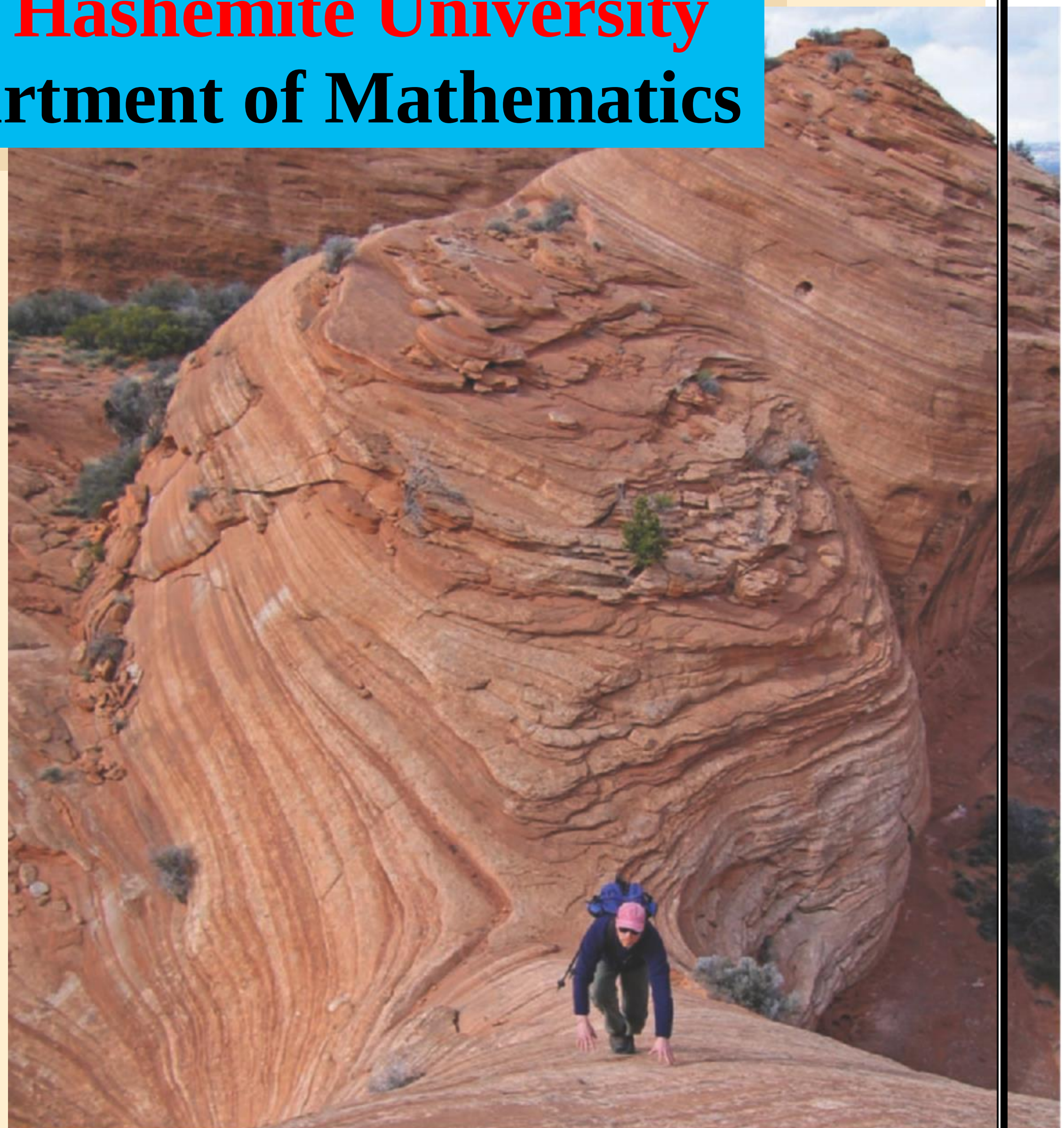
Calculus (3)

The Hashemite University
Department of Mathematics



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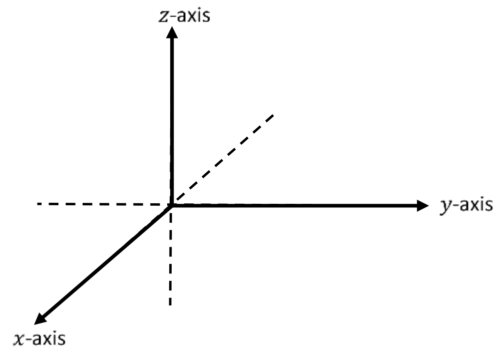
Chapter 12: Vectors and the Geometry of Space

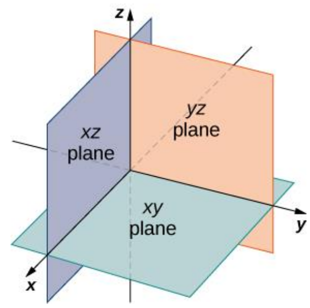
Section 12.1: Three-Dimensional Coordinate Systems

Definition 12.1.1:



- The space can be represented by sketching thee perpendicular axes called: the x-axis, the y-axis, and the z-axis that are intersected at a point called the origin $\mathbf{0}$.

- These axes (x-axis, y-axis, z-axis) are called the coordinate axes (المحاور الاحداثية).



