate) | 90 | 14.1 |
| Slag | 132 | 20.7 |
| Stone (including gravel) | 144 | 22.6 |
| Vermiculite and perlite aggregate, nonload-bearing | 25–50 | 3.9–7.9 |
| Other light aggregate, load-bearing | 70–105 | 11.0–16.5 |
| Concrete, reinforced | | |
| Cinder | 111 | 17.4 |
| Slag | 138 | 21.7 |
| Stone (including gravel) | 150 | 23.6 |
| Copper | 556 | 87.3 |
| Cork, compressed | 14 | 2.2 |
| Earth (not submerged) | | |
| Clay, dry | 63 | 9.9 |
| Clay, damp | 110 | 17.3 |
| Clay and gravel, dry | 100 | 15.7 |
| Silt, moist, loose | 78 | 12.3 |

**Table C3.1-2 (*Continued*)**

| Material | Density (lb/ft³) | Density (kN/m³) |
|---|---|---|
| Silt, moist, packed | 96 | 15.1 |
| Silt, flowing | 108 | 17.0 |
| Sand and gravel, dry, loose | 100 | 15.7 |
| Sand and gravel, dry, packed | 110 | 17.3 |
| Sand and gravel, wet | 120 | 18.9 |
| Earth (submerged) | | |
| Clay | 80 | 12.6 |
| Soil | 70 | 11.0 |
| River mud | 90 | 14.1 |
| Sand or gravel | 60 | 9.4 |
| Sand or gravel and clay | 65 | 10.2 |
| Glass | 160 | 25.1 |
| Gravel, dry | 104 | 16.3 |
| Gypsum, loose | 70 | 11.0 |
| Gypsum, wallboard | 50 | 7.9 |
| Ice | 57 | 9.0 |
| Iron | | |
| Cast | 450 | 70.7 |
| Wrought | 480 | 75.4 |
| Lead | 710 | 111.5 |
| Lime | | |
| Hydrated, loose | 32 | 5.0 |
| Hydrated, compacted | 45 | 7.1 |
| Masonry, ashlar stone | | |
| Granite | 165 | 25.9 |
| Limestone, crystalline | 165 | 25.9 |
| Limestone, oolitic | 135 | 21.2 |
| Marble | 173 | 27.2 |
| Sandstone | 144 | 22.6 |
| Masonry, brick | | |
| Hard (low absorption) | 130 | 20.4 |
| Medium (medium absorption) | 115 | 18.1 |
| Soft (high absorption) | 100 | 15.7 |
| Masonry, concrete[a] | | |
| Lightweight units | 105 | 16.5 |
| Medium weight units | 125 | 19.6 |
| Normal weight units | 135 | 21.2 |
| Masonry grout | 140 | 22.0 |
| Masonry, rubble stone | | |
| Granite | 153 | 24.0 |
| Limestone, crystalline | 147 | 23.1 |
| Limestone, oolitic | 138 | 21.7 |
| Marble | 156 | 24.5 |
| Sandstone | 137 | 21.5 |

*continues*

**Table C3.1-2 (*Continued*)**

| Material | Density (lb/ft³) | Density (kN/m³) |
|---|---|---|
| Mortar, cement or lime | 130 | 20.4 |
| Particleboard | 45 | 7.1 |
| Plywood | 36 | 5.7 |
| Riprap (not submerged) | | |
| Limestone | 83 | 13.0 |
| Sandstone | 90 | 14.1 |
| Sand | | |
| Clean and dry | 90 | 14.1 |
| River, dry | 106 | 16.7 |
| Slag | | |
| Bank | 70 | 11.0 |
| Bank screenings | 108 | 17.0 |
| Machine | 96 | 15.1 |
| Sand | 52 | 8.2 |
| Slate | 172 | 27.0 |
| Steel, cold-drawn | 492 | 77.3 |
| Stone, quarried, piled | | |
| Basalt, granite, gneiss | 96 | 15.1 |
| Limestone, marble, quartz | 95 | 14.9 |
| Sandstone | 82 | 12.9 |
| Shale | 92 | 14.5 |
| Greenstone, hornblende | 107 | 16.8 |
| Terra cotta, architectural | | |
| Voids filled | 120 | 18.9 |
| Voids unfilled | 72 | 11.3 |
| Tin | 459 | 72.1 |
| Water | | |
| Fresh | 62 | 9.7 |
| Sea | 64 | 10.1 |
| Wood, seasoned | | |
| Ash, commercial white | 41 | 6.4 |
| Cypress, southern | 34 | 5.3 |
| Fir, Douglas, coast region | 34 | 5.3 |
| Hem fir | 28 | 4.4 |
| Oak, commercial reds and whites | 47 | 7.4 |
| Pine, southern yellow | 37 | 5.8 |
| Redwood | 28 | 4.4 |
| Spruce, red, white, and Sitka | 29 | 4.5 |
| Western hemlock | 32 | 5.0 |
| Zinc, rolled sheet | 449 | 70.5 |

[a]Tabulated values apply to solid masonry and to the solid portion of hollow masonry.

**Table C3.1-1a Minimum Design Dead Loads (psf)[a]**

| Component | Load (psf) |
|---|---|
| CEILINGS | |
| Acoustical fiberboard | 1 |
| Gypsum board (per 1/8-in. thickness) | 0.55 |
| Mechanical duct allowance | 4 |
| Plaster on tile or concrete | 5 |
| Plaster on wood lath | 8 |
| Suspended steel channel system | 2 |
| Suspended metal lath and cement plaster | 15 |
| Suspended metal lath and gypsum plaster | 10 |
| Wood furring suspension system | 2.5 |
| COVERINGS, ROOF, AND WALL | |
| Asbestos-cement shingles | 4 |
| Asphalt shingles | 2 |
| Cement tile | 16 |
| Clay tile (for mortar add 10 psf) | |
| Book tile, 2-in. | 12 |
| Book tile, 3-in. | 20 |
| Ludowici | 10 |
| Roman | 12 |
| Spanish | 19 |
| Composition: | |
| Three-ply ready roofing | 1 |
| Four-ply felt and gravel | 5.5 |
| Five-ply felt and gravel | 6 |
| Copper or tin | 1 |
| Corrugated asbestos-cement roofing | 4 |
| Deck, metal, 20 gauge | 2.5 |
| Deck, metal, 18 gauge | 3 |
| Decking, 2-in. wood (Douglas fir) | 5 |
| Decking, 3-in. wood (Douglas fir) | 8 |
| Fiberboard, 1/2-in. | 0.75 |
| Gypsum sheathing, 1/2-in. | 2 |
| Insulation, roof boards (per inch thickness) | |
| Cellular glass | 0.7 |
| Fibrous glass | 1.1 |
| Fiberboard | 1.5 |
| Perlite | 0.8 |
| Polystyrene foam | 0.2 |
| Urethane foam with skin | 0.5 |
| Plywood (per 1/8-in. thickness) | 0.4 |
| Rigid insulation, 1/2-in. | 0.75 |
| Skylight, metal frame, 3/8-in. wire glass | 8 |
| Slate, 3/16-in. | 7 |
| Slate, 1/4-in. | 10 |
| Waterproofing membranes: | |
| Bituminous, gravel-covered | 5.5 |
| Bituminous, smooth surface | 1.5 |
| Liquid applied | 1 |
| Single-ply, sheet | 0.7 |
| Wood sheathing (per inch thickness) | 3 |
| Wood shingles | 3 |
| FLOOR FILL | |
| Cinder concrete, per inch | 9 |
| Lightweight concrete, per inch | 8 |
| Sand, per inch | 8 |
| Stone concrete, per inch | 12 |
| FLOORS AND FLOOR FINISHES | |
| Asphalt block (2-in.), 1/2-in. mortar | 30 |
| Cement finish (1-in.) on stone–concrete fill | 32 |
| Ceramic or quarry tile (3/4-in.) on 1/2-in. mortar bed | 16 |
| Ceramic or quarry tile (3/4-in.) on 1-in. mortar bed | 23 |
| Concrete fill finish (per inch thickness) | 12 |
| Hardwood flooring, 7/8-in. | 4 |
| Linoleum or asphalt tile, 1/4-in. | 1 |
| Marble and mortar on stone–concrete fill | 33 |

*continues*

**Table C3.1-1b Minimum Design Dead Loads (kN/m²)[a]**

| Component | Load (kN/m²) |
|---|---|
| CEILINGS | |
| Acoustical fiberboard | 0.05 |
| Gypsum board (per mm thickness) | 0.008 |
| Mechanical duct allowance | 0.19 |
| Plaster on tile or concrete | 0.24 |
| Plaster on wood lath | 0.38 |
| Suspended steel channel system | 0.10 |
| Suspended metal lath and cement plaster | 0.72 |
| Suspended metal lath and gypsum plaster | 0.48 |
| Wood furring suspension system | 0.12 |
| COVERINGS, ROOF, AND WALL | |
| Asbestos-cement shingles | 0.19 |
| Asphalt shingles | 0.10 |
| Cement tile | 0.77 |
| Clay tile (for mortar add 0.48 kN/m²) | |
| Book tile, 51 mm | 0.57 |
| Book tile, 76 mm | 0.96 |
| Ludowici | 0.48 |
| Roman | 0.57 |
| Spanish | 0.91 |
| Composition: | |
| Three-ply ready roofing | 0.05 |
| Four-ply felt and gravel | 0.26 |
| Five-ply felt and gravel | 0.29 |
| Copper or tin | 0.05 |
| Corrugated asbestos-cement roofing | 0.19 |
| Deck, metal, 20 gauge | 0.12 |
| Deck, metal, 18 gauge | 0.14 |
| Decking, 51-mm wood (Douglas fir) | 0.24 |
| Decking, 76-mm wood (Douglas fir) | 0.38 |
| Fiberboard, 13 mm | 0.04 |
| Gypsum sheathing, 13 mm | 0.10 |
| Insulation, roof boards (per mm thickness) | |
| Cellular glass | 0.0013 |
| Fibrous glass | 0.0021 |
| Fiberboard | 0.0028 |
| Perlite | 0.0015 |
| Polystyrene foam | 0.0004 |
| Urethane foam with skin | 0.0009 |
| Plywood (per mm thickness) | 0.006 |
| Rigid insulation, 13 mm | 0.04 |
| Skylight, metal frame, 10-mm wire glass | 0.38 |
| Slate, 5 mm | 0.34 |
| Slate, 6 mm | 0.48 |
| Waterproofing membranes: | |
| Bituminous, gravel-covered | 0.26 |
| Bituminous, smooth surface | 0.07 |
| Liquid applied | 0.05 |
| Single-ply, sheet | 0.03 |
| Wood sheathing (per mm thickness) | |
| Plywood | 0.0057 |
| Oriented strand board | 0.0062 |
| Wood shingles | 0.14 |
| FLOOR FILL | |
| Cinder concrete, per mm | 0.017 |
| Lightweight concrete, per mm | 0.015 |
| Sand, per mm | 0.015 |
| Stone concrete, per mm | 0.023 |
| FLOORS AND FLOOR FINISHES | |
| Asphalt block (51 mm), 13-mm mortar | 1.44 |
| Cement finish (25 mm) on stone–concrete fill | 1.53 |
| Ceramic or quarry tile (19 mm) on 13-mm mortar bed | 0.77 |
| Ceramic or quarry tile (19 mm) on 25-mm mortar bed | 1.10 |



## Example 1:

A three-ply asphalt felt and gravel roof over 2-in-thick insulation board is supported by 18-in-deep precast reinforced concrete beams with 3-ft- wide flanges (see Figure 2.2). If the insulation weighs 3 lb/ft$^2$ and the asphalt roofing weighs $5\frac{1}{2}$ lb/ft$^2$, determine the total dead load, per foot of length, each beam must support.

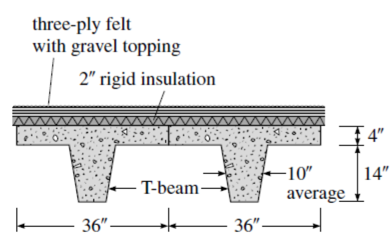



**Figure 2.2:** Cross section of reinforced concrete beams.

### Solution

Weight of beam is as follows:

Flange $\quad \frac{4}{12}\text{ ft} \times \frac{36}{12}\text{ ft} \times 1\text{ ft} \times 150\text{ lb/ft}^3 = 150\text{ lb/ft}$

Stem $\quad \frac{10}{12}\text{ ft} \times \frac{14}{12}\text{ ft} \times 1\text{ ft} \times 150\text{ lb/ft}^3 = 145\text{ lb/ft}$

Insulation $\quad 3\text{ lb/ft}^2 \times 3\text{ ft} \times 1\text{ ft} = 9\text{ lb/ft}$

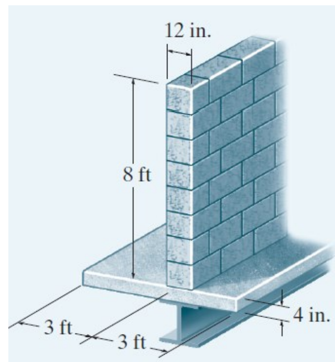
Roofing $\quad 5\frac{1}{2}\text{ lb/ft}^2 \times 3\text{ ft} \times 1\text{ ft} = 16.5\text{ lb/ft}$

$$\text{Total} = 320.5\text{ lb/ft,}$$

round to 0.321 kip/ft

## Example 2:

The floor beam in the figure used to support the 6-ft width of a lightweight plain concrete slab having a thickness of 4 in. The slab serves as a portion of the ceiling for the floor below, and therefore its bottom is coated with plaster. Furthermore, an 8-ft-high, 12-in.-thick lightweight solid concrete block wall is directly over the top flange of the beam. Determine the loading on the beam measured per foot of length of the beam.






## Solution:

**Concrete slab**: lightweight Plain concrete (Density=8 lb/ft$^2$ .in) (Table C3.1-1a -ASCE7-16 pp.426)

Load per foot= (8 lb/ft$^2$ .in. ) 4 in. *6 ft. = **192 lb/ft** .

**Plaster ceiling**: Plaster on tile or concrete (Density=5 lb/ft$^2$ ) (Table C3.1-1a -ASCE7-16 pp.426)

Load per foot= (5lb/ft$^2$ ) *6 ft. = **30 lb/ft**.

**Block wall:** Masonry, concrete Lightweight unit (Density=105 lb/ft$^3$ ) (Table C3.1-1a -ASCE7-16 pp.430)

Load per foot= (105 lb/ft$^3$ * 8ft.*1 ft.)=**840 lb/ft.**

**Total load**: (192+30+840= 1062 lb/ft.=1.06 K/ft).

## ❖ Live Loads

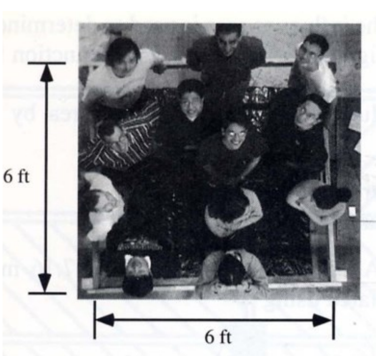
➢Loads that consist chiefly of occupancy loads in buildings. They may be either fully or partially in place or not present at all, and may also change in location.

➢Magnitude and distribution of live loads at any given time are uncertain, and even their maximum intensities throughout the lifetime of the structure are not known with precision.

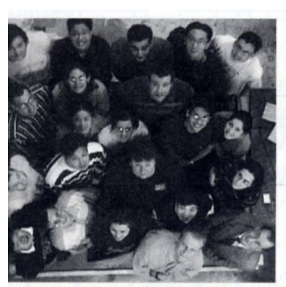
➢The minimum live loads for which the floors and roof of a building should be designed are usually specified in the building code.



## ❖ Live Loads




**50 psf (lb/ft²)**



**100 psf (lb/ft²)**

**150 psf (lb/ft²)**

Table 4.3-1 Minimum Uniformly Distributed Live Loads, $L_o$, and Minimum Concentrated Live Loads

| Occupancy or Use | Uniform, $L_o$ psf ($kN/m^2$) | Live Load Reduction Permitted? (Sec. No.) | Multiple-Story Live Load Reduction Permitted? (Sec. No.) | Concentrated lb (kN) | Also See Section |
|---|---|---|---|---|---|
| **Apartments (See Residential)** | | | | | |
| **Access floor systems** | | | | | |
| Office use | 50 (2.40) | Yes (4.7.2) | Yes (4.7.2) | 2,000 (8.90) | |
| Computer use | 100 (4.79) | Yes (4.7.2) | Yes (4.7.2) | 2,000 (8.90) | |
| **Armories and drill rooms** | 150 (7.18) | No (4.7.5) | No (4.7.5) | | |
| **Assembly areas** | | | | | |
| Fixed seats (fastened to floors) | 60 (2.87) | No (4.7.5) | No (4.7.5) | | |
| Lobbies | 100 (4.79) | No (4.7.5) | No (4.7.5) | | |
| Movable seats | 100 (4.79) | No (4.7.5) | No (4.7.5) | | |
| Platforms (assembly) | 100 (4.79) | No (4.7.5) | No (4.7.5) | | |
| Stage floors | 150 (7.18) | No (4.7.5) | No (4.7.5) | | |
| Reviewing stands, grandstands, and bleachers | 100 (4.79) | No (4.7.5) | No (4.7.5) | | 4.14 |
| Stadiums and arenas with fixed seats (fastened to the floor) | 60 (2.87) | No (4.7.5) | No (4.7.5) | | 4.14 |
| Other assembly areas | 100 (4.79) | No (4.7.5) | No (4.7.5) | | |
| **Balconies and decks** | 1.5 times the live load for the area served. Not required to exceed 100 psf (4.79 $kN/m^2$) | Yes (4.7.2) | Yes (4.7.2) | | |
| **Catwalks for maintenance access** | 40 (1.92) | Yes (4.7.2) | Yes (4.7.2) | 300 (1.33) | |
| **Corridors** | | | | | |
| First floor | 100 (4.79) | Yes (4.7.2) | Yes (4.7.2) | | |
| Other floors | Same as occupancy served except as indicated | | | | |
| **Dining rooms and restaurants** | 100 (4.79) | No (4.7.5) | No (4.7.5) | | |
| **Dwellings (See Residential)** | | | | | |
| **Elevator machine room grating** (on area of | | — | — | 300 (1.33) | |



## Live Loads Reduction

- For some types of buildings having very large floor areas, many codes will allow a **reduction** in the uniform live load for a floor.
- it is unlikely that the prescribed live load will occur simultaneously throughout the entire structure at any one time.
- ASCE 7-16 allows a reduction of live load on a member having an influence area ($K_{LL}$ $A_T$) of 400 ft$^2$ (37.2 m$^2$).

$$L = L_o\left(0.25 + \frac{15}{\sqrt{K_{LL}A_T}}\right) \quad \text{U.S. customary units}$$

$$L = L_o\left(0.25 + \frac{4.57}{\sqrt{K_{LL}A_T}}\right) \quad \text{SI units}$$

where
$L$ = reduced design live load per square foot or square meter of area supported by the member.
$L_o$ = unreduced design live load per square foot or square meter of area supported by the member.
$K_{LL}$ = live load element factor. For interior columns $K_{LL}$ = 4. ( Table 4.7-1 ASCE 7-16 pp.17)
$A_T$ = tributary area in square feet or square meters.*

## Table 4.7-1 Live Load Element Factor, $K_{LL}$

| Element | $K_{LL}$[a] |
|---|---|
| Interior columns | 4 |
| Exterior columns without cantilever slabs | 4 |
| Edge columns with cantilever slabs | 3 |
| Corner columns with cantilever slabs | 2 |
| Edge beams without cantilever slabs | 2 |
| Interior beams | 2 |
| All other members not identified, including | 1 |
| Edge beams with cantilever slabs | |
| Cantilever beams | |
| One-way slabs | |
| Two-way slabs | |
| Members without provisions for continuous shear transfer normal to their span | |

[a]In lieu of the preceding values, $K_{LL}$ is permitted to be calculated.



- ASCE 7-16 allows live reduction:

$K_{LL} * A_T \geq$ 400 ft$^2$ (37.2 m$^2$ )

$$L \geq \begin{cases} 50\% L_0 & \text{for members supporting one floor.} \\ 40\% L_0 & \text{for members supporting more than one floor.} \end{cases}$$

No reduction is allowed:

*if* $L_0 \geq 100$ lb/ft$^2$ (24.79 kN/m$^2$)

*or*

*if* The structures used for public assembly, garages, or roofs

The minimum uniformly distributed roof live loads are permitted to be reduced by ASCE standard as follows:

$$L_r = L_o R_1 R_2 \tag{2.2}$$

where $L_o$ = design roof live load

$L_r$ = reduced roof live load, with minimum of 12 psf $\leq L_r \leq$ 20 psf (0.58 m$^2 \leq L_r \leq$ 0.96 m$^2$ in SI units) for ordinary flat, pitched, and curved roofs

$R_1 = 1$ for $A_T \leq 200$ ft$^2$ (18.58 m$^2$); and $R_1 = 0.6$ for $A_T \geq 600$ ft$^2$ (55.74 m$^2$); $R_1 = 1.2 - 0.001A_T$ ($R_1 = 1.2 - 0.011A_T$ in SI units) for 200 ft$^2 < A_T <$ 600 ft$^2$ (18.58 m$^2 < A_T <$ 55.74m$^2$)

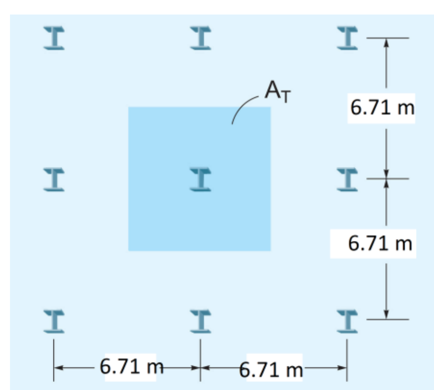
$R_2 = 1.0$ for flat roofs $F \leq 4$; $R_2 = 1.2 - 0.05F$ for $4 < F < 12$; and $R_2 = 0.6$ for $F \geq 12$; where $F$ = number of inches of rise per foot of roof slope for pitched roofs in SI: $F = 0.12 \times$ slope, with slope expressed in percentage)

For a column or beam supporting more than one floor, the term $A_T$ represents the sum of the tributary areas from all floors.

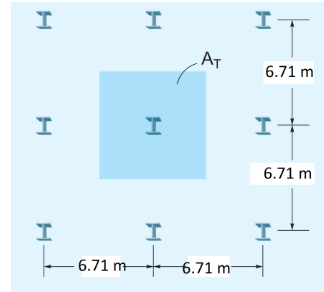


## Example 3:

A two-story **office building** shown in the photo has interior columns that are spaced 6.71 m apart in two perpendicular directions. If the (flat) roof loading is 0.96 kN/m². Determine the reduced live load supported by a typical interior column located at ground level.



The interior column has a tributary area or effective loaded area of
$A_T$= 6.71x6.71=45 $m^2$.
A ground floor column supports a roof live load of
$F_R$= 0.96 KN/$m^2$ x 45 $m^2$ = 43.2 kN
This load **cannot** be reduced since it is a roof load.




For the first floor, the Live Load is taken from table (4.3-1-ASCE 7-16 pp.14): $L_0$=2.4 kN/$m^2$

$L_0$=2.4 kN/$m^2$ < 24.79 KN/$m^2$ (OK.)

From table 4.7-1 4.3-1-ASCE 7-16 pp.14: $K_{LL}$=4

$K_{LL}$*$A_T$>=4*45=180$m^2$ > 37.2 $m^2$ (OK.)

$$L = L_0\left(0.25+\frac{4.57}{\sqrt{K_{LL}A_T}}\right) = 2.4\left(0.25+\frac{4.57}{\sqrt{4\times 45}}\right) = 2.4\times 0.59 = 1.42 \text{ kN/m}^2$$

$$\text{The load reduction here is } \frac{1.42}{2.4}\times 100\% = 59.1\% > 50\% \text{ (OK.)}$$

$$\text{The Floor load } F_F = 1.42\times 45 = 63.9\ kN$$

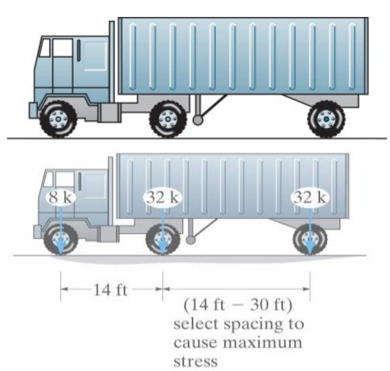
$$\text{The Roof load } F_R = 0.96\times 45 = 43.2\ kN \text{ (No Reduction in Roof load)}$$

$$F = F_R + F_F = 43.2 + 63.9 = 107.1 kN$$



## ◘ Highway Bridge loads

- Primary live loads are those due to traffic
- Specifications for truck loadings are reported in AASHTO *(American Association of State Highway and Transportation Officials)*
- For 2-axle truck, these loads are designated with H followed by the weight of truck in tons and another no. gives the year of the specifications that the load was reported.




HS 20–44 loading

## Railway Bridge loads

- Loadings are specified in AREMA
- A modern train having a 320kN (72k) loading on the driving axle of the engine is designated as an E-72 loading.

01_011





## Impact loads

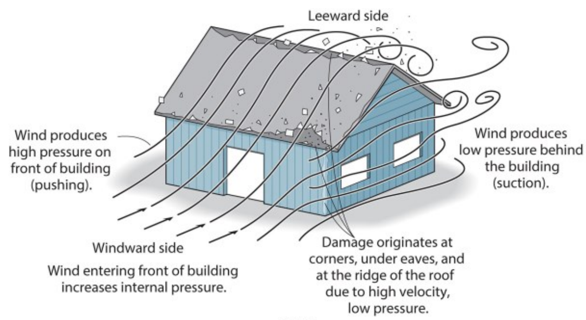
- Due to moving vehicles
- The % increase of the live loads due to impact is called the impact factor, I

$$I = \frac{50}{L+125} < 0.3 \text{ (US Units)}$$

$$I = \frac{15.24}{L+38.1} < 0.3 \text{ (SI Units)}$$

$L$ = length of the span in (ft., m) that is subjected to the live load

- **Wind loads**
  - Kinetic energy of the wind is converted into potential energy of pressure when structures block the flow of wind
  - Effects of wind depends on density & flow of air, angle of incidence, shape & stiffness of the structure & roughness of surface
  - For design, wind loadings can be treated as static or dynamic approach



01_012




Shear walls


Knee bracing
01_PH010



Diagonal bracing
01_PH011



Cross bracing
01_PH009

This pressure $q$ is defined by the air's kinetic energy per unit volume, $\boldsymbol{q = \frac{1}{2}\rho V2}$, where ρ is the density of the air and $V$ is its velocity.

According to the ASCE 7-16 Standard, this equation is modified to account for the **structure's height**, and **the terrain in which it is located**. Also the **importance of the structure** is considered, as it relates to the risk to human life or the public welfare if it is damaged or loses its functionality.

This modified equation is represented by the following equation

$$q_z = 0.00256 K_z K_{zt} K_d V^2 \ (\text{lb/ft}^2)$$
$$q_z = 0.613 K_z K_{zt} K_d V^2 \ (\text{N/m}^2)$$



where

$V$ = the velocity in mi/h (m/s) of a 3-second gust of wind measured 33 ft (10 m) above the ground. Specific values depend upon the "category" of the structure obtained from a specified wind map. For example, the interior portion of the continental United States reports a wind speed of 105 mi/h (47 m/s) if the structure is an agricultural or storage building, since it is of low risk to human life in the event of a failure. The wind speed is 120 mi/h (54 m/s) for cases where the structure is a hospital, since its failure would cause substantial loss of human life.

$K_z$ = the velocity pressure exposure coefficient, which is a function of height and depends upon the ground terrain. Table 1.5 lists values for a structure which is located in open terrain with scattered low-lying obstructions.

$K_{zt}$ = a factor that accounts for wind speed increases due to hills and escarpments. For flat ground $K_{zt} = 1.0$.

$K_d$ = a factor that accounts for the direction of the wind. It is used only when the structure is subjected to combinations of loads (see Sec. 1.4). For wind acting alone, $K_d = 1.0$.

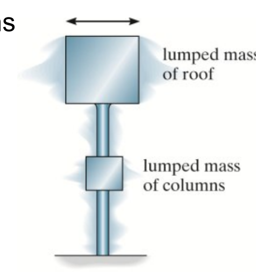
## Snow loads

- Design loadings depend on building's general shape & roof geometry, wind exposure, location and its importance
- Snow loads are determined from a zone map reporting 50-year recurrence interval



## Earthquake loads

- Earthquake produce loadings through its interaction with the ground & its response characteristics.
- Their magnitude depends on amount & type of ground acceleration, mass & stiffness of structure
- Top block is the lumped mass of the roof
- Middle block is the lumped stiffness of all the building's columns
- During earthquake, the ground vibrates both horizontally & vertically



- The horizontal accelerations create shear forces in the column that put the block in sequential motion with the ground.
- If the column is stiff & the block has a small mass, the period of vibration of the block will be short, the block will accelerate with the same motion as the ground & undergo slight relative displacements
- If the column is very flexible & the block has a large mass, induced motion will cause small accelerations of the block & large relative displacement



## Hydrostatic & Soil Pressure

- The pressure developed by these loadings when the structures are used to retain water or soil or granular materials
- E.g. tanks, dams, ships, bulkheads & retaining walls

**Other natural loads**

- Effect of blast
- Temperature changes
- Differential settlement of foundation

# Design Methods:

- **Allowable-stress design (ASD) method:**

(It was called Working Stress Design method prior 2005) includes both the material and load uncertainties into a single factor of safety.

required strength ≤ allowable strength

$$R_a \le \frac{R_n}{\Omega}$$

Nominal Strength

Factor of Safety

Required Strength (ASD)



- The many types of loads discussed previously can occur simultaneously on a structure, but it is very unlikely that the maximum of all these loads will occur at the same time. For example, both maximum wind and earthquake loads normally do not act simultaneously on a structure.
- For allowable-stress design the computed elastic stress in the material must not exceed the allowable stress for each of various load combinations. **Load combinations specified by the ASCE 7-016 Standard.**

1. $D$
2. $D + L$
3. $D +$ ($L_r$ or $S$ or $R$)
4. $D + 0.75L + 0.75$($L_r$ or $S$ or $R$)
5. $D + (0.6W)$
6. $D + 0.75L + 0.75(0.6W) + 0.75$($L_r$ or $S$ or $R$)
7. $0.6D + 0.6W$

- **Ultimate Strength Design method:**

Ultimate strength design is based on designing the ultimate strength of critical sections.

required strength ≤ design strength

Load factor $\boxed{> 1}$

$$R_u = \sum \gamma_i Q_{ni} \le \phi R_n$$

Required Strength

Load

Resistance Factor $\boxed{< 1}$

Nominal Strength/ Resistance



- This method uses load factors to the loads or combination of loads.

1. $1.4D$
2. $1.2D + 1.6L + 0.5(L_r \text{ or } S \text{ or } R)$
3. $1.2D + 1.6(L_r \text{ or } S \text{ or } R)+(L \text{ or } 0.5W)$
4. $1.2D + 1.0W + L + 0.5(L_r \text{ or } S \text{ or } R)$
5. $0.9D + 1.0W$
6. $1.2D + E_v + E_h + L + 0.2S$
7. $0.9D - E_v + E_h$

The Hashemite University
Faculty of Engineering
Department of Civil Engineering

# CE 315: Structural Analysis

## Analysis of Statically Determinate Structures

Dr. Ra'ed Al-Mazaidh

## Idealized Structure

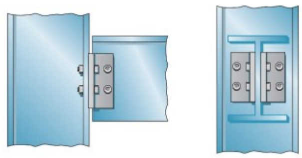
- In real sense exact analysis of a structure can never be carried out.
- Estimates have always to be made of the loadings and strength of materials.
- Furthermore, points of application for the loadings must be estimated.
- **Models or idealization should be made**.

## Support Connections

Structural members are joined together in various ways depending on the intent of the designer

**The three types of joints most often specified are:**

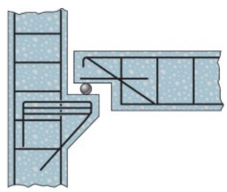
- **Pin connection**



typical "pin-supported" connection (metal)
(a)

**A pin-connected joint and a roller support allow some freedom for slight rotation**

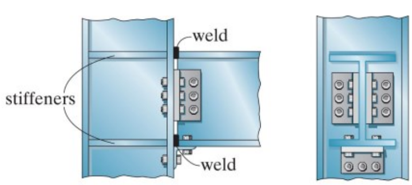
- **Roller support**,



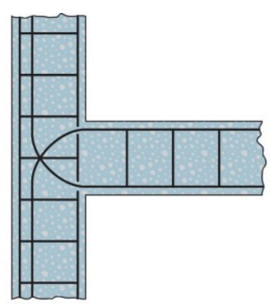
typical "roller-supported" connection (concrete)
(a)



- **Fixed joint**




typical "fixed-supported" connection (metal)
(b)



typical "fixed-supported" connection (concrete)
(b)
02_002b

**Fixed joint allows no relative rotation between the connected members. (Expensive)**

Idealized models are used in structural analysis to represent:

**1.**

pin support pin-connected joint

(a)

02_003a



**2.**

fixed support fixed-connected joint

(b)

02_003b





**3.**

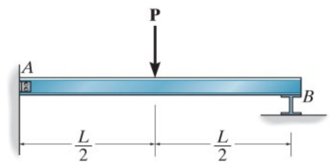

torsional spring support torsional spring joint

(c)

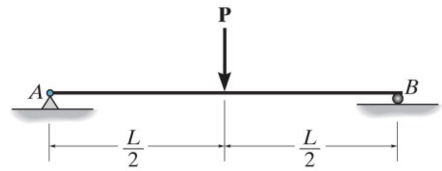
02_003c



**k = 0, the joint is a pin** **k = ∞, the joint is fixed.**

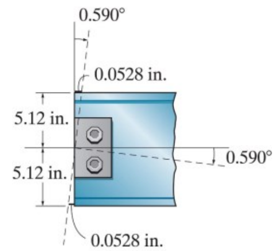



actual beam
(a)




idealized beam
(b)

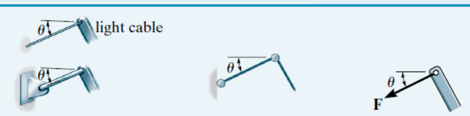
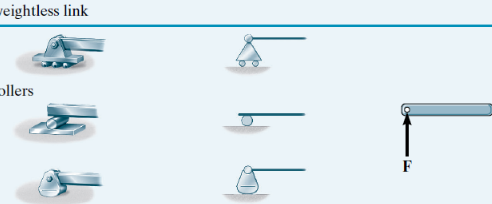
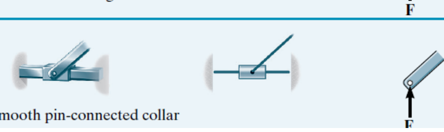
When selecting a particular model for each support or joint, <u>the engineer must be aware of how the assumptions will affect the actual performance of the member and whether the assumptions are reasonable for the structural design.</u>

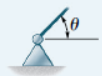
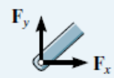
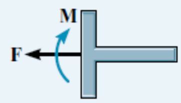
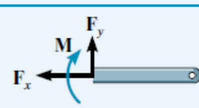



(c)

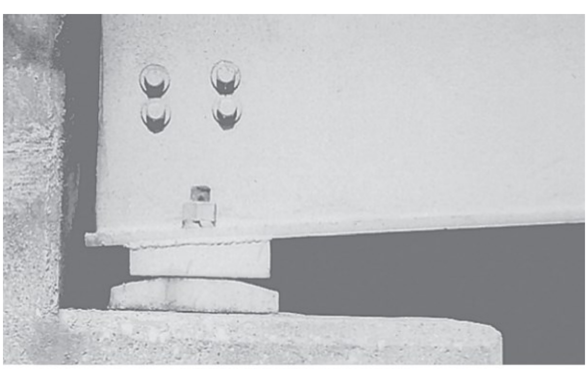


**TABLE 2.1 Supports for Coplanar Structures**

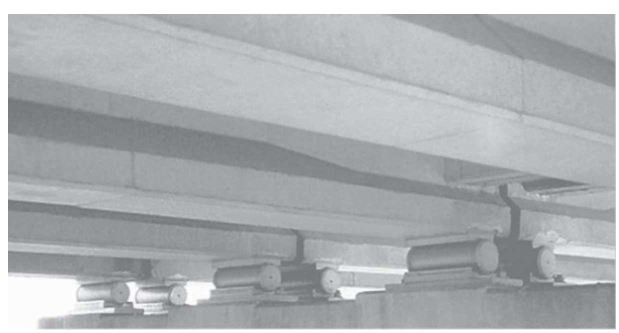
| Type of Connection | Idealized Symbol | Reaction | Number of Unknowns |
|---|---|---|---|
| (1) light cable; weightless link |  | | One unknown. The reaction is a force that acts in the direction of the cable or link. |
| (2) rollers; rocker |  | | One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact. |
| (3) smooth contacting surface |  | | One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact. |
| (4) smooth pin-connected collar |  | | One unknown. The reaction is a force that acts perpendicular to the surface at the point of contact. |

| | | | |
|---|---|---|---|
| (5)  smooth pin or hinge |  |  | Two unknowns. The reactions are two force components. |
| (6)  slider  fixed-connected collar |  |  | Two unknowns. The reactions are a force and a moment. |
| (7)  fixed support |  |  | Three unknowns. The reactions are the moment and the two force components. |

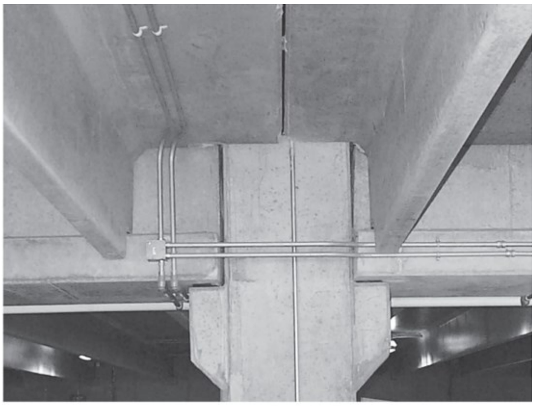




A typical rocker support used for a bridge girder.



Rollers and associated bearing pads are used to support the prestressed concrete girders of a highway bridge.



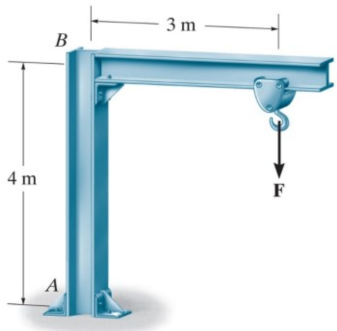
Concrete smooth or "roller" support.

The short link is used to connect the two girders of the highway bridge and allow for thermal expansion of the deck.



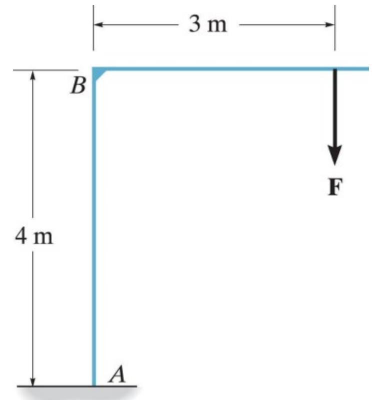
Steel pin support.

## Idealized Structure




actual structure

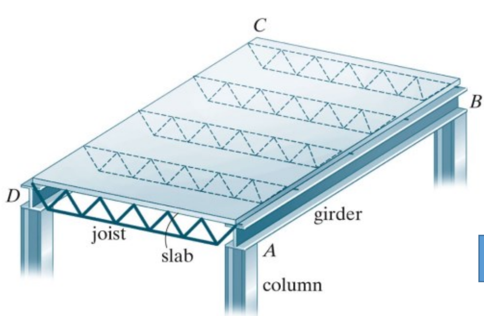
(a)



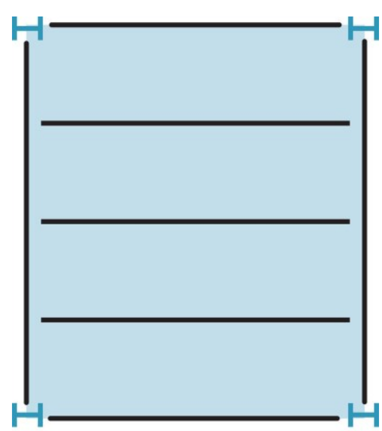

idealized structure

(b)



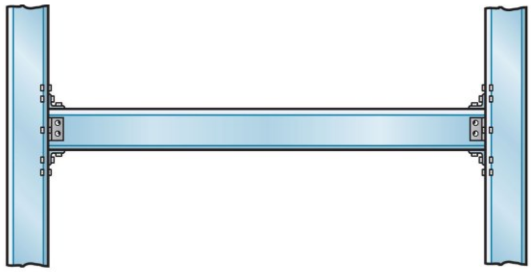



(a)



idealized framing plan

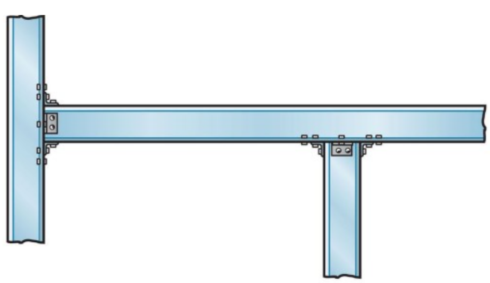
(b)



fixed-connected beam

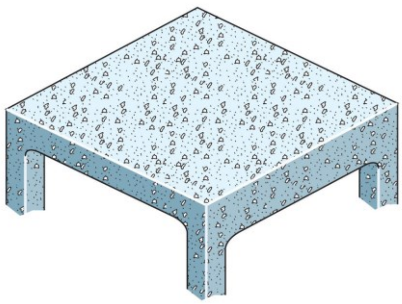
idealized beam



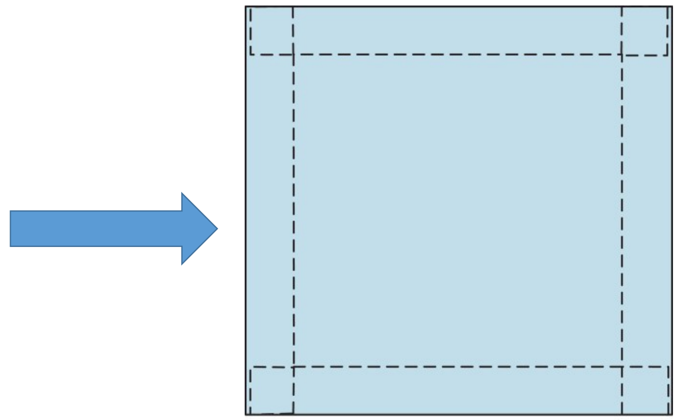


fixed-connected overhanging beam

idealized beam



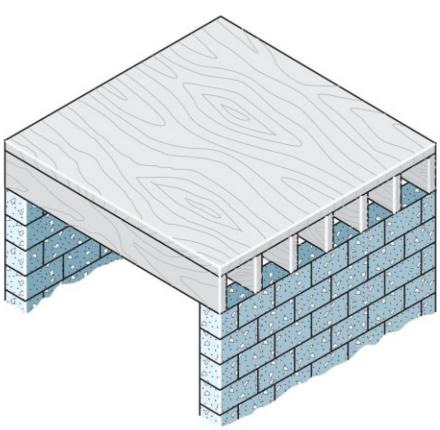
(a)



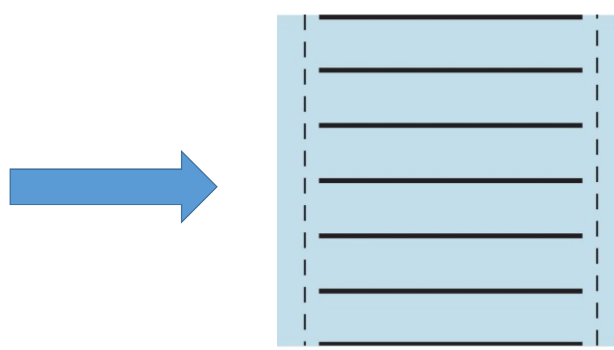
idealized framing plan

(b)





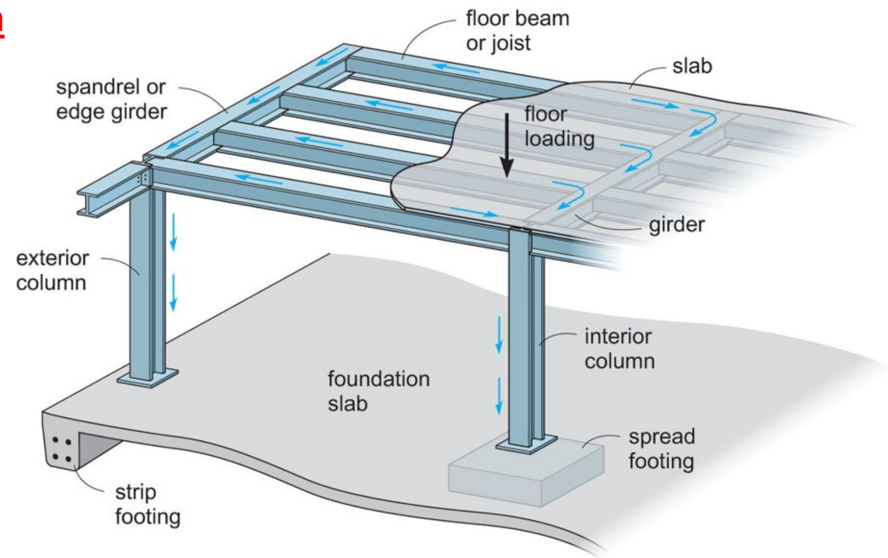
(a)



idealized framing plan

(b)

# Load Path




(a)

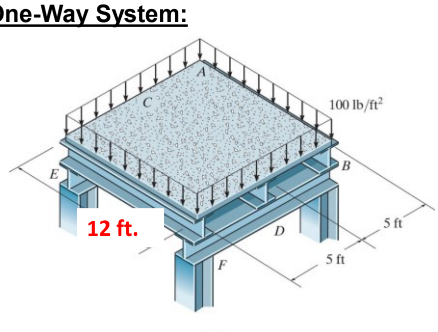


# Tributary Loadings:

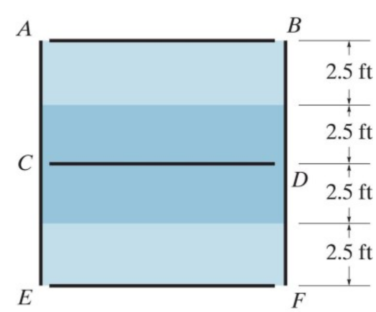
It is how the load on the surfaces of structural elements is transmitted to the other elements used for their support.

There are **two ways** depends on the geometry of the structural system, the material from which it is made, and the method of its construction.