- The plane that contains the:

x-axis and y-axis is called the xy-plane

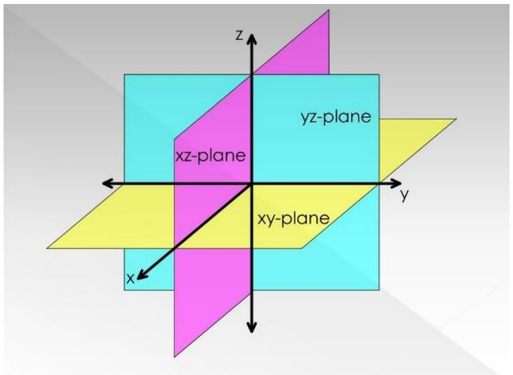
x-axis and z-axis is called the xz-plane

y-axis and z-axis is called the yz-plane

These planes are called the coordinate planes

Remark 12.1.2:

- We have 3 coordinate axes: x-axis, y-axis, z-axis
- We have 3 coordinate planes: xy-plane, xz-plane, yz-plane.
- The coordinate planes divide the space into 8 parts. Each part is called an octant. The first octant is the part that contains the positive parts of the coordinate axes.



Remark 12.1.3:

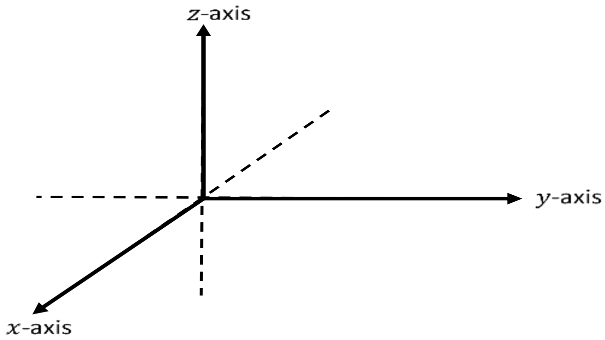
- A point (pt) P in the space is represented as $P(a, b, c)$, where:

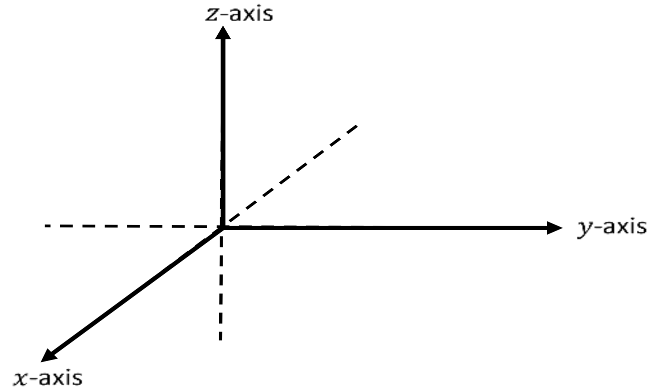
$$a = x\text{-coordinate of } P,\ b = y\text{-coordinate of } P,\ c = z\text{-coordinate of } P$$

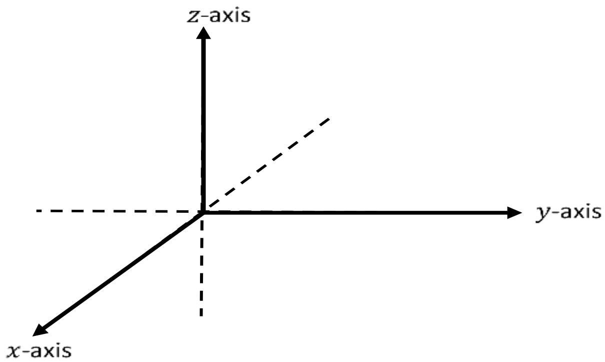
- The set of all numbers is $\mathbb{R} = (-\infty, \infty)$.
- The Cartesian product $\mathbb{R} \times \mathbb{R} = \{(x, y): x, y \in \mathbb{R}\}$ is called the 2-dimensional (2D or the plane) rectangular coordinate system. $\mathbb{R} \times \mathbb{R}$ is written as $\mathbb{R}^2$
- The Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R} = \{(x, y, z): x, y, z \in \mathbb{R}\}$ is called the 3-dimensional (3D or space) rectangular coordinate system. $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is written as $\mathbb{R}^3$.

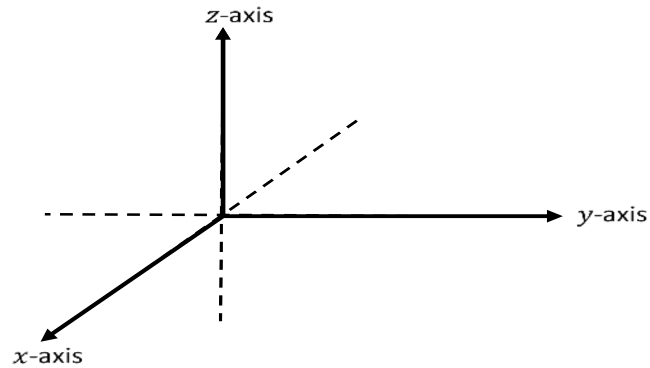
Example 12.1.4: Plot the following points in the space: $A(1,2,0), B(1,0,3), C(0,2,3), D(1,2,3),$
$E(-1,2,3), F(1,-2,3), G(1,2,-3), H(1,0,0), I(0,2,0), J(0,0,3), \boldsymbol{O}(0,0,0)$

Solution:









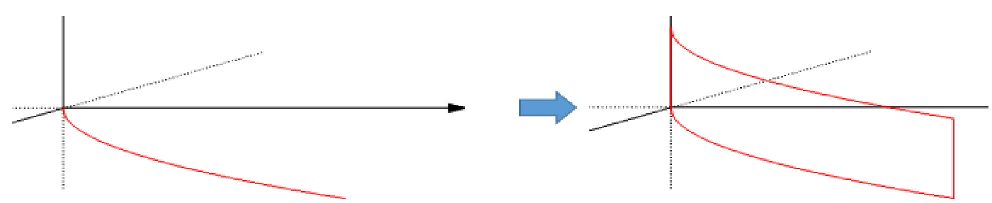
Remark 12.1.5:

- The graph of an equation in 2D (the plane $\mathbb{R}^2$) is a curve, for example if the equation $y = x^2$ is in the plane, then its graph is a curve.
- The graph of an equation in 3D (the space $\mathbb{R}^3$) is a surface, for example if the equation $y = x^2$ is in the space, then its graph is a surface.

Example 12.1.6: Sketch the graph of the surface whose equation is given by the equation:

(1) $y = x^2$ (2) $z = y$ (3) $z = \cos y$
(4) $z = 3$ (5) $x^2 = 4$ (6) $y = 1$

(1)



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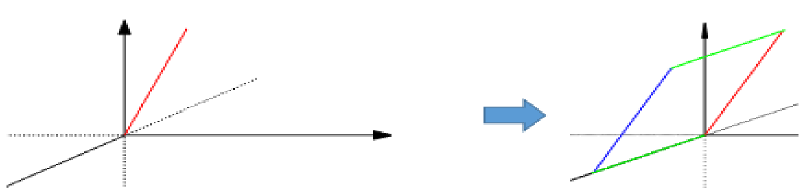
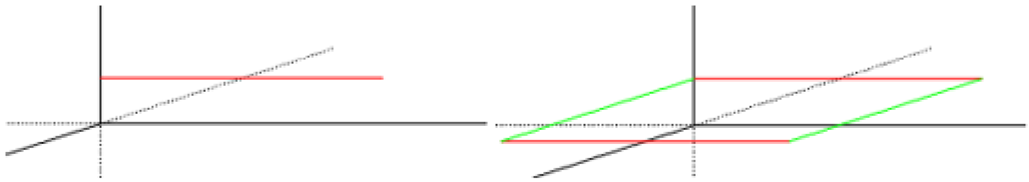


(3)

(4)



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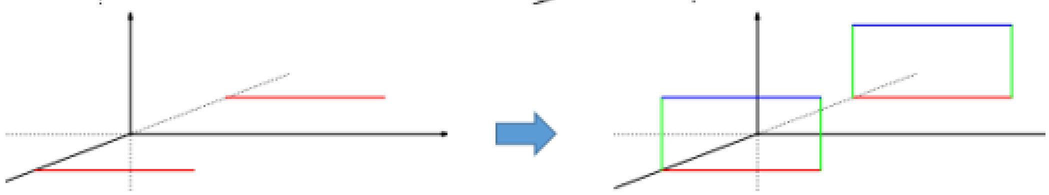

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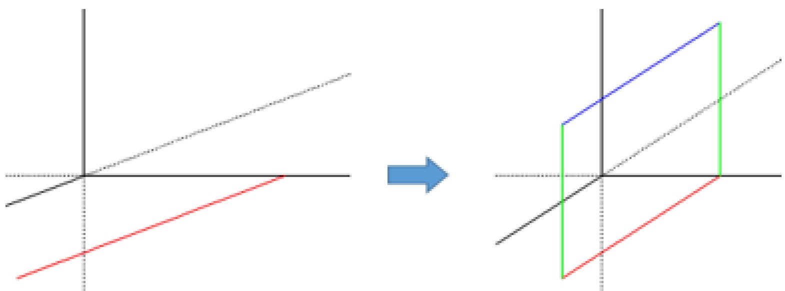
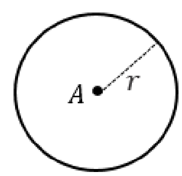


Remark 12.1.7:

(1) The equation of the xy-plane is $z = 0$
(2) The equation of the xz-plane is $y = 0$
(3) The equation of the yz-plane is $x = 0$
(4) In the plane, the equation of the circle centered at the pt. $A(a, b)$ of radius r is

$$(x - a)^2 + (y - b)^2 = r^2$$



Example 12.1.8:

(1) Identify and sketch the graph of the equation $x^2 + y^2 = 4$ in $\mathbb{R}^3$.

Solution:

(2) Which pts. (x, y, z) satisfy the equations $x^2 + y^2 = 4, z = 3$ in $\mathbb{R}^3$

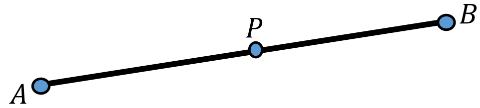
Solution:

Rule 12.1.9: Let $A(a,b,c)$ and $B(d,e,f)$ be two pts in $\mathbb{R}^3$.

(1) The distance between the pts A and B is $|AB| = \sqrt{(a-d)^2 + (b-e)^2 + (c-f)^2}$

(2) The midpoint (midpt.) of the **line segment** (قطعة مستقيمة) joining A and B is:

$$P\left(\frac{a+d}{2}, \frac{b+e}{2}, \frac{c+f}{2}\right)$$

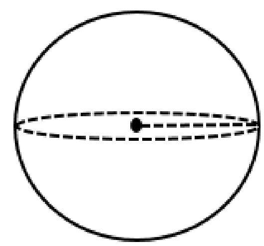


Example 12.1.10: Find the distance from the pt. $P(2,-1,4)$ to the pt. $Q(-2,0,1)$ and find the midpt. of the line segment (القطعة المستقيمة) joining P and Q.

Solution: The distance is $\text{dist}(P,Q) = \sqrt{(2-(-2))^2 + (-1-0)^2 + (4-1)^2} = \sqrt{26}$

The midpt. is $\left(\frac{2+(-2)}{2}, \frac{-1+0}{2}, \frac{4+1}{2}\right) = \left(0, \frac{-1}{2}, \frac{5}{2}\right)$

Rule 12.1.11: The standard form of the equation of the sphere centered at the pt. $A(a,b,c)$ of radius r is:



$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

When the center is the origin and the radius is 1, then the sphere $x^2 + y^2 + z^2 = 1$ is called the unit sphere.

The standard form of the equation of the sphere is:

$$x^2 + 2ax + y^2 + 2by + z^2 + 2cz + d = 0$$

with center $(-a,-b,-c)$ and radius $r = \sqrt{a^2 + b^2 + c^2 - d}$

Example 12.1.12: Which of the following is an equation of a sphere and write it in standard form and find its center and radius.

(1) $2x^2 - 12x + 3y^2 + 2z^2 + 8z = 1$
(2) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = -30$
(3) $x^2 - 6x + y^2 + z^2 + 4z = -13$
(4) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6$
(5) $2x^2 - 12x + 2z^2 + 8z = 6$

Solution:

(1) (Coefficient of x^2) $= 2$ and (Coefficient of y^2) $= 3 \Rightarrow$ not equal
The equation is not for a sphere

(2) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = -30 \Rightarrow x^2 - 6x + y^2 + z^2 + 4z = -15 \Rightarrow$

$$\underbrace{x^2 - 6x + 3^2 + y^2 + z^2 + 4z + 2^2 = -15 + 3^2 + 2^2}_{\text{اكمال المربع}}$$

$\Rightarrow\ (x-3)^2+y^2+(z+2)^2=\underbrace{-2}_{\text{مربع نصف القطر}}$ which is impossible (مربع نصف القطر مستحيل سالب)

The equation is not for any surface ⇒ The equation is not for a sphere.

(3) $x^2-6x+y^2+z^2+4z=-13 \Rightarrow x^2-6x+3^2+y^2+z^2+4z+2^2=-13+3^2+2^2$

$\Rightarrow\ (x-3)^2+y^2+(z+2)^2=0 \Rightarrow x=3, y=0, z=-2$ which is the point $(3,0,-2)$

The equation is not for a sphere it is not a surface but it is the point $(3,0,-2)$

(4) $2x^2-12x+2y^2+2z^2+8z=6 \Rightarrow x^2-6x+y^2+z^2+4z=3$

$\Rightarrow x^2-6x+3^2+y^2+z^2+4z+2^2=3+3^2+2^2$

$\Rightarrow\ (x-3)^2+y^2+(z+2)^2=16$

The equation is for a sphere centered at the point $(3,0,-2)$ of radius 4

The standard form of the sphere is $(x-3)^2+y^2+(z+2)^2=16$

(5) (Coefficient of x^2) $=2$ and (Coefficient of y^2) $=0\ \Rightarrow$ not equal

The equation is not for a sphere

Example 12.1.13: Find all values of a and b from which the equation is an equation of a sphere.

(1) $2x^2-12x+by^2+2z^2+8az=-30$

(2) $2x^2-12x+by^2+2z^2+8az=-6$

Solution: (Coefficient of x^2, y^2, and z^2 are equal) $\Rightarrow\ b=2$

(1) Equation $\Rightarrow\ 2x^2-12x+2y^2+2z^2+8az=-30$

$\Rightarrow\ x^2-6x+y^2+z^2+4az=-15$

$\Rightarrow (x-3)^2+y^2+(z+2a)^2=-15+9+4a^2 \Rightarrow -6+4a^2>0 \Rightarrow a^2>\frac{6}{4}$

$\Rightarrow \sqrt{a^2}>\sqrt{\frac{6}{4}} \Rightarrow |a|>\frac{\sqrt{6}}{2} \Rightarrow a>\frac{\sqrt{6}}{2}$ or $a<-\frac{\sqrt{6}}{2} \Rightarrow\ a\in\left(-\infty,-\frac{\sqrt{6}}{2}\right)\cup\left(\frac{\sqrt{6}}{2},\infty\right)$

(2) Exercise ($b=2$ and $a\in\mathbb{R}$

Example 12.1.14: Find the equation of the sphere centered at $A(0,-2,5)$ of radius $\sqrt{3}$

Solution: The equation is $(x-0)^2+(y-(-2))^2+(z-5)^2=\sqrt{3}^2$

$\Rightarrow\ x^2+(y+2)^2+(z-5)^2=3$

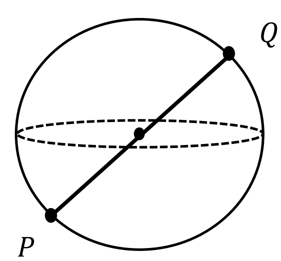
Example 12.1.15: Find the equation of the sphere if **one of its diameters** (أحد أقطارها) has end points $P(2,1,4)$ and $Q(2,-3,0)$.