- **One-Way System:**

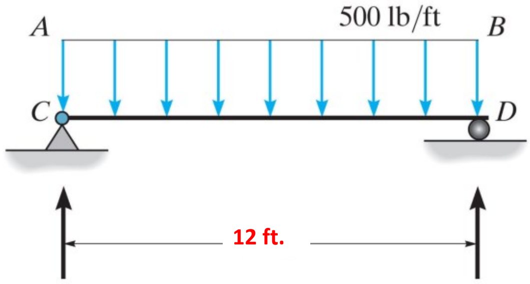



(a)

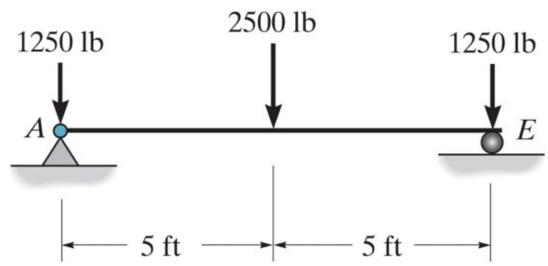



idealized framing plan

(b)



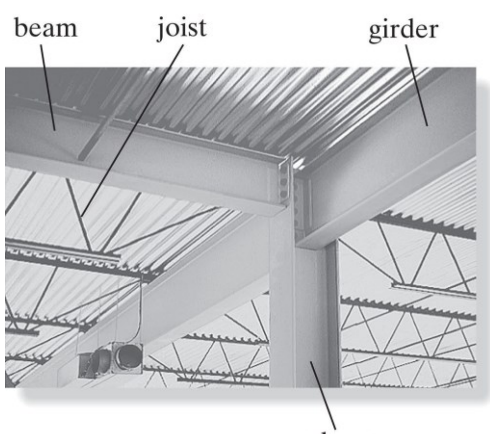

idealized beam
(c)



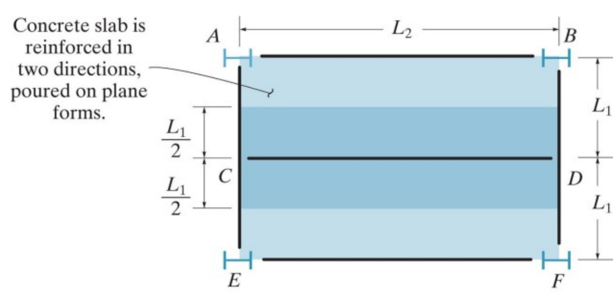

idealized girder
(d)

**ACI 318 code, if L2 > L1 and if the span ratio (L2/L1) > 2, the slab will behave as a one-way slab**






(a)

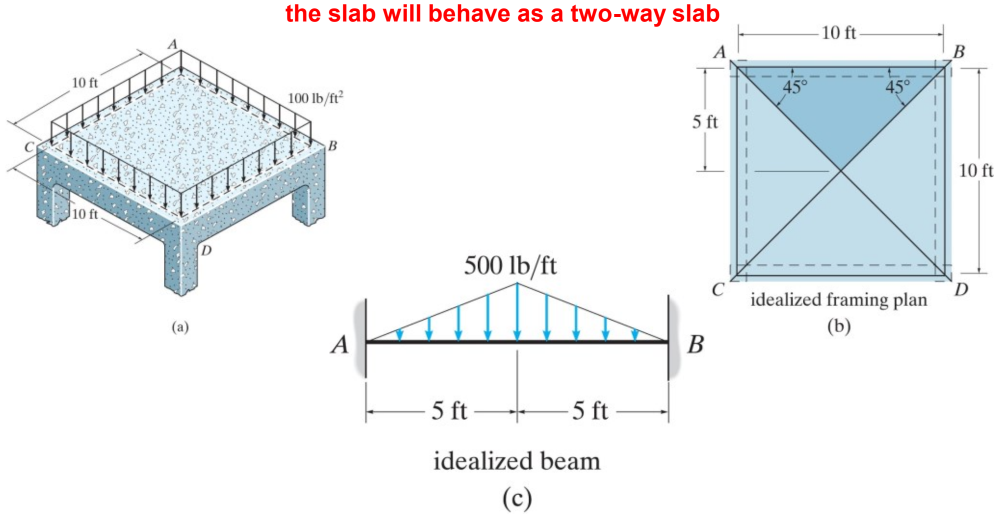



Idealized framing plan for one-way slab action requires $L_2/L_1 > 2$.

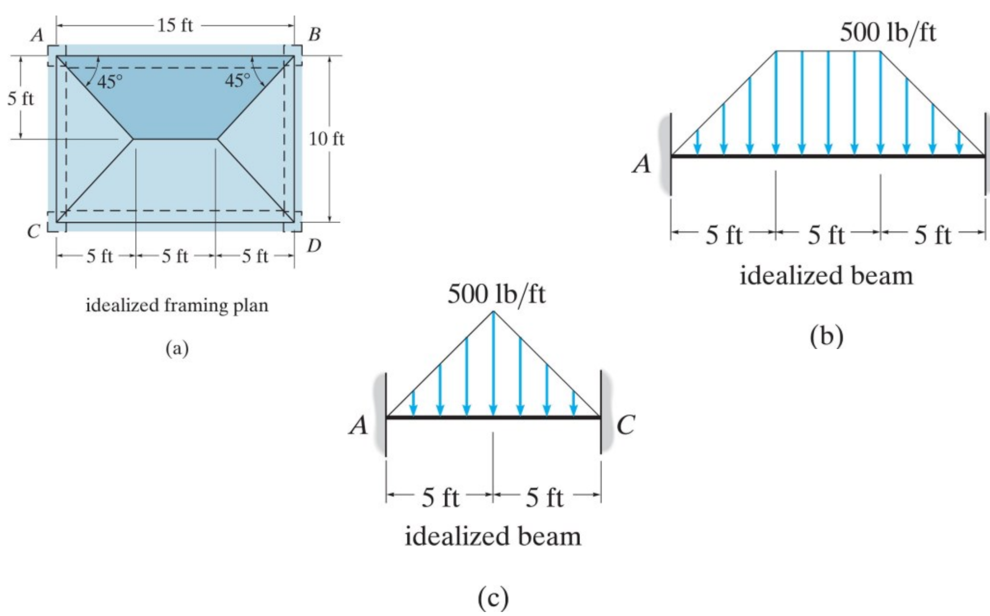
(b)

- **Two-Way System:**

**According to the ACI 318 concrete code, if L2 > L1 and the support ratio (L2 / L1) ≤ 2, the slab will behave as a two-way slab**





## EXAMPLE 1

The floor of a classroom is to be supported by the bar joists shown in the photo. Each joist is 15 ft long and they are spaced 2.5 ft apart. The floor itself is to be made from lightweight concrete that is 4 in. thick. Neglect the weight of the joists and the corrugated metal deck, and determine the load that acts along each joist.



## Solution:

The dead load on the floor is due to the weight of the concrete slab.

**Dead Load**: lightweight concrete slab (4 inch)

From ASCE7-16 Table C3.1-1 page 426:

| | |
|---|---|
| FLOOR FILL | |
| Cinder concrete, per inch | 9 |
| Lightweight concrete, per inch | 8 |
| Sand, per inch | 8 |
| Stone concrete, per inch | 12 |

**DL**= (8 lb/ft$^2$)X (4 in.)=**32 lb/ft$^2$.**

**Live Load:** From ASCE7-16 Table C3.1-1 page 15:

| | |
|---|---|
| **Schools** | |
| Classrooms | 40 (1.92) |
| Corridors above first floor | 80 (3.83) |
| First-floor corridors | 100 (4.79) |

**LL**= **40 lb/ft$^2$**

**Total Load**=DL+LL

= 32+40= **72 lb/ft²**

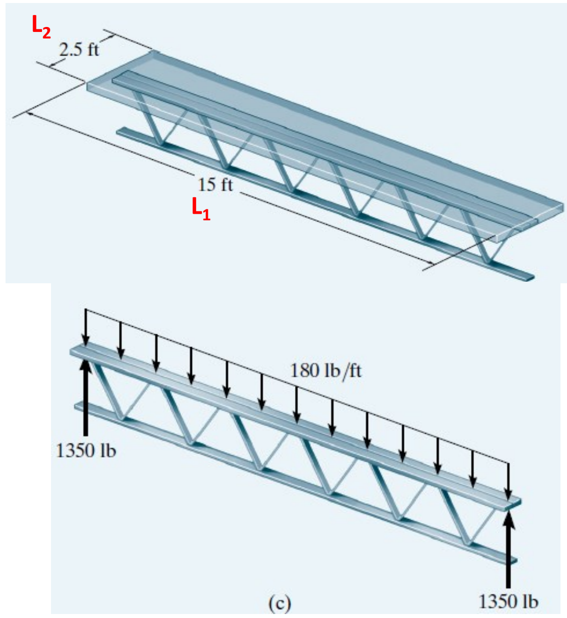
$L_2 = 15\ ft.$
$L_1 = 2.5\ ft.$

$$\rightarrow \frac{L_2}{L_1} = \frac{15}{2.5} = 6 > 2 \rightarrow \textbf{\textit{One way slab}}$$

The tributary area for each joist= 15x2.5= 37.5 ft²

Total Load on the tributary area= 72x37.5= 2700 lb

Load/ ft. on the joist= 2700/15 = **180 lb/ft.**

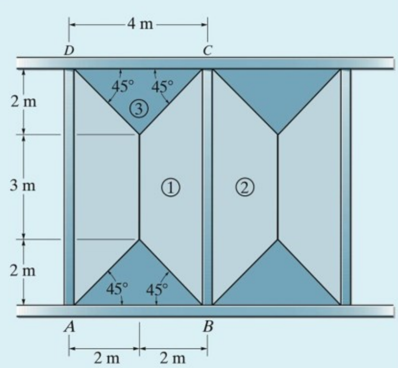


(c)



## EXAMPLE 2

The flat roof of the steel-frame building shown in the photo is intended to support a total load of 2 kN/m² over its surface. Determine the roof load within region *ABCD* that is transmitted to beams *BC* and *DC*. The dimensions are shown in Fig. 2–16*a*.

(a)

**Solution:**

$$L_2 = 7\ ft.$$
$$L_1 = 4\ ft.$$

$$\rightarrow \frac{L_2}{L_1} = \frac{7}{4} = 1.75 < 2 \rightarrow \textit{Two way slab}$$

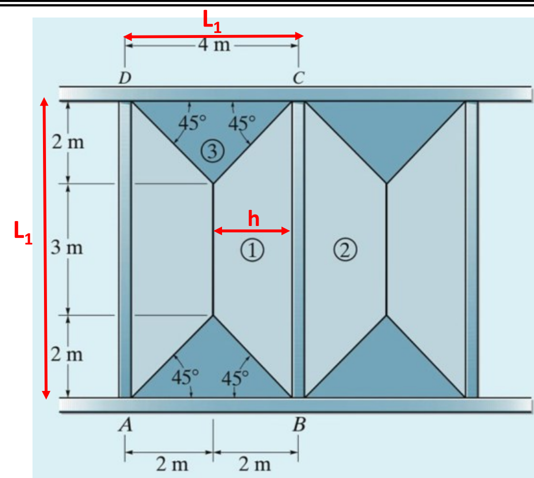
**Load on Beam BC** = Load of Tributary area (1)+ Load of Tributary area (2)

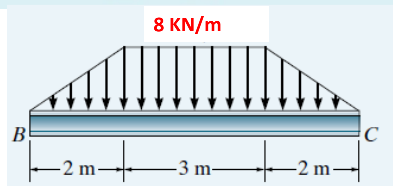
Tributary area (1)

The peak intensity of this loading=(Total load) X (h)
= 2 KN/m² X 2m= **4 KN/m**

Tributary area (2)= **4 KN/m** (Symmetric)

Total peak intensity loading= 4+4= **8 KN/m**

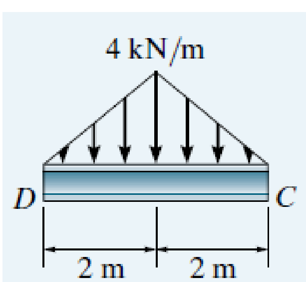


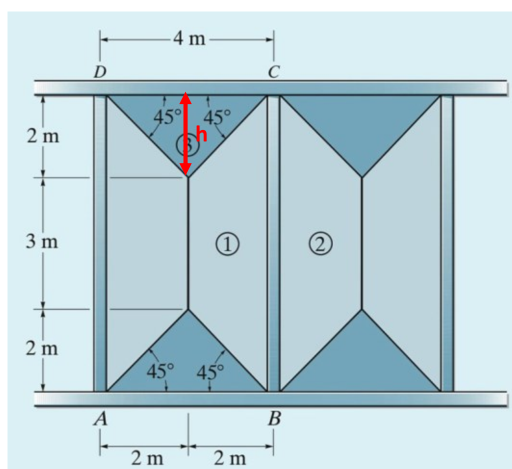






**Load on Beam DC** = Load of Tributary area (3)

The peak intensity of this loading=(Total load) X (h)
= 2 KN/m² X 2m= **4 KN/m**



## EXAMPLE 3

The concrete girders shown in the photo of the passenger car parking garage span 30 ft and are spaced 15 ft on center. If the floor slab is 5 in. thick and made of reinforced stone concrete, and the specified live load is 50 lb/ft$^2$, determine the distributed load the floor system transmits to each interior girder.



## Solution:

The dead load on the floor is due to the weight of the concrete slab.

**Dead Load**: reinforced stone concrete (5 inch)

From ASCE7-16 Table C3.1-2 page 430

| | | |
|---|---|---|
| Concrete, reinforced | | |
| Cinder | 111 | 17.4 |
| Slag | 138 | 21.7 |
| Stone (including gravel) | 150 | 23.6 |

**DL**= (150 lb/ft$^3$)X (5 in/12 in.)=**62.5 lb/ft.**

**Live Load:** 50 lb/ft$^2$ ( Given in the problem)

**LL= 50 lb/ft$^2$**

**Total Load**=DL+LL

= 62.5+50= **112.5 lb/ft²**

$L_2 = 30\ ft.$

$L_1 = 15\ ft.$

$\rightarrow \frac{L_2}{L_1} = \frac{30}{15} = 2 = 2 \rightarrow$ ***Two way slab***

**Load on Beam AB** = Load of Tributary area (1)+ Load of Tributary area (2)

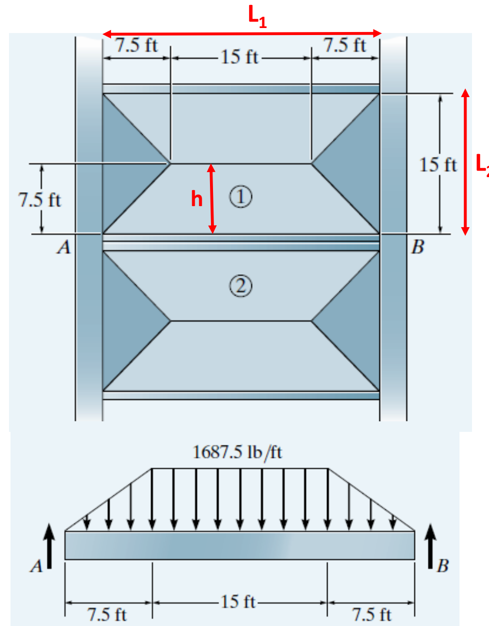
Tributary area (1)

The peak intensity of this loading=(Total load) X (h)

= 112.5 lb/ft.$^2$ X 7.5 ft.= **843.75 lb/ft.**

Tributary area (2)= **843.75 lb/ft.** (Symmetric)

Total peak intensity loading= 843.75+843.75= **1687.5 lb/ft.**






## Principle of Superposition

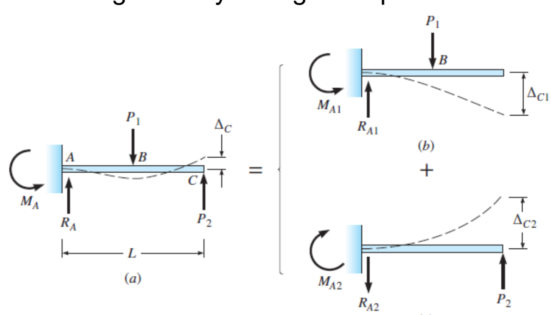
- The principle of superposition forms the basis for much of the theory of structural analysis. It may be stated as follows:

  ***"The total displacement or internal loadings (stress) at a point in a structure subjected to several external loadings can be determined by adding together the displacements or internal loadings (stress) caused by each of the external loads acting separately".***

- For this statement to be valid it is necessary that a linear relationship exist among the loads, stresses, and displacements.

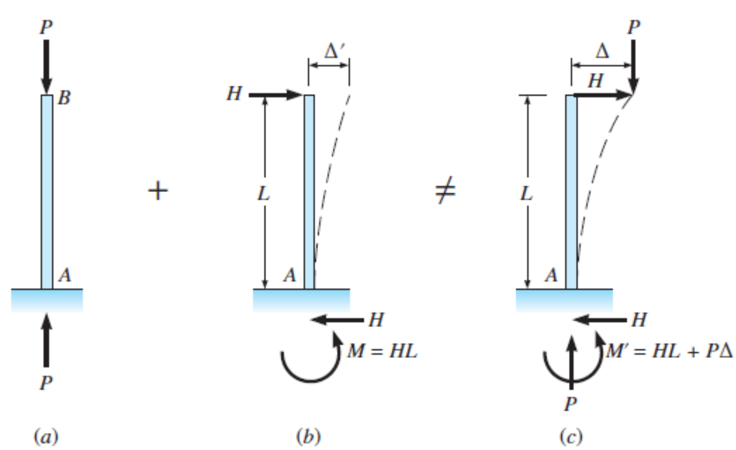
❖ **<u>TWO</u>** requirements for the principle to apply:

1. The material must behave in a linear-elastic manner, so that Hooke's law is valid, and therefore the load will be proportional to displacement.
2. The geometry of the structure must **not** undergo significant change when the loads are applied, i.e., small displacement theory applies. Large displacements will significantly change the position and orientation of the loads.




$$R_A = R_{A1} + R_{A2}$$
$$M_A = M_{A1} + M_{A2}$$
$$\Delta_C = \Delta_{C1} + \Delta_{C2}$$






Superposition not applicable:

## Equations of Equilibrium

- For equilibrium:

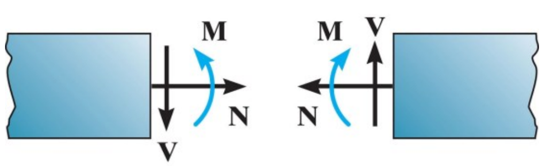
$$\sum F_x = 0 \qquad \sum F_y = 0 \qquad \sum F_z = 0$$
$$\sum M_x = 0 \qquad \sum M_y = 0 \qquad \sum M_z = 0$$

- The principal load-carrying portions of most structures, however, lie in a single plane, and since the loads are also coplanar, the above requirements for equilibrium reduce to

$$\sum F_x = 0$$
$$\sum F_y = 0$$
$$\sum M_o = 0$$



- If the internal loadings at a specified point in a member are to be determined, the method of sections must be used.




Internal loadings

## Determinacy

- Equilibrium equations provide both the necessary and sufficient conditions for equilibrium
- All forces can be determined strictly from these equations
- No. of unknown forces > equilibrium equations => **statically indeterminate**.
- This can be determined using a free body diagram.



❖ **For a coplanar structure**

$$r = 3n, \quad \underline{\text{statically determinate}} \quad \}$$
$$r > 3n, \quad \underline{\text{statically indeterminate}} \quad \}$$

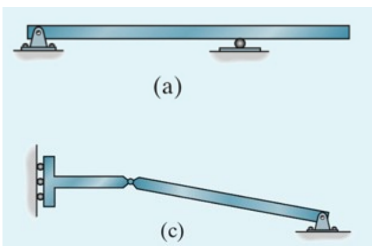
*r = number force and moment reaction components*

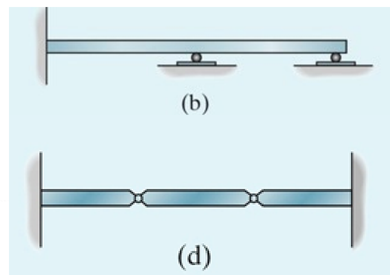
*n = number of parts*

❖ The additional equations needed to solve for the unknown equations are referred to as **compatibility equations**

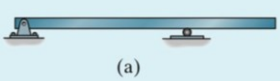
## EXAMPLE 4

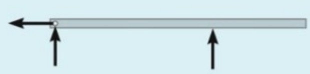
Classify each of the beams shown in Figs. 2–20*a* through 2–20*d* as statically determinate or statically indeterminate. If statically indeterminate, report the number of degrees of indeterminacy. The beams are subjected to external loadings that are assumed to be known and can act anywhere on the beams.



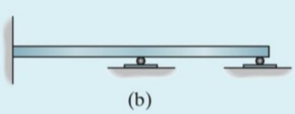





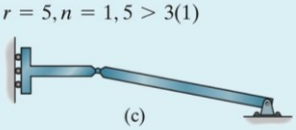



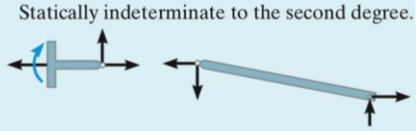



$r = 3, n = 1, 3 = 3(1)$ Statically determinate.

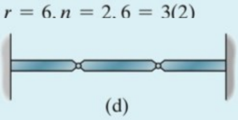


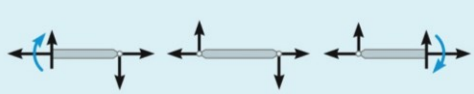

$r = 5, n = 1, 5 > 3(1)$ Statically indeterminate to the second degree.



$r = 6, n = 2, 6 = 3(2)$ Statically determinate.

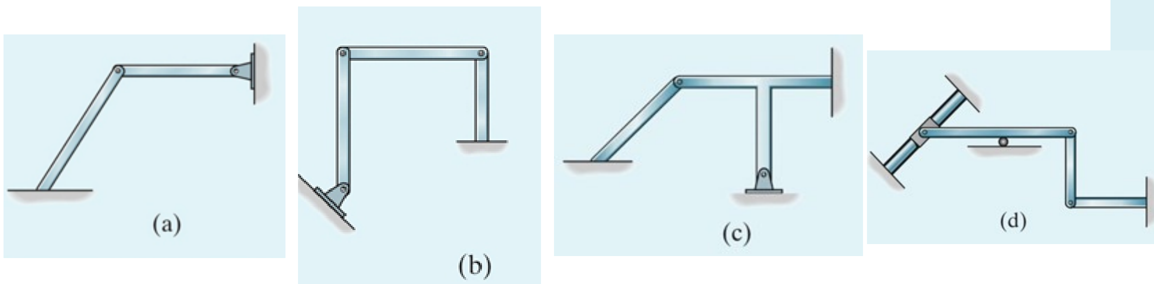


$r = 10, n = 3, 10 > 3(3)$ Statically indeterminate to the first degree. *Ans.*

## EXAMPLE 5

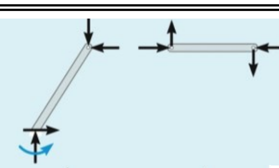
Classify each of the pin-connected structures shown in Figs. 2–21*a* through 2–21*d* as statically determinate or statically indeterminate. If statically indeterminate, report the number of degrees of indeterminacy. The structures are subjected to arbitrary external loadings that are assumed to be known and can act anywhere on the structures.





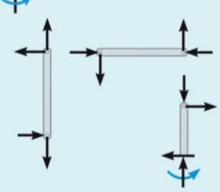

(a)

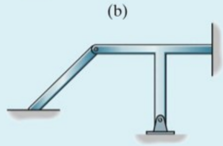
$r = 7, n = 2, 7 > 6$
Statically indeterminate to the first degree. *Ans.*



(b)

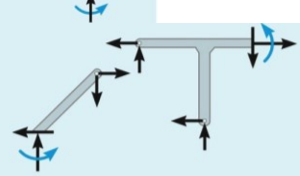
$r = 9, n = 3, 9 = 9$
Staticallv determinate. *Ans.*





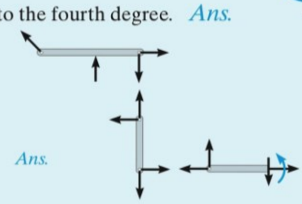
(c)

$r = 10, n = 2, 10 > 6$
Statically indeterminate to the fourth degree. *Ans.*



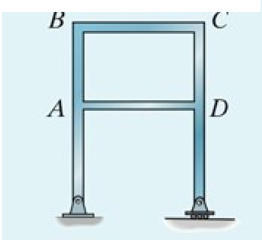
(d)

$r = 9, n = 3, 9 = 9$
Statically determinate. *Ans.*

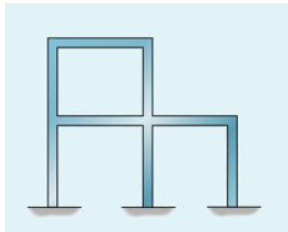


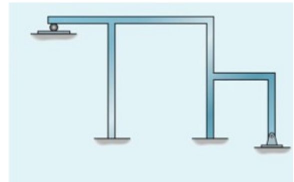
## EXAMPLE 6

Classify each of the frames shown in Figs. 2–22*a* through 2–22*c* as statically determinate or statically indeterminate. If statically indeterminate, report the number of degrees of indeterminacy. The frames are subjected to external loadings that are assumed to be known and can act anywhere on the frames.



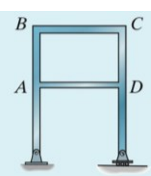
(a)


(b)

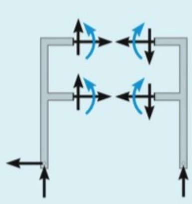

(This frame has no closed loops.)

(c)

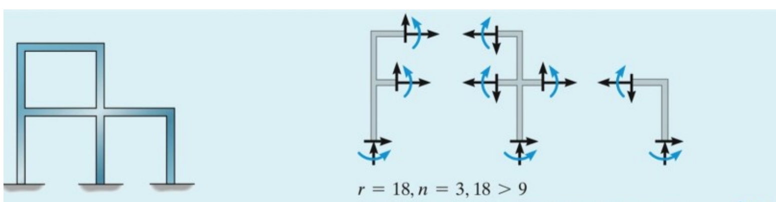




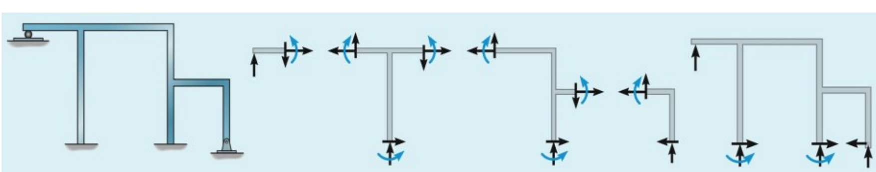
(a)



$r = 9, n = 2, 9 > 6$
Statically indeterminate to the third degree. *Ans.*


(b)

$r = 18, n = 3, 18 > 9$
Statically indeterminate to the ninth degree. *Ans.*


(This frame has no closed loops.)

(c)

$r = 18, n = 4, 18 > 12$
Statically indeterminate to the sixth degree. *Ans.*

$r = 9, n = 1, 9 > 3$
Statically indeterminate to the sixth degree. *Ans.*

# Stability

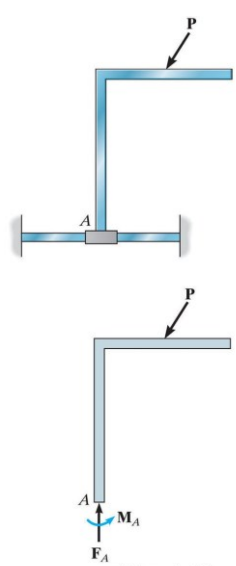
To ensure the equilibrium of a structure or its members, it is not only necessary to satisfy the equations of equilibrium, but the members must also be properly held or constrained by their supports regardless of how the structure is loaded.

- ❑ Two situations may occur where the conditions for proper constraint have not been met

**1. Partial Constraints**

Instability can occur if a structure or one of its members has *fewer* reactive forces than equations of equilibrium that must be satisfied



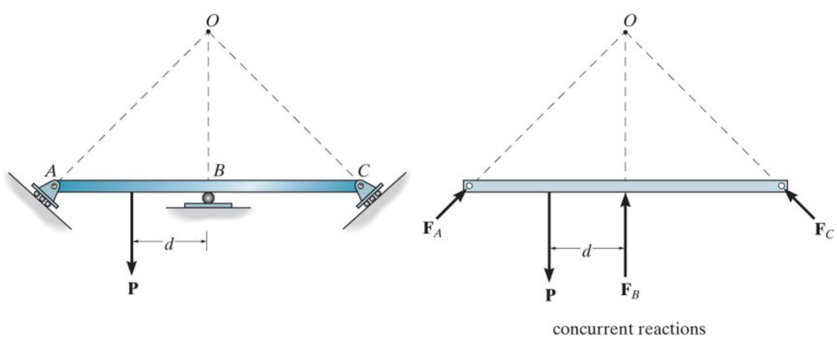



partial constraints

02_023

## 2. Improper Constraints

a) This can occur if all the support reactions are concurrent at a point.



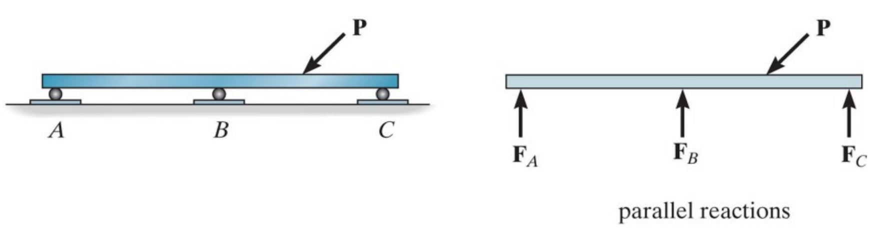

02_024



**The summation of moments about point *O* will *not* be equal to zero (*Pd* ≠0); thus rotation about point *O* will take place.**



b) Another way in which improper constraining leads to instability occurs when the **reactive forces are all parallel**.




02_025



**The summation of forces in the horizontal direction will not equal zero.**

| | |
|---|---|
| $r < 3n$ | unstable |
| $r \geq 3n$ | unstable if member reactions are concurrent or parallel or some of the components form a collapsible mechanism |



## EXAMPLE 7

Classify each of the structures shown in Figs. 2–26*a* through 2–26*d* as stable or unstable. The structures are subjected to arbitrary external loads that are assumed to be known.

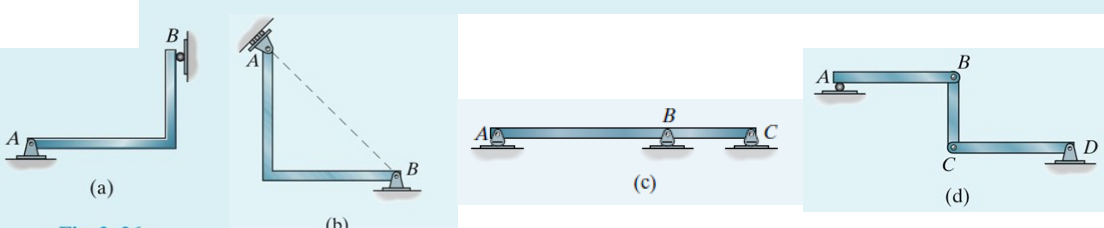
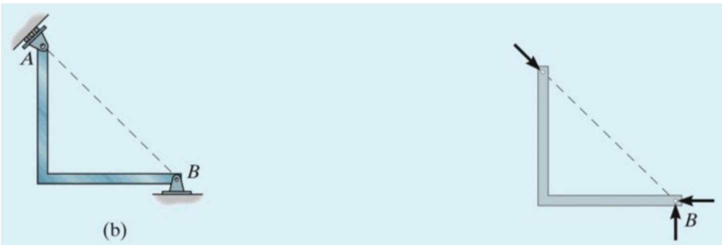


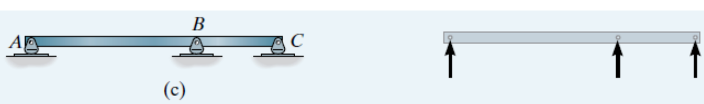

**Fig. 2–26**


The member is *stable* since the reactions are nonconcurrent and nonparallel. It is also statically determinate. *Ans.*

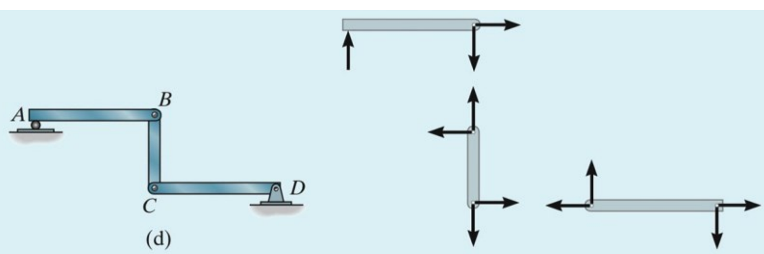



The member is *unstable* since the three reactions are concurrent at $B$.



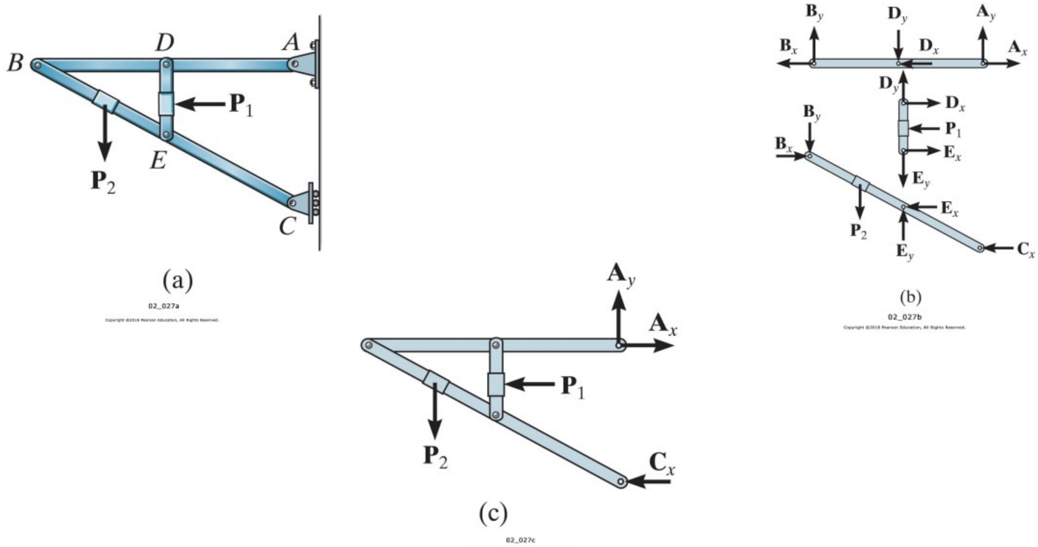



The beam is statically indeterminate, but *unstable* since the three reactions are all parallel. *Ans.*

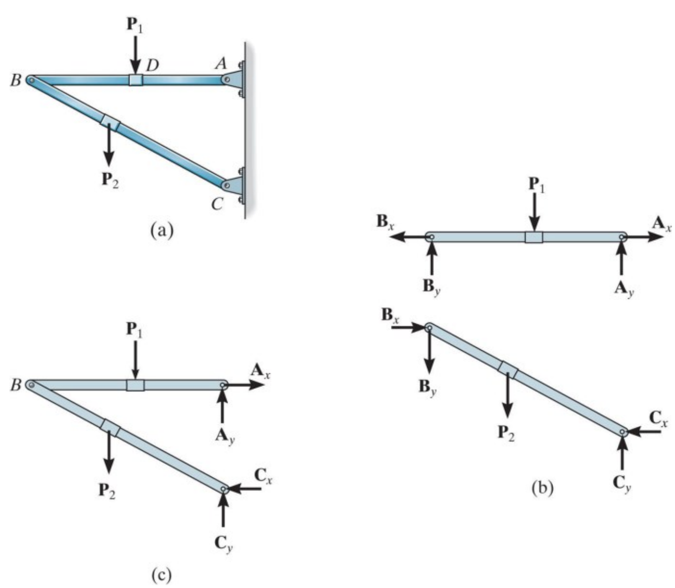



The structure is *unstable* since $r = 7$, $n = 3$, so that $r < 3n, 7 < 9$. Also, this can be seen by inspection, since $AB$ can move horizontally without restraint. *Ans.*

# Application of the Equations of Equilibrium



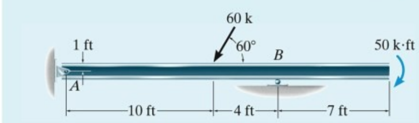


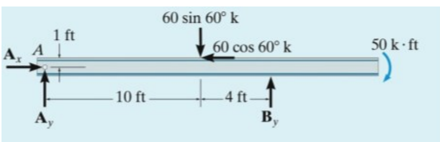

02_028

## EXAMPLE 8

Determine the reactions on the beam shown in Fig. 2–30*a*.




**Free-Body Diagram.**



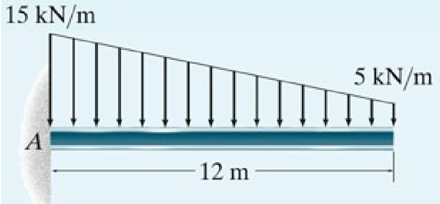

**Equations of Equilibrium.**

| | | | |
|---|---|---|---|
| $\overset{+}{\rightarrow}\Sigma F_x = 0;$ | $A_x - 60\cos 60° = 0$ | $A_x = 30.0$ k | *Ans.* |
| $\circlearrowleft + \Sigma M_A = 0;$ | $-60\sin 60°(10) + 60\cos 60°(1) + B_y(14) - 50 = 0$ | $B_y = 38.5$ k | *Ans.* |
| $+\uparrow \Sigma F_y = 0;$ | $-60\sin 60° + 38.5 + A_y = 0$ | $A_y = 13.4$ k | *Ans.* |



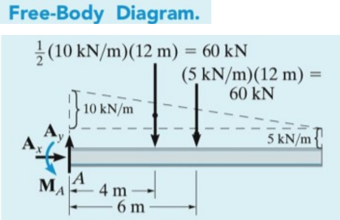
## EXAMPLE 9

Determine the reactions on the beam in Fig. 2–31*a*.




(a)

**Free-Body Diagram.**



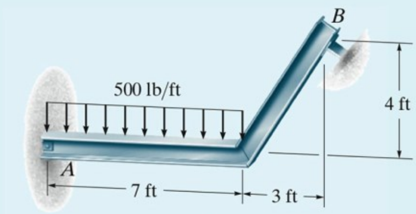

(b)

**Equations of Equilibrium.**

| | | | |
|---|---|---|---|
| $\overset{+}{\rightarrow}\Sigma F_x = 0;$ | $A_x = 0$ | | *Ans.* |
| $+\uparrow \Sigma F_y = 0;$ | $A_y - 60 - 60 = 0$ | $A_y = 120$ kN | *Ans.* |
| $\circlearrowleft + \Sigma M_A = 0;$ | $-60(4) - 60(6) + M_A = 0$ | $M_A = 600$ kN·m | *Ans.* |

## EXAMPLE 10

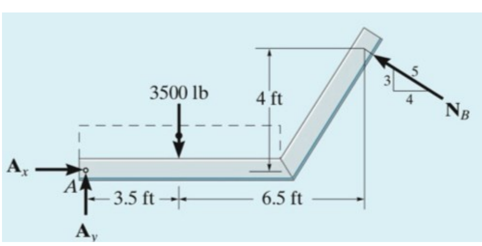
Determine the reactions on the beam in Fig. 2–32*a*. Assume $A$ is a pin and the support at $B$ is a roller (smooth surface).




(a)



**Free-Body Diagram.**




**Equations of Equilibrium.**

$$\circlearrowleft + \Sigma M_A = 0; \quad -3500(3.5) + \left(\tfrac{4}{5}\right)N_B(4) + \left(\tfrac{3}{5}\right)N_B(10) = 0 \qquad Ans.$$

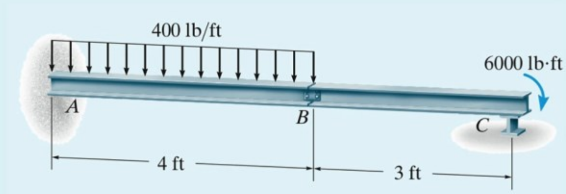
$$N_B = 1331.5 \text{ lb} = 1.33 \text{ k}$$

$$\xrightarrow{+} \Sigma F_x = 0; \quad A_x - \tfrac{4}{5}(1331.5) = 0 \qquad A_x = 1.07 \text{ k} \qquad Ans.$$

$$+\uparrow \Sigma F_y = 0; \quad A_y - 3500 + \tfrac{3}{5}(1331.5) = 0 \qquad A_y = 2.70 \text{ k} \qquad Ans.$$

## EXAMPLE 11

The compound beam in Fig. 2–33*a* is fixed at *A*. Determine the reactions at *A*, *B*, and *C*. Assume that the connection at *B* is a pin and *C* is a roller.



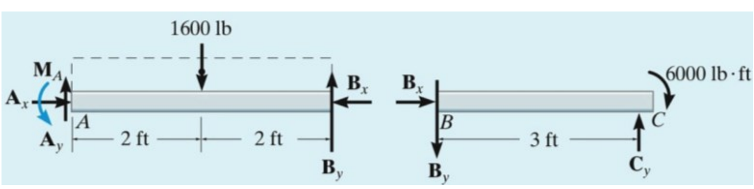

(a)

**Fig. 2–33**





### Free-Body Diagram.




### Equations of Equilibrium.

$\circlearrowleft + \Sigma M_C = 0;$ $\quad -6000 + B_y(3) = 0 \quad B_y = 2000\,\text{lb}$ *Ans.*

$+\uparrow \Sigma F_y = 0;$ $\quad -2000 + C_y = 0 \quad C_y = 2000\,\text{lb}$ *Ans.*

$\xrightarrow{+} \Sigma F_x = 0;$ $\quad B_x = 0$ *Ans.*

Segment *AB*:

$\circlearrowleft + \Sigma M_A = 0;$ $\quad M_A - 1600(2) + 2000(4) = 0$

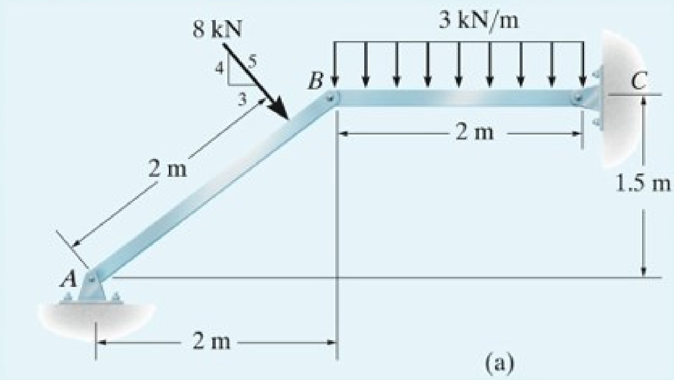
$M_A = -4.8\,\text{k}\cdot\text{ft}$ *Ans.*

$+\uparrow \Sigma F_y = 0;$ $\quad A_y - 1600 + 2000 = 0 \quad A_y = -400\,\text{lb}$ *Ans.*

$\xrightarrow{+} \Sigma F_x = 0;$ $\quad A_x - 0 = 0 \quad A_x = 0$ *Ans.*

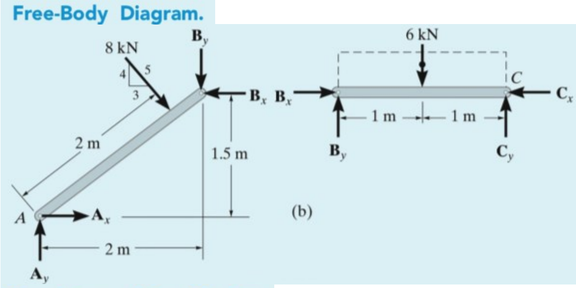
## EXAMPLE 12

Determine the horizontal and vertical components of reaction at the pins $A$, $B$, and $C$ of the two-member frame shown in Fig. 2–34*a*.






**Free-Body Diagram.**




**Equations of Equilibrium.**

Member $BC$:

$\circlearrowright + \Sigma M_C = 0; \quad -B_y(2) + 6(1) = 0 \qquad B_y = 3\text{ kN} \quad$ *Ans.*

Member $AB$:

$\circlearrowright + \Sigma M_A = 0; \quad -8(2) - 3(2) + B_x(1.5) = 0 \qquad B_x = 14.7\text{ kN} \quad$ *Ans.*

$\xrightarrow{+} \Sigma F_x = 0; \quad A_x + \frac{3}{5}(8) - 14.7 = 0 \qquad A_x = 9.87\text{ kN} \quad$ *Ans.*

$+\uparrow \Sigma F_y = 0; \quad A_y - \frac{4}{5}(8) - 3 = 0 \qquad A_y = 9.40\text{ kN} \quad$ *Ans.*

Member $BC$:

$\xrightarrow{+} \Sigma F_x = 0; \quad 14.7 - C_x = 0 \qquad C_x = 14.7\text{ kN} \quad$ *Ans.*

$+\uparrow \Sigma F_y = 0; \quad 3 - 6 + C_y = 0 \qquad C_y = 3\text{ kN} \quad$ *Ans.*

The Hashemite University
Faculty of Engineering
Department of Civil Engineering

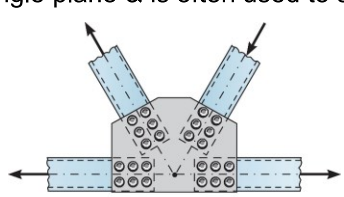
# CE 315: Structural Analysis

## Analysis of Statically Determinate Trusses

Dr. Ra'ed Al-Mazaidh

## Common Types of Trusses

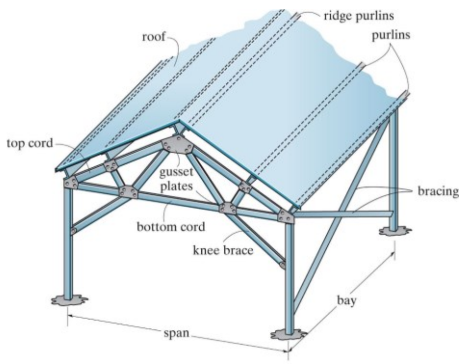
- A truss is one of the major types of engineering structures which provides a practical and economical solution for many engineering constructions, especially in the design of bridges and buildings that demand large spans.
- A truss is a structure composed of slender members joined together at their end points
- The joint connections are usually formed by bolting or welding the ends of the members to a common plate called **gusset plate**.
- Planar trusses lie in a single plane & is often used to support <u>roof or bridges</u>



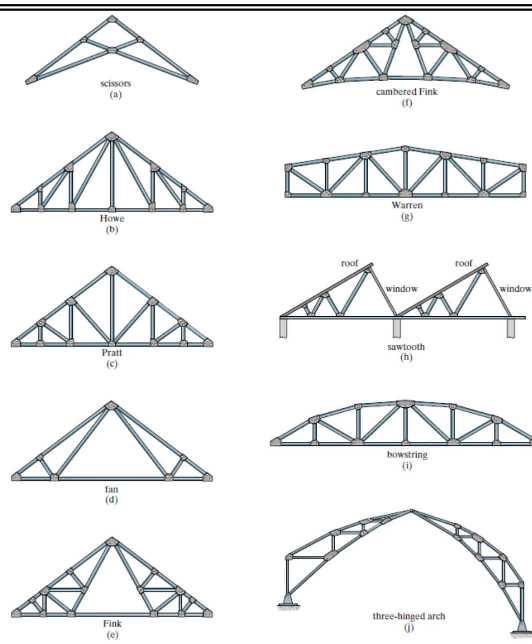
gusset plate

## Roof Trusses

- They are often used as part of an industrial building frame
- Roof load is transmitted to the truss at the joints by means of a series of purlins
- To keep the frame rigid & thereby capable of resisting horizontal wind forces, knee braces are sometimes used at the supporting column

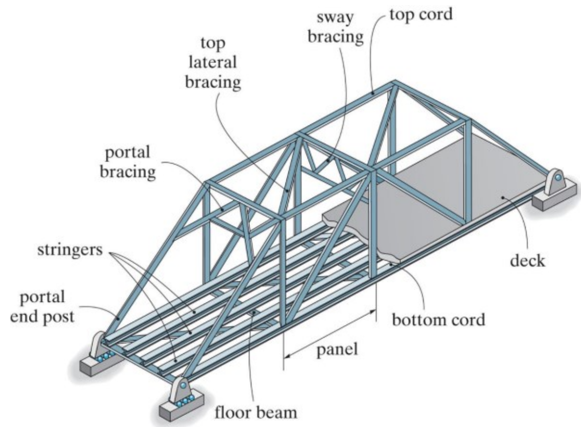




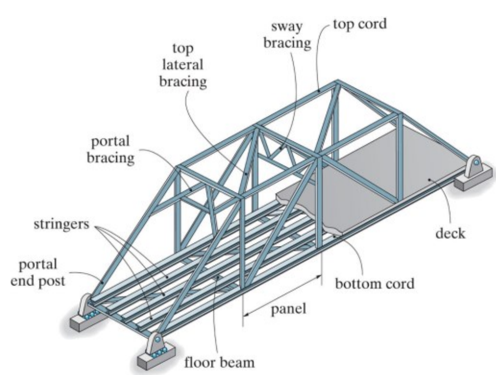
❖ **Bridge Trusses**

- The main structural elements of a typical bridge truss are shown in the figure below. Here it is seen that a load on the deck is first transmitted to stringers, then to floor beams, and finally to the joints of the two supporting side trusses.

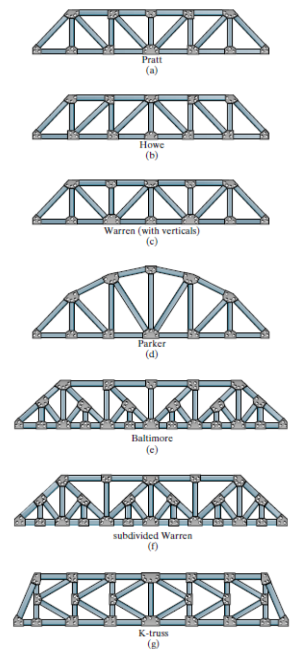





- The top and bottom cords of these side trusses are connected by top and bottom lateral bracing, which serves to resist the lateral forces caused by wind and the sidesway caused by moving vehicles on the bridge.
- Additional stability is provided by the portal and sway bracing. As in the case of many long-span trusses, a roller is provided at one end of a bridge truss to allow for thermal expansion



- In particular, the **Pratt**, **Howe**, and **Warren** trusses are normally used for spans up to 61 m in length. The most common form is the Warren truss with verticals.
- For larger spans, a truss with a polygonal upper cord, such as the **Parker** truss, is used for some savings in material.
- The Warren truss with verticals can also be fabricated in this manner for spans up to 91 m.

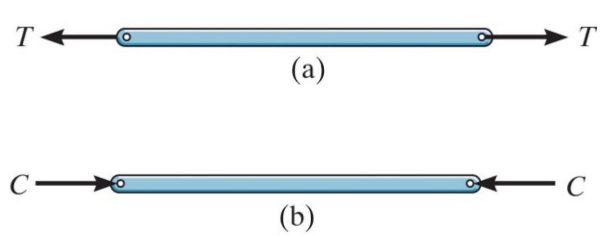





## ❖ <u>Assumptions for Design</u>

- The members are joined together by smooth pins
- All loadings are applied at the joints

**<u>Due to the 2 assumptions, each truss member acts as an axial force member</u>**

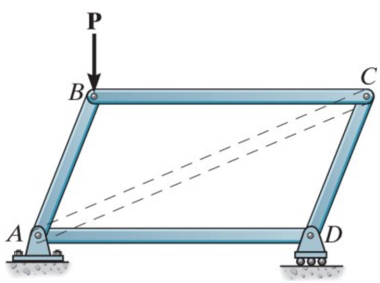


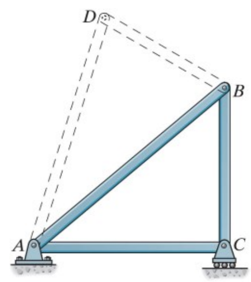
## Classification of Coplanar Trusses

- **Simple**
- **Compound or Complex Truss**

### Simple Truss

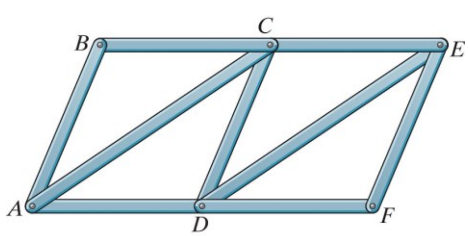
- To prevent collapse, the framework of a truss must be rigid
- The simplest framework that is rigid or stable is a triangle



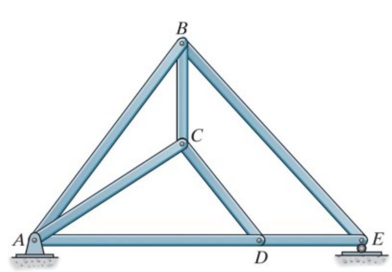






- The basic "stable" triangle element is ABC
- The remainder of the joints D, E & F are established in alphabetical sequence
- Simple trusses do **not** have to consist entirely of triangles



simple truss

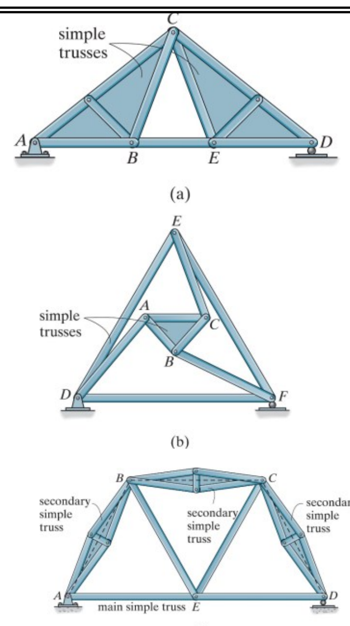


simple truss

- **Compound Truss**
  - It is formed by connecting 2 or more simple truss together.
  - Often, this type of truss is used to support loads acting over a larger span as it is cheaper to construct a lighter compound truss than a heavier simple truss



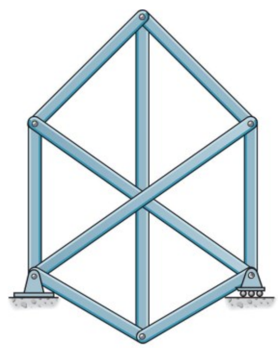
- Type 1
  - The trusses may be connected by a common joint & bar. Fig(a)
- Type 2
  - The trusses may be joined by 3 bars Fig(b)
- Type 3
  - The trusses may be joined where bars of a large simple truss, called the main truss, have been substituted by simple truss, called secondary trusses. Fig(c)



Various types of compound trusses

## Complex Truss

A complex truss is one that cannot be classified as being either simple or compound



Complex truss



## Determinacy

- The total number of unknowns includes the forces in **b** number of bars of the truss and the total number of external support reactions **r**.
- Since the truss members are all straight axial force members lying in the same plane, the force system acting at each joint is coplanar and concurrent.
- Consequently, rotational or moment equilibrium is automatically satisfied at the joint (or pin).

$$\boxed{\sum F_x = 0 \text{ and } \sum F_y = 0}$$

- By comparing the total unknowns with the total number of available equilibrium equations, we have:

$$\begin{array}{ll} b + r = 2j & \text{statically determinate} \\ b + r > 2j & \text{statically indeterminate} \end{array}$$

**Degree of indeterminacy**: $(b + r - 2j)$

**b:** The total number of bars of the truss.

**r :** The total number of external support reactions.

**j :** The total number of joints.

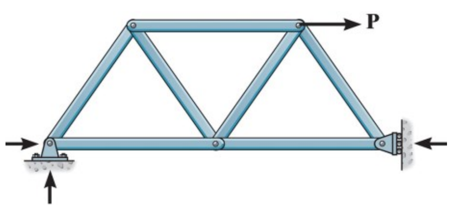


## Stability

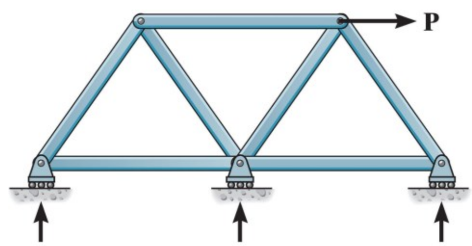
- If b + r < 2j => collapse (Unstable)
- A truss can be unstable if it is statically determinate or statically indeterminate
- Stability will have to be determined either through inspection or by force analysis

## External Stability

- A structure is externally unstable if all of its reactions are concurrent or parallel
- The trusses are externally unstable since the support reactions have lines of action that are either concurrent or parallel.




unstable—concurrent reactions

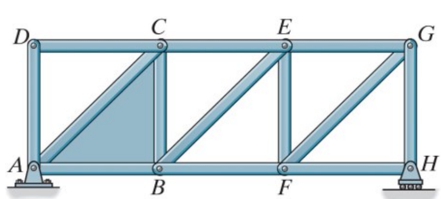


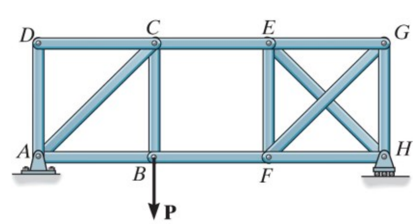

unstable—parallel reactions



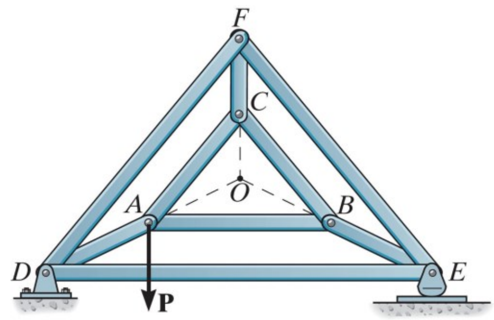
## Internal Stability

- The internal stability can be checked by careful inspection of the arrangement of its members
- If it can be determined that each joint is held fixed so that it cannot move in a "rigid body" sense with respect to the other joints, then the truss will be stable
- A simple truss will always be internally stable
- If a truss is constructed so that it does not hold its joints in a fixed position, it will be unstable or have a "critical form"



- To determine the internal stability of a compound truss, it is necessary to identify the way in which the simple truss are connected together
- The truss shown is unstable since the inner simple truss ABC is connected to DEF using 3 bars which are concurrent at point O. Thus an external load can be applied at A, B or C & cause the truss to rotate slightly.



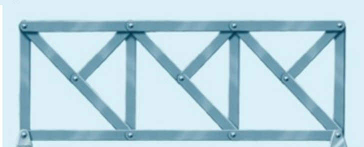



- For complex truss, it may not be possible to tell by inspection if it is stable
- The instability of any form of truss may also be noticed by using a computer to solve the 2j equations for the joints of the truss. If inconsistent results are obtained, the truss is unstable or have a critical form

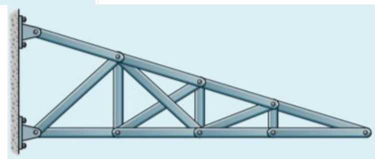
| | |
|---|---|
| $b + r < 2j$ | unstable |
| $b + r \geq 2j$ | unstable if truss support reactions are concurrent or parallel or if some of the components of the truss form a collapsible mechanism |

## Example 1

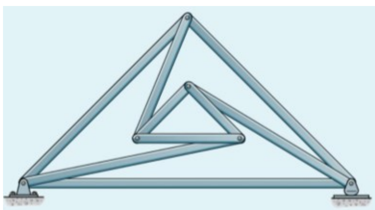
Classify each of the trusses in Fig. 3–18 as stable, unstable, statically determinate, or statically indeterminate. The trusses are subjected to arbitrary external loadings that are assumed to be known and can act anywhere on the trusses.



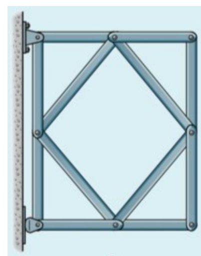
(a)



(b)



(c)



(d)



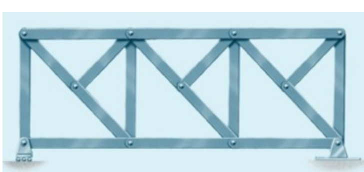
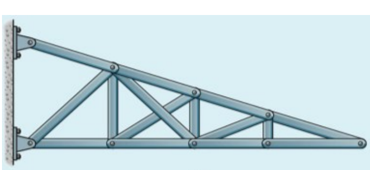


**Fig. 3–18*a*.** *Externally stable,* since the reactions are not concurrent or parallel. Since $b = 19$, $r = 3$, $j = 11$, then $b + r = 2j$ or $22 = 22$. Therefore, the truss is *statically determinate*. By inspection the truss is *internally stable*.



(b)

**Fig. 3–18*b*.** *Externally stable*. Since $b = 15$, $r = 4$, $j = 9$, then $b + r > 2j$ or $19 > 18$. The truss is *statically indeterminate* to the first degree. By inspection the truss is *internally stable*.

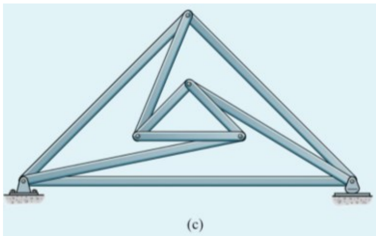


**Fig. 3–18c.** *Externally stable*. Since $b = 9$, $r = 3$, $j = 6$, then $b + r = 2j$ or $12 = 12$. The truss is *statically determinate*. By inspection the truss is *internally stable*.

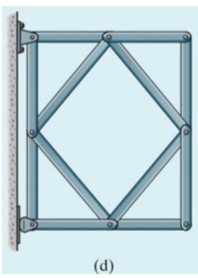


**Fig. 3–18d.** *Externally stable*. Since $b = 12$, $r = 3$, $j = 8$, then $b + r < 2j$ or $15 < 16$. The truss is *internally unstable*.

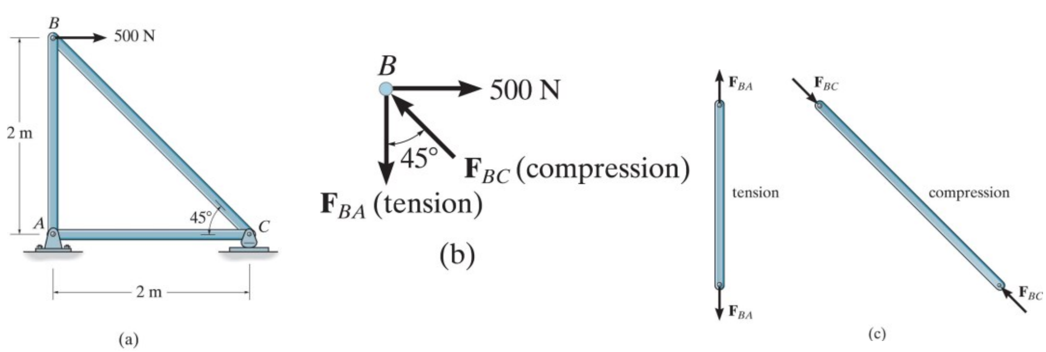


## Determination of the member forces

- **The Method of Joints**
- **The Method of Sections**

### The Method of Joints

- Satisfying the equilibrium equations for the forces exerted on the pin at each joint of the truss
- Applications of equations yields 2 algebraic equations that can be solved for the 2 unknowns

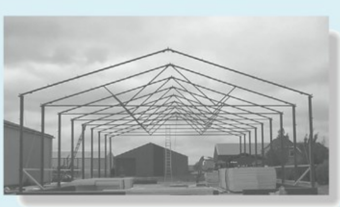


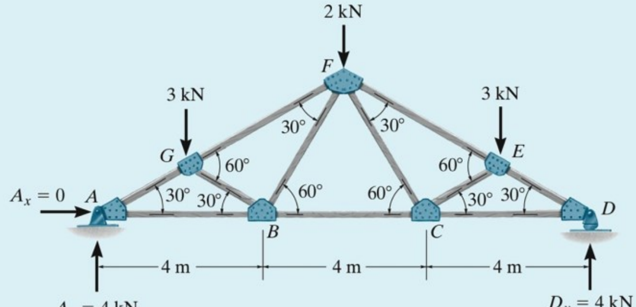
- Always assume the unknown member forces acting on the joint's free body diagram to be in tension
- Numerical solution of the equilibrium eqns will yield positive scalars for members in tension & negative for those in compression
- The correct sense of direction of an unknown member force can in many cases be determined by inspection
- A positive answer indicates that the sense is correct, whereas a negative answer indicates that the sense shown on the free-body diagram must be reversed.



## Example 2

Determine the force in each member of the roof truss shown in the photo. The dimensions and loadings are shown in Fig. 3–20*a*. State whether the members are in tension or compression. The reactions at the supports are given.



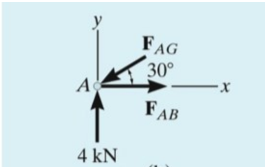


(a)

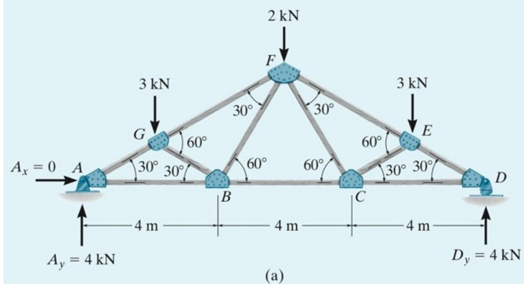
Only the forces in half the members have to be determined, since the truss is symmetric with respect to *both* loading and geometry.

**Joint A,**

**Free-Body Diagram.**



(b)



(a)

**Equations of Equilibrium.**

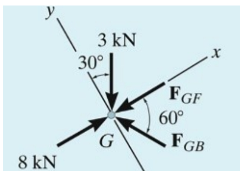
$\xrightarrow{+} \Sigma F_x = 0; \quad F_{AB} - 8\cos 30° = 0 \quad F_{AB} = 6.928 \text{ kN (T)}$ *Ans.*

$\xrightarrow{+} \Sigma F_x = 0; \quad F_{AB} - 8\cos 30° = 0 \quad F_{AB} = 6.928 \text{ kN (T)}$ *Ans.*



**Joint G,**

**Free-Body Diagram.**



(c)

**Equations of Equilibrium.**

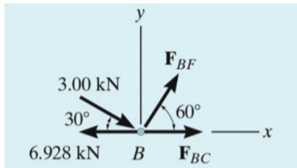
$+\nwarrow \Sigma F_y = 0; \quad F_{GB} \sin 60° - 3\cos 30° = 0$

$F_{GB} = 3.00 \text{ kN (C)}$ *Ans.*

$+\nearrow \Sigma F_x = 0; \quad 8 - 3\sin 30° - 3.00\cos 60° - F_{GF} = 0$

$F_{GF} = 5.00 \text{ kN (C)}$ *Ans.*

**Joint B,**



(d)

$+\uparrow \Sigma F_y = 0; \quad F_{BF} \sin 60° - 3.00\sin 30° = 0$

$F_{BF} = 1.73 \text{ kN (T)}$ *Ans.*

$\xrightarrow{+} \Sigma F_x = 0; \quad F_{BC} + 1.73\cos 60° + 3.00\cos 30° - 6.928 = 0$

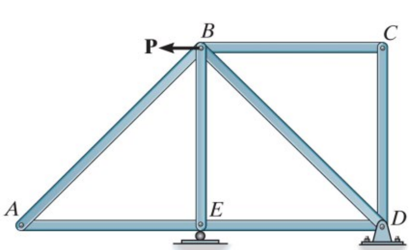
$F_{BC} = 3.46 \text{ kN (T)}$ *Ans.*

## Zero-Force Members

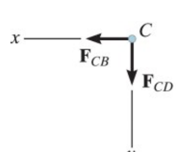
- Truss analysis using method of joints is greatly simplified if one is able to first determine those members that support no loading
- These zero-force members may be necessary for the stability of the truss during construction & to provide support if the applied loading is changed
- The zero-force members of a truss can generally be determined by inspection of the joints & they occur in 2 cases.



- **Case 1**
  - The 2 members at joint C are connected together at a right angle & there is no external load on the joint
  - The free-body diagram of joint C indicates that the force in each member must be zero in order to maintain equilibrium



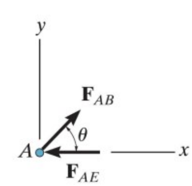

(a)




$\overset{+}{\leftarrow} \Sigma F_x = 0;\ F_{CB} = 0$

$+\downarrow \Sigma F_y = 0;\ F_{CD} = 0$

(b)




$+\uparrow \Sigma F_y = 0;\ F_{AB} \sin\theta = 0$

$F_{AB} = 0$ (since $\sin\theta \neq 0$)

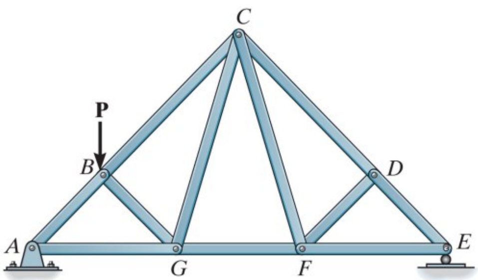
$\overset{+}{\leftarrow} \Sigma F_x = 0;\ -F_{AE} + 0 = 0$

$F_{AE} = 0$

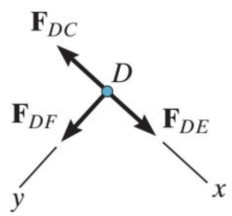
(c)

❖ **Case 2**

❑ Zero-force members also occur at joints having a geometry as joint D

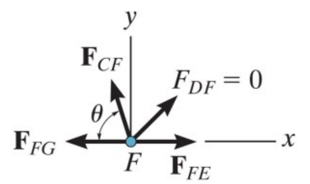


(a)



(b)

$+\swarrow \Sigma F_y = 0;\ F_{DF} = 0$



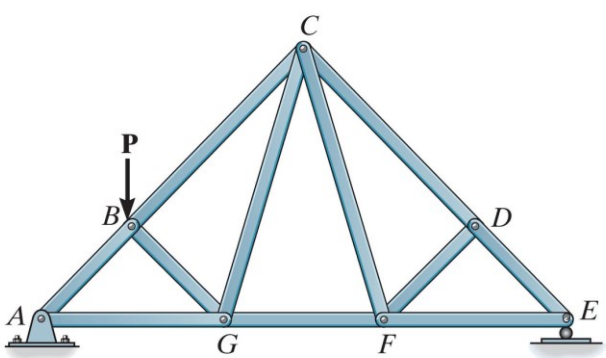
(c)

$+\uparrow \Sigma F_y = 0;\ F_{CF} \sin\theta + 0 = 0$

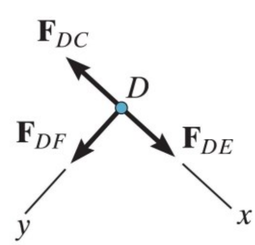
$F_{CF} = 0 \text{ (since } \sin\theta \neq 0)$



- ❑ No external load acts on the joint, so a force summation in the y-direction which is perpendicular to the 2 collinear members requires that $F_{DF} = 0$
- ❑ Using this result, FC is also a zero-force member, as indicated by the force analysis of joint F



(a)

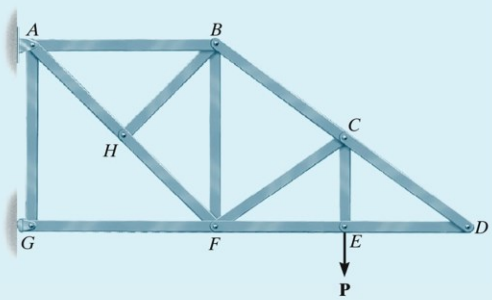


(b)

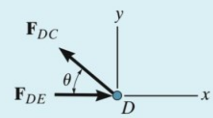
$+\swarrow \Sigma F_y = 0;\ F_{DF} = 0$

## Example 3

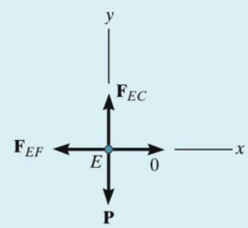
Find all the zero-force members of the truss shown in Fig. 3–24*a*.




(a)




(b)

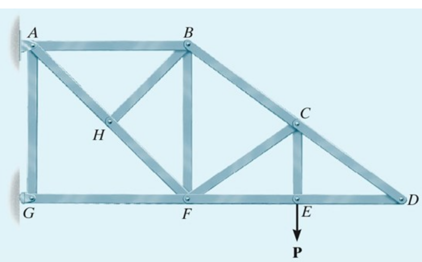





**Joint *D*, Fig. 3–24*b*.**

$+\uparrow \Sigma F_y = 0; \quad F_{DC} \sin \theta = 0 \quad F_{DC} = 0$ *Ans.*

$\xrightarrow{+} \Sigma F_x = 0; \quad F_{DE} + 0 = 0 \quad F_{DE} = 0$ *Ans.*

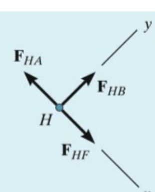



**Joint *E*, Fig. 3–24*c*.**

$\xleftarrow{+} \Sigma F_x = 0; \quad F_{EF} = 0$ *Ans.*

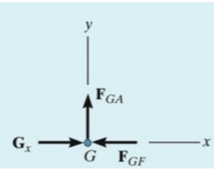
**Joint *H*, Fig. 3–24*d*.**

$+\nearrow \Sigma F_y = 0; \quad F_{HB} = 0$ *Ans.*



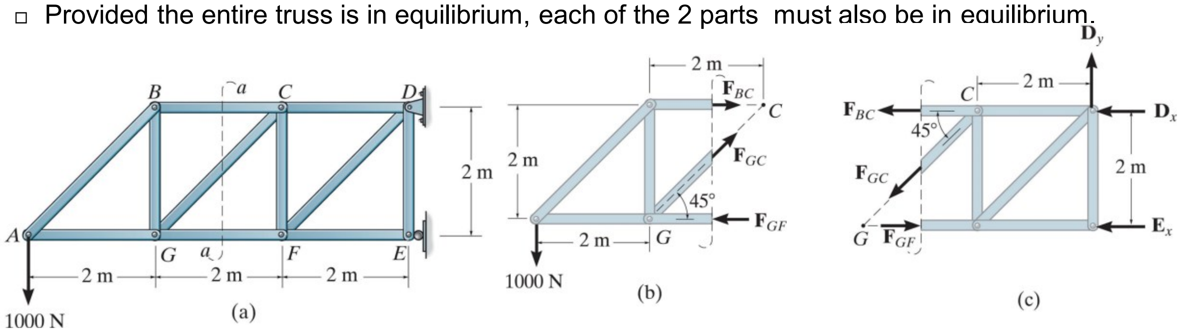

**Joint *G*, Fig. 3–24*e*.** The rocker support at *G* can only exert an *x* component of force on the joint, i.e., $\mathbf{G}_x$. Hence,

$+\uparrow \Sigma F_y = 0; \quad F_{GA} = 0$ *Ans.*



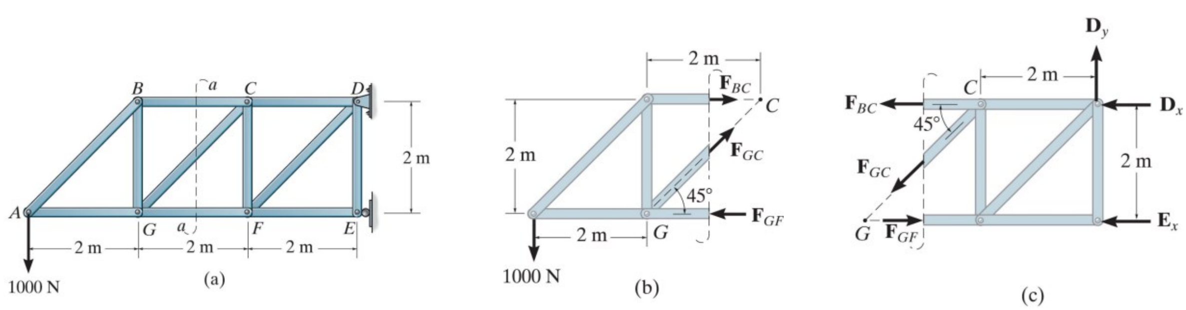
## The Method of Sections

- If the forces in only a few members of a truss are to be found, the method of sections generally provide the most direct means of obtaining these forces.
- This method consists of passing an imaginary section through the truss, thus cutting it into 2 parts.
- Provided the entire truss is in equilibrium, each of the 2 parts must also be in equilibrium.

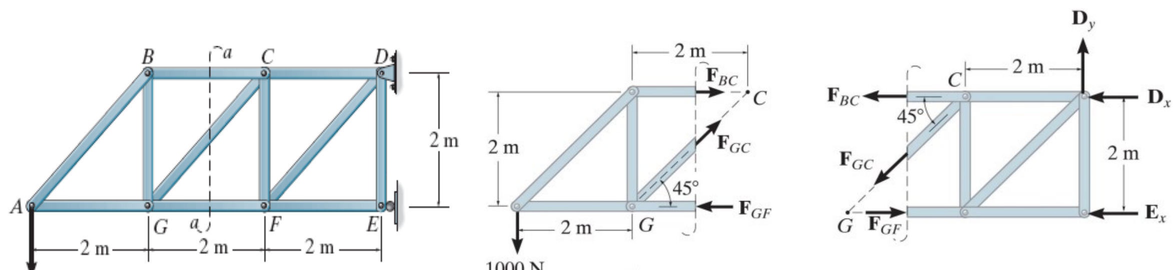





- The 3 eqns of equilibrium may be applied to either one of these 2 parts to determine the member forces at the "cut section"
- A decision must be made as to how to "cut" the truss
- In general, the section should pass through not more than 3 members in which the forces are unknown



- If the force in GC is to be determined, section a-a will be appropriate
- Also, the member forces acting on one part of the truss are equal but opposite
- The 3 unknown member forces, FBC, FGC & FGF can be obtained by applying the 3 equilibrium equations

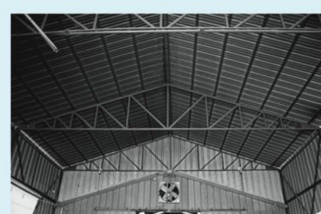


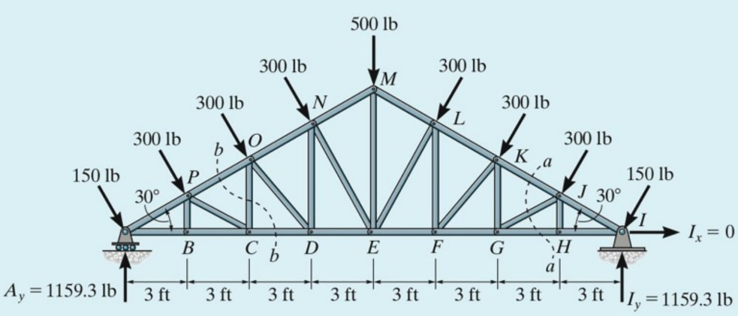

(a) (b) (c)



## Example 4

Determine the force in members *GJ* and *CO* of the roof truss shown in the photo. The dimensions and loadings are shown in Fig. 3–26*a*. State whether the members are in tension or compression. The reactions at the supports are given.

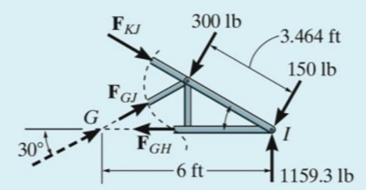





(a)

**Fig. 3–26**

**Member GJ.**
**Free-Body Diagram.**

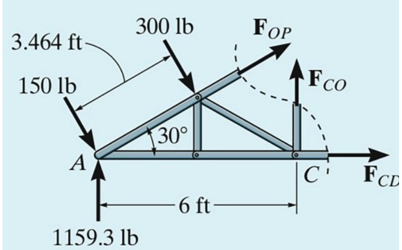



**Equations of Equilibrium.**

$$\circlearrowright + \Sigma M_I = 0; \qquad -F_{GJ} \sin 30°(6) + 300(3.464) = 0$$
$$F_{GJ} = 346 \text{ lb (C)}$$

**Member CO.**

**Free-Body Diagram.**



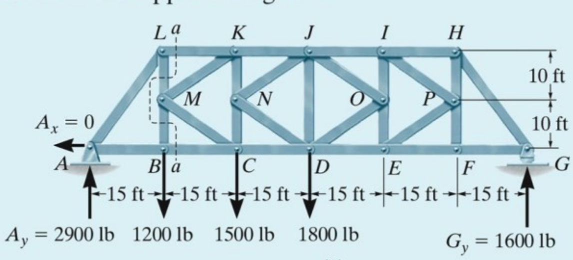

**Equations of Equilibrium.**

$$\circlearrowright + \Sigma M_A = 0; \qquad -300(3.464) + F_{CO}(6) = 0$$
$$F_{CO} = 173 \text{ lb (T)}$$



## Example 5

Determine the force in members $BC$ and $MC$ of the K-truss shown in Fig. 3–28$a$. State whether the members are in tension or compression. The reactions at the supports are given.




(a)

**Free-Body Diagram.**

**Equations of Equilibrium.**

$\circlearrowleft + \Sigma M_L = 0; \qquad -2900(15) + F_{BC}(20) = 0$

$F_{BC} = 2175 \text{ lb (T)}$

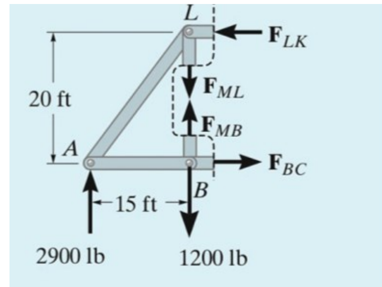
$\boldsymbol{To\ find\ F_{MB}}$

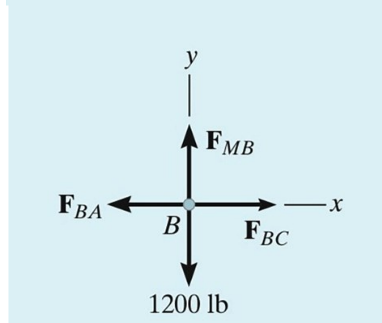
$\boldsymbol{From\ the\ F.B.D\ of\ Joint\ B}$

$\rightarrow \sum F_y = 0 \rightarrow F_{MB} = 1200\ lb$

$+\uparrow \Sigma F_y = 0; \qquad 2900 - 1200 + 1200 - F_{ML} = 0$

$F_{ML} = 2900 \text{ lb (T)}$






## Joint M

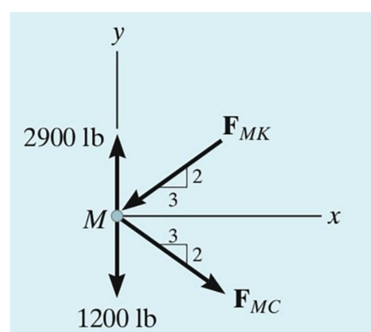
**Free-Body Diagram.**

**Equations of Equilibrium.**

$\xrightarrow{+} \Sigma F_x = 0; \qquad \left(\frac{3}{\sqrt{13}}\right)F_{MC} - \left(\frac{3}{\sqrt{13}}\right)F_{MK} = 0$

$+\uparrow \Sigma F_y = 0; \qquad 2900 - 1200 - \left(\frac{2}{\sqrt{13}}\right)F_{MC} - \left(\frac{2}{\sqrt{13}}\right)F_{MK} = 0$

$F_{MK} = 1532 \text{ lb (C)} \qquad F_{MC} = 1532 \text{ lb (T)}$ *Ans.*



The Hashemite University
Faculty of Engineering
Department of Civil Engineering

# CE 315: Structural Analysis

## Chapter 4: Internal Loadings Developed in Structural Members

Dr. Ra'ed Al-Mazaidh

## Internal loadings at a specified point

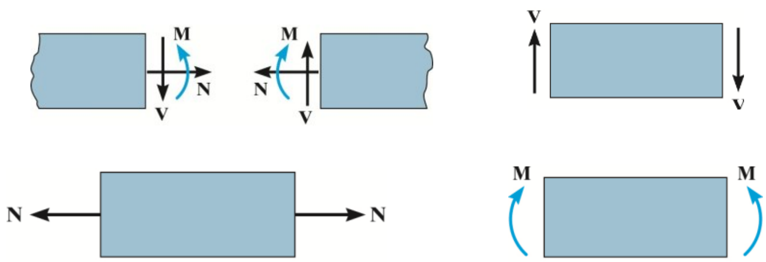
❖ The internal load at a specified point in a member can be determined by using the method of sections

❖ This consists of:

- **N, normal force**
- **V, shear force**
- **M, bending moment**

## ❖ Sign convention

Although the choice is **arbitrary**, the convention has been widely accepted in structural engineering






## Procedure for analysis

- Determine the support reactions before the member is "cut"
- If the member is part of a pin-connected structure, the pin reactions can be determine using the methods of section
- Keep all distributed loadings, couple moments & forces acting on the member in their exact location.
- Pass an imaginary section through the member, perpendicular to its axis at the point where the internal loading is to be determined
- Then draw a free-body diagram of the segment that has the least no. of loads on it
- Indicate the unknown resultants N, V & M acting in their positive directions
- Moments should be summed at the section about axes that pass through the centroid of the member's x-sectional area in order to eliminate N & V, thereby solving M.

➢ If the solution of the equilibrium equations yields a quantity having a negative magnitude, then the assumed directional sense of the quantity is opposite to that shown on the free-body diagram.



## EXAMPLE 1

Determine the internal shear and moment acting at a section passing through point $C$ in the beam shown in Fig. 4–3*a*.

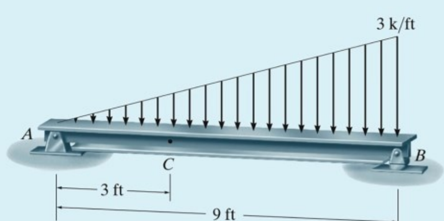
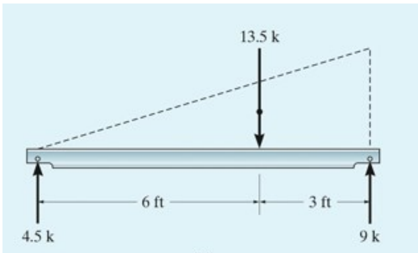



Fig. 4–3



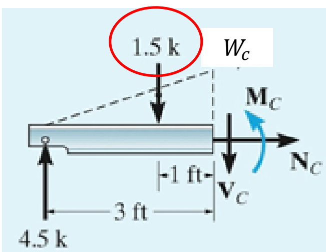

$= \frac{1}{2} * 3\,\frac{k}{ft} * 9\text{ ft} = 13.5\text{ k}$

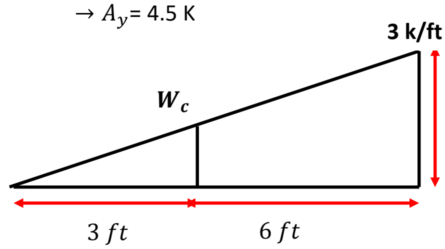
$$\sum M_A = -13.5\,(6) + B_y(9) = 0$$

$\rightarrow B_y$=9 K

$$\sum F_y = -13.5 + 9 + A_y = 0$$

$\rightarrow A_y$= 4.5 K



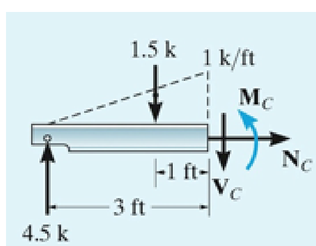




$$\frac{3}{9} = \frac{W_c}{3}$$

$\longrightarrow W_c = 1\ K/ft$

$$\boxed{= \frac{1}{2} * 1 * 3 = 1.5\ K}$$






$+\uparrow \Sigma F_y = 0; \quad 4.5 - 1.5 - V_C = 0 \qquad V_C = 3\text{ k}$

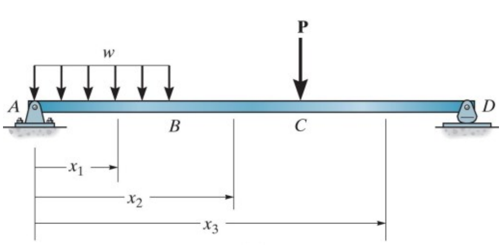
$\circlearrowleft + \Sigma M_C = 0; \quad -4.5(3) + 1.5(1) + M_C = 0 \qquad M_C = 12\text{ k}\cdot\text{ft}$

## Shear & Moment Functions

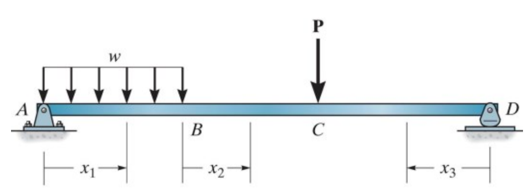
- Design of beam requires detailed knowledge of the variations of V & M
- **Internal N** is generally **not considered** as:
  - The **loads** applied to a beam act **perpendicular** to the beam's axis.
  - For design purpose, a beam's resistance to shear & bending is **more important** than its ability to resist normal force.
  - An exception is when it is subjected to **compressive axial force where buckling may occur.**



- In general, **the internal shear & moment functions** will be **discontinuous or their slope** will be **discontinuous** at points where:
  - The type or magnitude of the distributed load changes
  - Concentrated forces or couple moments are applied



(a)



(b)

## ❖ Procedure for Analysis

- Determine the support reactions on the beam.
- Resolve all the external forces into components acting perpendicular & parallel to beam's axis.
- Specify separate coordinates x and associated origins, extending into:
  - Regions of the beam between concentrated forces and/or couple moments
  - Discontinuity of distributed loading
- Section the beam perpendicular to its axis at each distance x
- From the free-body diagram of one of the segments, determine the unknowns V & M
- On the free-body diagram, V & M should be shown acting in their positive directions
- V is obtained from $\sum F_y = 0$
- M is obtained by $\sum M_s = 0$



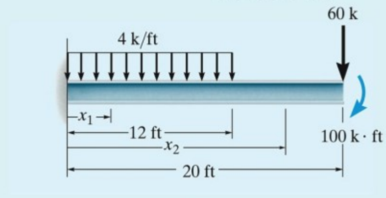
The results can be checked by noting that:

$$\frac{dM}{dx} = \text{v}$$

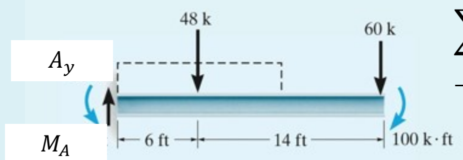
$$\frac{dV}{dx} = \text{w}$$

## EXAMPLE 2

Determine the shear and moment in the beam shown in Fig. 4–7*a* as a function of *x*.




(a)



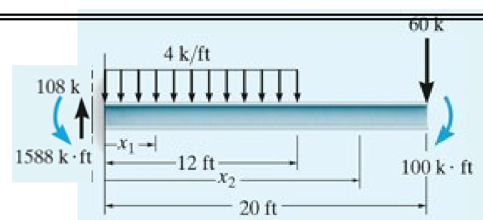

$$\sum M_A = -48\,(6) - 60(20) - 100 + M_A = 0$$

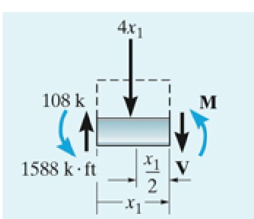
$\rightarrow M_A$=1588 K.ft

$$\sum F_y = -48 - 60 + A_y = 0$$

$\rightarrow A_y$= 108 K






$0 \leq x_1 \leq 12$ ft.

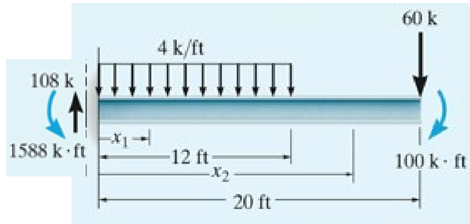
$+\uparrow \Sigma F_y = 0; \quad 108 - 4x_1 - V = 0, \quad V = 108 - 4x_1$

$\circlearrowleft + \Sigma M_S = 0; \quad 1588 - 108x_1 + 4x_1\left(\frac{x_1}{2}\right) + M = 0$

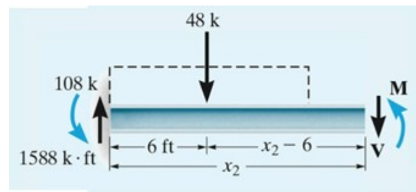
$$M = -1588 + 108x_1 - 2x_1^2$$

$$\frac{dM}{dx} = 108 - 4x_1 \quad = \quad \text{V}$$

$$\frac{dV}{dx} = -4 \text{ =w}$$




$12 \text{ ft} \leq x_2 \leq 20 \text{ ft},$




$$+\uparrow \Sigma F_y = 0; \qquad 108 - 48 - V = 0, \qquad V = 60$$

$$\circlearrowleft + \Sigma M_S = 0; \qquad 1588 - 108x_2 + 48(x_2 - 6) + M = 0$$

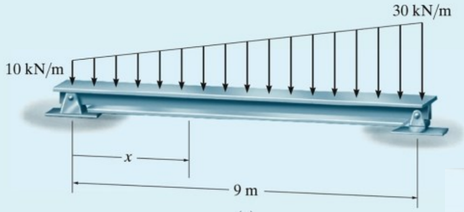
$$M = 60x_2 - 1300$$

$$\frac{dM}{dx} = 60 = V \qquad \frac{dV}{dx} = 0 = w$$



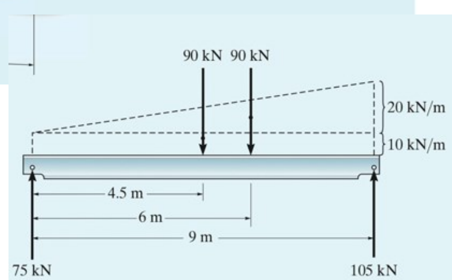
## EXAMPLE 3

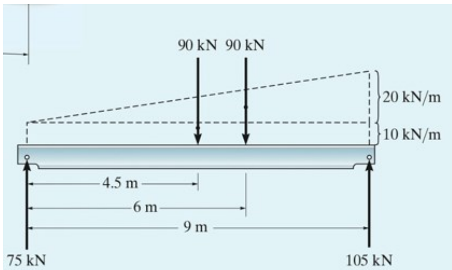
Determine the shear and moment in the beam shown in Fig. 4–8*a* as a function of *x*.