Solution: The radius is $r=\frac{1}{2}\text{dist}(P,Q)$

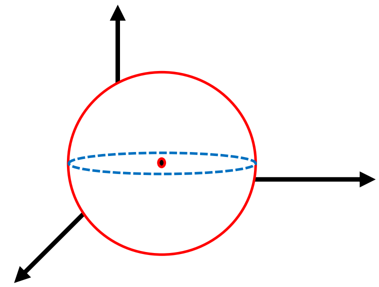
$=\frac{1}{2}\sqrt{(2-2)^2+(-3-1)^2+(0-4)^2}=\frac{1}{2}\sqrt{32}=\frac{1}{2}4\sqrt{2}=2\sqrt{2}$

The center is $midpt.=\left(\frac{2+2}{2},\frac{1+(-3)}{2},\frac{4+0}{2}\right)=(2,-1,2)$

The equation is $(x-2)^2+(y+1)^2+(z-2)^2=\left(2\sqrt{2}\right)^2=8$



Example 12.1.16: Find the equation of the sphere in the first octant of radius 5 that touches the coordinate planes.



Solution: The center is $(5,5,5) \Rightarrow$ The equation is:

$$(x-5)^2 + (y-5)^2 + (z-5)^2 = 25$$

Example 12.1.17:

(1) Find the equation of the spheres centered at the point $A(1,2,-1)$ that touches the sphere

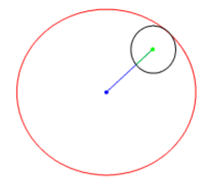
$$2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6$$

(2) Find the equation of the spheres centered at the point $A(-2,2,-1)$ that touches the sphere

$$2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6.$$

Solution:

(1) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6 \Rightarrow (x-3)^2 + y^2 + (z+2)^2 = 16$



The center is $B(3,0,-2)$ and radius 4

Distance between A and B is $\sqrt{(1-3)^2 + (2-0)^2 + \left(-1-(-2)\right)^2} = \sqrt{9} = 3$

لاحظ ان البعد بين المركزين للكرتين اصغر من نصف قطر احدهما و هذا يعني ان الكرتين احدهما داخل الأخرى

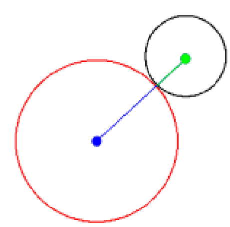
The radius of the required (مطلوب) sphere is $r = 4 - 3 = 1.$

The eq. is: $(x-1)^2 + (y-2)^2 + (z+1)^2 = 1^2 = 1$

(2) $2x^2 - 12x + 2y^2 + 2z^2 + 8z = 6 \Rightarrow (x-3)^2 + y^2 + (z+2)^2 = 16$

The center is $B(3,0,-2)$ and radius 4

Distance between A and B is $\sqrt{(-2-3)^2 + (2-0)^2 + \left(-1-(-2)\right)^2} = \sqrt{30}$



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The radius of the required (مطلوب) sphere is:

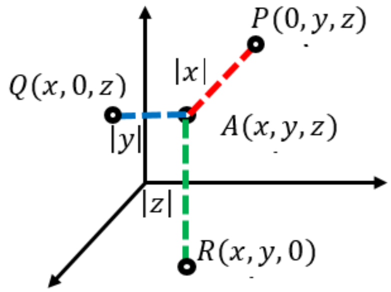
$r = \sqrt{30} - 4$. The eq. is $(x+2)^2 + (y-2)^2 + (z+1)^2 = (\sqrt{30} - 4)^2$

Rule 12.1.18: The distance from the pt. $A(x,y,z)$ to the:

(1) xy-plane is $|z|$

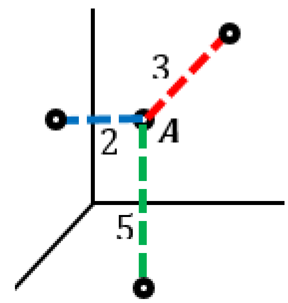
(2) xz-plane is $|y|$

(3) yz-plane is $|x|$



Example 12.1.19: Find the equation of the largest sphere in the first octant centered at the point $A(3,2,5)$.

Solution:



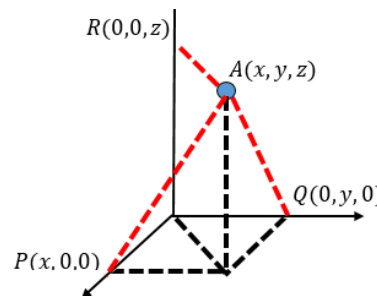
$D_1 = \text{Dist}(A, xy - \text{plane}) = 5$

$D_2 = \text{Dist}(A, xz - \text{plane}) = 2$

$D_3 = \text{Dist}(A, yz - \text{plane}) = 3$

The radius is $r = \min(D_1, D_2, D_3) = 2$

$\Rightarrow$ The equation is: $(x-3)^2 + (y-2)^2 + (z-5)^2 = 4$



Example 12.1.20: Find the equation of the sphere centered at $A(1,-2,-5)$ and touches the xz-plane.