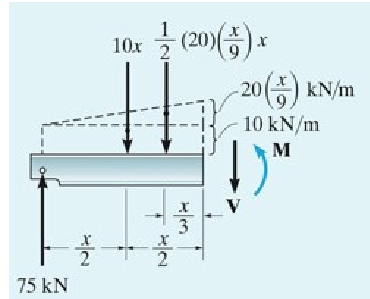


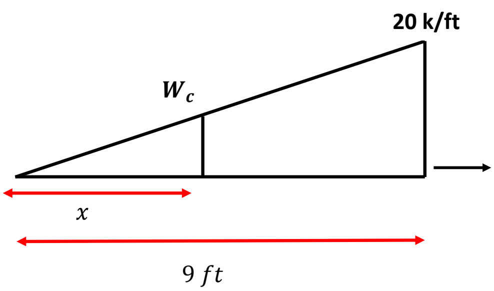

(a)

**Fig. 4–8**






$$\frac{20}{9} = \frac{W_c}{x}$$

$$W_c = (20)(\frac{x}{9})$$

$$+\uparrow\Sigma F_y = 0; \qquad 75 - 10x - \left[\frac{1}{2}(20)\left(\frac{x}{9}\right)x\right] - V = 0$$

$$V = 75 - 10x - 1.11x^2$$

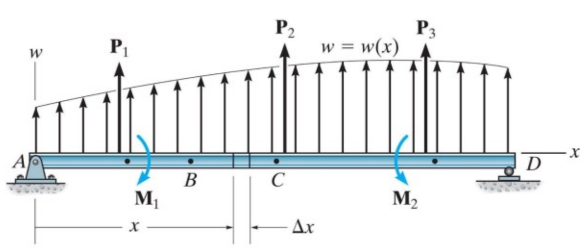
$$\curvearrowleft+\Sigma M_S = 0; \qquad -75x + (10x)\left(\frac{x}{2}\right) + \left[\frac{1}{2}(20)\left(\frac{x}{9}\right)x\right]\frac{x}{3} + M = 0$$

$$M = 75x - 5x^2 - 0.370x^3 \qquad \textit{Ans.}$$

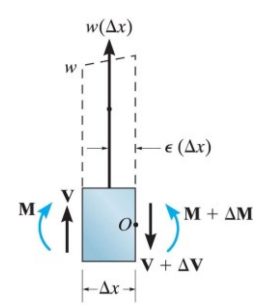


## Shear & Moment Diagrams for a Beam

☐ If the variations of V & M are plotted, **the graphs are termed the shear diagram and moment diagram**

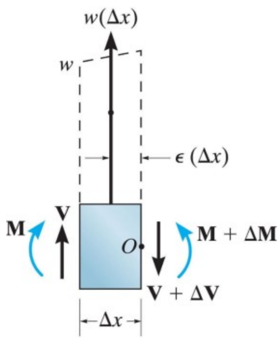



(a)




(b)

- Applying the equation of equilibrium, we have:




(b)

$$+\uparrow \sum F_y = 0;$$

$$V + w(x)\Delta x - (V + \Delta V) = 0$$
$$\boldsymbol{\Delta V = w(x)\Delta x}$$

With anti-clockwise moments as +ve:
$$\sum M_o = 0;$$

$$-V\Delta x - M - w(x)\Delta x\,\varepsilon(\Delta x) + (M + \Delta M) = 0$$
$$\boldsymbol{\Delta M = V\Delta x + w(x)\varepsilon(\Delta x)^2}$$



- Dividing by Δx & taking the limit as Δx →∞, the previous equations become:

$$\frac{dV}{dx} = w(x) \quad , \quad \frac{dM}{dx} = V$$

- Integrating from one point to another between concentrated forces or couples in which case

$$\Delta V = \int w(x)dx \quad , \quad \Delta M = \int V(x)dx$$

$$\frac{dV}{dx} = w$$

$$\left.\begin{array}{r}\text{Slope of}\\ \text{Shear Diagram}\end{array}\right\} = \left\{\begin{array}{l}\text{Intensity of}\\ \text{Distributed Load}\end{array}\right.$$

$$\frac{dM}{dx} = V$$

$$\left.\begin{array}{r}\text{Slope of}\\ \text{Moment Diagram}\end{array}\right\} = \{\text{Shear}$$

$$\Delta V = \int w\,dx$$

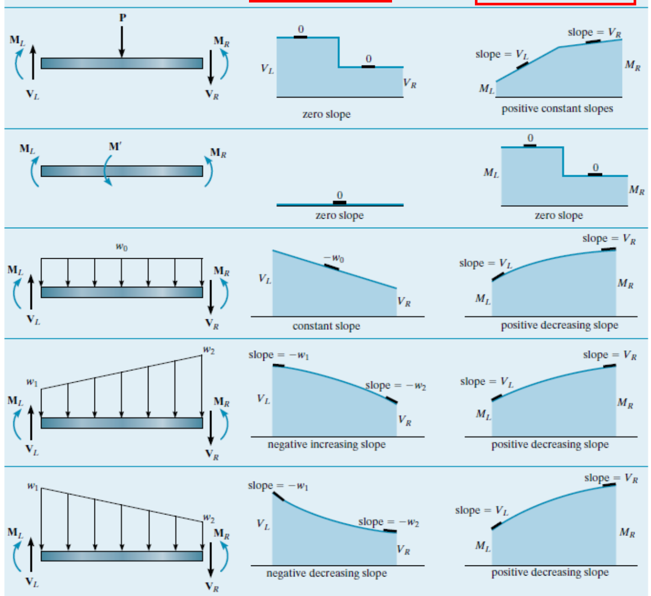
$$\left.\begin{array}{r}\text{Change in}\\ \text{Shear}\end{array}\right\} = \left\{\begin{array}{l}\text{Area under}\\ \text{Distributed Loading}\\ \text{Diagram}\end{array}\right.$$

$$\Delta M = \int V\,dx$$

$$\left.\begin{array}{r}\text{Change in}\\ \text{Moment}\end{array}\right\} = \left\{\begin{array}{l}\text{Area under}\\ \text{Shear Diagram}\end{array}\right.$$

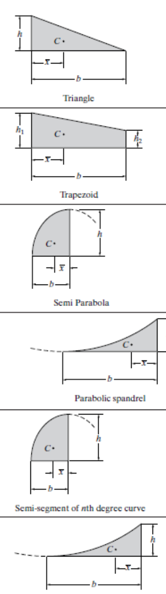
**TABLE 4.1 Relationship between Loading, Shear, and Moment**

| Loading | Shear Diagram, $\frac{dV}{dx} = w$ | Moment Diagram, $\frac{dM}{dx} = V$ |
|---|---|---|



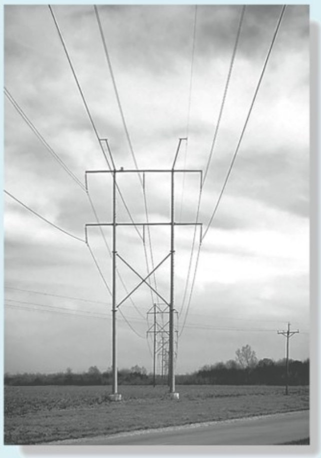



**Geometric Properties of Areas**

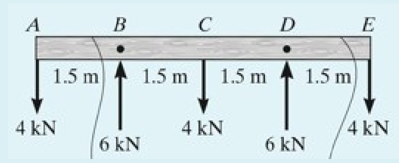


| | |
|---|---|
| Triangle | $A = \frac{1}{2}bh$, $\bar{x} = \frac{1}{3}b$ |
| Trapezoid | $A = \frac{1}{2}b(h_1 + h_2)$, $\bar{x} = \frac{b(2h_2 + h_1)}{3(h_1 + h_2)}$ |
| Semi Parabola | $A = \frac{2}{3}bh$, $\bar{x} = \frac{3}{8}b$ |
| Parabolic spandrel | $A = \frac{1}{3}bh$, $\bar{x} = \frac{1}{4}b$ |
| Semi-segment of *n*th degree curve | $A = bh\left(\frac{n}{n+1}\right)$, $\bar{x} = \frac{b(n+1)}{2(n+2)}$ |
| Spandrel of *n*th degree curve | $A = bh\left(\frac{1}{n+1}\right)$, $\bar{x} = \frac{b}{n+2}$ |

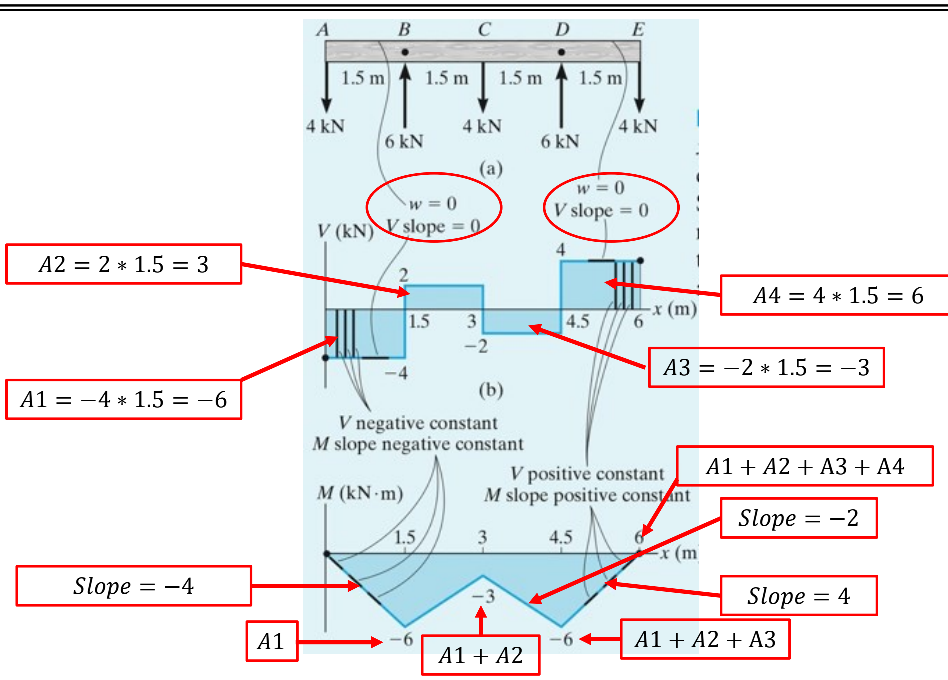
## EXAMPLE 4



Each of the two horizontal members of the powerline support frame is subjected to the cable loadings shown in Fig. 4–11*a*. Draw the shear and moment diagrams for these members.





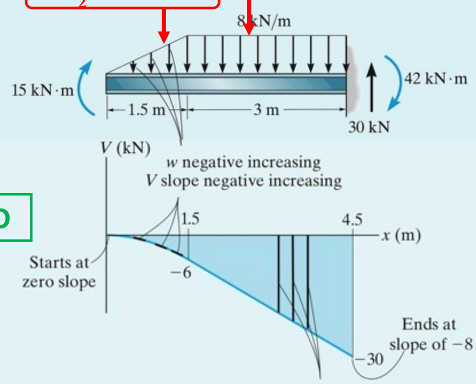



## EXAMPLE 5

Draw the shear and moment diagrams for the beam shown in Fig. 4–13. The support reactions have been calculated.

$R1 = \frac{1}{2} * -8 * 1.5 = -6$

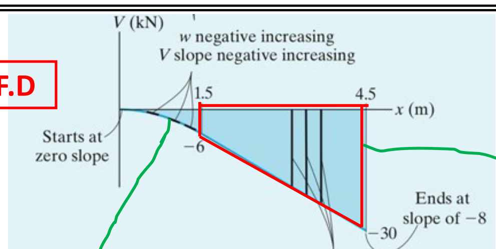
$R2 = -8 * 3 = -24$



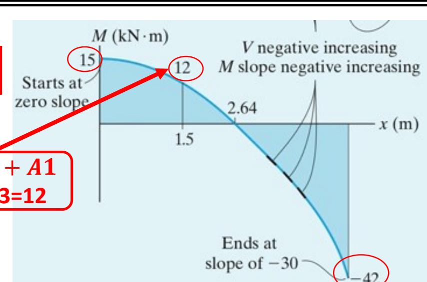

**S.F.D**



**S.F.D**

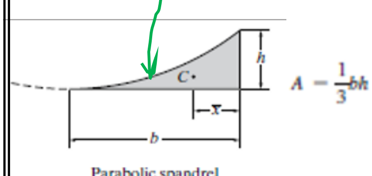



**B.M.D**

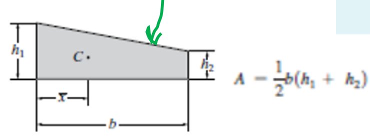



$= 15 + A1$
$= 15 - 3 = 12$

$= 15 + A1 + A2$
$= 15 - 3 - 54 = -42$




Parabolic spandrel




$$A1 = \frac{1}{3} * b * h - \frac{1}{3} * 1.5 * 6 = -3$$

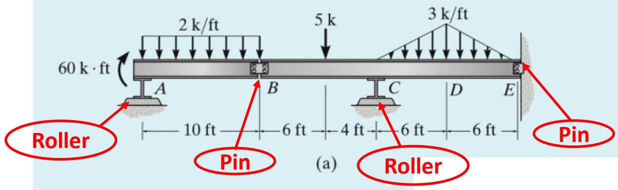
$$A2 = \frac{1}{2} b(h1 + h2)$$
$$= \frac{1}{2} * 3 * (-30 \ - 6) = -54$$

**Clockwise (+ value)**

**Counter-clockwise (- value)**

## EXAMPLE 6

Draw the shear and moment diagrams for the compound beam shown in Fig. 4–15*a*. Assume the supports at *A* and *C* are rollers and *B* and *E* are pin connections.



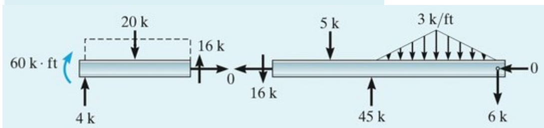

(a)

$n = 2 \rightarrow 3n = 6$

$r = 6$

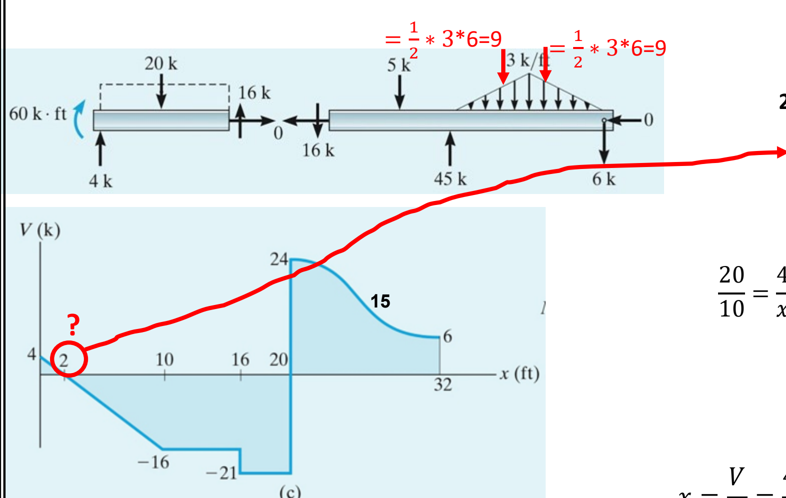
$r = 3n = 6$

**Statically determinate**

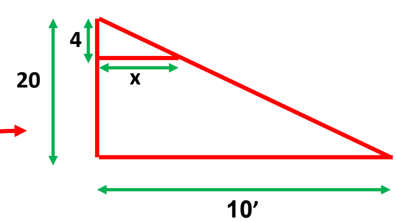



**Exercise**



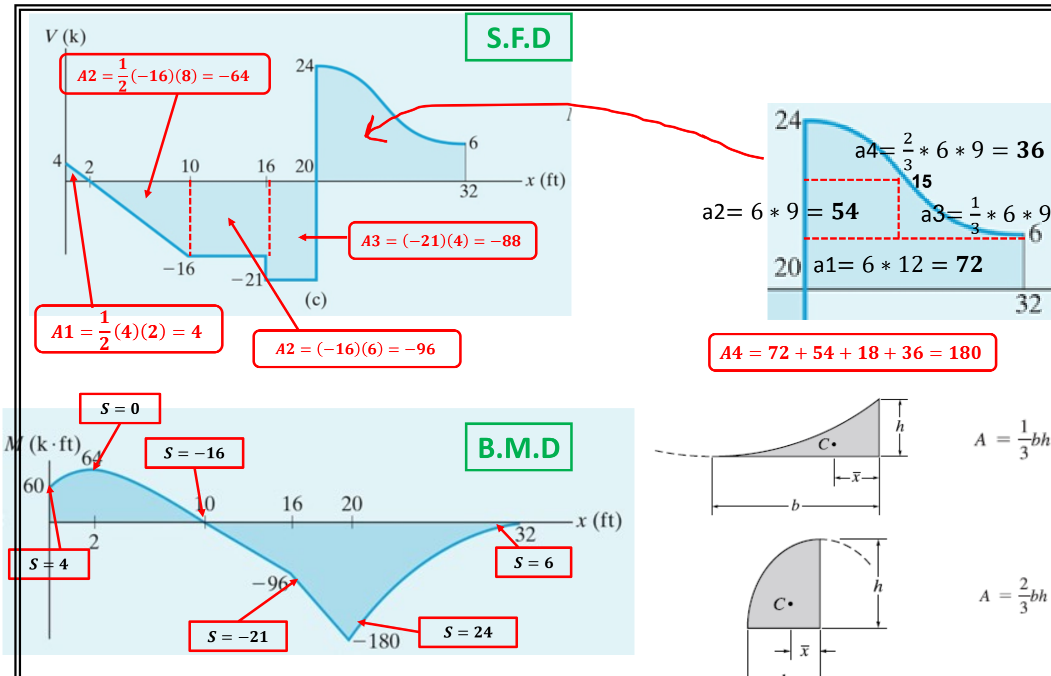



(c)




$$\frac{20}{10} = \frac{4}{x} \rightarrow x = 2$$

$$x = \frac{V}{w} = \frac{4}{2} = 2$$

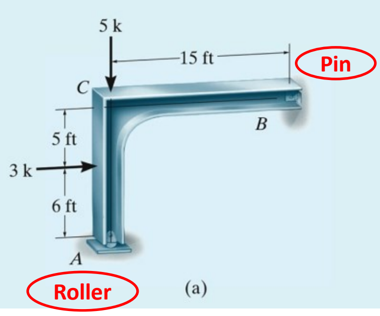




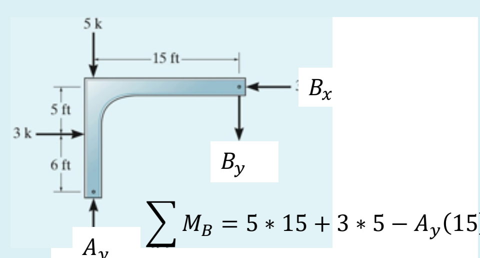

# Shear and Moment Diagrams for a Frame

## EXAMPLE 7

Draw the moment diagram for the frame shown in Fig. 4–16*a*. Assume the support at *A* is a roller and *B* is a pin.



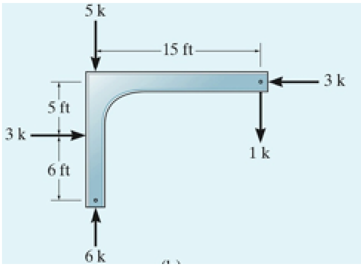

(a)

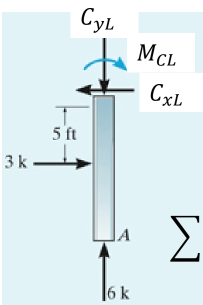



$$\sum M_B = 5*15 + 3*5 - A_y(15) = 0 \rightarrow A_y = 6\ K$$

$$\sum F_y = -5 + 6 - B_y = 0 \rightarrow B_y = 1\ K$$

$$\sum F_x = -B_x + 3 = 0 \rightarrow B_x = 3\ K$$

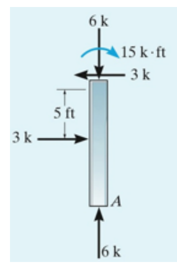


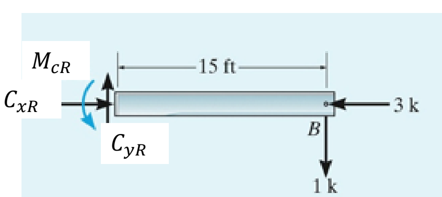

$$\sum M_c = 3 * 5 - M_{cL} = 0 \rightarrow M_{cL} = 15K.ft$$

$$\sum F_x = 3 - C_{xL} = 0 \rightarrow C_{xL} = 3\ K$$

$$\sum F_y = 6 - C_{yL} = 0 \rightarrow C_{yL} = 6\ K$$



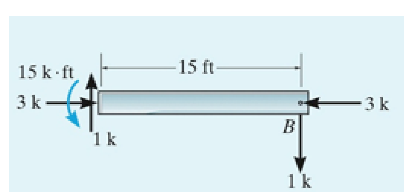


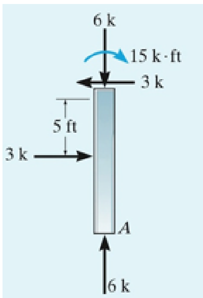



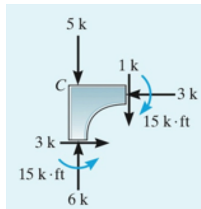

$$\sum M_c = -1 * 15 + M_{cR} = 0 \rightarrow M_{cR} = 15K.ft$$

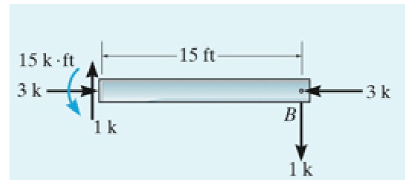
$$\sum F_x = -3 + C_{xR} = 0 \rightarrow C_{xR} = 3\ K$$

$$\sum F_y = -1 + C_{yR} = 0 \rightarrow C_{yR} = 1\ K$$

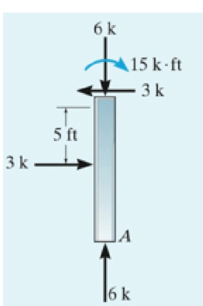


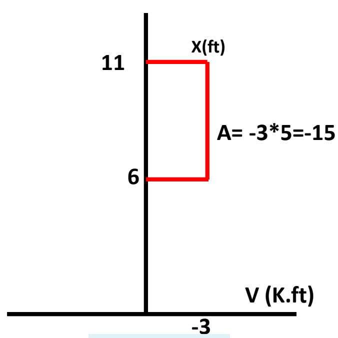

6 k
15 k·ft
3 k
5 ft
3 k
A
6 k


5 k
1 k
C
3 k
15 k·ft
3 k
15 k·ft
6 k

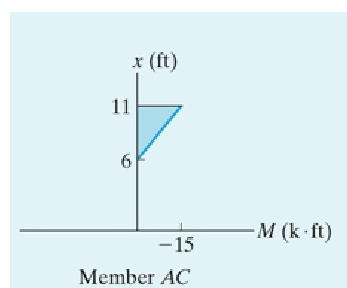

15 ft
15 k·ft
3 k
1 k
B
3 k
1 k

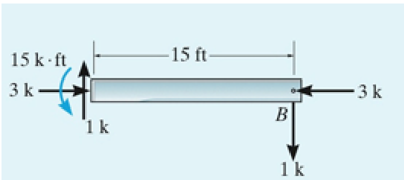


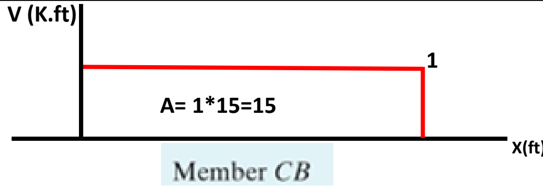

6 k
15 k·ft
3 k
5 ft
3 k
A
6 k

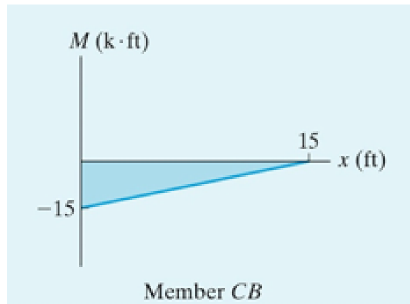

X(ft)
11
A= -3*5=-15
6
V (K.ft)
-3

Member *AC*


x (ft)
11
6
−15
M (k·ft)
Member AC



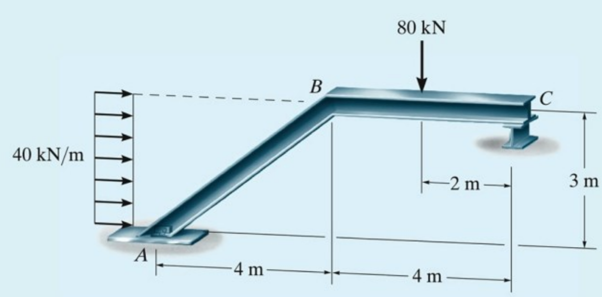






## EXAMPLE 8

Draw the shear and moment diagrams for the frame shown in Fig. 4–17*a*. Assume *A* is a pin, *C* is a roller, and *B* is a rigid joint.



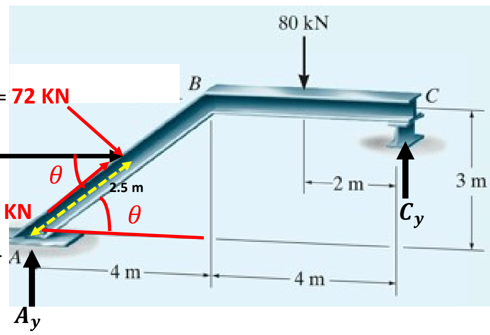

(a)

$tan^{-1}\left(\frac{3}{4}\right) = 36.87^o$

=120 KN*sin (36.87)= **72 KN**

R =40 KN/m *3= **120 KN**

=120 KN*cos (36.87)= **96 KN**

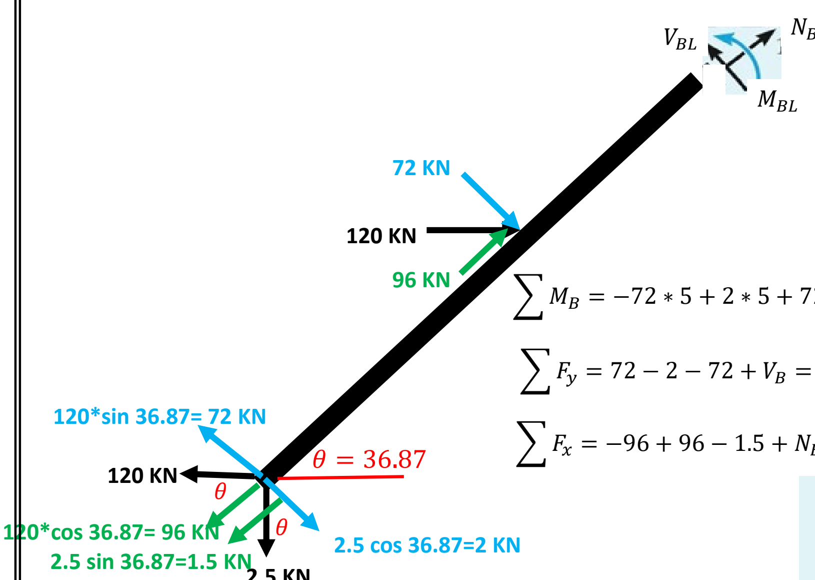



$$\sum M_A = -72 * 2.5 - 80 * 6 + C_y * 8 = 0 \rightarrow C_y = 82.5\ KN$$

$$\sum F_y = -80 + 82.5 + A_y = 0 \rightarrow A_y = -2.5\ KN$$

$$\sum F_x = 120 + A_x = 0 \rightarrow A_x = -120\ KN$$

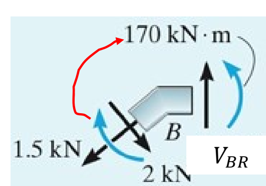


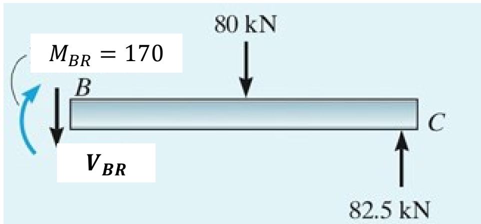



$$\sum M_B = -72 * 5 + 2 * 5 + 72 * 2.5 + M_{BL} = 0 \rightarrow M_{BL} = 170\ KN.m$$

$$\sum F_y = 72 - 2 - 72 + V_B = 2 \rightarrow V_{BL} = 2\ KN$$

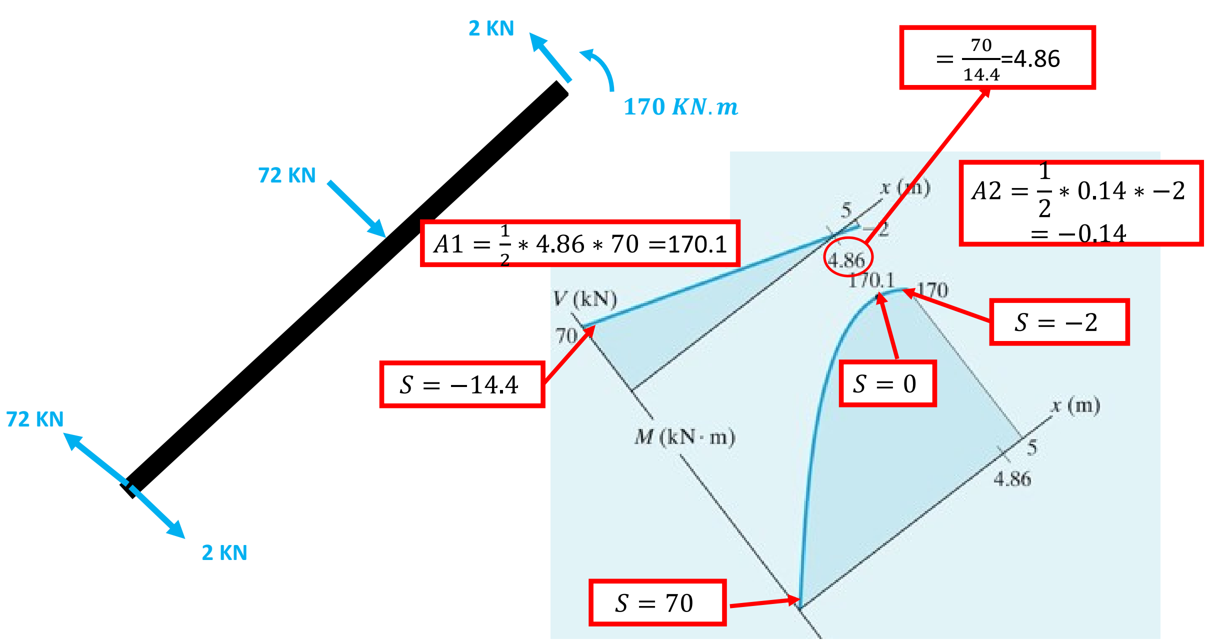
$$\sum F_x = -96 + 96 - 1.5 + N_B = 0 \rightarrow N_B = 1.5\ KN$$

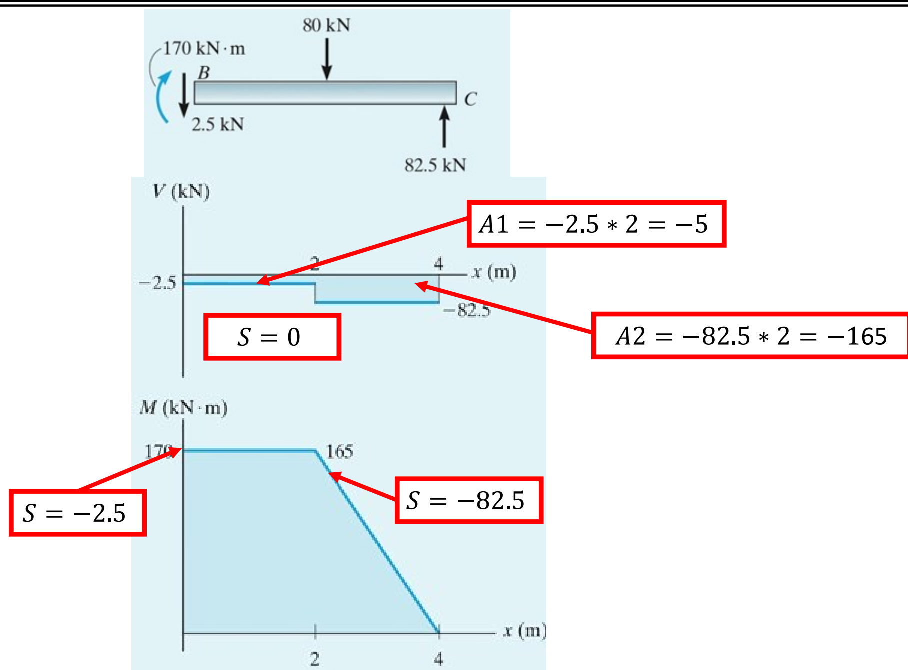





$$\sum F_y = -80 + 82.5 - V_B = 2 \rightarrow V_B = 2.5\ KN$$



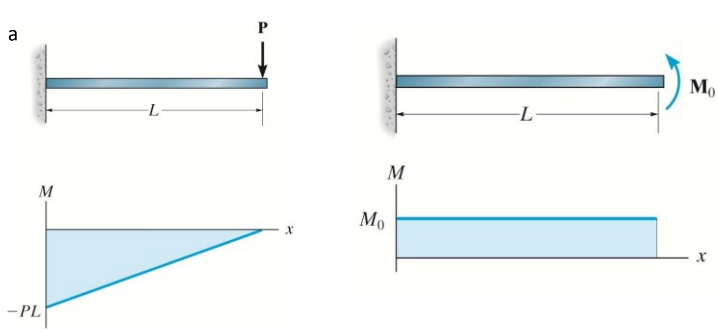


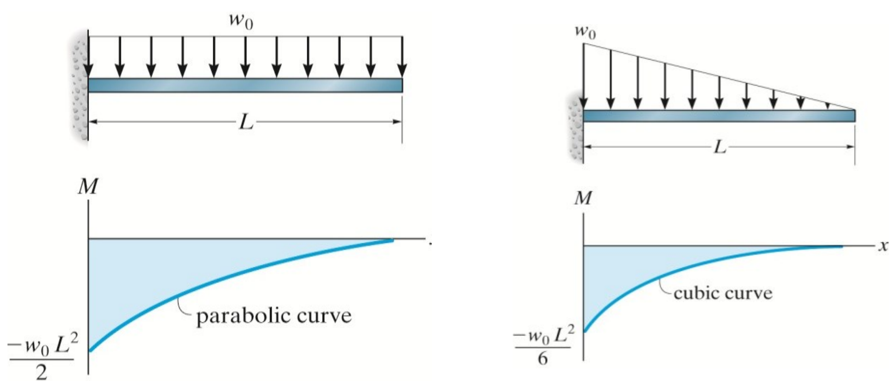





## Moment Diagrams Constructed by the Method of Superposition

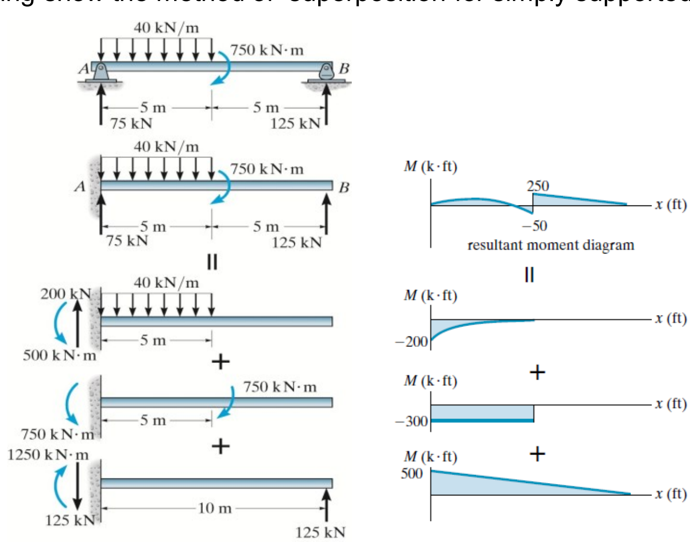
- Beams are used primarily to resist **bending stress**, it is important that the moment diagram accompany **the solution for their design**.
- Most loadings on beams in structural analysis will be a **combination of the loading**s as shown.








Following show the method of superposition for simply supported beam.



The Hashemite University
Faculty of Engineering
Department of Civil Engineering

# CE 315: Structural Analysis

# Chapter 5- Deflections

Dr. Ra'ed Al-Mazaidh

## Deflections

Question: What are **Structural Deflections?**

Answer: The deformations or movements of a structure and its components, such as beams and trusses, from their original positions.

- It is as important for the designer to determine deflections and strains as it is to know the stresses caused by loads.
- Deflection is caused by many sources, such as, loads, temperature, construction error, and settlements.
- It is important to include the calculation of deflections into the design procedure to prevent cracking of attached brittle materials (concrete or plaster walls or roofs) or to solve indeterminate problems.

## Calculation of Deflections

- **Double integration method: (Direct integration)**

  Equations which define the slope and the elastic curve

- **Geometrical methods**: The strain of an elastic structure is used to determine the deflection.

They are used to obtain the slope and deflection at specific points on the beam.

1. **The moment-area theorems method.**
2. **The conjugate-beam method.**

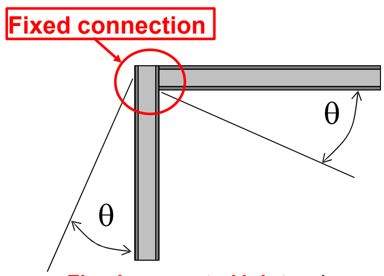
- **Energy methods** :are based on the principle of conservation of energy.

1. **The method of virtual work.**
2. **Castigliano's theorem method.**

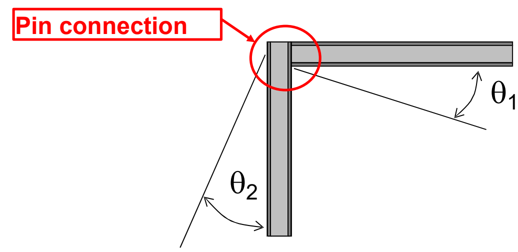


## Deflection diagrams & the elastic curve

- In this topic**, only linear elastic material** response is considered
- This means a structure subjected to load will return to its original undeformed position after the load is removed.
- Usually, before the slope and deflection are calculated, it is important to sketch the shape of the structure when loaded (**deflected shape**).
- To do this, we need to know how different connections rotate, $\theta$, and deflect, $\Delta$ ,as a response to loading.

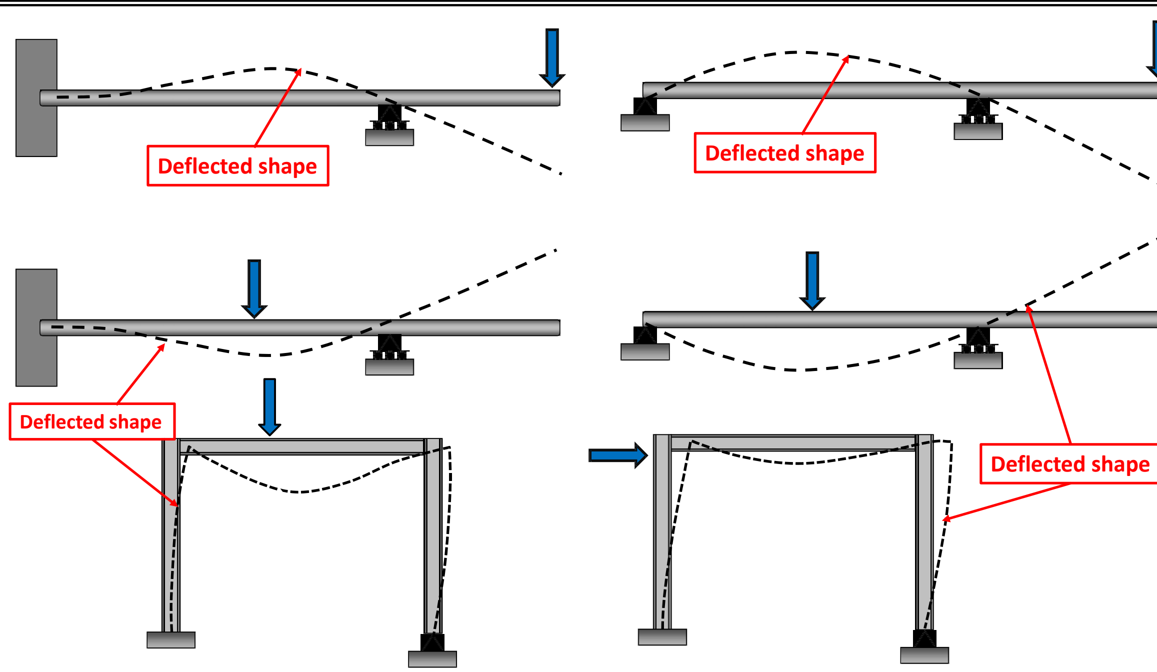




**Fixed-connected joint** – since the connection cannot rotate, <u>the slope of the each member is the same</u>

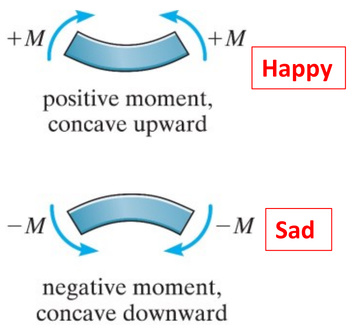


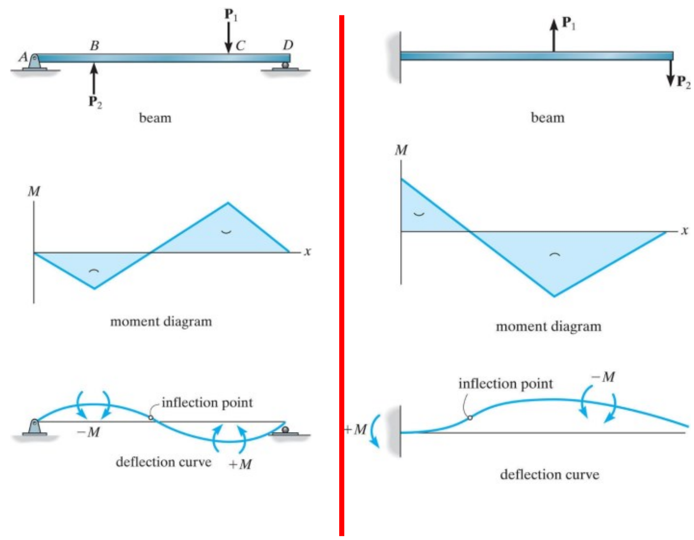

**Pin-connected joint** – the pinned joint may rotate which results in <u>different</u> slopes for each member same





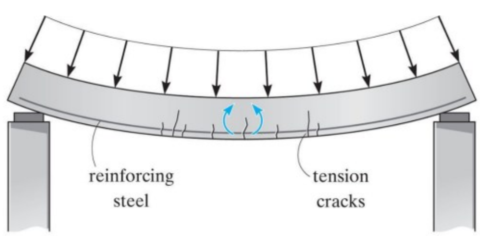
- If you have a difficult time drawing the deflected shape from the elastic response, <u>try to construct the moment  diagram</u> and then <u>use the sign of the moment to determine the curvature of the structure</u>



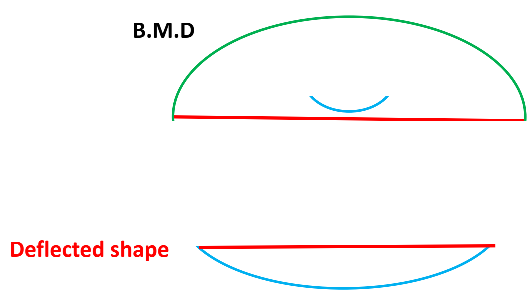



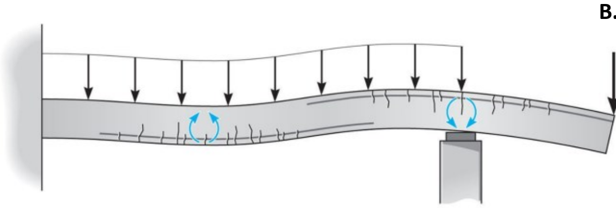

**Inflection point**: The point where the curve changes from concave down to concave up.



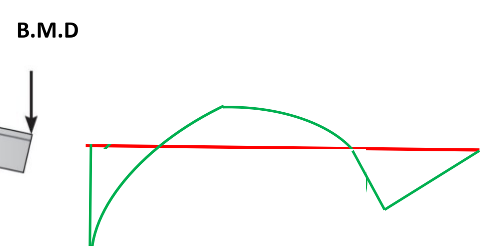


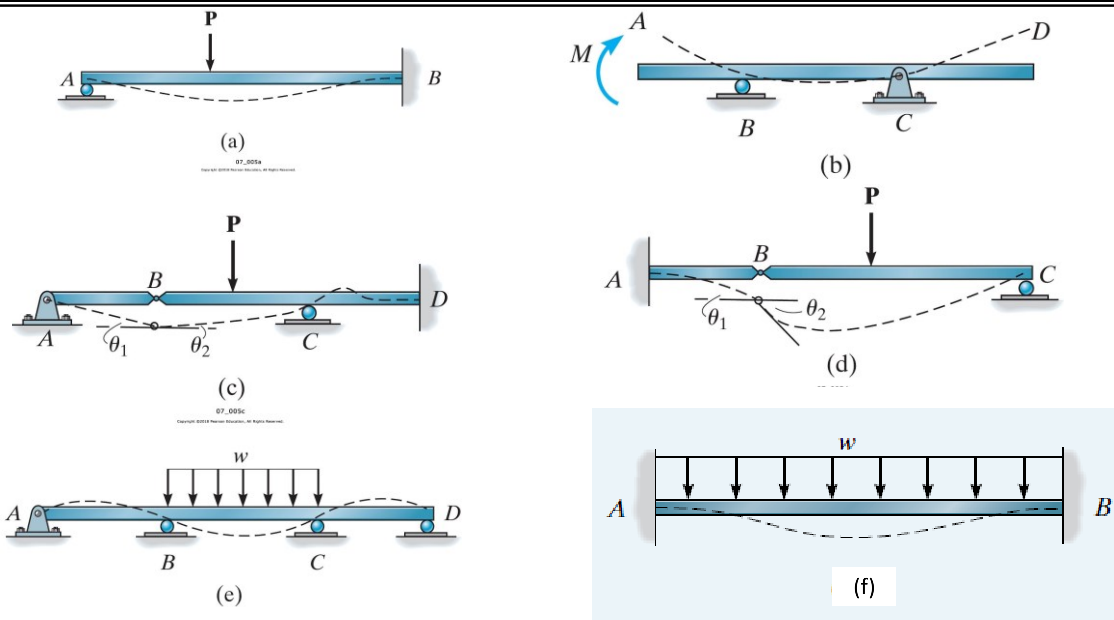

simply supported beam



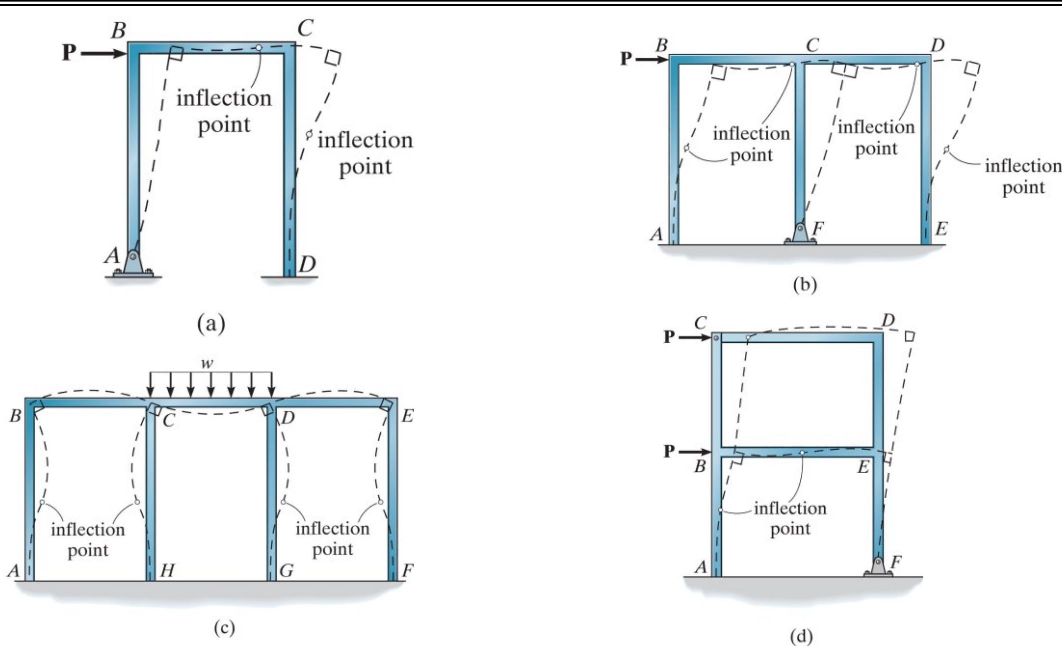



overhang beam




P
A
B
(a)
M
A
D
B
C
(b)
P
B
A
D
C
$\theta_1$
$\theta_2$
(c)
P
A
B
C
$\theta_1$
$\theta_2$
(d)
w
A
B
C
D
(e)
w
A
B
(f)

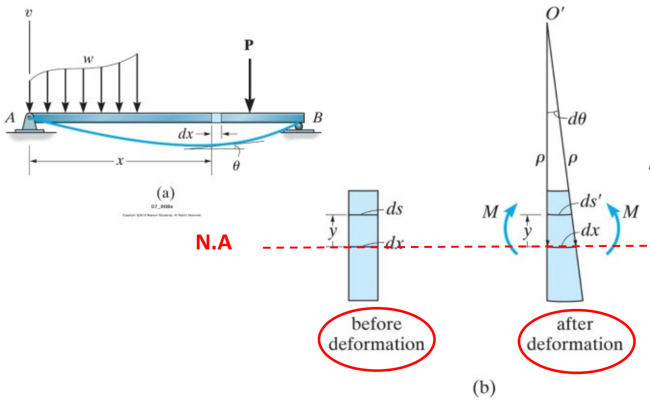



P
B
C
inflection point
inflection point
A
D
(a)
P
B
C
D
inflection point
inflection point
inflection point
A
F
E
(b)
w
B
C
D
E
inflection point
inflection point
A
H
G
F
(c)
P
C
D
P
B
E
inflection point
A
F
(d)

# Elastic-Beam Theory

The relationship between the internal moment and the deflected shape is derived.