Solution:

The radius is $r = \text{dist}(A, xz - \text{plane}) = |-2| = 2$

The equation is $(x-1)^2 + (y+2)^2 + (z+5)^2 = 4$

Rule 12.1.21: The distance from the pt. $A(x,y,z)$ to the:

(1) x-axis is $\sqrt{y^2+z^2}$ (2) y-axis is $\sqrt{x^2+z^2}$ (3) z-axis is $\sqrt{x^2+y^2}$

Example 12.1.22: Find the distance from the pt. $A(1,4,-3)$ to the:

(1) x-axis (2) y-axis (3) z-axis

Solution:

(1) The distance $\text{dist}(A, x-axis) = \sqrt{4^2+(-3)^2} = 5$

(2) The distance $\text{dist}(A, y-axis) = \sqrt{1^2+(-3)^2} = \sqrt{10}$

(3) The distance $\text{dist}(A, z-axis) = \sqrt{1^2+4^2} = \sqrt{17}$

Example 12.1.23: What region in $\mathbb{R}^3$ is represented by the inequalities:

(1) $x^2+y^2+z^2 > 6z$ (2) $x^2+y^2+z^2 \leq 6z$ (3) $x^2 \geq 4$

(4) $x < 2$ (5) $z < 0$ (6) $y^2+z^2 < 4$

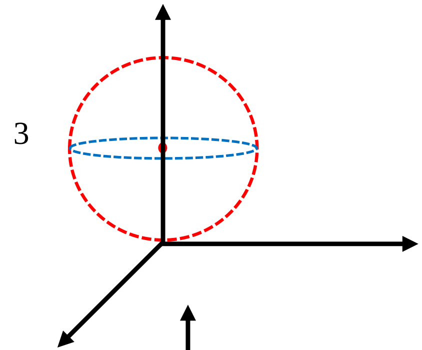
Solution:

(1) $x^2+y^2+z^2>6z: \Rightarrow\ x^2+y^2+z^2=6z$

$\Rightarrow\ x^2+y^2+z^2-6z=0$

$\Rightarrow\ x^2+y^2+\underbrace{z^2-6z+9}_{\text{اكمال المربع}}=0+9$

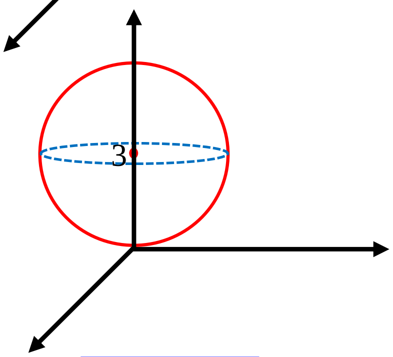
$\Rightarrow\ x^2+y^2+(z-3)^2=9$



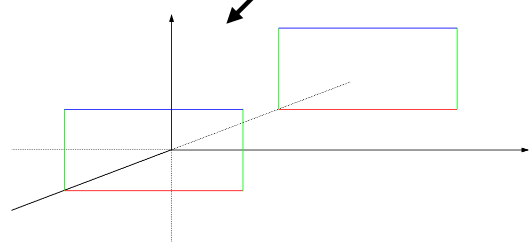
(2) $x^2+y^2+z^2\le 6z: \Rightarrow\ x^2+y^2+z^2=6z$

$\Rightarrow\ x^2+y^2+z^2-6z=0$

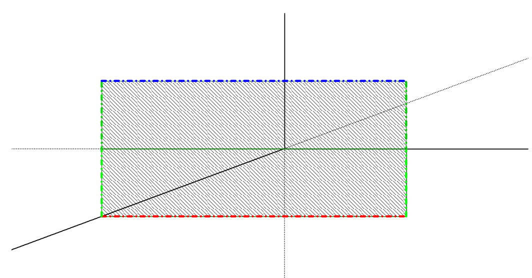
$\Rightarrow\ x^2+y^2+\underbrace{z^2-6z+9}_{\text{اكمال المربع}}=0+9$



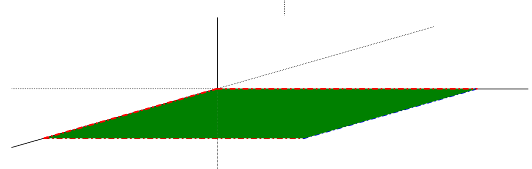
(3) $x^2\ge 4\ \Rightarrow\ x^2=4 \Rightarrow\ x=-2$ or $x=2$



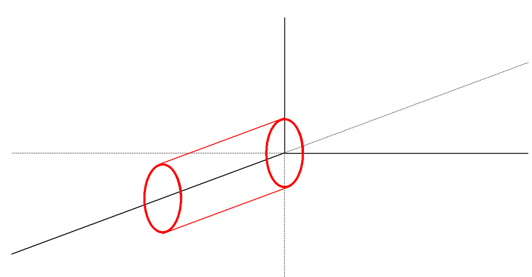
(4) $x<2\ \Rightarrow\ x=2$



(5) $z<0\ \Rightarrow\ z=0$

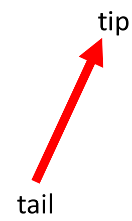


(6) $y+z^2<4\ \Rightarrow\ y^2+z^2=4$



Section 12.2: Vectors

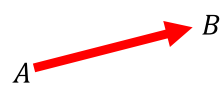
Definition 12.2.1: A vector $\vec{v}$ is a quantity that has both: $\underbrace{\text{magnitude}}_{①}$ (sometimes called length) written as $|\vec{v}|$ and $\underbrace{\text{direction}}_{②}$.