Consider a straight elastic beam deformed by a set of applied loads.




$$\epsilon = (ds' - ds)/ds. \quad \text{......(1)}$$

$$ds = dx = \rho\, d\theta \quad \text{......(2)}$$

$\rho$: The radius of curvature which is measured from the center of curvature O' to dx

$$ds' = (\rho - y)\, d\theta. \quad \text{......(3)}$$

Substitute (2) and (3) in (1)

$$\epsilon = \frac{(\rho - y)\, d\theta - \rho\, d\theta}{\rho\, d\theta} \quad \text{......(4)}$$

$$\frac{1}{\rho} = -\frac{\epsilon}{y} \quad \text{......(5)}$$



$$\epsilon = \sigma/E. \quad \text{......(6)}$$

$$\sigma = -My/I. \Rightarrow y = -\frac{\sigma I}{M} \quad \text{..........(7)}$$

Substitute (6) and (7) in (5)

$$\frac{1}{\rho} = \frac{M}{EI} \quad \text{..........(8)}$$

Here

$\rho$ = the radius of curvature at a specific point on the elastic curve ($1/\rho$ is referred to as the *curvature*)

$M$ = the internal moment in the beam at the point where $\rho$ is to be determined

$E$ = the material's modulus of elasticity

$I$ = the beam's moment of inertia computed about the neutral axis

The product $EI$ in this equation is referred to as the *flexural rigidity*,

From Eq.2 $dx = \rho\, d\theta, \Rightarrow \rho = \dfrac{dx}{d\theta}$ Then substitute it in Eq. (8)

$$\Rightarrow \boxed{d\theta = \frac{M}{EI}\, dx}$$

If we can express the curvature $(\frac{1}{\rho})$ in terms of $x$ and v, we can then determine the elastic curve for the beam.

$$\frac{1}{\rho} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}} \quad ..........(9)$$ From calculus

Substitute (9) in (8)

$$\frac{M}{EI} = \frac{d^2v/dx^2}{\left[1 + (dv/dx)^2\right]^{3/2}} \quad ..........(10)$$

Nonlinear second-order differential equation Its solution, v = f(x), gives the exact shape of the elastic curve—assuming, of course, that beam deflections occur only due to bending

Since the slope of the elastic curve for most structures is very small, we will use small deflection theory and assume $\frac{dv}{dx} \approx 0$

$$\boxed{\frac{d^2v}{dx^2} = \frac{M}{EI}}$$



## The double integration method

$$\boxed{\frac{d^2v}{dx^2} = \frac{M}{EI}}$$

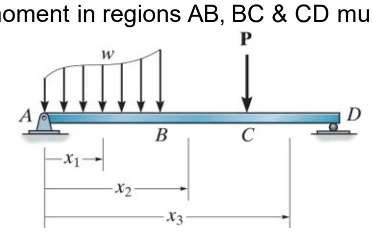
- Once $M$ is expressed as a function of position $x$, then **successive integrations** of the above equation will yield the beam's slope.

$$\theta \approx tan\theta = \frac{d_v}{d_x} = \int \frac{M}{EI} dx$$ Recall: $$d\theta = \frac{M}{EI} dx$$

- The equation of the elastic curve:

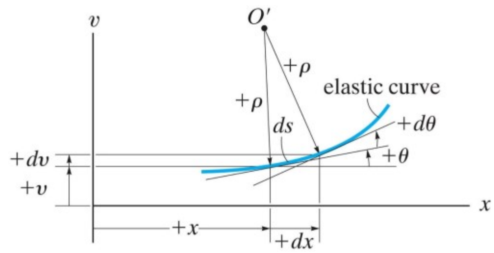
$$v = \int (\int \frac{M}{EI} dx) d_x$$

The internal moment in regions AB, BC & CD must be written in terms of $x_1$, $x_2$ and $x_3$



- Once these <u>functions are integrated</u> & the <u>constants determined</u>, the functions will give the **<u>slope</u>** & **<u>deflection</u>** for each region of the beam.
- It is important to use the proper sign for M as established by the sign convention used in derivation.
  - **+ve $v$ is upward**.
  - **+ve slope angle**, will be measured **counterclockwise from the x-axis**.

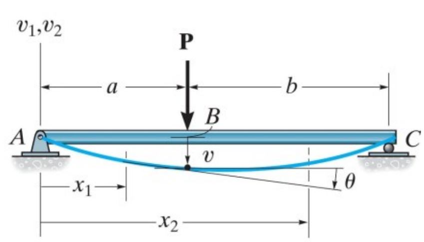

(a)




(b)



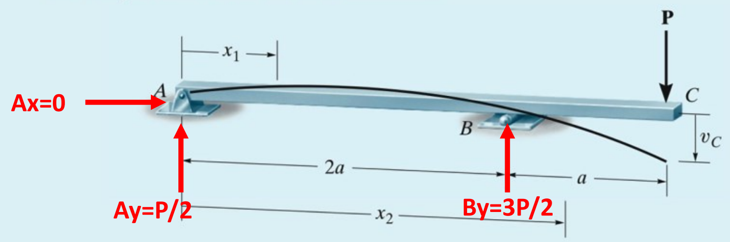
- The constants of integration are determined <u>by evaluating the functions for slope or displacement at a particular point on the beam where the value of the function is known.</u> These values are called **boundary conditions.**



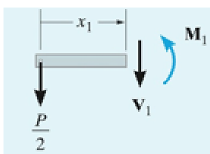

Once **the functions for the slope & deflections are obtained**, they must give **the same values for slope & deflection at point B.**

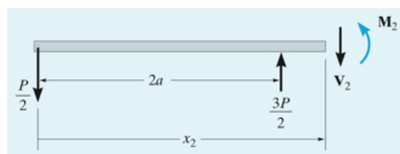
# Example 1

The beam in Fig. 7–14*a* is subjected to a load **P** at its end. Determine the displacement at *C*. *EI* is constant.

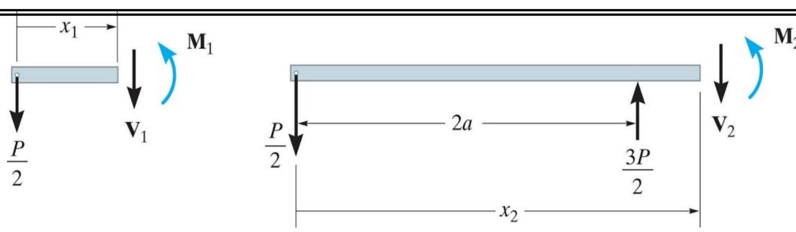



(a)






**Moment Functions.**

$$M_1 = -\frac{P}{2}x_1 \qquad 0 \le x_1 \le 2a$$

$$M_2 = -\frac{P}{2}x_2 + \frac{3P}{2}(x_2 - 2a)$$

$$= Px_2 - 3Pa \qquad 2a \le x_2 \le 3a$$

**Slope and Elastic Curve.**

$$\boxed{\frac{d^2v}{dx^2} = \frac{M}{EI}}$$

For $x_1$,

$$EI\frac{d^2v_1}{dx_1^2} = -\frac{P}{2}x_1$$

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1 \qquad (1)$$

$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \qquad (2)$$

For $x_2$, $$EI\frac{d^2v_2}{dx_2^2} = Px_2 - 3Pa$$

$$EI\frac{dv_2}{dx_2} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3 \qquad (3)$$

$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \qquad (4)$$

**C1,C2,C3, and C4??**

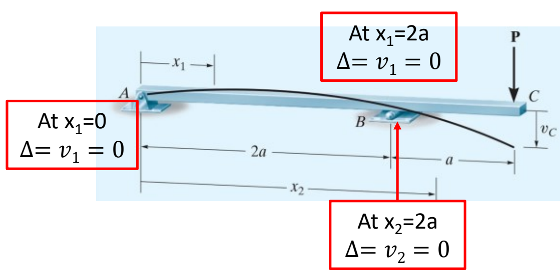
**4 unknown ⟶ 4 Equations**

**Boundary Conditions (B.C)**



$$EIv_1 = -\frac{P}{12}x_1^3 + C_1x_1 + C_2 \quad \text{........(2)}$$

$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \quad \text{........(4)}$$



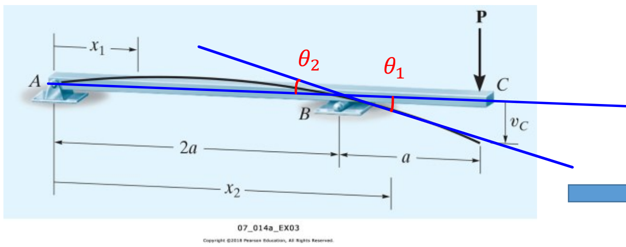

$v_1 = 0$ at $x_1 = 0$; $\quad 0 = 0 + 0 + C_2$

$$\boxed{C_2 = 0}$$

$v_1 = 0$ at $x_1 = 2a$; $\quad 0 = -\frac{P}{12}(2a)^3 + C_1(2a) + C_2$

$$\boxed{C_1 = \frac{Pa^2}{3}}$$

$v_2 = 0$ at $x_2 = 2a$; $\quad 0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$




$\theta_1 = \theta_2$

$$\frac{dv_1}{dx_1} = \frac{dv_2}{dx_2}$$ at X1=X2=2a

$$EI\frac{dv_1}{dx_1} = -\frac{P}{4}x_1^2 + C_1$$ ........(1)

$$EI\frac{dv_2}{dx_2} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3$$ ........(3)

$$\boxed{C_1 = \frac{Pa^2}{3}}$$ Previous slide

$$\frac{dv_1(2a)}{dx_1} = \frac{dv_2(2a)}{dx_2}; \quad -\frac{P}{4}(2a)^2 + C_1 = \frac{P}{2}(2a)^2 - 3Pa(2a) + C_3$$

$$\boxed{C_3 = \frac{10}{3}Pa^2}$$

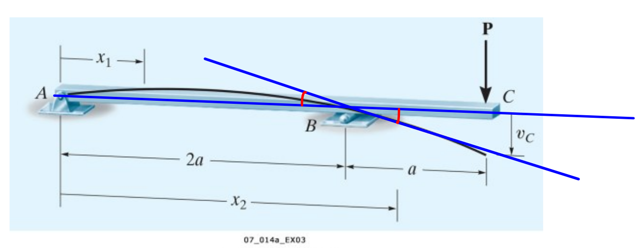


$$v_2 = 0 \text{ at } x_2 = 2a; \quad 0 = \frac{P}{6}(2a)^3 - \frac{3}{2}Pa(2a)^2 + C_3(2a) + C_4$$

$$\boxed{C_4 = -2Pa^3}$$

$$C_1 = \frac{Pa^2}{3} \quad C_2 = 0 \quad C_3 = \frac{10}{3}Pa^2 \quad C_4 = -2Pa^3$$

**To find the displacement at C (X=3a) , use eq (4)**




$$EIv_2 = \frac{P}{6}x_2^3 - \frac{3}{2}Pax_2^2 + C_3x_2 + C_4 \qquad (4)$$

$$v_2 = \frac{P}{6EI}x_2^3 - \frac{3Pa}{2EI}x_2^2 + \frac{10Pa^2}{3EI}x_2 - \frac{2Pa^3}{EI}$$

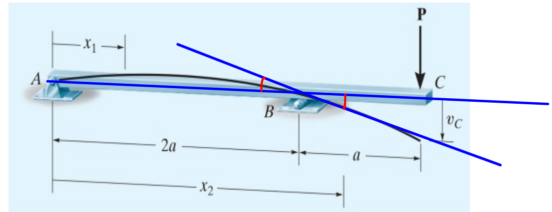
The displacement at $C$ is determined by setting $x_2 = 3a$. We get

$$\boxed{v_C = -\frac{Pa^3}{EI}}$$

(-) Downward

## $\theta_C$?????

$$EI\frac{dv_2}{dx_2} = \frac{P}{2}x_2^2 - 3Pax_2 + C_3 \qquad (3)$$




$$EI\,\frac{dv_2}{dx_2} = \frac{P}{2}x_2^2 - 3Pax_2 + \frac{10}{3}Pa^2$$

The slope at C is determined by setting $x = 3a$

$$EI\,\frac{dv_2}{dx_2} = \frac{P}{2}(3a)^2 - 3Pa(3a) + \frac{10}{3}Pa^2$$

$$\rightarrow \frac{dv_2}{dx_2} = \theta_C = -\frac{\boldsymbol{7Pa^2}}{\boldsymbol{6EI}}$$

**(-) Clockwise from the x-axis**



## Conjugate-Beam Method

**Heinrich Müller-Breslau**
**(1851–1925)**

**German Civil Engineer.**

**He developed the Conjugate-Beam Method in 1885**

**Also known as "Method of Elastic Weights"**

# Conjugate-Beam Method

- ❑ The basis for the method comes from **similarity equations**
- ❑ To show this similarity, we can write these equations as shown:

$$\frac{dV}{dx} = w \qquad \frac{d^2M}{dx^2} = w$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \qquad \frac{d^2v}{dx^2} = \frac{M}{EI}$$

Or integrating,

$$V = \int w\, dx \qquad M = \int\left[\int w\, dx\right] dx$$

$$\updownarrow \quad \updownarrow \qquad\qquad \updownarrow \quad \updownarrow$$

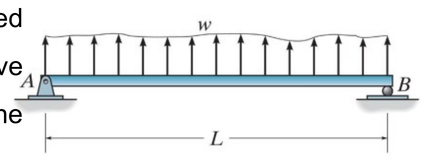
$$\theta = \int\left(\frac{M}{EI}\right) dx \qquad v = \int\left[\int\left(\frac{M}{EI}\right) dx\right] dx$$

Here the shear **V** compares with the **slope $\theta$**, the moment **M** compares with the **displacement $v$**, and the **external load $w$** compares with the $\frac{M}{EI}$ **diagram**.

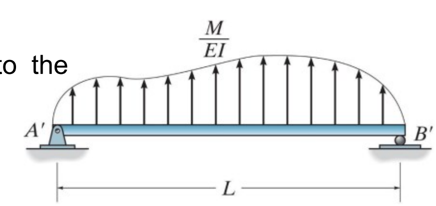


To make use of this comparison we will now consider a beam having the same length as the real beam but referred to as the **"conjugate beam",**

The conjugate beam is loaded with the M/EI diagram derived from the load **w** on the real beam From the above comparisons, we can state **2 theorems** related to the conjugate beam:



real beam



conjugate beam

## Theorem 1

The slope at a point in the real beam is numerically equal to the shear at the corresponding point in the conjugate beam.

## Theorem 2

The displacement of a point in the real beam is numerically equal to the moment at the corresponding point in the conjugate beam.

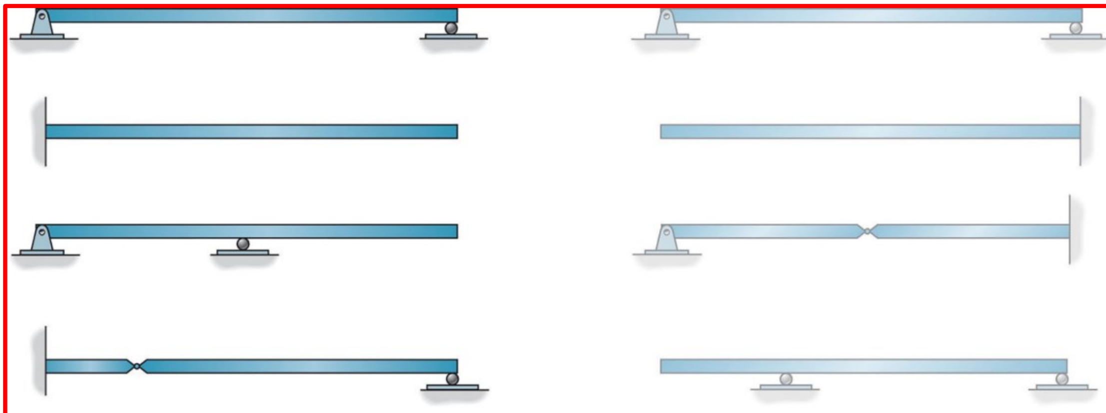
## Conjugate-Beam Supports

- When drawing the conjugate beam, it is important that the shear & moment developed at the supports of the conjugate beam **account** for the corresponding slope & displacement of the real beam at its supports.
- Consequently, from Theorem 1 & 2, the conjugate beam must be supported by a pin or roller since this support has zero moment but has a shear or end reaction.
- When the real beam is fixed supported, both beam has a free end since at this end there is zero shear & moment



**TABLE 8.2**

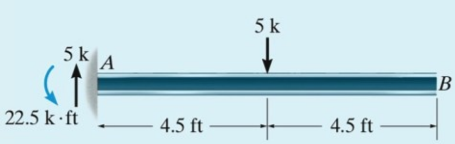
| | Real Beam | | Conjugate Beam | |
|---|---|---|---|---|
| 1) | $\theta$, $\Delta = 0$ | pin | $V$, $M = 0$ | pin |
| 2) | $\theta$, $\Delta = 0$ | roller | $V$, $M = 0$ | roller |
| 3) | $\theta = 0$, $\Delta = 0$ | fixed | $V = 0$, $M = 0$ | free |
| 4) | $\theta$, $\Delta$ | free | $V$, $M$ | fixed |
| 5) | $\theta$, $\Delta = 0$ | internal pin | $V$, $M = 0$ | hinge |
| 6) | $\theta$, $\Delta = 0$ | internal roller | $V$, $M = 0$ | hinge |
| 7) | $\theta$, $\Delta$ | hinge | $V$, $M$ | internal roller |





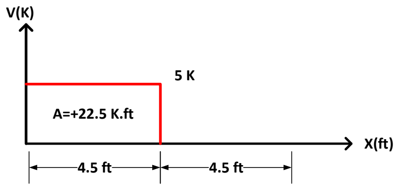

## Example 2

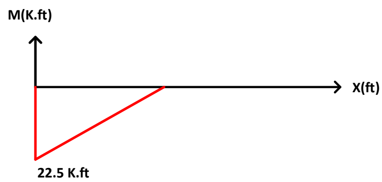
Determine the slope and displacement at point $B$ of the steel beam shown in Fig. 7–25*a*. The reactions are given. $E = 29(10^3)$ ksi, $I = 800$ in$^4$.




real beam
(a)

**Fig. 7–25**

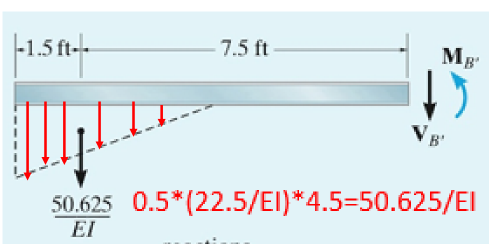




conjugate beam





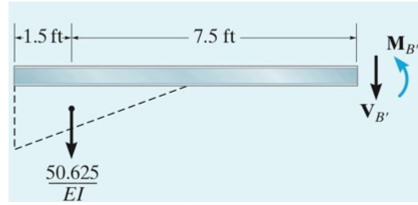

$$+\uparrow \Sigma F_y = 0; \qquad -\frac{50.625\ \text{k} \cdot \text{ft}^2}{EI} - V_{B'} = 0$$

$$\theta_B = V_{B'} = -\frac{50.625\ \text{k} \cdot \text{ft}^2}{EI}$$

$$= \frac{-50.625\ \text{k} \cdot \text{ft}^2}{[29(10^3)\ \text{k/in}^2\ (12\ \text{in.})^2/\text{ft}^2]\ [800\ \text{in}^4(1\ \text{ft}^4/12\ \text{in}^4)]}$$
$$= -0.000314\ \text{rad}$$

***-θ: Clockwise***

$$\circlearrowleft + \Sigma M_{B'} = 0; \qquad \frac{50.625 \text{ k}\cdot\text{ft}^2}{EI}(7.5 \text{ ft}) + M_{B'} = 0$$




$$\Delta_B = M_{B'} = -\frac{379.69 \text{ k}\cdot\text{ft}^3}{EI}$$

$$= \frac{-379.69 \text{ k}\cdot\text{ft}^3}{[29(10^3) \text{ k/in}^2(12 \text{ in.})^2/\text{ft}^2][800 \text{ in}^4(1 \text{ ft}^4/12 \text{ in}^4)]}$$

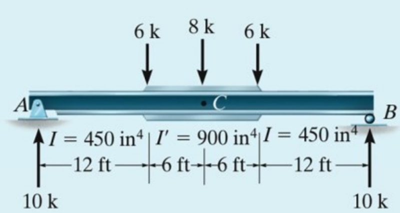
$$= -0.002357 \text{ ft} = -0.0283 \text{ in.}$$

-Δ: ***Downward***




# Example 3

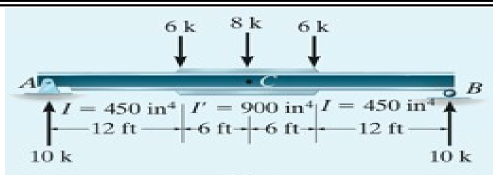
The girder in Fig. 7–27*a* is made from a uniform beam and reinforced at its center with cover plates where its moment of inertia is larger. The 12-ft end segments have a moment of inertia of $I = 450 \text{ in}^4$, and the center portion has a moment of inertia of $I' = 900 \text{ in}^4$. Determine the displacement at the center $C$. Take $E = 29(10^3)$ ksi. The reactions are given.

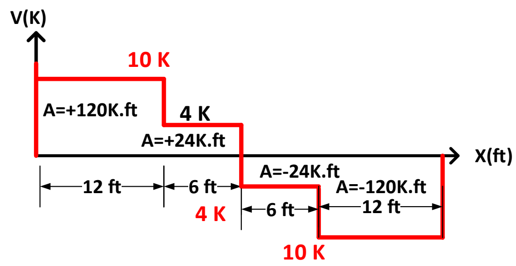


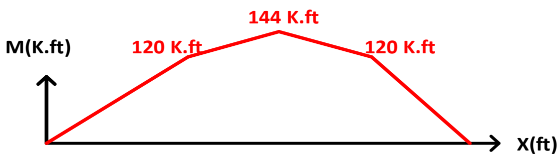

real beam

(a)

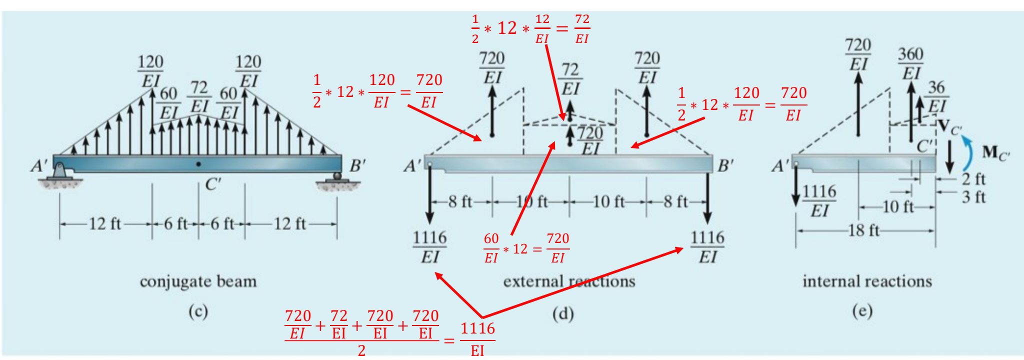
**Fig. 7–27**





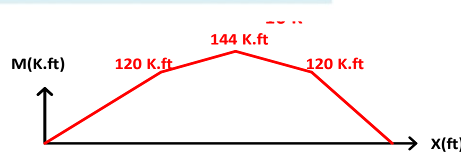




$$\circlearrowleft + \Sigma M_{C'} = 0; \quad \frac{1116}{EI}(18) - \frac{720}{EI}(10) - \frac{360}{EI}(3) - \frac{36}{EI}(2) + M_{C'} = 0$$

$$M_{C'} = -\frac{11\,736 \text{ k}\cdot\text{ft}^3}{EI}$$

Substituting the numerical data for $EI$ and converting feet to inches, we have

$$\Delta_C = M_{C'} = -\frac{11\,736 \text{ k}\cdot\text{ft}^3(1728 \text{ in}^3/\text{ft}^3)}{29(10^3)\text{ k/in}^2(450 \text{ in}^4)} = -1.55 \text{ in.} \quad \textit{Ans.}$$



The Hashemite University
Faculty of Engineering
Department of Civil Engineering

# CE 315: Structural Analysis

## Chapter 6: Deflections Using Energy Methods

Dr. Ra'ed Al-Mazaidh

## External Work and Strain Energy

- For more complicated loadings or for structures such as trusses & frames, it is suggested that **energy methods** be used for the computations.
- Most energy methods are based on the conservation of energy principal:

**" Work done by all external forces acting on a structure, $U_e$ , is transformed into internal work or strain energy $U_i$"**

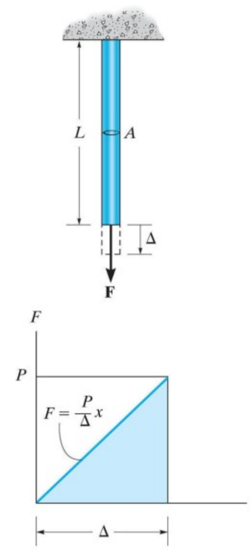
$$\boxed{U_e = U_i}$$

- If the material's elastic limit is not exceeded, the elastic strain energy will return the structure to its undeformed state when the loads are removed.

## External Work—Force.

- ❑ When a force ***F*** undergoes a displacement $dx$ in the <u>same direction as the force</u>, the **work** done is: $dU_e = F\,dx.$
- ❑ If the total displacement is $x$, the work becomes: $U_e = \int_0^x F\,dx$
- ❑ Consider the effect caused by an axial force applied to the end of a bar
- ❑ F is gradually increased from 0 to some limiting value $F = P$
- ❑ The final elongation of the bar becomes Δ
- ❑ If the material has a linear elastic response, then F = (P/Δ)x
- ❑ Substituting into equation $U_e = \int_0^x F\,dx$ and integrating from 0 to Δ , we get

$$\boxed{U_e = \tfrac{1}{2}P\Delta}$$




(a)

08_001a





## External Work—Moment

- ❑ The work of a moment = magnitude of the moment (M) x the angle ($d\theta$) through which it rotates.

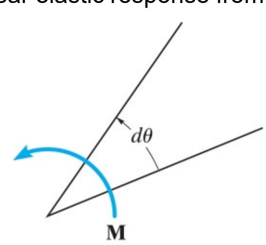
$$dU_e = M\,d\theta$$

- ❑ If the total angle of rotation is $\theta$ rad, the work becomes:

$$U_e = \int_0^\theta M\,d\theta$$

- ❑ If the moment is applied gradually to a structure having a linear elastic response from 0 to M, then the work done is:

$$\boxed{U_e = \tfrac{1}{2}M\theta}$$



## Strain Energy—Axial Force.

- When an axial force N is applied gradually to the bar, it will strain the material such that the external work done by N **will be converted** into strain energy.
- Provided the material is linearly elastic, Hooke's Law is valid

$$\sigma = E\epsilon, \quad \text{..........(1)}$$

- If the bar has a constant x-sectional area (A) and length (L)

$$\sigma = N/A \quad \text{.............(2)}$$

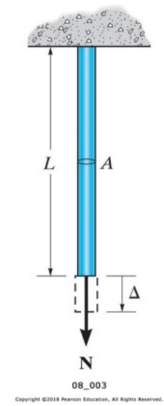
$$\epsilon = \Delta/L \quad \text{............(3)}$$

Substitute Eq. (2) and Eq. (3) into Eq. (1)

$$\longrightarrow \quad \Delta = \frac{NL}{AE} \quad \text{.....................(4)}$$

But $\quad U_e = \frac{1}{2}P\Delta$ ......................(5)

Substitute Eq. (4) into Eq. (5) with P=N $\Rightarrow$ $\boxed{U_i = \frac{N^2L}{2AE}}$





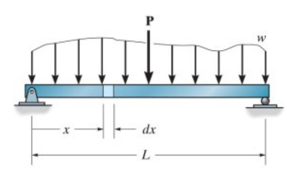

## Strain Energy—Bending

- Consider the beam in Fig. (a), $P$ & $w$ are gradually apply
- These loads create an **internal moment M** in the beam at a section located a distance $x$ from the left support.
- Consequently, the strain energy or work stored in the element can be determined since the internal moment is gradually developed
- The resulting rotation of the differential element $dx$, Fig.(b), can be found from the following equation, that is:

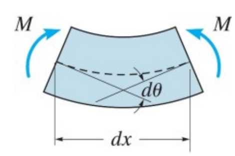
$$d\theta = (M/EI)\,dx.$$

- Since the internal moment is gradually developed , the strain energy, or work stored in the element, is determined from eq. $U_e = \frac{1}{2}M\theta$

Hence, $$dU_i = \frac{M^2\,dx}{2EI}$$




(a)




(b)

- The strain energy for the beam is determined by integrating this result over the beam's entire length *L*. The result is

$$\boxed{U_i = \int_0^L \frac{M^2\,dx}{2EI}}$$

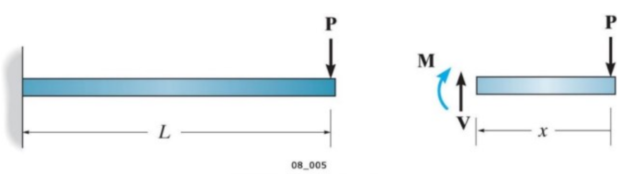


## Principle of Work and Energy

- Consider finding the displacement Δ at a point where the force P is applied to the cantilever beam.
- The external work:

$$U_e = \tfrac{1}{2}P\Delta$$

- To obtain the resulting strain energy, we must first determine the internal moment as a function of position x in the beam and then apply. $U_i = \int_0^L \frac{M^2\,dx}{2EI}$




- In this case $M = -Px$, so that:

$$U_i = \int_0^L \frac{M^2\,dx}{2EI} = \int_0^L \frac{(-Px)^2\,dx}{2EI} = \frac{1}{6}\frac{P^2L^3}{EI}$$

- The external work:

$$U_e = \tfrac{1}{2}P\Delta$$

$$U_e = U_i$$

$$\frac{1}{2}P\Delta = \frac{1}{6}\frac{P^2L^3}{EI}$$

$$\Delta = \frac{PL^3}{3EI}$$

**Limitations**:

- ✓ Although the solution here is quite direct, application of this method is limited to only a few select problems.
- ✓ It will be noted that only one load may be applied to the structure.
- ✓ If more than one load were applied, there would be an unknown displacement under each load, and yet it is possible to write only one "work" equation for the beam
- ✓ Only the displacement under the force can be obtained, since the external work depends upon both the force and its corresponding displacement.

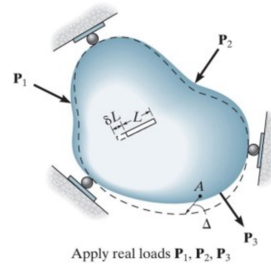


## Principle of Virtual Work

- Developed by John Bernoulli in 1717 and is sometimes referred to as the **unit-load method.**
- It provides a general means of obtaining the displacement and slope at a specific point on a structure, be it a beam, frame, or truss.
- If we take a deformable structure of any shape or size & apply a series of external loads P to it, it will cause internal loads $u$ at points throughout the structure
- As a consequence of these loadings, external displacement $\Delta$ will occur at the P loads & internal displacement $\delta$ will occur at each point of internal loads $u$
- In general, these displacement do not have to be elastic, & they may not be related to the loads
- **In general, then, the principle of work and energy states**

$$\underset{\substack{\text{Work of}\\ \text{External Loads}}}{\Sigma P\Delta} = \underset{\substack{\text{Work of}\\ \text{Internal Loads}}}{\Sigma u\delta}$$

- Consider the structure (or body) to be of arbitrary shape.
- Suppose it is necessary to determine the displacement $\Delta$ of point A on the body caused by the "real loads" $P_1$, $P_2$ and $P_3$
- It is to be understood that these loads cause no movement of the supports
- They can strain the material beyond the elastic limit.
- Since no external load acts on the body at A and in the direction of $\Delta$, the displacement $\Delta$ can be determined by first placing on the body a "**virtual**" load such that this force P' acts in the same direction as $\Delta$
- We will choose P' to have a unit magnitude, **P' =1.** Once the virtual loadings are applied, then the body is subjected to the real loads $P_1$, $P_2$ and $P_3$.
- Point A will be displaced an amount $\Delta$ causing the element to deform an amount $dL$



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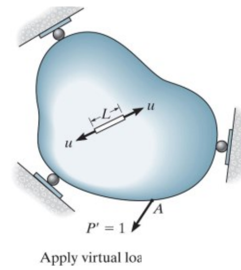


(b)



- As a result, the external virtual force P' & internal load $u$ "ride along" by $\Delta$ and $dL$ & therefore, perform external virtual work of $1.\Delta$ on the body and internal virtual work of $u.dL$ on the element.

virtual loadings

$$1 \cdot \Delta = \Sigma u \cdot dL$$

real displacements

where

$P' = 1$ = external virtual unit load acting in the direction of $\Delta$.

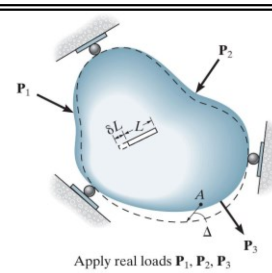
$u$ = internal virtual load acting on the element in the direction of $dL$.

$\Delta$ = external displacement caused by the real loads.

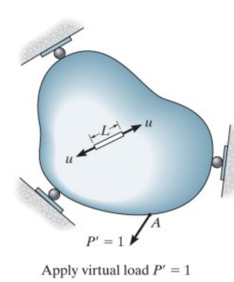
$dL$ = internal deformation of the element caused by the real loads.

- By choosing P' = 1, it can be seen from the solution for $\Delta$ follows directly since

$$(1)\Delta = \Sigma u \, dL.$$



(a)



(b)

- A virtual couple moment $M'$ having a unit magnitude is applied at this point
- This couple moment causes a virtual load $u_\theta$ in one of the elements of the body.
- Assuming that the real loads deform the element an amount $dl$, the rotation $\theta$ can be found from the virtual – work equation.

virtual loadings

$$1 \cdot \theta = \Sigma u_\theta \cdot dL$$

real displacements

where

$M' = 1$ = external virtual unit couple moment acting in the direction of $\theta$.

$u_\theta$ = internal virtual load acting on an element in the direction of $dL$.

$\theta$ = external rotational displacement or slope in radians caused by the real loads.

$dL$ = internal deformation of the element caused by the real loads.



## Method of Virtual Work: Trusses

We can use the method of virtual work to determine the **displacement of a truss joint** when the truss is subjected to an external loading, temperature change, or fabrication errors.

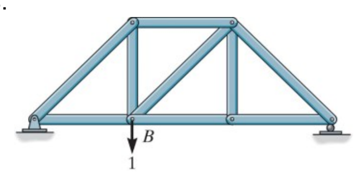
**External Loading**

- For the purpose of explanation let us consider the vertical displacement of joint *B* of the truss in Fig(a).
- A typical element of the truss would be one of its members having a length $L$
- If the applied loadings P1 and P2 cause a linear elastic material response, then this element deforms an amount:

$$\Delta L = \frac{NL}{AE}$$

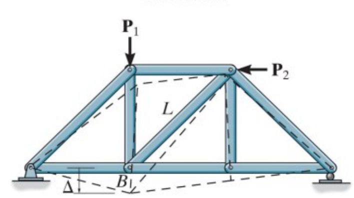
where N is the normal or axial force in the member, caused by the loads

$$1 \cdot \Delta = \Sigma u \cdot dL$$




Apply virtual unit load to $B$

(a)




Apply real loads $\mathbf{P}_1$, $\mathbf{P}_2$

(b)

$$1 \cdot \Delta = \sum \frac{nNL}{AE}$$

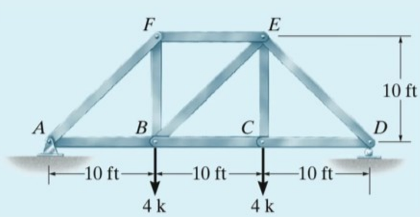
where

$1$ = external virtual unit load acting on the truss joint in the stated direction of $\Delta$.
$n$ = internal virtual normal force in a truss member caused by the external virtual unit load.
$\Delta$ = external joint displacement caused by the real loads on the truss.
$N$ = internal normal force in a truss member caused by the real loads.
$L$ = length of a member.
$A$ = cross-sectional area of a member.
$E$ = modulus of elasticity of a member.

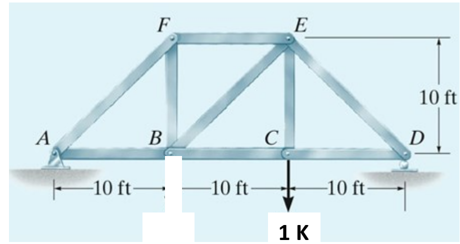


## Example 1

Determine the vertical displacement of joint $C$ of the steel truss shown in Fig. 8–8*a*. The cross-sectional area of each member is $A = 0.5\ \text{in}^2$ and $E = 29(10)^3$ ksi.

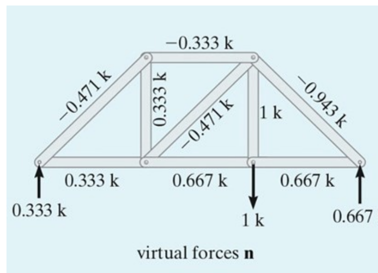


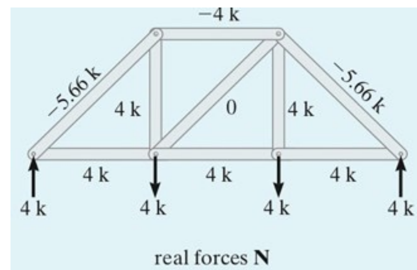
1. Apply 1 unit force (virtual load) (1 K) at point C
No real loads are applied




2. Analyze this truss by method of section and/or method of joints to find the virtual forces n

3. Analyze the real truss by method of section and/or method of joints to find the real forces N






**Virtual-Work Equation.**

$$1\text{ k}\cdot\Delta_{C_v} = \sum\frac{nNL}{AE}$$

4. Construct a table contains the following for each member:

| Member | $n$ (k) | $N$ (k) | $L$ (ft) | $nNL$ (k$^2\cdot$ft) |
|---|---|---|---|---|
| $AB$ | 0.333 | 4 | 10 | 13.33 |
| $BC$ | 0.667 | 4 | 10 | 26.67 |
| $CD$ | 0.667 | 4 | 10 | 26.67 |
| $DE$ | −0.943 | −5.66 | 14.14 | 75.42 |
| $FE$ | −0.333 | −4 | 10 | 13.33 |
| $EB$ | −0.471 | 0 | 14.14 | 0 |
| $BF$ | 0.333 | 4 | 10 | 13.33 |
| $AF$ | −0.471 | −5.66 | 14.14 | 37.71 |
| $CE$ | 1 | 4 | 10 | 40 |
| | | | | $\Sigma = 246.47$ |

5. Substitute :the value from the table and the values of E,A in the equation

$$1\text{ k}\cdot\Delta_{C_v} = \sum\frac{nNL}{AE} = \frac{246.47\text{ k}^2\cdot\text{ft}}{AE}$$

$$1\text{ k}\cdot\Delta_{C_v} = \frac{(246.47\text{ k}^2\cdot\text{ft})\,(12\text{ in./ft})}{(0.5\text{ in}^2)(29(10^3)\text{ k/in}^2)}$$

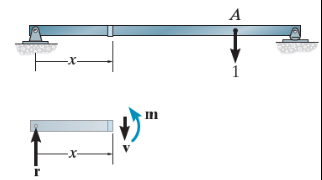
$$\Delta_{C_v} = 0.204\text{ in.}$$

## Method of virtual work: Beams & Frames

- ❖ The method of virtual work can also be applied to deflection problems involving **beams and frames**.
- ❖ Strains due **to bending** are the primary cause of beam or frame deflections.
- ➢ To compute Δ, **a virtual unit load** acting in the direction of is placed on the beam at A.
- ➢ The **internal virtual moment** $\boldsymbol{m}$ is determined by the method of sections at an arbitrary location $x$ from the left support
- ➢ When point A is displaced , the element $dx$ deforms or rotates $d\theta = \left(\frac{M}{EI}\right) dx$
- ➢ The **internal virtual work** done by the moment $\boldsymbol{m}$ is $md\theta = m\left(\frac{M}{EI}\right) dx$.

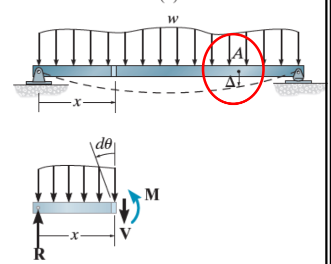
$$\overbrace{1 \cdot \Delta = \Sigma u}^{\text{virtual loadings}} \cdot dL$$

$$1 \cdot \underbrace{\Delta = \Sigma u \cdot dL}_{\text{real displacements}}$$




Apply virtual unit load to point *A*

(a)




Apply real load *w*

(b)



$$\boxed{1 \cdot \Delta = \int_0^L \frac{mM}{EI} dx}$$

where

1 = external virtual unit load acting on the beam or frame in the direction of Δ.

$m$ = internal virtual moment in the beam or frame, expressed as a function of $x$ and caused by the external virtual unit load.

Δ = external displacement of the point caused by the real loads acting on the beam or frame.

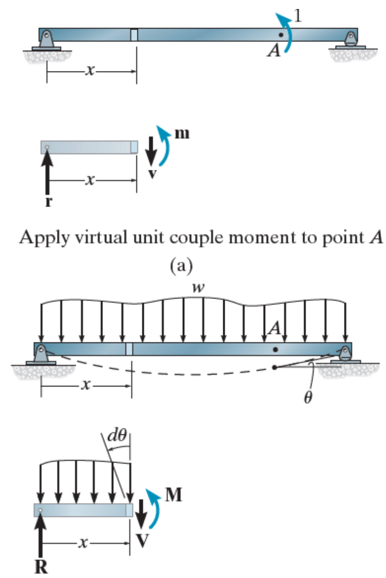
$M$ = internal moment in the beam or frame, expressed as a function of $x$ and caused by the real loads.

$E$ = modulus of elasticity of the material.

$I$ = moment of inertia of cross-sectional area, computed about the neutral axis.

- If the tangent rotation or **slope angle $\theta$** at a point on the beam's elastic curve is to be determined, **a unit couple moment** is applied at the point.
- The corresponding internal moment $m_\theta$ m have to be determined.

$$1 \cdot \theta = \int_0^L \frac{m_\theta M}{EI} dx$$

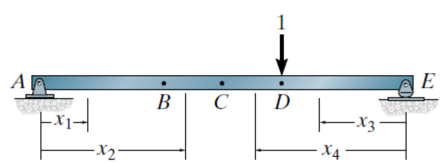


Apply virtual unit couple moment to point $A$

(a)

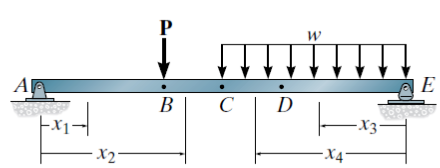


- ✓ If concentrated **forces or couple moments act on the beam** or the **distributed load is discontinuous**, **separate x coordinates** will have to chosen within regions that have no discontinuity of loading.
- ✓ It **is not necessary** that each $x$ have the same origin.
- ✓ The $x$ selected for determining the real moment $M$ in a particular region **must be the same** $x$ as that selected for determining the virtual moment $m$ or $m_\theta$ within the same region.



Apply virtual unit load

(a)



Apply real loads

(b)

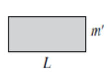
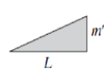
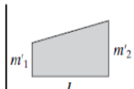
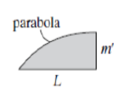
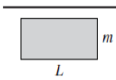
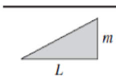
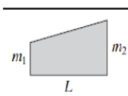
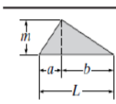
## Integration Using Tables

➢ When the structure is subjected to a **relatively simple loading**, and yet the **solution for a displacement requires several integrations**, a **tabular method** may be used to perform these integrations.

- The **moment diagrams** for each member are drawn first for <u>both</u> the **real and virtual loadings.**
- By matching these diagrams for $m$ and $M$ with those given in the table below (**It is on the inside front cover of the textbook**) , the integral $\int mM.dx$ can be determined from the appropriate formula

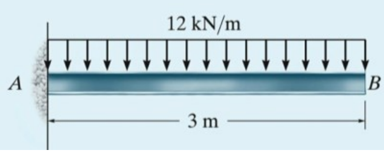


### Table for Evaluating $\int_0^L m\,m'\,dx$

| $\int_0^L m\,m'\,dx$ |   |   |   |   |   |
|---|---|---|---|---|---|
|   | $mm'L$ | $\frac{1}{2}mm'L$ | $\frac{1}{2}m(m'_1+m'_2)L$ | $\frac{2}{3}mm'L$ | $\frac{1}{3}mm'L$ |
|   | $\frac{1}{2}mm'L$ | $\frac{1}{3}mm'L$ | $\frac{1}{6}m(m'_1+2m'_2)L$ | $\frac{5}{12}mm'L$ | $\frac{1}{4}mm'L$ |
|   | $\frac{1}{2}m'(m_1+m_2)L$ | $\frac{1}{6}m'(m_1+2m_2)L$ | $\frac{1}{6}[m'_1(2m_1+m_2)+m'_2(m_1+2m_2)]L$ | $\frac{1}{12}[m'(3m_1+5m_2)]L$ | $\frac{1}{12}m'(m_1+3m_2)L$ |
|   | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'(L+a)$ | $\frac{1}{6}m[m'_1(L+b)+m_2(L+a)]$ | $\frac{1}{12}mm'\left(3+\frac{3a}{L}-\frac{a^2}{L^2}\right)L$ | $\frac{1}{12}mm'(1+\frac{a}{L}+\frac{a^2}{L^2})L$ |
|   | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'L$ | $\frac{1}{6}m(2m'_1+m'_2)L$ | $\frac{1}{4}mm'L$ | $\frac{1}{12}mm'L$ |

## Example 2

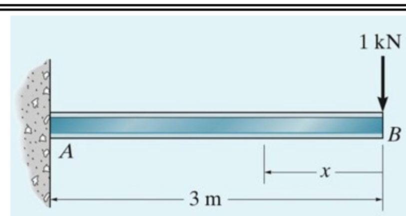
Determine the displacement of point $B$ of the steel beam shown in Fig. 8–17$a$. Take $E = 200$ GPa, $I = 71.1(10^6)$ mm$^4$.





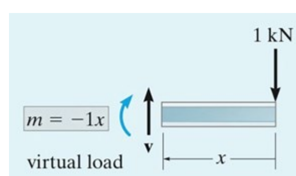

1. **Apply 1 unit force (virtual load) (1 KN) at point B**

**No real loads are applied**




2. **Take a section at a distance x from point B**

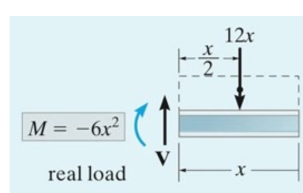
$$m = -1x$$




3. **Take a section at the same distance x (same reference) with real loads**

**No virtual load are applied**

$$M = -6x^2$$



$$1\ \text{kN} \cdot \Delta_B = \int_0^L \frac{mM}{EI}\, dx$$

$$m = -1x \qquad M = -6x^2$$

$$\int_0^3 \frac{(-1x)(-6x^2)\, dx}{EI}$$

$$1\ \text{kN} \cdot \Delta_B = \frac{121.5\ \text{kN}^2 \cdot \text{m}^3}{EI}$$

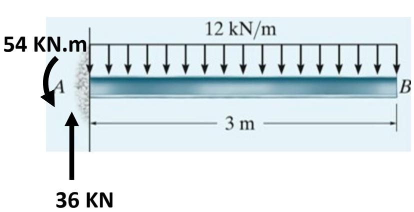
$$\Delta_B = \frac{121.5\ \text{kN} \cdot \text{m}^3}{200(10^6)\ \text{kN/m}^2[71.1(10^6)\ \text{mm}^4](10^{-12}\ \text{m}^4/\text{mm}^4)}$$

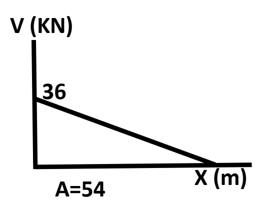
$$= 0.008544\ \text{m} = 8.54\ \text{mm}$$

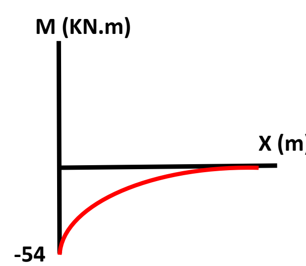


## Another method : Using Tables

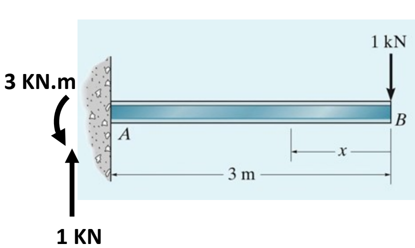
**1. Draw B.M.D for real loads**

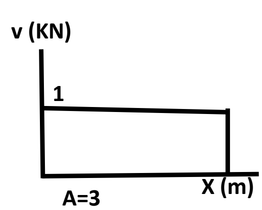


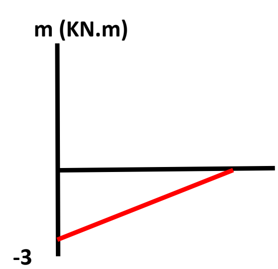




**2. Draw B.M.D for real loads**



### 3. Select the formula from the table

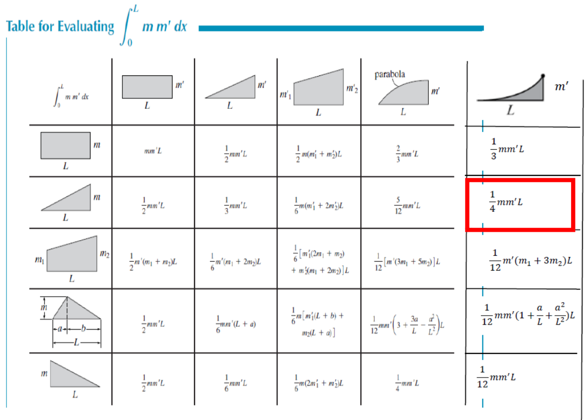
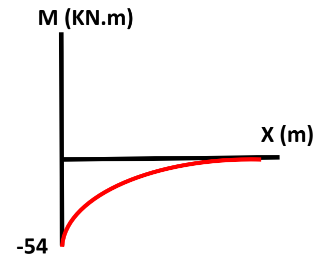
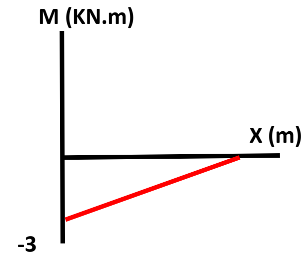


Table for Evaluating $\int_0^L m\, m'\, dx$

| $\int_0^L m m' dx$ | rectangle $m'$, $L$ | triangle $m'$, $L$ | trapezoid $m'_1$, $m'_2$, $L$ | parabola $m'$, $L$ | $m'$, $L$ |
|---|---|---|---|---|---|
| rectangle $m$, $L$ | $mm'L$ | $\frac{1}{2}mm'L$ | $\frac{1}{2}m(m'_1+m'_2)L$ | $\frac{2}{3}mm'L$ | $\frac{1}{3}mm'L$ |
| triangle $m$, $L$ | $\frac{1}{2}mm'L$ | $\frac{1}{3}mm'L$ | $\frac{1}{6}m(m'_1+2m'_2)L$ | $\frac{5}{12}mm'L$ | $\boxed{\frac{1}{4}mm'L}$ |
| trapezoid $m_1$, $m_2$, $L$ | $\frac{1}{2}m'(m_1+m_2)L$ | $\frac{1}{6}m'(m_1+2m_2)L$ | $\frac{1}{6}[m'_1(2m_1+m_2)+m'_2(m_1+2m_2)]L$ | $\frac{1}{12}[m'(3m_1+5m_2)]L$ | $\frac{1}{12}m'(m_1+3m_2)L$ |
| triangle $m$, $a$, $b$, $L$ | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'(L+a)$ | $\frac{1}{6}m[m'_1(L+b)+m'_2(L+a)]$ | $\frac{1}{12}mm'\left(3+\frac{3a}{L}-\frac{a^2}{L^2}\right)L$ | $\frac{1}{12}mm'(1+\frac{a}{L}+\frac{a^2}{L^2})L$ |
| triangle $m$, $L$ | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'L$ | $\frac{1}{6}m(2m'_1+m'_2)L$ | $\frac{1}{4}mm'L$ | $\frac{1}{12}mm'L$ |




$$= \int_0^L mm'.dx = \frac{1}{4}mm'L = \frac{1}{4} * -54 * -3 * 3 = 121.5$$




$$1\ \text{kN} \cdot \Delta_B = \frac{121.5\ \text{kN}^2 \cdot \text{m}^3}{EI}$$

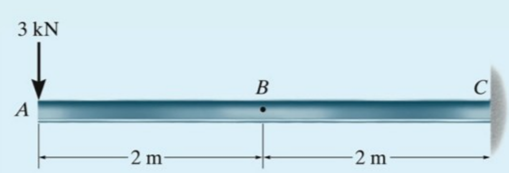
$$\Delta_B = \frac{121.5\ \text{kN} \cdot \text{m}^3}{200(10^6)\ \text{kN/m}^2[71.1(10^6)\ \text{mm}^4](10^{-12}\ \text{m}^4/\text{mm}^4)}$$

$$= 0.008544\ \text{m} = 8.54\ \text{mm}$$



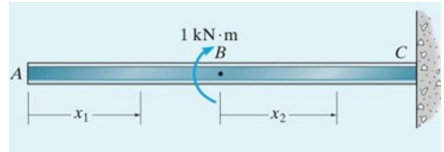
## Example 3

Determine the slope $\theta$ at point $B$ of the steel beam shown in Fig. 8–18*a*. Take $E = 200$ GPa, $I = 60(10^6)$ mm$^4$.

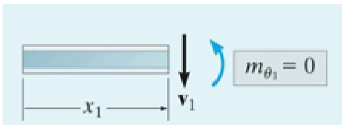


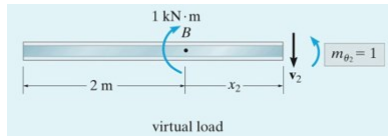
(a)

1. **Apply 1 unit moment (virtual moment) (1 KN.m) at point B (No real loads are applied)**

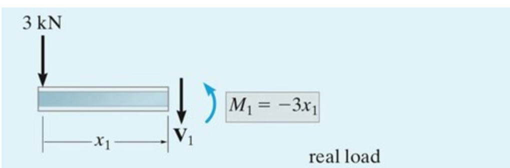


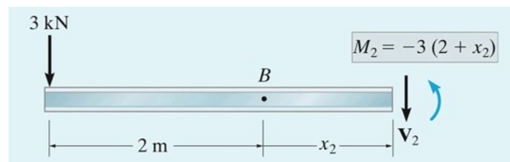

2. **Take 2 sections (why?)**




3. **Take 2 section at the same distances (X1,X2) (same reference) with real loads No virtual moment are applied**






**Virtual-Work Equation.** The slope at $B$ is thus

$$(1\ \text{kN}\cdot\text{m})\cdot\theta_B = \int_0^L \frac{m_\theta M}{EI}\,dx$$

$$= \int_0^2 \frac{(0)(-3x_1)\,dx_1}{EI} + \int_0^2 \frac{(1)[-3(2+x_2)]\,dx_2}{EI}$$

$$\theta_B = \frac{-18\ \text{kN}\cdot\text{m}^2}{EI}$$

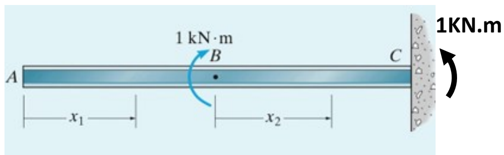
$$= \frac{-18\ \text{kN}\cdot\text{m}^2}{200(10^6)\ \text{kN/m}^2\left[60(10^6)\ \text{mm}^4\right](10^{-12}\ \text{m}^4/\text{mm}^4)}$$

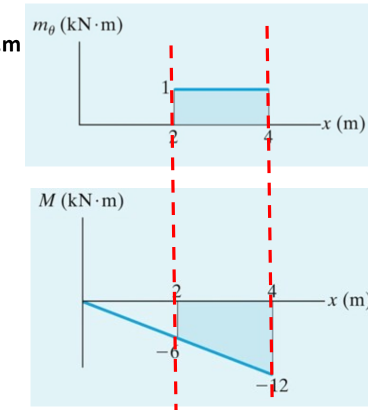
$$= -0.00150\ \text{rad} \qquad \textit{Ans.}$$

The *negative sign* indicates $\theta_B$ is *opposite* to the direction of the virtual couple moment

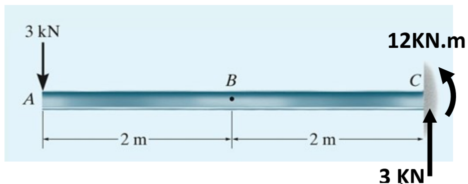
## Another method : Using Tables

**1. Draw B.M.D for Vrtual load**



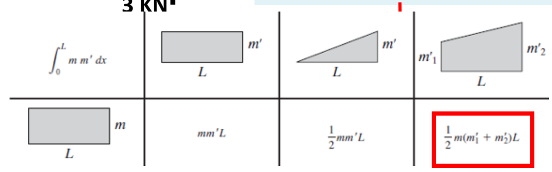

**2. Draw B.M.D for real loads**




**3. Use the tables**

$$= 0 + \int_0^L \frac{1}{2} m(m_1' + m_2')L.\,dx$$

$$= \frac{1}{2} * 1 * (-6 - 12) * 2 = -18$$



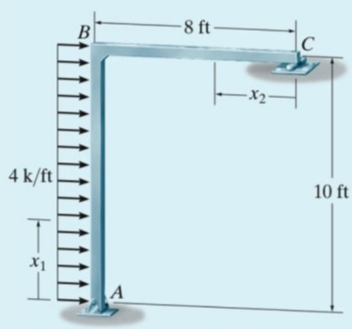

$$= \frac{-18 \text{ kN}\cdot\text{m}^2}{200(10^6)\text{ kN/m}^2[60(10^6)\text{ mm}^4](10^{-12}\text{ m}^4/\text{mm}^4)}$$

$$= -0.00150 \text{ rad} \qquad \textit{Ans.}$$



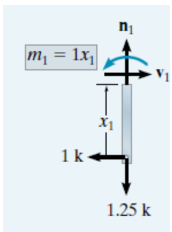
## Example 4

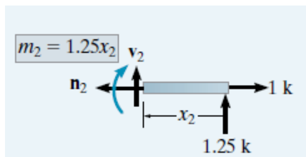
Determine the horizontal displacement of point $C$ on the frame shown in Fig. 8–20$a$. Take $E = 29(10^3)$ ksi and $I = 600$ in$^4$ for both members.

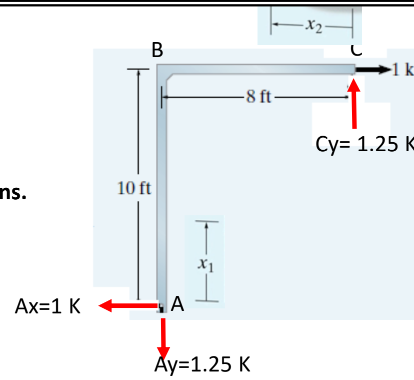


1. Apply 1 unit load (virtual load) (1 KN) at point B (No real loads are applied)

2. Take 2 sections (why?) then find moments at these sections.




$m_1 = 1x_1$ $\quad$ $m_2 = 1.25x_2$

$$\circlearrowright^{+} \sum M_A = -1(10) + Cy(8) = 0 \rightarrow Cy = 1.25K \uparrow$$

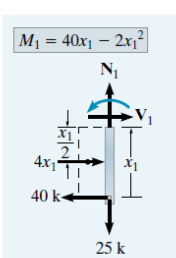
$$+\uparrow \sum F_y = -Ay + 1.25 = 0 \rightarrow Ay = 1.25K \downarrow$$

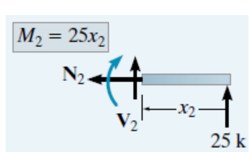
$$\xrightarrow{+} \sum F_X = -Ax + 1 = 0 \rightarrow Ax = 1\ K \leftarrow$$

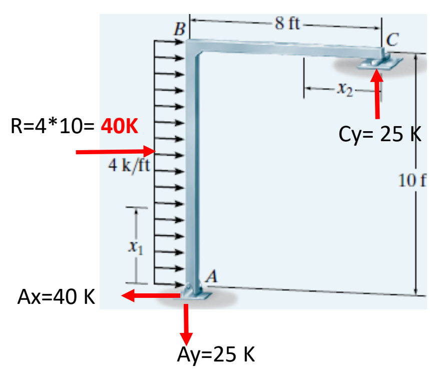


3. Apply the real loads (No virtual load are applied)

4. Take 2 sections , then find moments at these sections.




$M_1 = 40x_1 - 2x_1^2$ $\quad$ $M_2 = 25x_2$

$$\circlearrowright^{+} \sum M_A = -40(50) + Cy(8) = 0 \rightarrow Cy = 25\ K \uparrow$$

$$+\uparrow \sum F_y = -Ay + 50 = 0 \rightarrow Ay = 25\ K \downarrow$$

$$\xrightarrow{+} \sum F_X = -Ax + 40 = 0 \rightarrow Ax = 40\ K \leftarrow$$

$$1 \cdot \Delta_{C_h} = \int_0^L \frac{mM}{EI}dx \qquad = \int_0^{10} \frac{m_1 . M_1}{EI} . dx + \int_0^{8} \frac{m_2 . M_2}{EI} . dx$$

$m_1 = 1x_1$ $\qquad$ $M_1 = 40x_1 - 2x_1^2$

$m_2 = 1.25x_2$ $\qquad$ $M_2 = 25x_2$

$$= \int_0^{10} \frac{(1x_1)(40x_1 - 2x_1^2)dx_1}{EI} + \int_0^{8} \frac{(1.25x_2)(25x_2)dx_2}{EI}$$

$$\Delta_{C_h} = \frac{8333.3}{EI} + \frac{5333.3}{EI} = \frac{13\,666.7 \text{ k}\cdot\text{ft}^3}{EI}$$

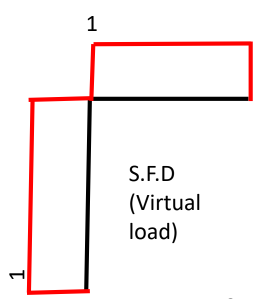
$$\Delta_{C_h} = \frac{13\,666.7 \text{ k}\cdot\text{ft}^2}{[29(10^3)\text{ k/in}^2((12)^2\text{ in}^2/\text{ft}^2)][600\text{ in }(\text{ft}^4/(12)^4\text{ in}^4)]}$$

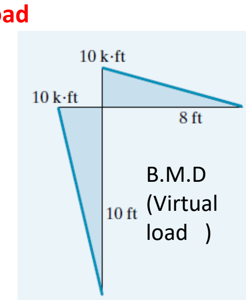
$$= 0.113 \text{ ft} = 1.36 \text{ in.}$$

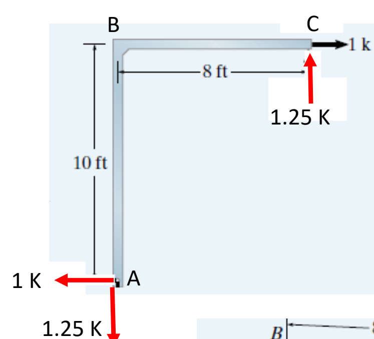


## Another method : Using Tables

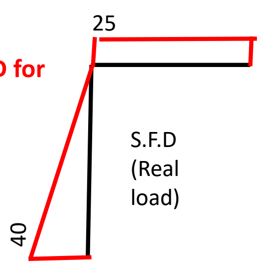
### 1. Draw B.M.D for Virtual load

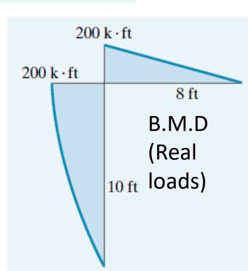


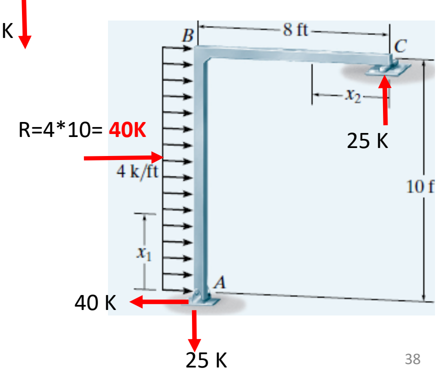



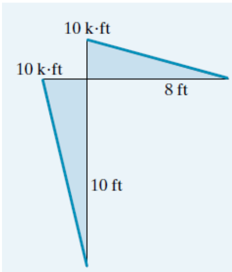




### 2. Draw B.M.D for the real loads





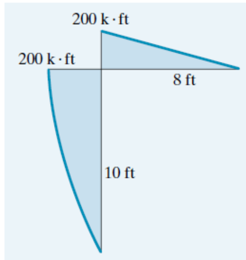




Table for Evaluating $\int_0^L m\, m'\, dx$

| $\int_0^L m\, m'\, dx$ | $m'$, $L$ | $m'$, $L$ | $m'_1$, $m'_2$, $L$ | parabola, $m'$, $L$ | $m'$, $L$ |
|---|---|---|---|---|---|
| $m$, $L$ | $mm'L$ | $\frac{1}{2}mm'L$ | $\frac{1}{2}m(m'_1 + m'_2)L$ | $\frac{2}{3}mm'L$ | $\frac{1}{3}mm'L$ |
| $m$, $L$ | $\frac{1}{2}mm'L$ | $\frac{1}{3}mm'L$ | $\frac{1}{6}m(m'_1 + 2m'_2)L$ | $\frac{5}{12}mm'L$ | $\frac{1}{4}mm'L$ |
| $m_1$, $m_2$, $L$ | $\frac{1}{2}m'(m_1 + m_2)L$ | $\frac{1}{6}m'(m_1 + 2m_2)L$ | $\frac{1}{6}[m'_1(2m_1 + m_2) + m'_2(m_1 + 2m_2)]L$ | $\frac{1}{12}[m'(3m_1 + 5m_2)]L$ | $\frac{1}{12}m'(m_1 + 3m_2)L$ |
| $m$, $a$, $b$, $L$ | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'(L + a)$ | $\frac{1}{6}m[m'_1(L + b) + m'_2(L + a)]$ | $\frac{1}{12}mm'\left(3 + \frac{3a}{L} - \frac{a^2}{L^2}\right)L$ | $\frac{1}{12}mm'(1 + \frac{a}{L} + \frac{a^2}{L^2})L$ |
| $m$, $L$ | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'L$ | $\frac{1}{6}m(2m'_1 + m'_2)L$ | $\frac{1}{4}mm'L$ | $\frac{1}{12}mm'L$ |

$$\int mM\, dx = \frac{5}{12}mm'L + \frac{1}{3}mm'L$$

$$= \frac{5}{12}(10)(200)(10) + \frac{1}{3}(10)(200)(8)$$

$$= 8333.3 + 5333.3 = 13\,666.7 \text{ k}^2 \cdot \text{ft}^3$$

$$\Delta_{C_h} = \frac{13\,666.7 \text{ k} \cdot \text{ft}^2}{[29(10^3)\text{ k/in}^2((12)^2\text{ in}^2/\text{ft}^2)][600 \text{ in }(\text{ft}^4/(12)^4\text{ in}^4)]}$$

$$= 0.113 \text{ ft} = 1.36 \text{ in.}$$

**The Hashemite University**
**Faculty of Engineering**
**Department of Civil Engineering**

# CE 315: Structural Analysis

## Chapter 7: Analysis of Statically Indeterminate Structures by the Force Method

**Dr. Ra'ed Al-Mazaidh**

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## Statically indeterminate structure:

A structure of any type is classified as **statically indeterminate** when the number of unknown reactions or internal forces **exceeds** the number of equilibrium equations available for its analysis.

- Most of the structures designed today are statically indeterminate. For example, reinforced concrete buildings are almost always statically indeterminate since the columns and beams are poured as continuous members through the joints and over supports.

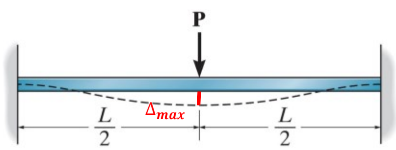
### Advantages & Disadvantages

- For a given loading, the max stress and deflection of an indeterminate structure are generally **smaller** than those of its statically determinate counterpart.
- Statically indeterminate structure has a tendency to redistribute its load to its redundant supports in cases of faulty designs or overloading.

$$M_{max} = \frac{PL}{8}$$

$$\sigma = \frac{MC}{I} = \frac{PL.C}{8EI}$$

$$\Delta_{max} = \frac{PL^3}{192EI}$$

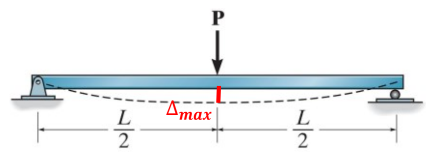



(a)

$$M_{max} = \frac{PL}{4}$$

$$\sigma = \frac{MC}{I} = \frac{PL.C}{4EI}$$

$$\Delta_{max} = \frac{PL^3}{48EI}$$




(b)

The fixed-supported beam has one fourth the deflection and one half the stress at its center of the one that is simply supported.



- Although statically <u>indeterminate structure</u> can support loading with <u>thinner members</u> & with <u>increased stability</u> compared to their statically determinate counterpart, the cost savings in material must be compared with the added cost to fabricate the structure since often it becomes <u>more costly to construct the supports & joints of an indeterminate structure</u>

- Because statically indeterminate structures have redundant support reactions, one has to be very careful to prevent <u>differential displacement</u> of the supports, since this effect will introduce internal stress in the structure.

## ☐ Method of analysis

When analyzing any **indeterminate structure**, it is necessary to satisfy **equilibrium, compatibility**, and **force-displacement requirements** for the structure.

**Equilibrium** is satisfied when the reactive forces hold the structure at rest, and **compatibility** is satisfied when the various segments of the structure fit together without intentional breaks or overlaps.

**The force-displacement requirements** depend upon the way the material responds; (Here we assume linear elastic response).



### ❖ Force Method:

The force method consists of writing equations that satisfy the compatibility and force-displacement requirements for the structure in order to determine the **redundant forces**. Once these forces have been determined, the remaining reactive forces on the structure are determined by satisfying the equilibrium requirements.

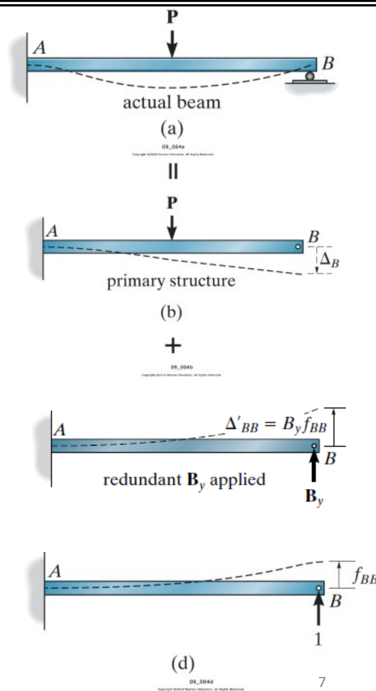
### ❖ Displacement Method:

The displacement method of analysis is based on first writing force-displacement relations for the members and then satisfying the equilibrium requirements for the structure. In this case the unknowns in the equations are displacements. Once the displacements are obtained, the forces are determined from the compatibility and force-displacement equations

| | Unknowns | Equations used for solution | Coefficients of the unknowns |
|---|---|---|---|
| Force Method | Forces | Compatibility and force-displacement | Flexibility coefficients |
| Displacement Method | Displacements | Equilibrium and force-displacement | Stiffness coefficients |

## Force Method of Analysis: General Procedure

- From free-body diagram, there would be 4 unknown support reactions
- 3 equilibrium equations
- Beam is indeterminate to first degree
- Use principle of superposition & consider the compatibility of displacement at one of the supports
- Choose one of the support reactions as **redundant** & temporarily removing its effect on the beam
- This will allow the beam to be statically determinate & stable
- Here, we will remove the roller at B
- As a result, the load P will cause B to be displaced downward
- By superposition, the unknown reaction at B causes the beam at B to be displaced upward



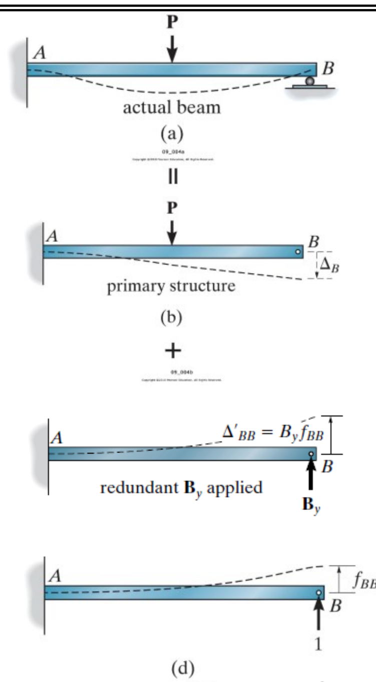



- Assuming positive displacements act upward, then we can write the necessary **compatibility equation** at the roller as

$$(+\uparrow) \qquad 0 = -\Delta_B + \Delta'_{BB}$$

- Here the first letter in this double-subscript notation refers to the point (B) where the deflection is specified, and the second letter refers to the point (B) where the unknown reaction acts.
- Let us denote the displacement at B caused by a unit load acting in the direction of $B_y$ as the **linear flexibility coefficient $f_{BB}$**.
- Since the material behaves in a linear-elastic manner, a force of $B_y$ acting at B, instead of the unit load, will cause a proportionate increase in $f_{BB}$ .

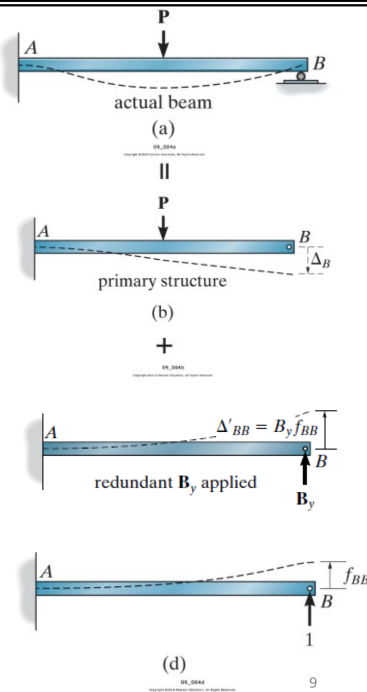
$$\Delta'_{BB} = B_y f_{BB}$$



- The linear flexibility coefficient $f_{BB}$ is a measure of the deflection per unit force, and so its units are m/N, ft./lb .
- The compatibility equation above can therefore be written in terms of the unknown $B_y$ as

$$0 = -\Delta_B + B_y f_{BB}$$

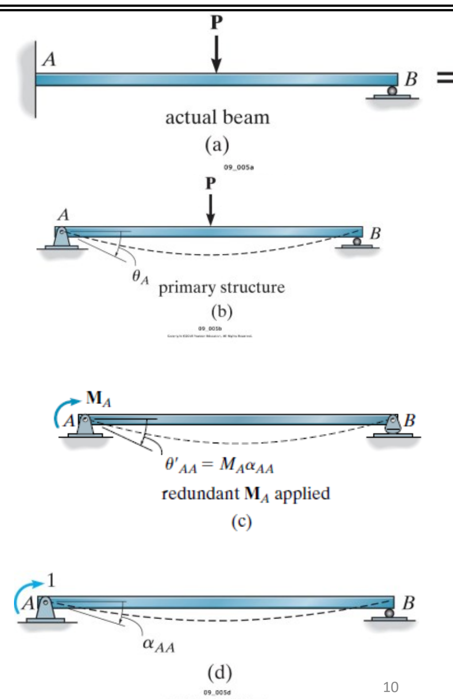
- Using the method of virtual work the appropriate load-displacement relations for the deflection $\Delta_B$ and the flexibility coefficient $f_{BB}$, can be obtained and the solution for $B_y$ can be determined.
- Once this is accomplished, the three reactions at the wall A can then be found from the equations of equilibrium.
- The choice of redundant is arbitrary



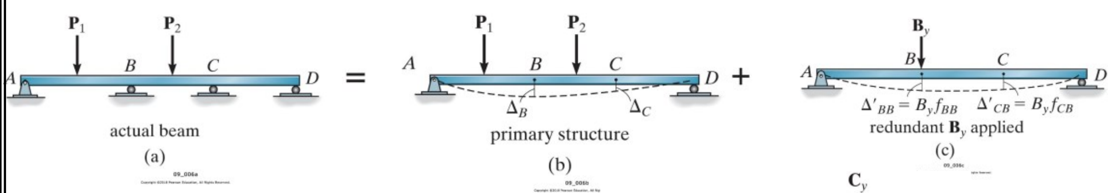


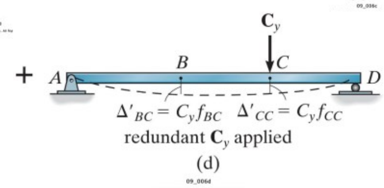

- The moment at A can be determined directly by removing the capacity of the beam to support moment at A, replacing fixed support by pin support
- The rotation at A caused by P is $\theta_A$
- The rotation at A caused by the **redundant** $\boldsymbol{M_A}$ at A is $\theta'_{AA}$
- If we denote an **angular flexibility coefficient** $\boldsymbol{\alpha_{AA}}$ as the angular displacement at A caused by a unit couple moment applied to A, then $\theta'_{AA} = M_A \alpha_{AA}$
- Thus, the angular flexibility coefficient $\alpha_{AA}$ measures the angular displacement per unit couple moment, and therefore it has units of rad/N, rad/lb. **The compatibility equation** for rotation at A therefore requires

$$(\circlearrowleft +) \qquad 0 = \theta_A + M_A \alpha_{AA}$$



## The beam is indeterminate to the <u>second degree</u>:



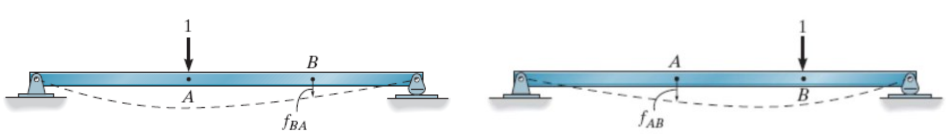

$$(+\downarrow) \qquad 0 = \Delta_B + B_y f_{BB} + C_y f_{BC}$$

$$(+\downarrow) \qquad 0 = \Delta_C + B_y f_{CB} + C_y f_{CC}$$



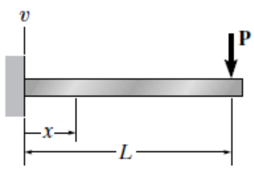
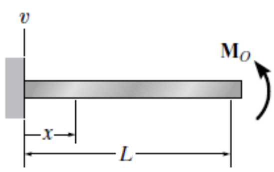

## Maxwell's Theorem of Reciprocal Displacements: Betti's Law

- The displacement of a point B on a structure due to a unit load acting at point A is equal to the displacement of point A when the load is acting at point B
- Proof of this theorem is easily demonstrated using the principle of virtual work

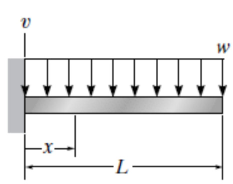
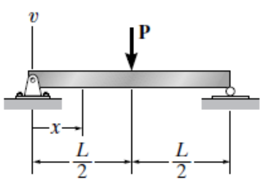
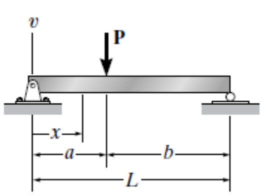


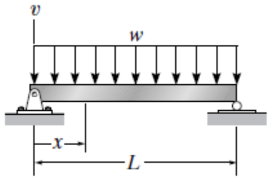
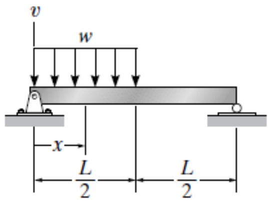
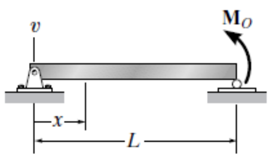

$$\boxed{f_{BA} = f_{AB}}$$

## Beam Deflections and Slopes

| Loading | $v$ + ↑ | $\theta$ + ↺ | Equation + ↑ + ↺ |
|---|---|---|---|
|  | $v_{\max} = -\dfrac{PL^3}{3EI}$ at $x = L$ | $\theta_{\max} = -\dfrac{PL^2}{2EI}$ at $x = L$ | $v = \dfrac{P}{6EI}(x^3 - 3Lx^2)$ |
|  | $v_{\max} = \dfrac{M_O L^2}{2EI}$ at $x = L$ | $\theta_{\max} = \dfrac{M_O L}{EI}$ at $x = L$ | $v = \dfrac{M_O}{2EI}x^2$ |

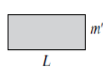
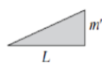
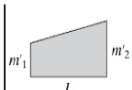
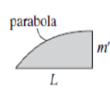
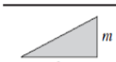
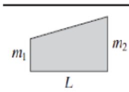
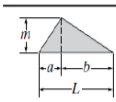
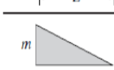


| | | | |
|---|---|---|---|
|  | $v_{\max} = -\dfrac{wL^4}{8EI}$ at $x = L$ | $\theta_{\max} = -\dfrac{wL^3}{6EI}$ at $x = L$ | $v = -\dfrac{w}{24EI}(x^4 - 4Lx^3 + 6L^2x^2)$ |
|  | $v_{\max} = -\dfrac{PL^3}{48EI}$ at $x = L/2$ | $\theta_{\max} = \pm\dfrac{PL^2}{16EI}$ at $x = 0$ or $x = L$ | $v = \dfrac{P}{48EI}(4x^3 - 3L^2x)$, $0 \le x \le L/2$ |
|  | | $\theta_L = -\dfrac{Pab(L+b)}{6LEI}$ $\theta_R = \dfrac{Pab(L+a)}{6LEI}$ | $v = -\dfrac{Pbx}{6LEI}(L^2 - b^2 - x^2)$ $0 \le x \le a$ |

| | | | |
|---|---|---|---|
|  | $v_{\max} = -\dfrac{5wL^4}{384EI}$ at $x = \dfrac{L}{2}$ | $\theta_{\max} = \pm\dfrac{wL^3}{24EI}$ | $v = -\dfrac{wx}{24EI}(x^3 - 2Lx^2 + L^3)$ |
|  | | $\theta_L = -\dfrac{3wL^3}{128EI}$ $\theta_R = \dfrac{7wL^3}{384EI}$ | $v = -\dfrac{wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $0 \le x \le L/2$ $v = -\dfrac{wL}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$ $L/2 \le x \le L$ |
|  | $v_{\max} = -\dfrac{M_OL^2}{9\sqrt{3}EI}$ | $\theta_L = -\dfrac{M_OL}{6EI}$ $\theta_R = \dfrac{M_OL}{3EI}$ | $v = -\dfrac{M_Ox}{6EIL}(L^2 - x^2)$ |

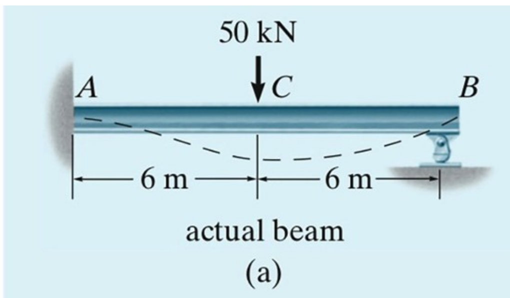


## Table for Evaluating $\int_0^L m\,m'\,dx$

| $\int_0^L m\,m'\,dx$ |  |  |  |  |  |
|---|---|---|---|---|---|
|  | $mm'L$ | $\frac{1}{2}mm'L$ | $\frac{1}{2}m(m'_1 + m'_2)L$ | $\frac{2}{3}mm'L$ | $\frac{1}{3}mm'L$ |
|  | $\frac{1}{2}mm'L$ | $\frac{1}{3}mm'L$ | $\frac{1}{6}m(m'_1 + 2m'_2)L$ | $\frac{5}{12}mm'L$ | $\frac{1}{4}mm'L$ |
|  | $\frac{1}{2}m'(m_1 + m_2)L$ | $\frac{1}{6}m'(m_1 + 2m_2)L$ | $\frac{1}{6}[m'_1(2m_1 + m_2) + m'_2(m_1 + 2m_2)]L$ | $\frac{1}{12}[m'(3m_1 + 5m_2)]L$ | $\frac{1}{12}m'(m_1 + 3m_2)L$ |
|  | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'(L + a)$ | $\frac{1}{6}m[m'_1(L + b) + m_2(L + a)]$ | $\frac{1}{12}mm'\left(3 + \frac{3a}{L} - \frac{a^2}{L^2}\right)L$ | $\frac{1}{12}mm'(1 + \frac{a}{L} + \frac{a^2}{L^2})L$ |
|  | $\frac{1}{2}mm'L$ | $\frac{1}{6}mm'L$ | $\frac{1}{6}m(2m'_1 + m'_2)L$ | $\frac{1}{4}mm'L$ | $\frac{1}{12}mm'L$ |

## Example 1

Determine the reaction at the roller support *B* of the beam shown in Fig. 9–9*a*. *EI* is constant.





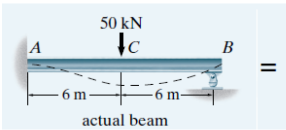

## How to calculate Δ and *f* :

**1) Tables (Beam deflections and slope)……….. (direct) or,**

**2) Double integration method, or**

**3) Conjugate –beam Method, or**

**4) Virtual work method, or**

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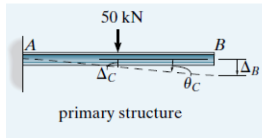
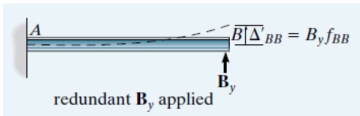


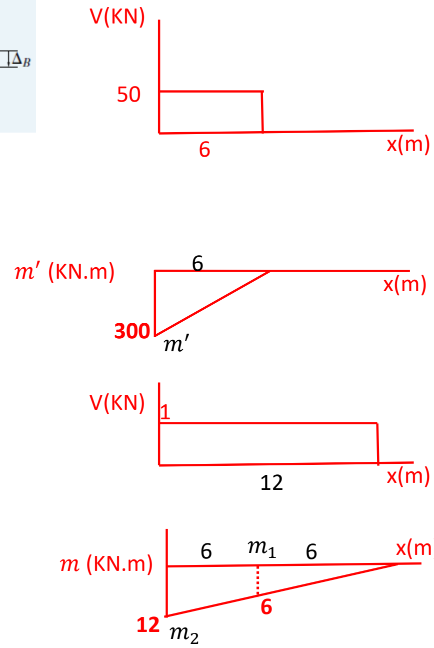




Compatibility equation

$$(+\uparrow) \qquad 0 = -\Delta_B + B_y f_{BB}$$

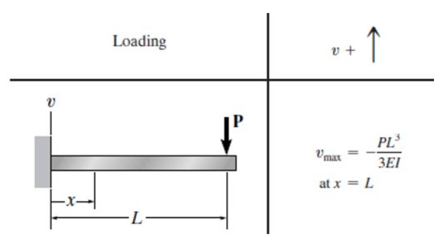
$$\Delta_B = \int mm'.dx \qquad \frac{1}{6}m'(m_1 + 2m_2)L$$

$$= \frac{1}{6}(300)(6 + 2 * 12)(6) = \frac{9000}{\text{EI}} \downarrow$$



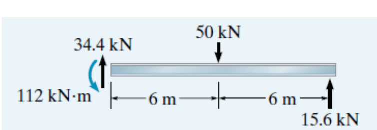


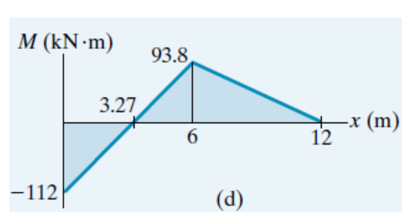

$$f_{BB} = \frac{PL^3}{3EI} = \frac{1(12\text{ m})^3}{3EI} = \frac{576\text{ m}^3}{EI}\uparrow$$




$$(+\uparrow) \qquad 0 = -\Delta_B + B_y f_{BB}$$

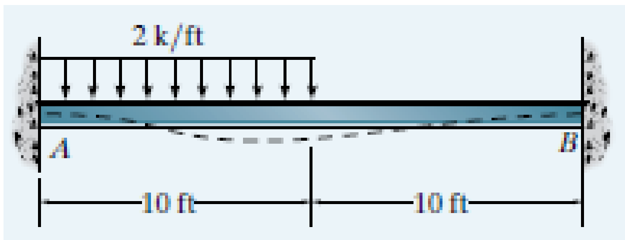
$$(+\uparrow) \qquad 0 = -\frac{9000}{EI} + B_y\left(\frac{576}{EI}\right) \qquad B_y = 15.6\text{ kN}$$



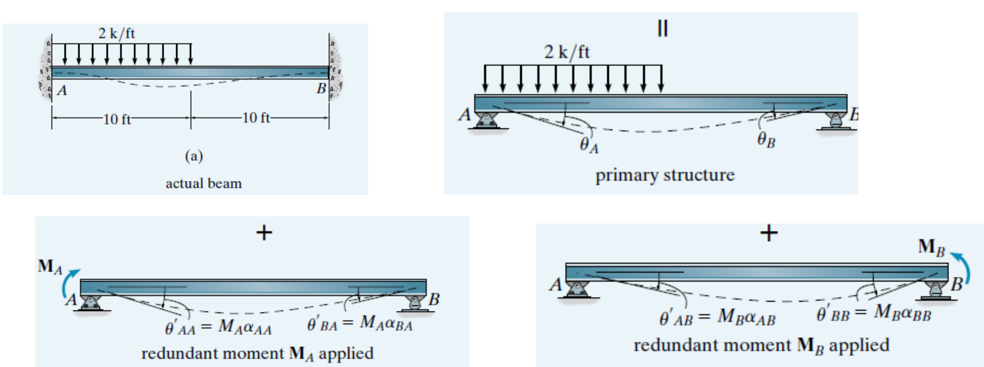



## Example 2

Draw the shear and moment diagrams for the beam shown in Fig. 9–11*a*. *EI* is constant. Neglect the effects of axial load.

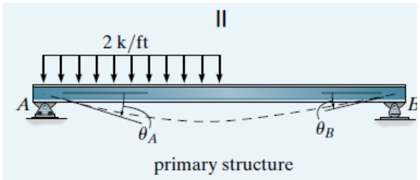


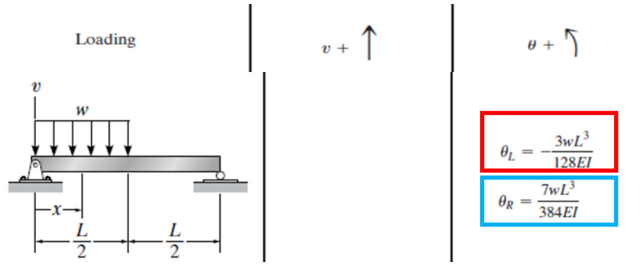



$$(\circlearrowleft +) \qquad 0 = \theta_A + M_A\alpha_{AA} + M_B\alpha_{AB} \qquad (1)$$

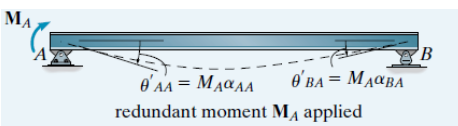
$$(\circlearrowright +) \qquad 0 = \theta_B + M_A\alpha_{BA} + M_B\alpha_{BB} \qquad (2)$$

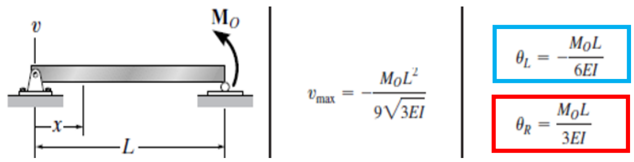



$$\theta_A = \frac{3wL^3}{128EI} = \frac{3(2)(20)^3}{128EI} = \frac{375}{EI}$$

$$\theta_B = \frac{7wL^3}{384EI} = \frac{7(2)(20)^3}{384EI} = \frac{291.7}{EI}$$



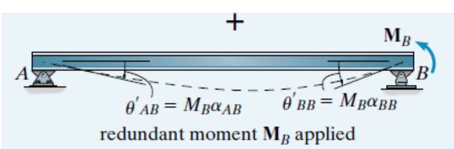



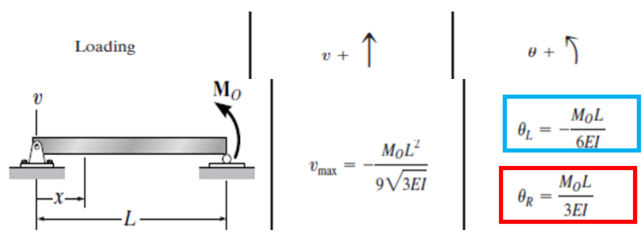

$$\alpha_{AA} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{BA} = \frac{ML}{6EI} = \frac{1(20)}{6EI} = \frac{3.33}{EI}$$



+




$$\alpha_{BB} = \frac{ML}{3EI} = \frac{1(20)}{3EI} = \frac{6.67}{EI}$$

$$\alpha_{AB} = \frac{ML}{6EI} = \frac{1(20)}{6EI} = \frac{3.33}{EI}$$

$$\alpha_{AB} = \alpha_{BA} = \frac{ML}{6EI}$$

**Why?**

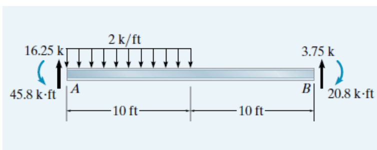
**Maxwell's Theorem of Reciprocal Displacements: Betti's Law**

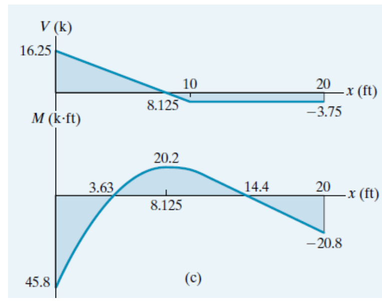
Substituting the data into the compatibility equations:

$$0 = \frac{375}{EI} + M_A\left(\frac{6.67}{EI}\right) + M_B\left(\frac{3.33}{EI}\right)$$

$$0 = \frac{291.7}{EI} + M_A\left(\frac{3.33}{EI}\right) + M_B\left(\frac{6.67}{EI}\right)$$

$$M_A = -45.8 \text{ k}\cdot\text{ft} \qquad M_B = -20.8 \text{ k}\cdot\text{ft}$$





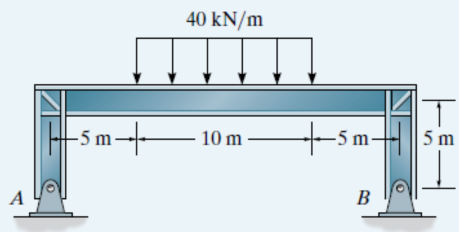

## Force Method of Analysis: Frames

The force method is <u>very useful</u> for solving problems involving <u>statically indeterminate frames</u> that have a <u>single story and unusual geometry</u>.