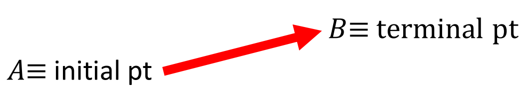
Remark 12.2.2:

(1) A graph of a vector is given by a row:
- ➢ The magnitude of a vector is the distance from its tail to its tip
- ➢ The direction is indicated by the row.

(2) If we move from a pt. A to a pt. B, then the displacement vector (متجه الازاحة) $\vec{v}$ is given by $\vec{v} = \overrightarrow{AB}$. In this case $|\vec{v}| = \text{dist}(A, B)$



(3) When we write $\vec{v} = \overrightarrow{AB}$, then the point A is called the initial point and B is called the terminal point.



Definition 12.2.3: The zero vector $\vec{0}$ is the vector of length 0 but in any direction ⇒ $|\vec{0}| = 0$

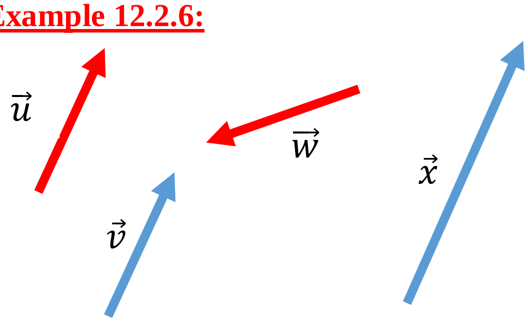
Remark 12.2.4:

(1) The zero vector $\vec{0}$ is be defined as a vector for which its initial and terminal points are the same.

(2) If $A, B,$ and C are pints, then $\vec{0} = \overrightarrow{AA} = \overrightarrow{BB} = \overrightarrow{CC} \Rightarrow |\overrightarrow{AA}| = |\overrightarrow{BB}| = |\overrightarrow{CC}| = 0.$

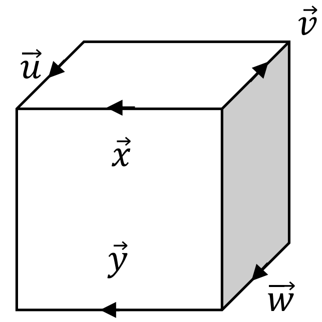
Definition 12.2.5: Two vectors $\vec{u}$ and $\vec{v}$ are equal, written as $\vec{u} = \vec{v}$, if they have the same magnitude and the same direction.

Example 12.2.6:



$\vec{u} = \vec{v}$ (the same length and the same direction)
$\vec{u} \neq \vec{w}$ (different directions)
$\vec{v} \neq \vec{x}$ (different length)

Example 12.2.7:



$\vec{u} = \vec{w}$ (the same length and the same direction)
$\vec{u} \neq \vec{v}$ (different directions)
$\vec{x} = y$ (the same length and the same direction)
$\vec{v} \neq \vec{x}$ (different directions)

12.2.8 Scalar multiplication of vectors (ضرب المتجه بعدد):

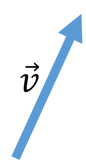
Let c be a scalar (عدد) and $\vec{v}$ be a vector. Then $c\vec{v}$ is a vector of:

- ❖ $\underbrace{\text{length}}_{①}$ $|c\vec{v}| = |c|\,|\vec{v}|$

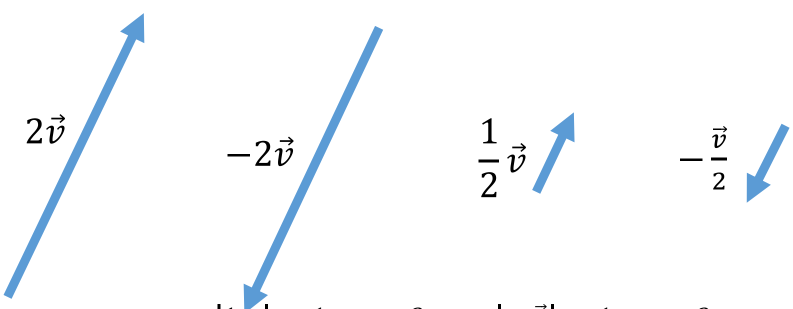
and its

- ❖ $\underbrace{\text{direction}}_{②}$ is: $\begin{cases} \text{in the same direction of } \vec{v}, \text{if } c > 0 \\ \text{in the opposite direction of } \vec{v}, \text{if } c < 0 \end{cases}$

Example 12.2.9: If $\vec{v}$ is the vector given in the figure with $|\vec{v}| = 3$.
Plot the graph of the following vectors: $2\vec{v}$, $-2\vec{v}$, $\frac{1}{2}\vec{v}$, and $-\frac{\vec{v}}{2}$.
Also, find the lengths of these vectors.