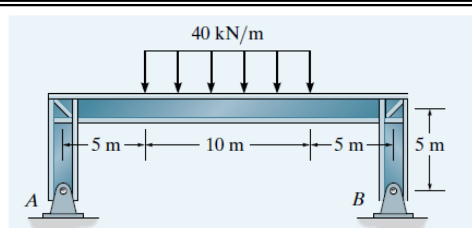
Problems involving multistory frames, or those with a <u>high degree of indeterminacy</u>, are best solved using <u>the slope-deflection</u>, <u>**moment-distribution**</u>, or the <u>**stiffness method**</u>.

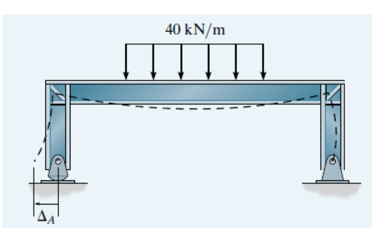
# Example 3

The saddle bent shown in the photo is used to support the bridge deck. Assuming $EI$ is constant, a drawing of it along with the dimensions and loading is shown in Fig. 10–13$a$. Determine the horizontal support reaction at $A$.



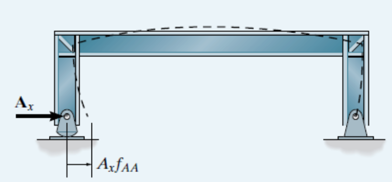


Primary structure

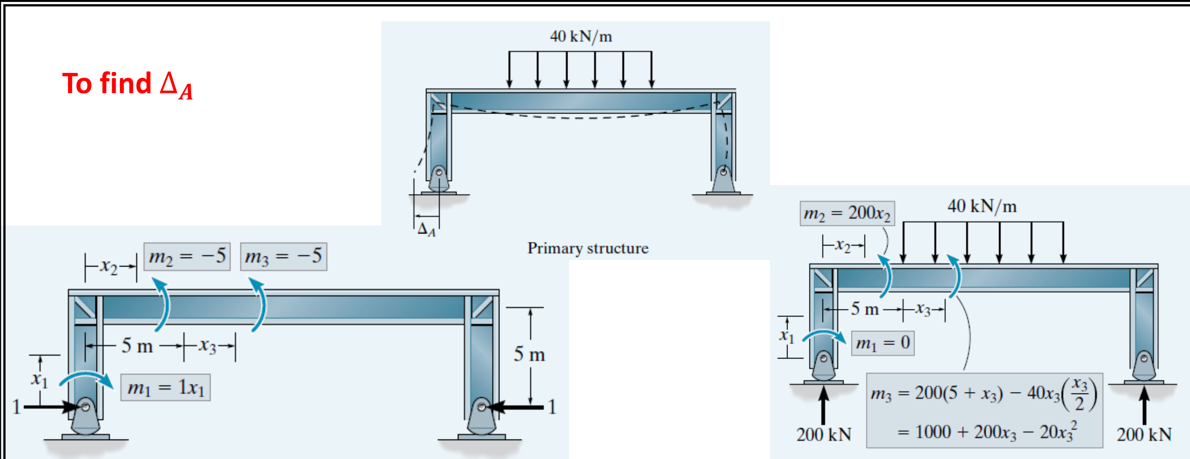
+



Redundant force $\mathbf{A}_x$ applied

$$(\overset{+}{\rightarrow}) \qquad 0 = \Delta_A + A_x f_{AA}$$

## To find $\Delta_A$



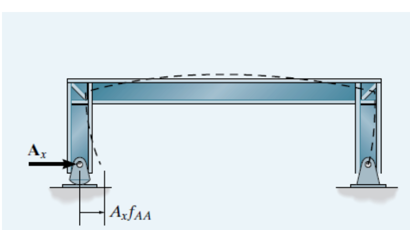

$$\Delta_A = \int_0^L \frac{Mm}{EI}dx = 2\int_0^5 \frac{(0)(1x_1)dx_1}{EI} + 2\int_0^5 \frac{(200x_2)(-5)dx_2}{EI}$$

$$+ 2\int_0^5 \frac{(1000 + 200x_3 - 20x_3^2)(-5)dx_3}{EI}$$

$$= 0 - \frac{25\,000}{EI} - \frac{66\,666.7}{EI} = -\frac{91\,666.7}{EI}$$



## To find $f_{AA}$




Redundant force $A_x$ applied

$$f_{AA} = \int_0^L \frac{mm}{EI}dx = 2\int_0^5 \frac{(1x_1)^2 dx_1}{EI} + 2\int_0^5 (5)^2 dx_2 + 2\int_0^5 (5)^2 d$$

$$= \frac{583.33}{EI}$$

$$= \frac{583.33}{EI}$$

$$0 = \frac{-91\,666.7}{EI} + A_x\left(\frac{583.33}{EI}\right)$$

$$A_x = 157 \text{ kN}$$

## Force Method of Analysis: Trusses

- The degree of indeterminacy of a truss can usually be determined by inspection; however, if this becomes difficult, use the following equation:

$$\boxed{b + r > 2j}$$

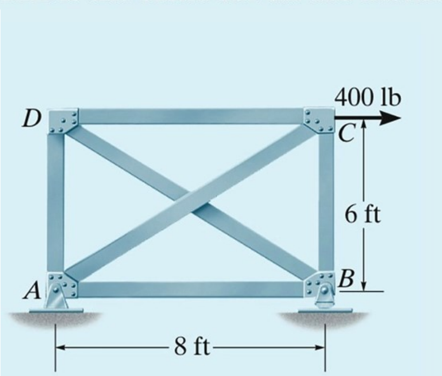
Here the unknowns are represented **by the number of bar forces (*b*)** plus **the support reactions (*r*)**, and the number of available equilibrium equations is $2j$ since two equations can be written for each of the **( *j*) joints**

- The force method is quite suitable for analyzing trusses that are statically indeterminate to the first or second degree.

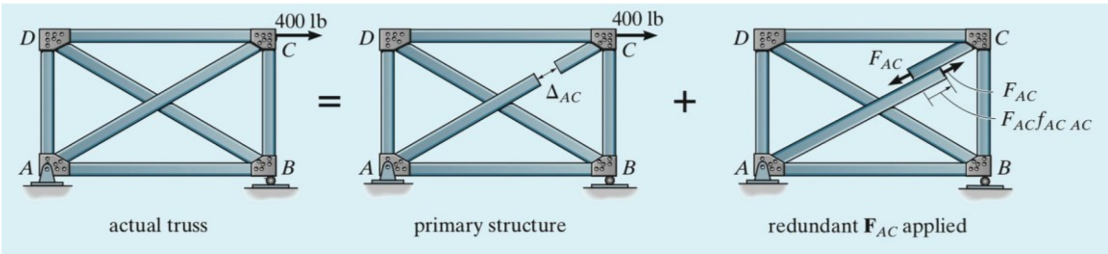


## Example 4

Determine the force in member $AC$ of the truss shown in Fig. 9–15$a$. $AE$ is the same for all the members.




.., $b + r > 2j$ or $6 + 3 > 2(4)$, $9 > 8$, $9 - 8 =$ 1st degree.




**Compatibility Eq.**

$$\boxed{0 = \Delta_{AC} + F_{AC}f_{AC\,AC}}$$



## To find $\Delta_{AC\,AC}$

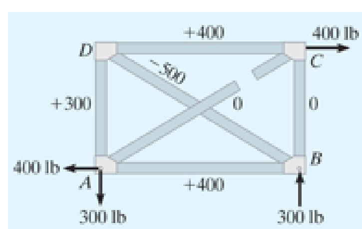
$$\Delta_{AC} = \sum \frac{nNL}{AE}$$

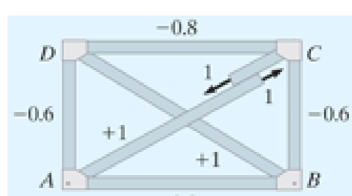
$$\Delta_{AC} = \sum \frac{nNL}{AE}$$

$$= 2\left[\frac{(-0.8)(400)(8)}{AE}\right] + \frac{(-0.6)(0)(6)}{AE} + \frac{(-0.6)(300)(6)}{AE}$$

$$+ \frac{(1)(-500)(10)}{AE} + \frac{(1)(0)(10)}{AE}$$

$$= -\frac{11\,200}{AE}$$



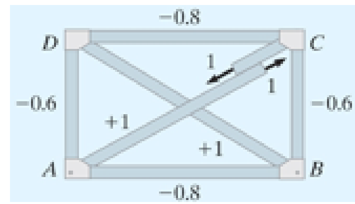



## To find $f_{AC}$

$$f_{AC\,AC} = \sum \frac{n^2 L}{AE}$$

$$= 2\left[\frac{(-0.8)^2(8)}{AE}\right] + 2\left[\frac{(-0.6)^2(6)}{AE}\right] + 2\left[\frac{(1)^2 10}{AE}\right]$$

$$= \frac{34.56}{AE}$$




$$\boxed{0 = \Delta_{AC} + F_{AC} f_{AC\,AC}}$$

$$0 = -\frac{11\,200}{AE} + \frac{34.56}{AE} F_{AC}$$

$$F_{AC} = 324 \text{ lb (T)}$$

The Hashemite University
Faculty of Engineering
Department of Civil Engineering

# CE 315: Structural Analysis

## Chapter 8: Displacement Method of Analysis( Moment Distribution)

Dr. Ra'ed Al-Mazaidh

## General Principles & Definition

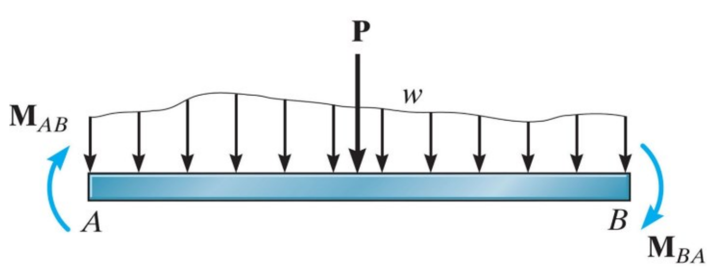
- Displacement method requires satisfying equilibrium equations for the structures.
- The unknowns displacement are written in terms of the loads by using the load-displacement relations.
- These equations are solved for the displacement.
- Once the displacement are obtained, the unknown loads are determined from the compatibility equations using the load displacement relations.

- **Moment distribution** is a method of successive approximations that may be carried out to any desired degree of accuracy.
- The method begins by assuming each joint of a structure is **fixed**.
- By unlocking and locking each joint in succession, the internal moments at the joints are "distributed" & balanced until the joints have rotated to their final or nearly final positions.



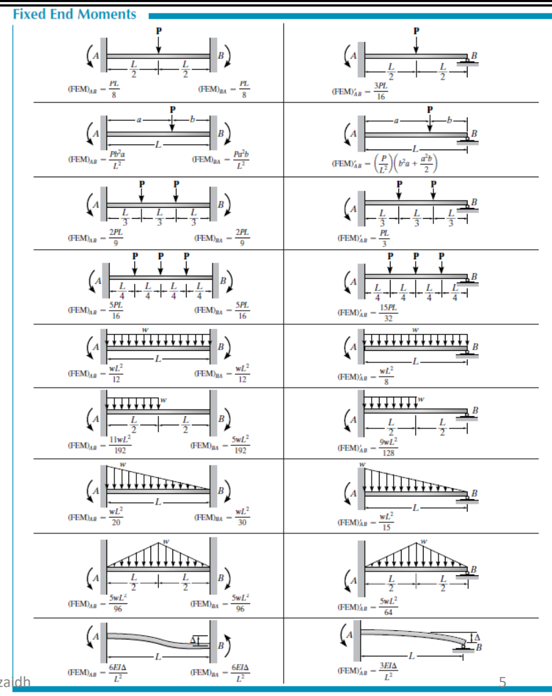
**1) Sign Convention**

- **Clockwise moments** that act on the member are considered **positive**, whereas counterclockwise moments are negative.



## 2) Fixed-End Moments (FEMs)

The moments at the "walls" or fixed joints of a loaded member are called **fixed-end moments.** These moments can be determined from the table on the inside back cover of the textbook, depending upon **the type of loading on the member**.



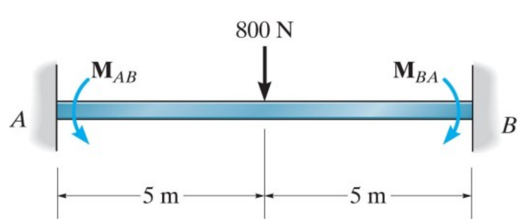



For example, the beam loaded as shown in figure below has fixed-end moments of

$$FEM = \frac{PL}{8} = \frac{800(10)}{8} = 1000\ N.m.$$

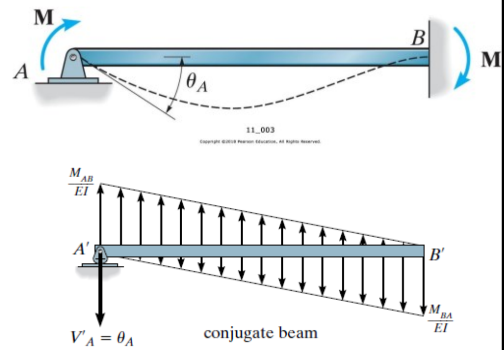
Noting the action of these moments on the beam and applying our sign convention, it is seen that

$$M_{AB} = -1000 N.m. \text{ and } M_{BA} = 1000 N.m.$$



## 3) Member stiffness factor

Consider the beam in the figure, which is pinned at one end and fixed at the other. Application of the moment **M** causes the end $A$ to rotate through an angle $\theta_A$ .Using the conjugate-beam method M can be related to $\theta_A$ as follows:



(b)

$$\zeta + \Sigma M_{A'} = 0; \quad \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{2L}{3} = 0$$

$$\zeta + \Sigma M_{B'} = 0; \quad \left[\frac{1}{2}\left(\frac{M_{BA}}{EI}\right)L\right]\frac{L}{3} - \left[\frac{1}{2}\left(\frac{M_{AB}}{EI}\right)L\right]\frac{2L}{3} + \theta_A L = 0$$

$$\boxed{M_{AB} = \frac{4EI}{L}\theta_A}$$

$$\boxed{K = \frac{4EI}{L}}$$
Far End Fixed

***K*** is referred to as the **stiffness factor** at $A$ and can be defined as the amount of moment $M$ required to rotate the end $A$ of the beam $\theta_A$=1 rad.

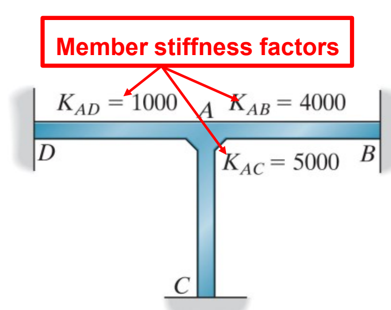


## 4) Joint stiffness factor

If several members are fixed connected to joint and **each of their far ends is fixed**, then by the principle of superposition, **the total stiffness factor** at the joint is the sum of the member stiffness factors at the joint.

- The total stiffness factor of joint A is

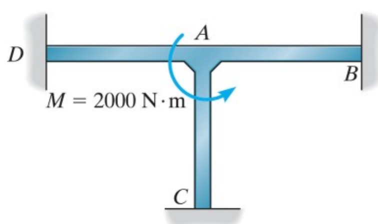
$$K_T = \sum K = 4000 + 5000 + 1000 = 10000$$



(a)

## 5) Distribution Factor (DF)

If a moment $M$ is applied to a fixed connected joint, <u>the connecting members will each supply a portion of the resisting moment necessary to **satisfy** moment equilibrium at the joint</u>. That fraction of the total resisting moment supplied by the member is called **the distribution factor (DF).**



(b)

$$DF_i = \frac{M_i}{M} = \frac{K_i\theta}{\theta \Sigma K_i}$$

$$\boxed{DF = \frac{K}{\Sigma K}}$$



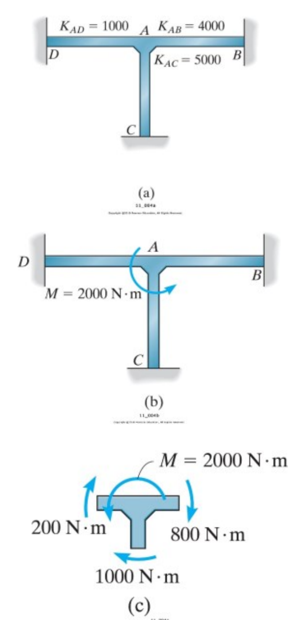
For example, the distribution factors for members *AB*, *AC*, and *AD* at joint *A*

$$\left.\begin{aligned} DF_{AB} &= 4000/10\,000 = 0.4 \\ DF_{AC} &= 5000/10\,000 = 0.5 \\ DF_{AD} &= 1000/10\,000 = 0.1 \end{aligned}\right\}$$

The sum of the distribution factors for all members at the same Joint must equal 1.0

As a result, if $M$ = 2000 N . m acts at joint *A*, Fig. (*b)*, the equilibrium moments exerted by the members on the joint, *c*, are

$$M_{AB} = 0.4(2000) = 800\ \text{N}\cdot\text{m}$$
$$M_{AC} = 0.5(2000) = 1000\ \text{N}\cdot\text{m}$$
$$M_{AD} = 0.1(2000) = 200\ \text{N}\cdot\text{m}$$



## 6) Member relative stiffness factor

- Quite often a continuous beam or a frame will be made from the same material E will therefore be constant
- In the case, the common factor $4E$ will cancel from the numerator and denominator when the distribution factor for a joint is determined.

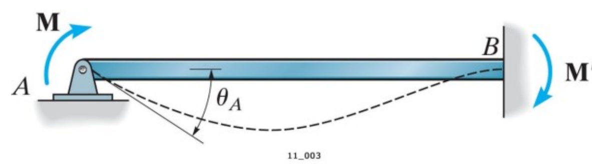
$$K_R = \frac{I}{L}$$

Far End Fixed



## 7) Carry-over (CO) factor

$$M_{AB} = \frac{4EI}{L}\theta_A \qquad M_{BA} = \frac{2EI}{L}\theta_A$$




Solving for $\theta_A$ and equating these equations,

$$M_{BA} = 0.5M_{AB}$$

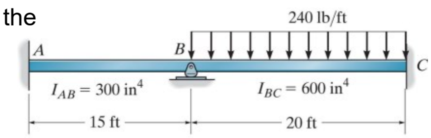
The moment $M$ at the pin induces a moment of $M' = 0.5M$ at the wall

**In the case of a beam with the far end fixed, the CO factor is +0.5**

## Moment Distribution for Beams

Moment distribution is based on the principle of successively locking and unlocking the joints of a structure in order to allow the moments at the joints to be distributed and balanced. **The best way to explain the method is by examples**.

- Consider the beam with a constant modulus of elasticity E and having the dimensions and loading shown in Fig(a).




**STEP 1: Determine the member stiffness factors on either side of B.**

$$K_{BA} = \frac{4E(300)}{15} = 4E(20)\ \text{in}^4/\text{ft} \qquad K_{BC} = \frac{4E(600)}{20} = 4E(30)\ \text{in}^4/\text{ft}$$

**STEP 2: Determine the distribution factors at the two ends of each span.**

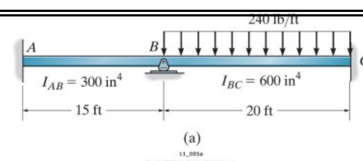
$$\text{DF}_{BA} = \frac{4E(20)}{4E(20) + 4E(30)} = 0.4 \quad \text{DF}_{BC} = \frac{4E(30)}{4E(20) + 4E(30)} = 0.6 \quad \text{DF}_{AB} = \frac{4E(20)}{\infty + 4E(20)} = 0 \quad \text{DF}_{CB} = \frac{4E(30)}{\infty + 4E(30)} = 0$$

Recall: Member relative stiffness factor $K_R = \dfrac{I}{L}$

The wall stiffness factor is infinite.

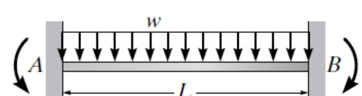


**STEP 3:** Determine the FEMs




$$(\text{FEM})_{BC} = -\frac{wL^2}{12} = -\frac{240(20)^2}{12} = -8000\ \text{lb}\cdot\text{ft} \quad \text{Counterclockwise}$$

$$(\text{FEM})_{CB} = \frac{wL^2}{12} = \frac{240(20)^2}{12} = 8000\ \text{lb}\cdot\text{ft} \quad \text{Clockwise}$$

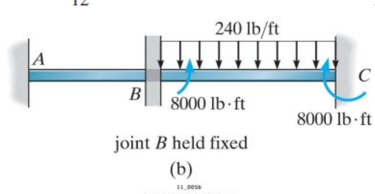



$$(\text{FEM})_{AB} = \frac{wL^2}{12} \qquad (\text{FEM})_{BA} = \frac{wL^2}{12}$$

**STEP 4: Determine the unbalanced moment acting at the initially "locked" joint (B)**

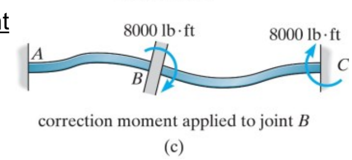
✓ Assume joint B is **fixed or locked** (Fig.b)

✓ The fixed-end moment at B does not represent the actual equilibrium situation at B, since the moments on each side of this joint must be equal but opposite.



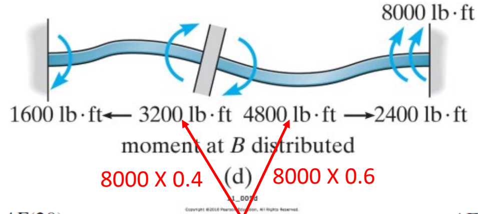

joint $B$ held fixed
(b)

**STEP 5: Unlock** the joint and apply an equal but opposite unbalanced moment to correct the equilibrium (Fig.c)



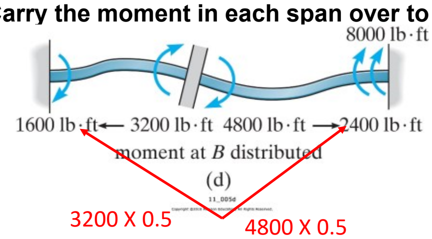

correction moment applied to joint $B$
(c)

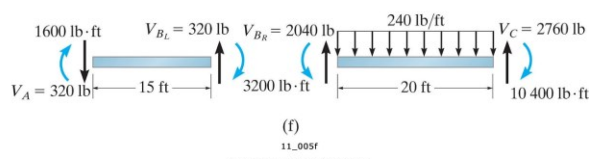
**STEP 6:** **Distribute the moment among the connecting spans based on their DFs**




$$\mathrm{DF}_{BA} = \frac{4E(20)}{4E(20) + 4E(30)} = 0.4 \quad \boxed{\textbf{8000 lb.ft}} \quad \mathrm{DF}_{BC} = \frac{4E(30)}{4E(20) + 4E(30)} = 0.6$$

**STEP 7:** **Carry the moment in each span over to its other end**



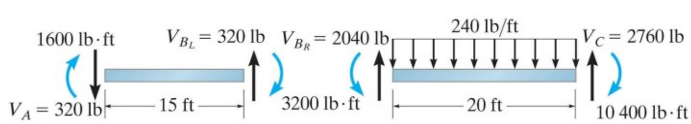



| | Joint | A | B | | C |
|---|---|---|---|---|---|
| 1 | Joint | A | B | | C |
| 2 | Member | AB | BA | BC | CB |
| 3 | DF | 0 | 0.4 | 0.6 | 0 |
| 4 | FEM | | | −8000 | 8000 |
| | Dist.CO | 1600 | 3200 | 4800 | 2400 |
| | ΣM | 1600 | 3200 | −3200 | 10 400 |

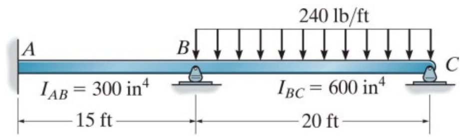
**Unbalanced Moment based on DFs**

**Carry-Over**

(e)

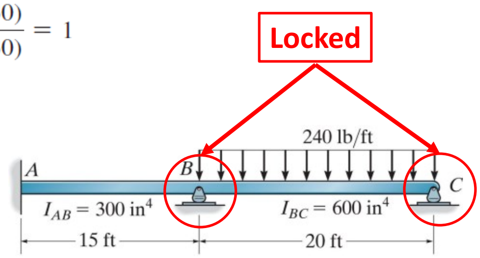



(f)




(a)

$$\mathrm{DF}_{CB} = \frac{4E(30)}{4E(30)} = 1$$



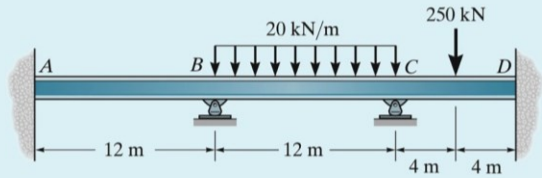

(a)



| Joint | *A* | *B* | | *C* | |
|---|---|---|---|---|---|
| Member | *AB* | *BA* | *BC* | *CB* | |
| DF | 0 | 0.4 | 0.6 | 1 | 1 |
| FEM | | | −8000 | 8000 | 2 |
| Dist. | | 3200 | 4800 | −8000 | 3 |
| CO | | | −4000 | 2400 | 4 |
| Dist. | 1600 | 1600 | 2400 | −2400 | 5 |
| CO | | | −1200 | 1200 | 6 |
| Dist. | 800 | 480 | 720 | −1200 | 7 |
| CO | | | −600 | 360 | 8 |
| Dist. | 240 | 240 | 360 | −360 | 9 |
| CO | | | −180 | 180 | 10 |
| Dist. | 120 | 72 | 108 | −180 | 11 |
| CO | | | −90 | 54 | 12 |
| Dist. | 36 | 36 | 54 | −54 | 13 |
| CO | | | −27 | 27 | 14 |
| Dist. | 18 | 10.8 | 16.2 | −27 | 15 |
| CO | | | −13.5 | 8.1 | 16 |
| Dist. | 5.4 | 5.4 | 8.1 | −8.1 | 17 |
| CO | | | −4.05 | 4.05 | 18 |
| Dist. | 2.7 | 1.62 | 2.43 | −4.05 | 19 |
| CO | | | −2.02 | 1.22 | 20 |
| Dist. | 0.81 | 0.80 | 1.22 | −1.22 | 21 |
| CO | | | −0.61 | 0.61 | 22 |
| Dist. | 0.40 | 0.24 | 0.37 | −0.61 | 23 |
| $\Sigma M$ | 2823 | 5647 | −5647 | 0 | 24 |

## EXAMPLE 1

Determine the internal moments at each support of the beam shown in Fig. 11–7*a*. *EI* is constant.






$$K_{AB} = \frac{4EI}{12}$$

$$K_{BC} = \frac{4EI}{12}$$

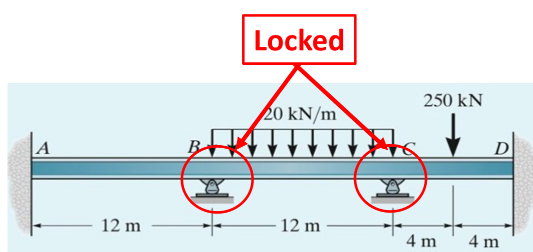
$$K_{CD} = \frac{4EI}{8}$$

$$\mathrm{DF}_{AB} = \mathrm{DF}_{DC} = 0$$

$$\mathrm{DF}_{BA} = \mathrm{DF}_{BC} = \frac{4EI/12}{4EI/12 + 4EI/12} = 0.5$$

$$\mathrm{DF}_{CB} = \frac{4EI/12}{4EI/12 + 4EI/8} = 0.4$$

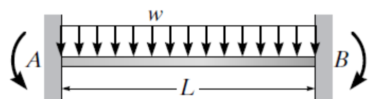
$$\mathrm{DF}_{CD} = \frac{4EI/8}{4EI/12 + 4EI/8} = 0.6$$



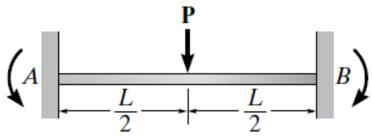
The fixed-end moments are

$$(\mathrm{FEM})_{BC} = -\frac{wL^2}{12} = \frac{-20(12)^2}{12} = -240\ \mathrm{kN\cdot m} \qquad (\mathrm{FEM})_{CB} = \frac{wL^2}{12} = \frac{20(12)^2}{12} = 240\ \mathrm{kN\cdot m}$$

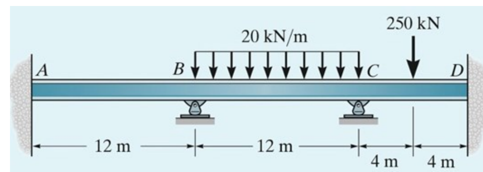
$$(\mathrm{FEM})_{CD} = -\frac{PL}{8} = \frac{-250(8)}{8} = -250\ \mathrm{kN\cdot m} \qquad (\mathrm{FEM})_{DC} = \frac{PL}{8} = \frac{250(8)}{8} = 250\ \mathrm{kN\cdot m}$$




$(\mathrm{FEM})_{AB} = \frac{wL^2}{12}$ $\qquad$ $(\mathrm{FEM})_{BA} = \frac{wL^2}{12}$



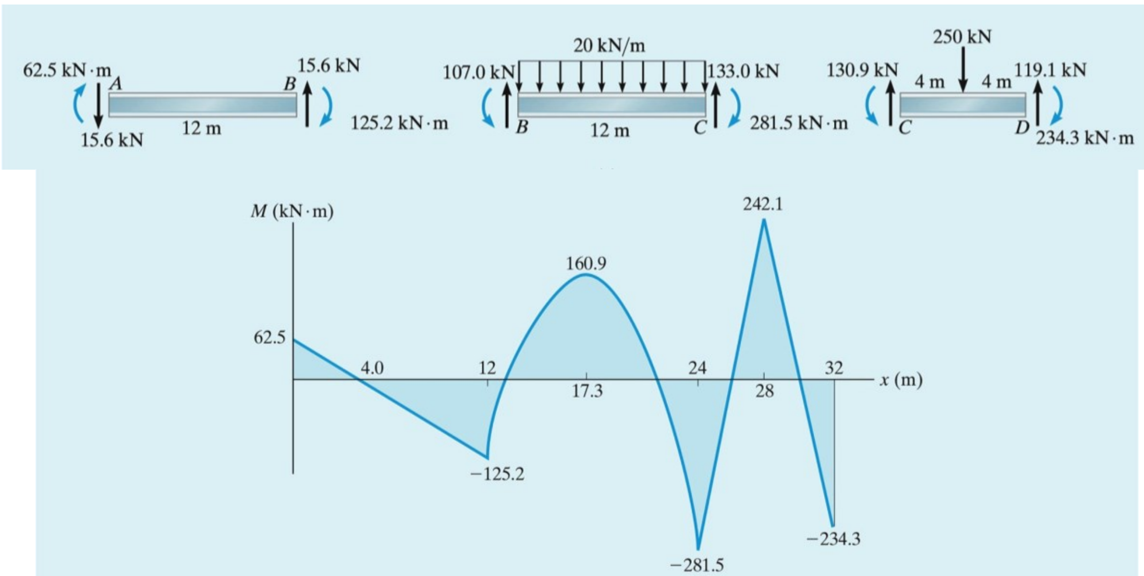

$(\mathrm{FEM})_{AB} = \frac{PL}{8}$ $\qquad$ $(\mathrm{FEM})_{BA} = \frac{PL}{8}$






Locked

| Joint | A | B | | C | | D | 1 |
|---|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | DC | 2 |
| DF | 0 | 0.5 | 0.5 | 0.4 | 0.6 | 0 | 3 |
| FEM | | | −240 | 240 | −250 | 250 | 4 |
| Dist. | | 120 | 120 | 4 | 6 | | 5 |
| CO | 60 | | 2 | 60 | | 3 | 6 |
| Dist. | | −1 | −1 | −24 | −36 | | 7 |
| CO | −0.5 | | −12 | −0.5 | | −18 | 8 |
| Dist. | | 6 | 6 | 0.2 | 0.3 | | 9 |
| CO | 3 | | 0.1 | 3 | | 0.2 | 10 |
| Dist. | | −0.05 | −0.05 | −1.2 | −1.8 | | 11 |
| CO | −0.02 | | −0.6 | −0.02 | | −0.9 | 12 |
| Dist. | | 0.3 | 0.3 | 0.01 | 0.01 | | 13 |
| $\Sigma M$ | 62.5 | 125.2 | −125.2 | 281.5 | −281.5 | 234.3 | 14 |






## Stiffness-Factor Modifications

### 1) Member Pin Supported at Far End

$$\circlearrowleft +\Sigma M_{B'} = 0; \quad V'_A(L) - \frac{1}{2}\left(\frac{M}{EI}\right)L\left(\frac{2}{3}L\right) = 0$$

$$V'_A = \theta = \frac{ML}{3EI}$$

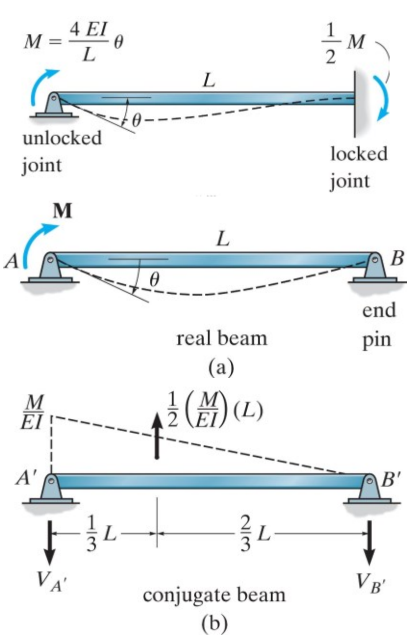
or

$$M = \frac{3EI}{L}\theta$$

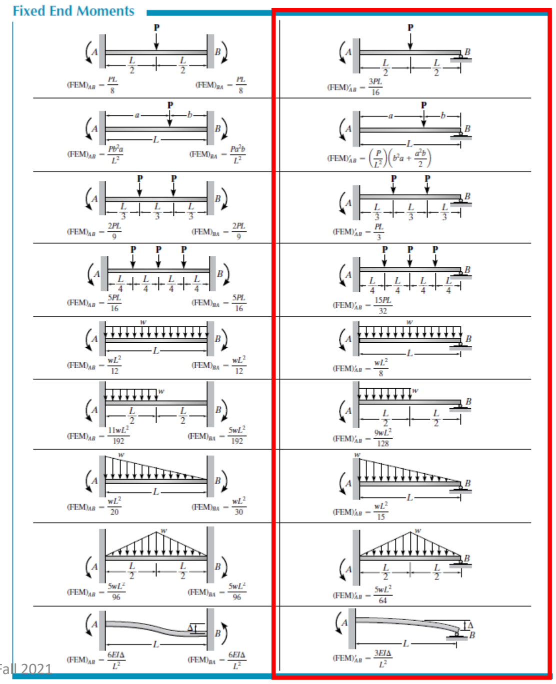
Thus, the stiffness factor for this beam is

$$\boxed{K = \frac{3EI}{L}}$$

Far End Pinned
or Roller Supported




- The carry-over factor is zero, since the pin at B does not support a moment.






## 2) Symmetric Beam and Loading

$$\circlearrowleft + \Sigma M_{C'} = 0; \qquad -V_{B'}(L) + \frac{M}{EI}(L)\left(\frac{L}{2}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{2EI}$$

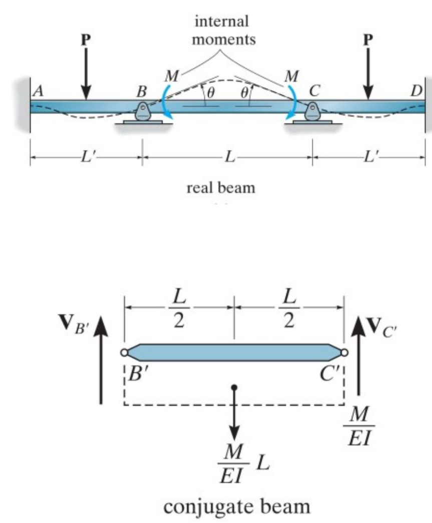
or

$$M = \frac{2EI}{L}\theta$$

The stiffness factor for the center span is therefore

$$\boxed{K = \frac{2EI}{L}}$$

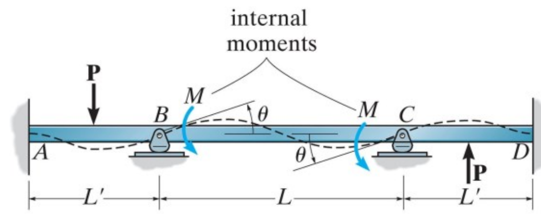
Symmetric Beam and Loading




conjugate beam

(b)

## 3) Symmetric Beam with Antisymmetric Loading




real beam

(a)

$$\circlearrowleft +\Sigma M_{C'} = 0; \quad -V_{B'}(L) + \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{5L}{6}\right) - \frac{1}{2}\left(\frac{M}{EI}\right)\left(\frac{L}{2}\right)\left(\frac{L}{6}\right) = 0$$

$$V_{B'} = \theta = \frac{ML}{6EI}$$

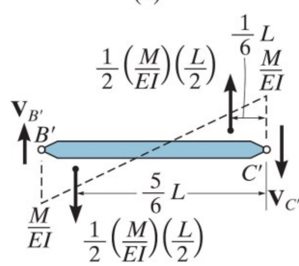
or

$$M = \frac{6EI}{L}\theta$$

The stiffness factor for the center span is, therefore,

$$K = \frac{6EI}{L} \qquad (12\text{–}6)$$

Symmetric Beam with Antisymmetric Loading



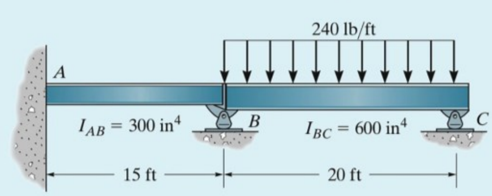

conjugate beam

(b)



## EXAMPLE 2

Determine the internal moments at the supports of the beam shown in Fig. 11–14*a*. The moment of inertia of the two spans is shown in the figure.




(a)

$$K_{AB} = \frac{4EI}{L} = \frac{4E(300)}{15} = 80E$$

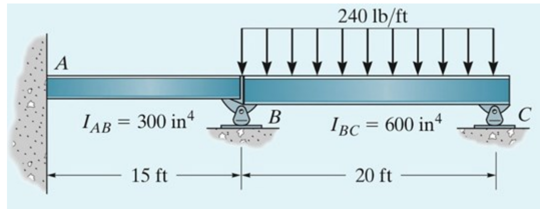
$$K_{BC} = \frac{3EI}{L} = \frac{3E(600)}{20} = 90E$$

$$\text{DF}_{AB} = \frac{80E}{\infty + 80E} = 0$$

$$\text{DF}_{BA} = \frac{80E}{80E + 90E} = 0.4706$$

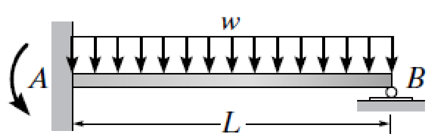
$$\text{DF}_{BC} = \frac{90E}{80E + 90E} = 0.5294$$

$$\text{DF}_{CB} = \frac{90E}{90E} = 1$$



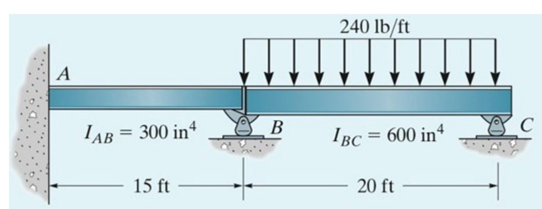



$$(\text{FEM})_{BC} = -\frac{wL^2}{8} = -\frac{240(20)^2}{8} = -12\,000 \text{ lb} \cdot \text{ft}$$




$$(\text{FEM})'_{AB} = \frac{wL^2}{8}$$

| Joint | $A$ | $B$ | | $C$ |
|---|---|---|---|---|
| Member | $AB$ | $BA$ | $BC$ | $CB$ |
| DF | 0 | 0.4706 | 0.5294 | 1 |
| FEM | | | −12 000 | |
| Dist. | | 5647.2 | 6352.8 | |
| CO | 2823.6 | | | |
| $\Sigma M$ | 2823.6 | 5647.2 | −5647.2 | 0 |



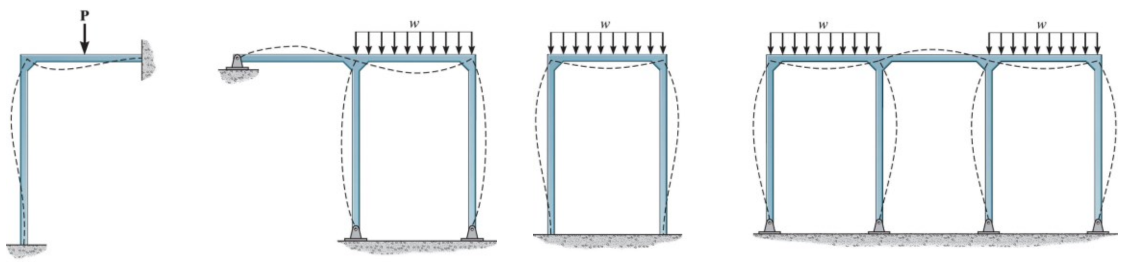
## No Sidesway and Sidesway frames

### No Sidesway frame

- The frame will not be **displaced** to the right or left. frame does not sidesway if:

1) It is **restrained** against sidesway.

2) The frame **geometry and loading is symmetrical.**



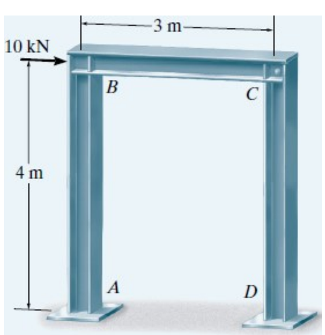

Restrained frames

Symmetrical frames

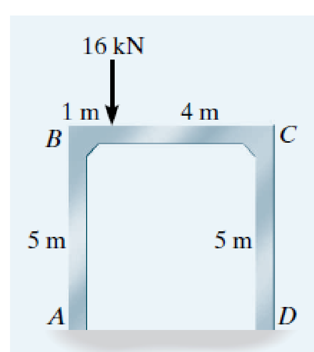


### Sidesway frame

A frame will sidesway, or be displaced to the side, when it is **not restrained** against sidesway or the loading acting on it is **nonsymmetrical**



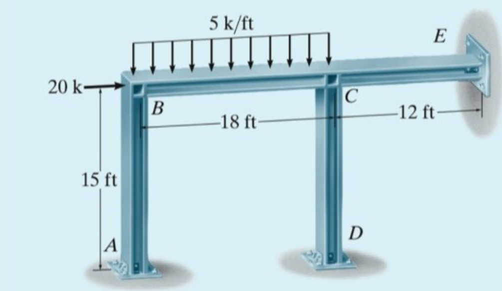

Not restrained against sidesway




Nonsymmetrical loading

## EXAMPLE 3

Determine the internal moments at the joints of the frame shown in Fig. 11–15*a*. There is a pin at *E* and *D* and a fixed support at *A*. *EI* is constant.






$$K_{AB} = \frac{4EI}{15} \quad K_{BC} = \frac{4EI}{18} \quad K_{CD} = \frac{3EI}{15} \quad K_{CE} = \frac{3EI}{12}$$

$$\text{DF}_{AB} = 0$$

$$\text{DF}_{BA} = \frac{4EI/15}{4EI/15 + 4EI/18} = 0.545$$

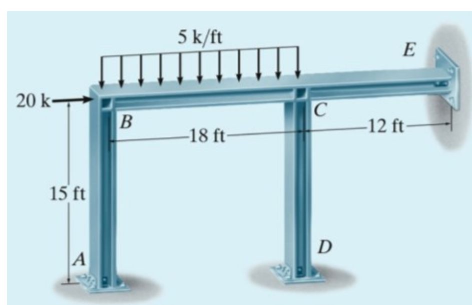
$$\text{DF}_{BC} = 1 - 0.545 = 0.455$$

$$\text{DF}_{CE} = \frac{3EI/12}{4EI/18 + 3EI/15 + 3EI/12} = 0.372$$

$$\text{DF}_{CD} = \frac{3EI/15}{4EI/18 + 3EI/15 + 3EI/12} = 0.298$$

$$\text{DF}_{CB} = 1 - 0.372 - 0.298 = 0.330$$

$$\text{DF}_{DC} = 1 \qquad \text{DF}_{EC} = 1$$

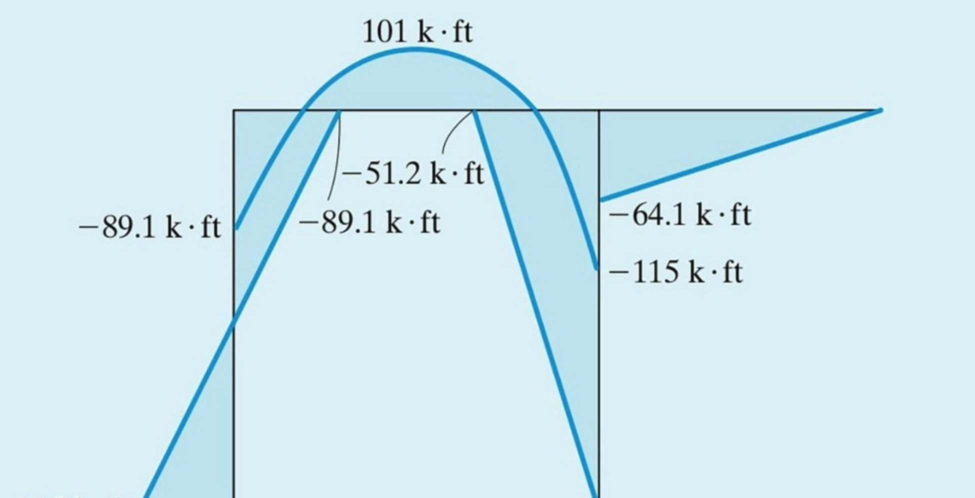


$$(FEM)_{BC} = -\frac{wL^2}{12} = -\frac{5(18)^2}{12} = -135\ \text{k}\cdot\text{ft}$$

$$(FEM)_{CB} = \frac{wL^2}{12} = \frac{5(18)^2}{12} = 135\ \text{k}\cdot\text{ft}$$

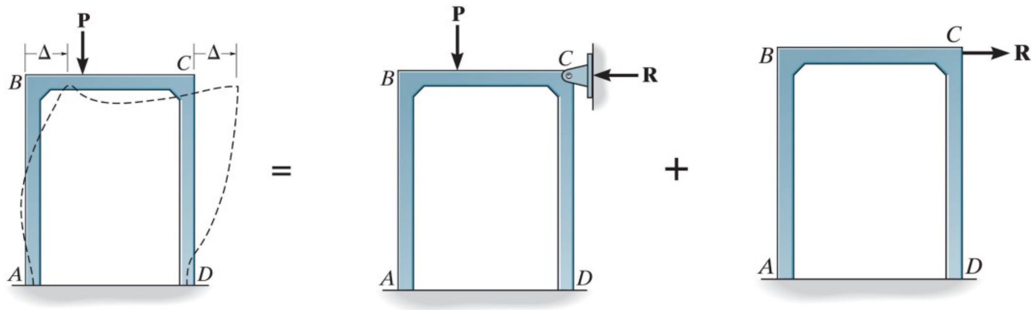
| Joint | A | B | | C | | | D | E |
|---|---|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | CE | DC | EC |
| DF | 0 | 0.545 | 0.455 | 0.330 | 0.298 | 0.372 | 1 | 1 |
| FEM | | | −135 | 135 | | | | |
| Dist. | | 73.6 | 61.4 | −44.6 | −40.2 | −50.2 | | |
| CO | 36.8 | | −22.3 | 30.7 | | | | |
| Dist. | | 12.2 | 10.1 | −10.1 | −9.1 | −11.5 | | |
| CO | 6.1 | | −5.1 | 5.1 | | | | |
| Dist. | | 2.8 | 2.3 | −1.7 | −1.5 | −1.9 | | |
| CO | 1.4 | | −0.8 | 1.2 | | | | |
| Dist. | | 0.4 | 0.4 | −0.4 | −0.4 | −0.4 | | |
| CO | 0.2 | | −0.2 | 0.2 | | | | |
| Dist. | | 0.1 | 0.1 | −0.1 | 0.0 | −0.1 | | |
| ΣM | 44.5 | 89.1 | −89.1 | 115 | −51.2 | −64.1 | | |






(c)

## Moment Distribution for Frames: Sidesway

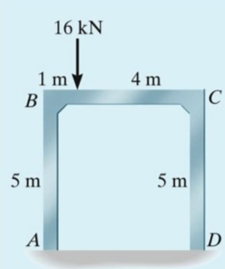



artificial joint applied
(no sidesway)

artificial joint removed
(sidesway)

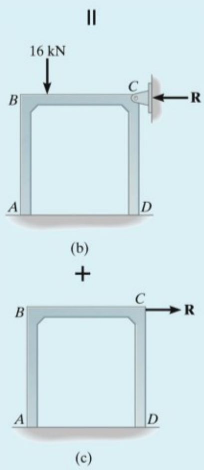


## EXAMPLE 3




(a)

Determine the moments at each joint of the frame shown in Fig. 11–18*a*. *EI* is constant.



(b)

(c)

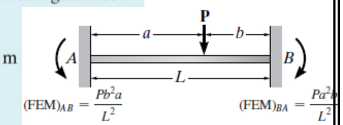
**Fig. 11–18**

## SOLUTION

First we consider the frame held from sidesway as shown in Fig. 11–18*b*. We have

$$(\text{FEM})_{BC} = -\frac{16(4)^2(1)}{(5)^2} = -10.24\ \text{kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{16(1)^2(4)}{(5)^2} = 2.56\ \text{kN}\cdot\text{m}$$




$$K_{AB} = K_{BC} = K_{CD} = \frac{4EI}{L} = \frac{4EI}{5}$$

$DF_{(AB)} = 0$ (Fixed end)

$DF_{(BA)} = K_{AB}/(K_{AB}+K_{BC}) = \frac{4EI}{5}/(\frac{4EI}{5}+\frac{4EI}{5})$ =0.5

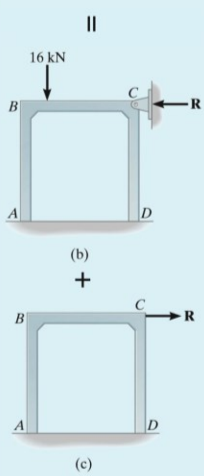
$DF_{(BC)} = 1 - DF_{(BA)} = 0.50$

$DF_{(CB)} = K_{BC}/(K_{BC}+K_{CD}) = \frac{4EI}{5}/(\frac{4EI}{5}+\frac{4EI}{5})$ =0.5

$DF_{(CD)} = 1 - DF_{(CB)} = 0.50$

$DF_{(DC)} = 0$ (Fixed end)





(b)

(c)

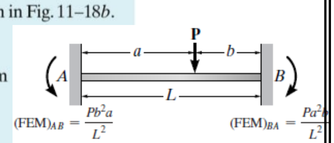
**Fig. 11–18**

## SOLUTION

First we consider the frame held from sidesway as shown in Fig. 11–18*b*. We have

$$(\text{FEM})_{BC} = -\frac{16(4)^2(1)}{(5)^2} = -10.24\ \text{kN}\cdot\text{m}$$

$$(\text{FEM})_{CB} = \frac{16(1)^2(4)}{(5)^2} = 2.56\ \text{kN}\cdot\text{m}$$




$$K_{AB} = K_{BC} = K_{CD} = \frac{4EI}{L} = \frac{4EI}{5}$$

$DF_{(AB)} = 0$ (Fixed end)

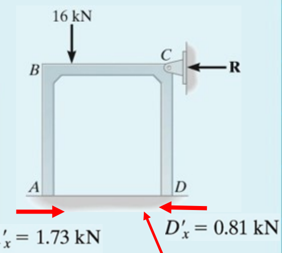
$DF_{(BA)} = K_{AB}/(K_{AB}+K_{BC}) = \frac{4EI}{5}/(\frac{4EI}{5}+\frac{4EI}{5})$ =0.5

$DF_{(BC)} = 1 - DF_{(BA)} = 0.50$

$DF_{(CB)} = K_{BC}/(K_{BC}+K_{CD}) = \frac{4EI}{5}/(\frac{4EI}{5}+\frac{4EI}{5})$ =0.5

$DF_{(CD)} = 1 - DF_{(CB)} = 0.50$

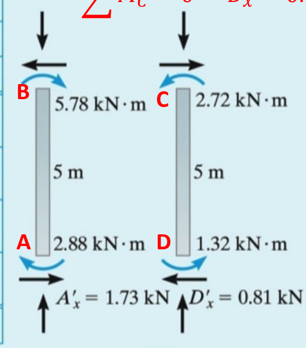
$DF_{(DC)} = 0$ (Fixed end)




| Joint | A | B | | C | | D |
|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | DC |
| DF | 0 | 0.5 | 0.5 | 0.5 | 0.5 | 0 |
| FEM | | | −10.24 | 2.56 | | |
| Dist. | | 5.12 | 5.12 | −1.28 | −1.28 | |
| CO | 2.56 | | −0.64 | 2.56 | | −0.64 |
| Dist. | | 0.32 | 0.32 | −1.28 | −1.28 | |
| CO | 0.16 | | −0.64 | 0.16 | | −0.64 |
| Dist. | | 0.32 | 0.32 | −0.08 | −0.08 | |
| CO | 0.16 | | −0.04 | 0.16 | | −0.04 |
| Dist. | | 0.02 | 0.02 | −0.08 | −0.08 | |
| ΣM | 2.88 | 5.78 | −5.78 | 2.72 | −2.72 | −1.32 |

(d)

$\sum M_C = 0 \rightarrow D'_x = 0.81\ KN$

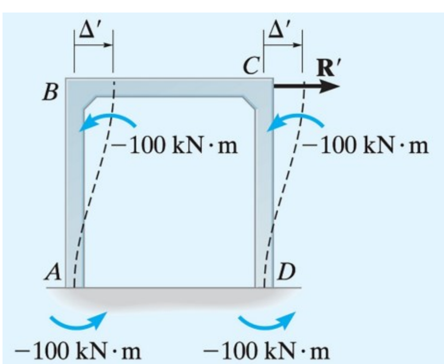


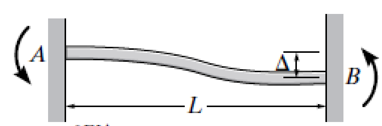

(e)

$\sum M_B = 0 \rightarrow A'_x = 1.73\ KN$

$$\boxed{\Sigma F_x = 0; \qquad R = 1.73\text{ kN} - 0.81\text{ kN} = 0.92\text{ kN}}$$



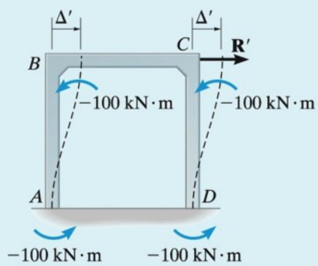



$$(FEM)_{AB} = \frac{6EI\Delta}{L^2} \qquad (FEM)_{BA} = \frac{6EI\Delta}{L^2}$$

$$\frac{6EI\Delta}{L^2} = 100\ KN.m$$

$$(FEM)_{AB} = (FEM)_{BA} = (FEM)_{CD} = (FEM)_{DC} = -100\text{ kN}\cdot\text{m}$$



(f)

| Joint | A | B | | C | | D |
|---|---|---|---|---|---|---|
| Member | AB | BA | BC | CB | CD | DC |
| DF | 0 | 0.5 | 0.5 | 0.5 | 0.5 | 0 |
| FEM | −100 | −100 | | | −100 | −100 |
| Dist. | | 50 | 50 | 50 | 50 | |
| CO | 25 | | 25 | 25 | | 25 |
| Dist. | | −12.5 | −12.5 | −12.5 | −12.5 | |
| CO | −6.25 | | −6.25 | −6.25 | | −6.25 |
| Dist. | | 3.125 | 3.125 | 3.125 | 3.125 | |
| CO | 1.56 | | 1.56 | 1.56 | | 1.56 |
| Dist. | | −0.78 | −0.78 | −0.78 | −0.78 | |
| CO | −0.39 | | −0.39 | −0.39 | | −0.39 |
| Dist. | | 0.195 | 0.195 | 0.195 | 0.195 | |
| $\Sigma M$ | −80.00 | −60.00 | 60.00 | 60.00 | −60.00 | −80.00 |

(g)



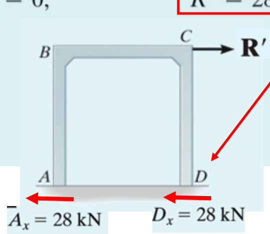
Since *both* $B$ and $C$ happen to be displaced the same amount $\Delta'$, and $AB$ and $DC$ have the *same* $E$, $I$, and $L$, the FEM in $AB$ will be the *same* as that in $DC$. As shown in Fig. 11–18*f*, we will arbitrarily *assume* this fixed-end moment to be

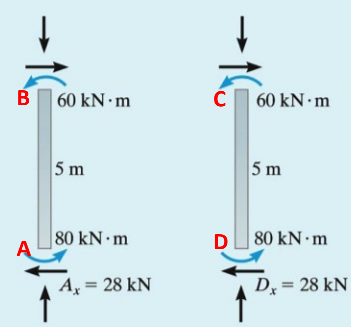
$$(\text{FEM})_{AB} = (\text{FEM})_{BA} = (\text{FEM})_{CD} = (\text{FEM})_{DC} = -100\ \text{kN}\cdot\text{m}$$

A *negative sign* is necessary since the moment must act *counterclockwise* on the column for deflection $\Delta'$ to the right. The value of $\mathbf{R}'$ associated with this $-100$ kN · m moment can now be determined. The moment distribution of the FEMs is shown in Fig. 11–18*g*. From equilibrium, the horizontal reactions at $A$ and $D$ are calculated, Fig. 11–18*h*. Thus, for the entire frame we require

$\Sigma F_x = 0;$ $\boxed{R' = 28 + 28 = 56.0\ \text{kN}}$

$\sum M_C = 0 \rightarrow D_x = 28\ KN$




$\sum M_B = 0 \rightarrow A_x = 28\ KN$

(h)

Hence, $R' = 56.0$ kN creates the moments tabulated in Fig. 11–18*g*. Corresponding moments caused by $R = 0.92$ kN can now be determined by proportion. The resultant moment in the frame, Fig. 11–18*a*, is therefore equal to the *sum* of those calculated for the frame in Fig. 11–18*b* plus the proportionate amount of those for the frame in Fig. 11–18*c*. We have

$$M_{xy} = M_{xy}(no - sidesway) + \frac{R}{R'} * M_{xy}(sidesway)$$

$$M_{AB} = 2.88 + \tfrac{0.92}{56.0}(-80) = 1.57\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BA} = 5.78 + \tfrac{0.92}{56.0}(-60) = 4.79\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{BC} = -5.78 + \tfrac{0.92}{56.0}(60) = -4.79\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CB} = 2.72 + \tfrac{0.92}{56.0}(60) = 3.71\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{CD} = -2.72 + \tfrac{0.92}{56.0}(-60) = -3.71\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

$$M_{DC} = -1.32 + \tfrac{0.92}{56.0}(-80) = -2.63\ \text{kN}\cdot\text{m} \qquad \textit{Ans.}$$

The Hashemite University
Faculty of Engineering
Department of Civil Engineering

# CE 315: Structural Analysis

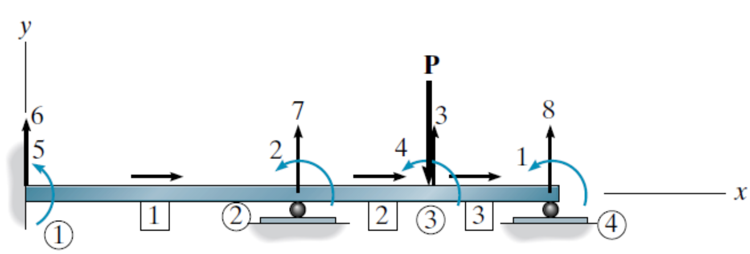
## Chapter 10: Beam and Frame Analysis Using the Stiffness Method

**Dr. Ra'ed Al-Mazaidh**

## Member and Node Identification

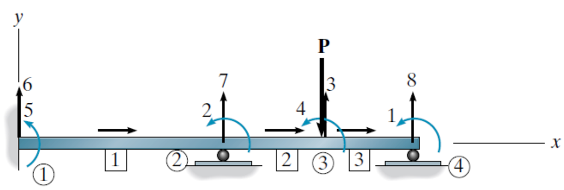
- Each **element** must be free from load and have a prismatic cross section.
- The **nodes** of each element are located at a **support** or at **points where members are connected together**, where an **external force is applied**, where the **cross sectional area suddenly changes**, or where **the vertical or rotational displacement at a point is to be determined**.

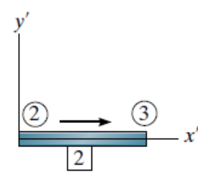
✓ **3 Elements**
✓ **4 Nodes**



## Global and Member Coordinates

- The global coordinate system will be identified using ***x, y, z*** **axes** that generally have their origin at a node and are positioned so that the nodes at other points on the beam all have positive coordinates,
- The local or member *x'*, *y'*, *z'* coordinates have their origin at the "near" end of each element, and the positive *x* axis is directed towards the "far" end.
- In both cases we have used a right-handed coordinate system, so that if the fingers of the right hand are curled from the x (x') axis towards the y (y') axis, the thumb points in the positive direction of the z (z') axis, which is directed out of the page.

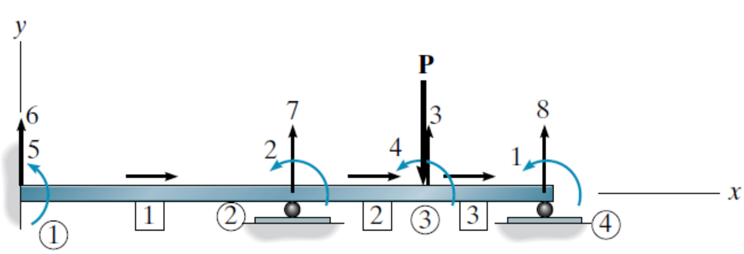




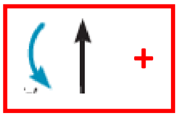

## Degrees of Freedom

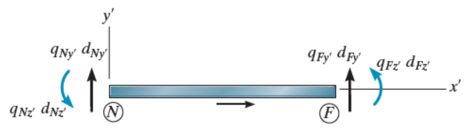
- Consider the effects of both **bending** and **shear**, then each node on a beam can have two degrees of freedom, namely, a **vertical displacement** and a **rotation.**
- **The lowest code** numbers will be used to identify the unknown displacements (**unconstrained degrees of freedom**), and the highest numbers are used to identify the known displacements (**constrained degrees of freedom**)

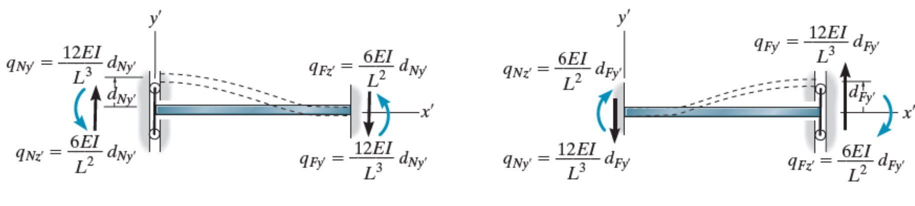


# Beam-Member Stiffness Matrix

➤ **Displacements:**



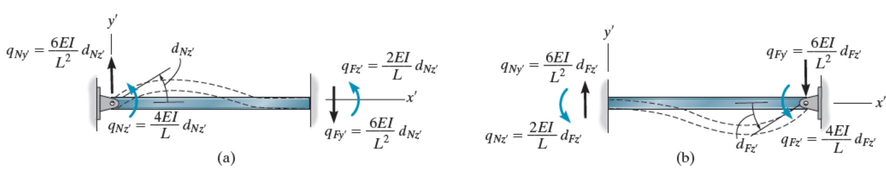



(a) $y'$ displacements (b)



➤ **Rotations:**




$$
\begin{bmatrix} q_{Ny'} \\ q_{Nz'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} N_{y'} & N_{z'} & F_{y'} & F_{z'} \end{array} \\
\begin{bmatrix}
\frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
\frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\end{array}
\begin{bmatrix} d_{Ny'} \\ d_{Nz'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix}
$$

These equations can also be written in abbreviated form as

$$\mathbf{q} = \mathbf{k}\mathbf{d}$$

## Beam-Structure Stiffness Matrix

Once all the member stiffness matrices have been found, we must assemble them into the structure stiffness matrix **K**. This process depends on first knowing the location of each element in the member stiffness matrix. Here the rows and columns of each **k** matrix are identified by the two code numbers at the near end of the member $(N_{y'}, N_{z'})$ followed by those at the far end $(F_{y'}, F_{z'})$. Therefore, when assembling the matrices, each element must be placed in the same location of the **K** matrix.

## Member forces

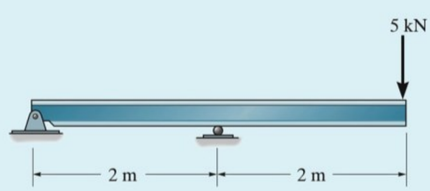
$$\mathbf{q} = \mathbf{k}\mathbf{d} + \mathbf{q}_0$$

**$q_o$ :Fixed-end reactions**



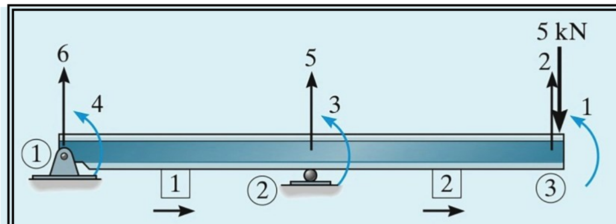
## Example 1:

Determine the reactions at the supports of the beam shown in Fig. 15–8*a*. *EI* is constant.




(a)

Fig. 15–8




$$K=\begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

## Member Stiffness Matrices.

**Node (1)........(2)**

$$\mathbf{k}_1 = EI \begin{array}{c} \begin{matrix} 6 & 4 & 5 & 3 \end{matrix} \\ \begin{bmatrix} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{bmatrix} \end{array} \begin{matrix} 6 \\ 4 \\ 5 \\ 3 \end{matrix}$$

$$\frac{12EI}{L^3} = EI\frac{12}{2^3} = 1.5 \text{ EI}$$

$$\frac{6EI}{L^2} = EI\frac{6}{2^2} = 1.5 \text{ EI}$$

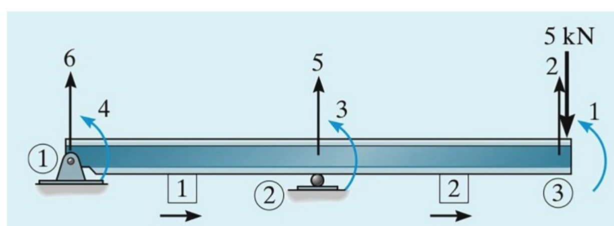
$$\frac{4EI}{L} = EI\frac{4}{2} = 2 \text{ EI}$$

$$\frac{2EI}{L} = EI\frac{2}{2} = 1.0 \text{ EI}$$



**Node (2)........(3)**

$$\mathbf{k}_2 = EI \begin{array}{c} \begin{matrix} 5 & 3 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{bmatrix} \end{array} \begin{matrix} 5 \\ 3 \\ 2 \\ 1 \end{matrix}$$



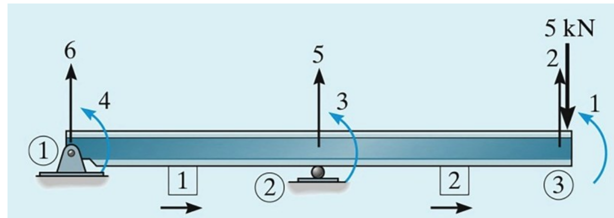

**K=k1+k2**

$$\mathbf{k}_1 = EI \begin{array}{c} \begin{matrix} 6 & 4 & 5 & 3 \end{matrix} \\ \begin{bmatrix} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{bmatrix} \end{array} \begin{matrix} 6 \\ 4 \\ 5 \\ 3 \end{matrix}$$

$$\mathbf{k}_2 = EI \begin{array}{c} \begin{matrix} 5 & 3 & 2 & 1 \end{matrix} \\ \begin{bmatrix} 1.5 & 1.5 & -1.5 & 1.5 \\ 1.5 & 2 & -1.5 & 1 \\ -1.5 & -1.5 & 1.5 & -1.5 \\ 1.5 & 1 & -1.5 & 2 \end{bmatrix} \end{array} \begin{matrix} 5 \\ 3 \\ 2 \\ 1 \end{matrix}$$

$$EI \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \left[\begin{array}{cccc:cc} 2 & -1.5 & 1 & 0 & 1.5 & 0 \\ -1.5 & 1.5 & -1.5 & 0 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 2 & -1.5 & 1.5 \\ \hdashline 1.5 & -1.5 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 1.5 & 1.5 & -1.5 & 1.5 \end{array}\right] \end{array}$$

$$\mathbf{Q} = \begin{bmatrix} Q_1 = 0 \\ Q_2 = -5 \\ Q_3 = 0 \\ Q_4 = 0 \\ Q_5 \\ Q_6 \end{bmatrix} \qquad \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 = 0 \\ D_6 = 0 \end{bmatrix}$$




$$\mathbf{Q} = \mathbf{KD}$$

$$\begin{bmatrix} Q_1 = 0 \\ Q_2 = -5 \\ Q_3 = 0 \\ Q_4 = 0 \\ Q_5 \\ Q_6 \end{bmatrix} = EI \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccc:cc} 2 & -1.5 & 1 & 0 & 1.5 & 0 \\ -1.5 & 1.5 & -1.5 & 0 & -1.5 & 0 \\ 1 & -1.5 & 4 & 1 & 0 & 1.5 \\ 0 & 0 & 1 & 2 & -1.5 & 1.5 \\ \hdashline 1.5 & -1.5 & 0 & -1.5 & 3 & -1.5 \\ 0 & 0 & 1.5 & 1.5 & -1.5 & 1.5 \end{array}\right] \end{array} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 = 0 \\ D_6 = 0 \end{bmatrix}$$



$$0 = 2D_1 - 1.5D_2 + D_3 + 0$$

$$-\frac{5}{EI} = -1.5D_1 + 1.5D_2 - 1.5D_3 + 0$$

$$0 = D_1 - 1.5D_2 + 4D_3 + D_4$$

$$0 = 0 + 0 + D_3 + 2D_4$$

Solving,

$$D_1 = -\frac{16.67}{EI}$$

$$D_2 = -\frac{26.67}{EI}$$

$$D_3 = -\frac{6.67}{EI}$$

$$D_4 = \frac{3.33}{EI}$$

$$Q_5 = 1.5EI\left(-\frac{16.67}{EI}\right) - 1.5EI\left(-\frac{26.67}{EI}\right) + 0 - 1.5EI\left(\frac{3.33}{EI}\right)$$

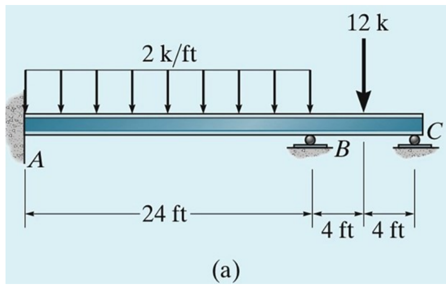
$$= 10\text{ kN}$$

$$Q_6 = 0 + 0 + 1.5EI\left(-\frac{6.67}{EI}\right) + 1.5EI\left(\frac{3.33}{EI}\right)$$

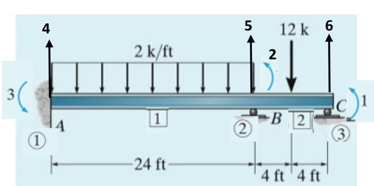
$$= -5\text{ kN}$$

## Example 2:

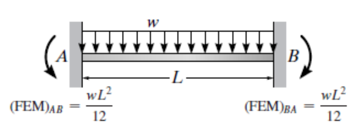
Determine the moment developed at support $A$ of the beam shown in Fig. 15–11$a$. Assume the roller supports can pull down or push up on the beam. Take $E = 29(10^3)$ ksi, $I = 510$ in$^4$.

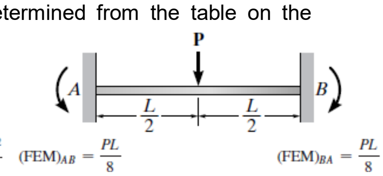





The matrix analysis requires that **the external loading be applied at the nodes,** and therefore the distributed and concentrated loads are replaced by their **equivalent fixed-end moments,** which are determined from the table on the inside back cover.




$$\frac{wl^2}{12} = \frac{(2)(24^2)}{12} = 96 \text{ K.ft.}$$

$$\frac{PL}{8} = \frac{(12)(8)}{8} = 12K.ft.$$

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 = 0 \\ D_4 = 0 \\ D_5 = 0 \\ D_6 = 0 \end{bmatrix}$$

## FEMs

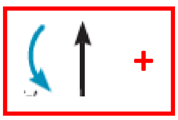
$$\frac{wl^2}{12} = \frac{(2)(24^2)}{12} = 96\ \text{K.ft.}$$

$$\frac{PL}{8} = \frac{(12)(8)}{8} = 12K.ft.$$


=12K.ft.X(1ft/12in)=144 K.ft.

96-12=84 K.ft.

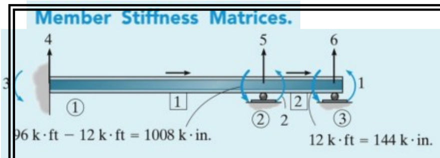
=84 K.ft.X(1ft/12in)=1008 K.ft.




$$\begin{bmatrix} Q_1 = 144 \\ Q_2 = 1008 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}$$



## Member Stiffness Matrices.




beam to be analyzed by stiffness method

$$K = \begin{bmatrix} \frac{12EI}{L^3} & \frac{6EI}{L^2} & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{4EI}{L} & -\frac{6EI}{L^2} & \frac{2EI}{L} \\ -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ \frac{6EI}{L^2} & \frac{2EI}{L} & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

### Member 1.

**Node (1)........(2)**

$$\frac{12EI}{L^3} = \frac{12(29)(10^3)(510)}{[24(12)]^3} = 7.430$$

$$\frac{6EI}{L^2} = \frac{6(29)(10^3)(510)}{[24(12)]^2} = 1069.9$$

$$\frac{4EI}{L} = \frac{4(29)(10^3)(510)}{24(12)} = 205\,417$$

$$\frac{2EI}{L} = \frac{2(29)(10^3)(510)}{24(12)} = 102\,708$$

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccc} 4 & 3 & 5 & 2 \end{array} \\ \begin{bmatrix} 7.430 & 1069.9 & -7.430 & 1069.9 \\ 1069.9 & 205\,417 & -1069.9 & 102\,708 \\ -7.430 & -1069.9 & 7.430 & -1069.9 \\ 1069.9 & 102\,708 & -1069.9 & 205\,417 \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 3 \\ 5 \\ 2 \end{array}$$

**Node (2)........(3)**

### Member 2.

$$\frac{12EI}{L^3} = \frac{12(29)(10^3)(510)}{[8(12)]^3} = 200.602$$

$$\frac{6EI}{L^2} = \frac{6(29)(10^3)(510)}{[8(12)]^2} = 9628.91$$

$$\frac{4EI}{L} = \frac{4(29)(10^3)(510)}{8(12)} = 616\,250$$

$$\frac{2EI}{L} = \frac{2(29)(10^3)(510)}{8(12)} = 308\,125$$

$$\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccc} 5 & 2 & 6 & 1 \end{array} \\ \begin{bmatrix} 200.602 & 9628.91 & -200.602 & 9628.91 \\ 9628.91 & 616\,250 & -9628.91 & 308\,125 \\ -200.602 & -9628.91 & 200.602 & -9628.91 \\ 9628.91 & 308\,125 & -9628.91 & 616\,250 \end{bmatrix} \end{array} \begin{array}{c} 5 \\ 2 \\ 6 \\ 1 \end{array}$$

**K=K1+K2**

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6
\end{array}
$$

$$
\left[\begin{array}{cc:cccc}
616\,250 & 308\,125 & 0 & 0 & 9628.91 & -9628.91 \\
308\,125 & 821\,667 & 102\,708 & 1069.9 & 8559.03 & -9628.91 \\
\hdashline
0 & 102\,708 & 205\,417 & 1069.9 & -1069.9 & 0 \\
0 & 1069.9 & 1069.9 & 7.430 & -7.430 & 0 \\
9628.91 & 8559.03 & -1069.9 & -7.430 & 208.032 & -200.602 \\
-9628.91 & -9628.91 & 0 & 0 & -200.602 & 200.602
\end{array}\right]
$$

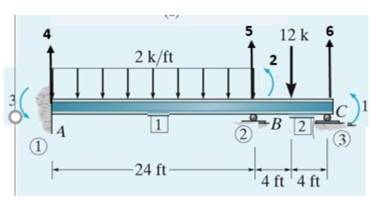
$$\mathbf{Q} = \mathbf{KD}$$

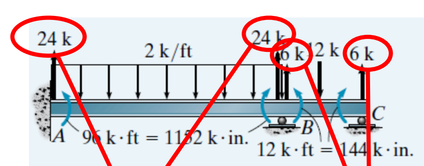
$$
\begin{bmatrix} 144 \\ 1008 \\ \hdashline Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\left[\begin{array}{cc:cccc}
616\,250 & 308\,125 & 0 & 0 & 9628.91 & -9628.91 \\
308\,125 & 821\,667 & 102\,708 & 1069.9 & 8559.03 & -9628.91 \\
\hdashline
0 & 102\,708 & 205\,417 & 1069.9 & -1069.9 & 0 \\
0 & 1069.9 & 1069.9 & 7.430 & -7.430 & 0 \\
9628.91 & 8559.03 & -1069.9 & -7.430 & 208.032 & -200.602 \\
-9628.91 & -9628.91 & 0 & 0 & -200.602 & 200.602
\end{array}\right]
\end{array}
\begin{bmatrix} D_1 \\ D_2 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}
$$

Solving in the usual manner,

$$144 = 616\,250D_1 + 308\,125D_2$$
$$1008 = 308\,125D_1 + 821\,667D_2$$
$$D_1 = -0.4673(10^{-3})\text{ rad}$$
$$D_2 = 1.40203(10^{-3})\text{ rad}$$





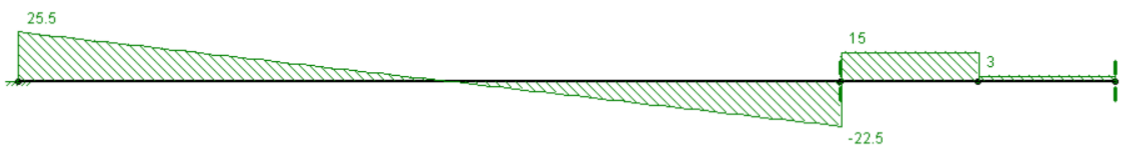




$$\frac{wl}{2} = \frac{(2)(24)}{2} = 24\,K \qquad \frac{P}{2} = \frac{12}{2} = 6\,K$$

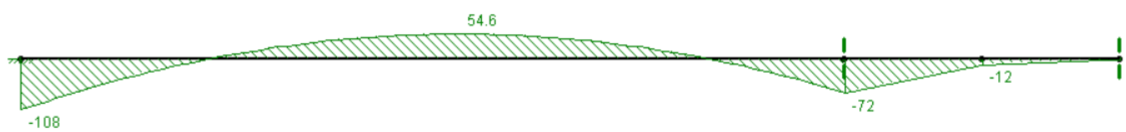
$$\mathbf{q}_1 = \mathbf{k}_1\mathbf{d}_1 + (\mathbf{q}_0)_1$$

$$
\begin{bmatrix} q_4 \\ q_3 \\ q_5 \\ q_2 \end{bmatrix}
=
\begin{array}{c}
\begin{array}{cccc} 4 & 3 & 5 & 2 \end{array} \\
\begin{bmatrix}
7.430 & 1069.9 & -7.430 & 1069.9 \\
1069.9 & 205\,417 & -1069.9 & 102\,708 \\
-7.430 & -1069.9 & 7.430 & -1069.9 \\
1069.9 & 102\,708 & -1069.9 & 205\,417
\end{bmatrix}
\end{array}
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1.40203 \end{bmatrix}(10^{-3}) +
\begin{bmatrix} 24 \\ 1152 \\ 24 \\ -1152 \end{bmatrix}
=
\begin{bmatrix} 25\,K \\ 1296\,KN.in = 108K.ft \\ 22.5KN \\ -864\,KN.in = -72\,K.ft \end{bmatrix}
$$

## S.F.D

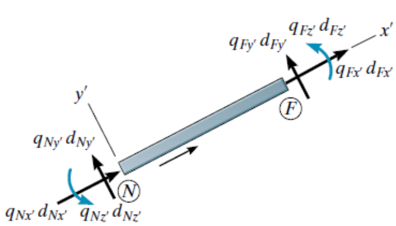



## B.M.D






## Frame-Member Stiffness Matrix




$$
\begin{array}{cc}
 & \begin{array}{cccccc} N_{x'} & N_{y'} & N_{z'} & F_{x'} & F_{y'} & F_{z'} \end{array} \\
\begin{bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nz'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix} =
& \begin{bmatrix}
\frac{AE}{L} & 0 & 0 & -\frac{AE}{L} & 0 & 0 \\
0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & -\frac{12EI}{L^3} & \frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{2EI}{L} \\
-\frac{AE}{L} & 0 & 0 & \frac{AE}{L} & 0 & 0 \\
0 & -\frac{12EI}{L^3} & -\frac{6EI}{L^2} & 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\
0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & -\frac{6EI}{L^2} & \frac{4EI}{L}
\end{bmatrix}
\begin{bmatrix} d_{Nx'} \\ d_{Ny'} \\ d_{Nz'} \\ d_{Fx'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix}
\end{array}
$$

$$\mathbf{q} = \mathbf{k}'\mathbf{d}$$

$$\boldsymbol{k'}$$

## Frame- Displacement and Force Transformation Matrices

$$\mathbf{d} = \mathbf{T}\mathbf{D}$$

**T=**

$$\begin{bmatrix} d_{Nx'} \\ d_{Ny'} \\ d_{Nz'} \\ d_{Fx'} \\ d_{Fy'} \\ d_{Fz'} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 0 & -\sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} D_{Nx} \\ D_{Ny} \\ D_{Nz} \\ D_{Fx} \\ D_{Fy} \\ D_{Fz} \end{bmatrix}$$



## Frame- Force Transformation Matrix

$$\begin{bmatrix} Q_{Nx} \\ Q_{Ny} \\ Q_{Nz} \\ Q_{Fx} \\ Q_{Fy} \\ Q_{Fz} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & 0 & \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_{Nx'} \\ q_{Ny'} \\ q_{Nz'} \\ q_{Fx'} \\ q_{Fy'} \\ q_{Fz'} \end{bmatrix}$$

$$\mathbf{Q} = \mathbf{T}^T\mathbf{q}$$

## Frame-Member Global Stiffness Matrix

$$\boxed{\mathbf{q} = \mathbf{k}'\mathbf{T}\mathbf{D}}$$

$$(\mathbf{Q} = \mathbf{T}^T\mathbf{q})$$

$$\mathbf{Q} = \mathbf{T}^T\mathbf{k}'\mathbf{T}\mathbf{D}$$

$$\boxed{\mathbf{Q} = \mathbf{k}\mathbf{D}}$$

$$\mathbf{k} = \mathbf{T}^T\mathbf{k}'\mathbf{T}$$



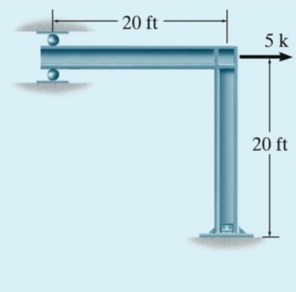
$$\mathbf{k} = \begin{array}{cc}
\begin{array}{cccccc} N_x & N_y & N_z & F_x & F_y & F_z \end{array} & \\
\begin{bmatrix}
\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & -\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y \\
\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} \\
-\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & \left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y \\
-\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x \\
-\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{4EI}{L}
\end{bmatrix} & \begin{array}{c} N_x \\ N_y \\ N_z \\ F_x \\ F_y \\ F_z \end{array}
\end{array}$$

$$\lambda_x = \cos(\theta)$$

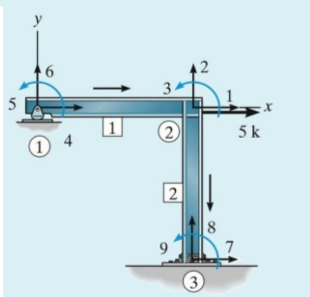
$$\lambda_y = \sin(\theta)$$

# Example 3:

Determine the loadings at the joints of the two-member frame shown in Fig. 16–4*a*. Take $I = 500\text{ in}^4$, $A = 10\text{ in}^2$, and $E = 29(10^3)$ ksi for both members.




(a)




$$\mathbf{D}_k = \begin{array}{c} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} 6 \\ 7 \\ 8 \\ 9 \end{array} \qquad \mathbf{Q}_k = \begin{array}{c} \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{array}\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}$$



$$\mathbf{k} = \begin{array}{c} \begin{array}{cccccc} N_x & N_y & N_z & F_x & F_y & F_z \end{array} \\ \begin{bmatrix} \left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y & -\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\frac{6EI}{L^2}\lambda_y \\ \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & \frac{6EI}{L^2}\lambda_x \\ -\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} \\ -\left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y & \left(\frac{AE}{L}\lambda_x^2 + \frac{12EI}{L^3}\lambda_y^2\right) & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \frac{6EI}{L^2}\lambda_y \\ -\left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & -\left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x & \left(\frac{AE}{L} - \frac{12EI}{L^3}\right)\lambda_x\lambda_y & \left(\frac{AE}{L}\lambda_y^2 + \frac{12EI}{L^3}\lambda_x^2\right) & -\frac{6EI}{L^2}\lambda_x \\ -\frac{6EI}{L^2}\lambda_y & \frac{6EI}{L^2}\lambda_x & \frac{2EI}{L} & \frac{6EI}{L^2}\lambda_y & -\frac{6EI}{L^2}\lambda_x & \frac{4EI}{L} \end{bmatrix} \end{array} \begin{array}{c} \\ N_x \\ N_y \\ N_z \\ F_x \\ F_y \\ F_z \end{array}$$

$$\frac{AE}{L} = \frac{10[29(10^3)]}{20(12)} = 1208.3 \text{ k/in.}$$

$$\frac{12EI}{L^3} = \frac{12[29(10^3)(500)]}{[20(12)]^3} = 12.6 \text{ k/in.}$$

$$\frac{6EI}{L^2} = \frac{6[29(10^3)(500)]}{[20(12)]^2} = 1510.4 \text{ k}$$

$$\frac{4EI}{L} = \frac{4[29(10^3)(500)]}{20(12)} = 241.67(10^3) \text{ k} \cdot \text{in.}$$

$$\frac{2EI}{L} = \frac{2[29(10^3)(500)]}{20(12)} = 120.83(10^3) \text{ k} \cdot \text{in.}$$



**Member 1.**

Node (1)........(2) $\cos(0) = 1$ $sin(0) = 0$

Substituting the data into Eq. 16–10, we have

$$\mathbf{k}_1 = \begin{array}{c} \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.67(10^3) & 0 & -1510.4 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83(10^3) & 0 & -1510.4 & 241.67(10^3) \end{bmatrix} \end{array} \begin{array}{c} 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array}$$

**Node (2)……..(3)** $\cos(270) = 0$ $\quad sin(270) = -1$

Substituting the data into Eq. 16–10 yields

$$
\mathbf{k}_2 = \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{cccccc}
12.6 & 0 & 1510.4 & -12.6 & 0 & 1510.4 \\
0 & 1208.3 & 0 & 0 & -1208.3 & 0 \\
1510.4 & 0 & 241.67(10^3) & -1510.4 & 0 & 120.83(10^3) \\
-12.6 & 0 & -1510.4 & 12.6 & 0 & -1510.4 \\
0 & -1208.3 & 0 & 0 & 1208.3 & 0 \\
1510.4 & 0 & 120.83(10^3) & -1510.4 & 0 & 241.67(10^3)
\end{array}\right] \end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 7 \\ 8 \\ 9 \end{array}
$$



The structure stiffness matrix is determined by assembling $\mathbf{k}_1$ and $\mathbf{k}_2$. The result, $\mathbf{Q} = \mathbf{KD}$, is

$$
\left[\begin{array}{c} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ \hdashline Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{array}\right]
=
\begin{array}{c} \begin{array}{ccccc:cccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\ \left[\begin{array}{ccccc:cccc}
1220.9 & 0 & 1510.4 & -1208.3 & 0 & 0 & -12.6 & 0 & 1510.4 \\
0 & 1220.9 & -1510.4 & 0 & -1510.4 & -12.6 & 0 & -1208.3 & 0 \\
1510.4 & -1510.4 & 483.33(10^3) & 0 & 120.83(10^3) & 1510.4 & -1510.4 & 0 & 120.83(10^3) \\
-1208.3 & 0 & 0 & 1208.3 & 0 & 0 & 0 & 0 & 0 \\
0 & -1510.4 & 120.83(10^3) & 0 & 241.67(10^3) & 1510.4 & 0 & 0 & 0 \\ \hdashline
0 & -12.6 & 1510.4 & 0 & 1510.4 & 12.6 & 0 & 0 & 0 \\
-12.6 & 0 & 1510.4 & 0 & 0 & 0 & 12.6 & 0 & 1510.4 \\
0 & -1208.3 & 0 & 0 & 0 & 0 & 0 & 1208.3 & 0 \\
1510.4 & 0 & 120.83(10^3) & 0 & 0 & 0 & 1510.4 & 0 & 241.67(10^3)
\end{array}\right] \end{array}
\left[\begin{array}{c} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ \hdashline 0 \\ 0 \\ 0 \\ 0 \end{array}\right] \quad (1)
$$

**Displacements and Loads.** Expanding to determine the displacements yields

$$\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1220.9 & 0 & 1510.4 & -1208.3 & 0 \\ 0 & 1220.9 & -1510.4 & 0 & -1510.4 \\ 1510.4 & -1510.4 & 483.33(10^3) & 0 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 \\ 0 & -1510.4 & 120.83(10^3) & 0 & 241.67(10^3) \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix}$$



Solving, we obtain

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \end{bmatrix} = \begin{bmatrix} 0.696 \text{ in.} \\ -1.55(10^{-3}) \text{ in.} \\ -2.488(10^{-3}) \text{ rad} \\ 0.696 \text{ in.} \\ 1.234(10^{-3}) \text{ rad} \end{bmatrix}$$

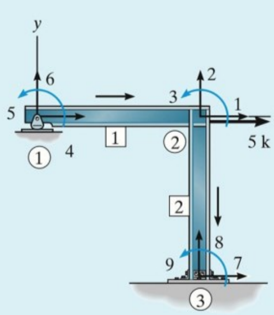
Using these results, the support reactions are determined from Eq. (1) as follows:

$$\begin{bmatrix} Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{bmatrix} = \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & -12.6 & 1510.4 & 0 & 1510.4 \\ -12.6 & 0 & -1510.4 & 0 & 0 \\ 0 & -1208.3 & 0 & 0 & 0 \\ 1510.4 & 0 & 120.83(10^3) & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} 0.696 \\ -1.55(10^{-3}) \\ -2.488(10^{-3}) \\ 0.696 \\ 1.234(10^{-3}) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.87 \text{ k} \\ -5.00 \text{ k} \\ 1.87 \text{ k} \\ 750 \text{ k} \cdot \text{in.} \end{bmatrix} \quad \textit{Ans.}$$

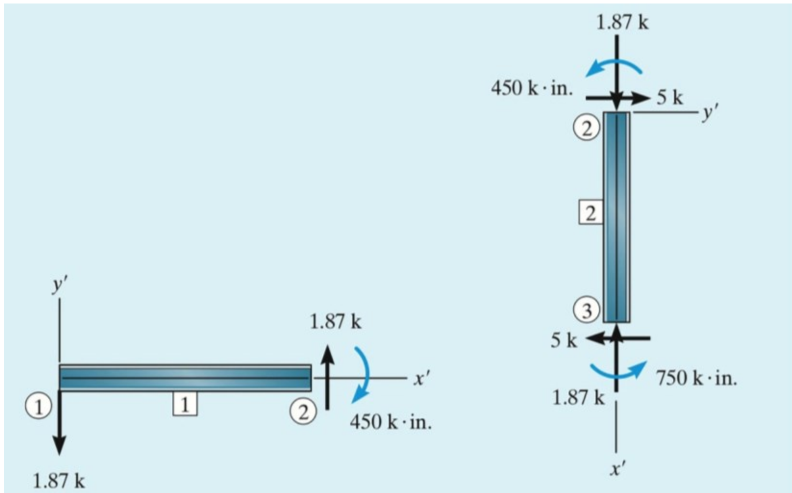
$$\mathbf{q}_1 = \mathbf{k}'_1\mathbf{T}_1\mathbf{D} = \begin{array}{c} \begin{array}{cccccc} 4 & 6 & 5 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 1208.3 & 0 & 0 & -1208.3 & 0 & 0 \\ 0 & 12.6 & 1510.4 & 0 & -12.6 & 1510.4 \\ 0 & 1510.4 & 241.67(10^3) & 0 & -1510.4 & 120.83(10^3) \\ -1208.3 & 0 & 0 & 1208.3 & 0 & 0 \\ 0 & -12.6 & -1510.4 & 0 & 12.6 & -1510.4 \\ 0 & 1510.4 & 120.83(10^3) & 0 & -1510.4 & 241.67(10^3) \end{bmatrix} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.696 \\ 0 \\ 1.234(10^{-3}) \\ 0.696 \\ -1.55(10^{-3}) \\ -2.488(10^{-3}) \end{bmatrix} \begin{array}{c} 4 \\ 6 \\ 5 \\ 1 \\ 2 \\ 3 \end{array}$$



Solving yields




$$\begin{bmatrix} q_4 \\ q_6 \\ q_5 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.87\text{ k} \\ 0 \\ 0 \\ 1.87\text{ k} \\ -450\text{ k}\cdot\text{in.} \end{bmatrix}$$ *Ans.*


1.87 k
450 k · in.
5 k
y'
2
2
3
5 k
750 k · in.
1.87 k
x'
y'
1.87 k
x'
1
1
2
450 k · in.
1.87 k