Solution:



$\Rightarrow |2\vec{v}| = 2|\vec{v}| = 6$, $|-2\vec{v}| = 2|\vec{v}| = 6$, $\left|\frac{1}{2}\vec{v}\right| = \frac{1}{2}|\vec{v}| = \frac{3}{2}$, and $\left|-\frac{\vec{v}}{2}\right| = \frac{1}{2}|\vec{v}| = \frac{3}{2}$

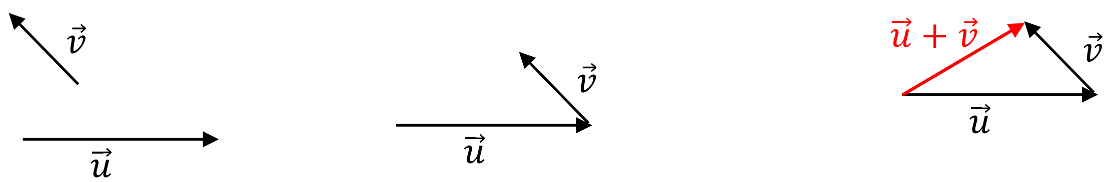
Remark 12.2.10: If $\vec{v} = \overrightarrow{AB}$, then $-\vec{v} = \overrightarrow{BA}$ that is $-\overrightarrow{AB} = \overrightarrow{BA}$. Also, $|\vec{v}| = |-\vec{v}|$

The Sum Rule 12.2.11: Let $\vec{u}$ and $\vec{v}$ be vectors in which the terminal point of $\vec{u}$ is the initial point of $\vec{v}$. The sum of two vectors $\vec{u}$ and $\vec{v}$ written as $\vec{u} + \vec{v}$ is the vector with initial point as that of $\vec{u}$ and terminal point as that of $\vec{v}$, that is if $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{BC}$, then

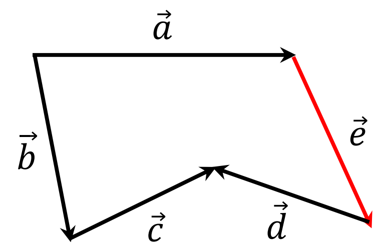
$$\vec{u} + \vec{v} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

Remark 12.2.12: To plot the graph of $\vec{u}+\vec{v}$

Step 1 Step 2



Example 12.2.13: Write the vector $\vec{e}$ as a sum of the vectors $\vec{a},\vec{b},\vec{c}$, and $\vec{d}$ given in the figure



Solution: $\vec{e}=-\vec{a}+\vec{b}+\vec{c}-\vec{d}$

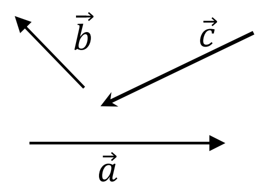
ابسط صورة

Example 12.2.14: Let A, B, and C be three points. Write $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}$ in simplicit form.

Solution: $\overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\overrightarrow{AC}-\overrightarrow{AC}=\overrightarrow{AC}+\left(-\overrightarrow{AC}\right)=\overrightarrow{AC}+\overrightarrow{CA}=\overrightarrow{AA}=\vec{0}$

$\Rightarrow \overrightarrow{AB}+\overrightarrow{BC}-\overrightarrow{AC}=\vec{0}$

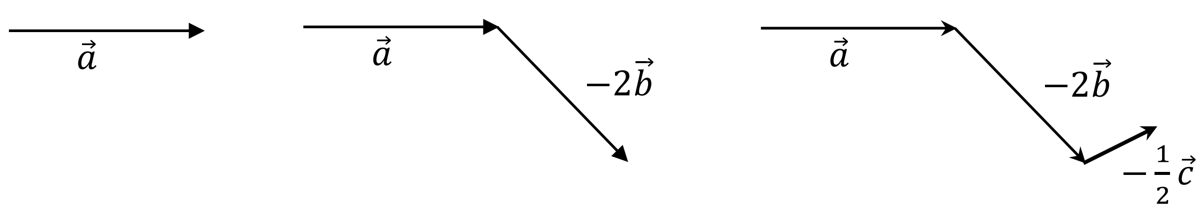
Example 12.2.15: Draw the vector $\vec{a}-2\vec{b}-\frac{1}{2}\vec{c}$, where the vectors $\vec{a},\vec{b}$, and $\vec{c}$ are given in the figure.



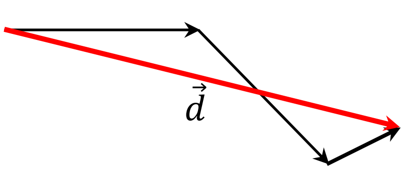
Solution: We deal with the vectors: $\vec{a},\ -2\vec{b},\ -\frac{1}{2}\vec{c}$:

Let $\vec{d}=\vec{a}-2\vec{b}-\frac{1}{2}\vec{c}$

Step 1 **Step 2** **Step 3**



Step 4



Properties of Vectors 12.2.16: Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors and let c and d be salars. Then

(1) $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ (2) $\vec{u}+\vec{0}=\vec{0}+\vec{u}=\vec{u}$ (3) $\vec{u}-\vec{u}=\vec{0}$

(4) $(\vec{u}+\vec{v})+\vec{w}=\vec{v}+(\vec{u}+\vec{w})$ (5) $(c+d)\vec{u}=c\vec{u}+d\vec{u}$ (6) $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$

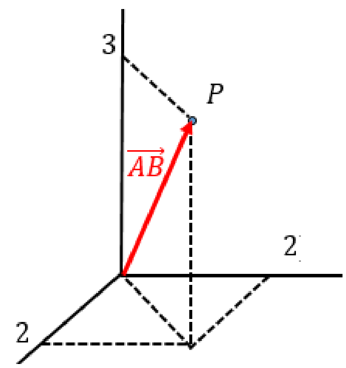
(7) $0\vec{u}=\vec{0}$

Component Form of Vectors 12.2.17: Let $A(a,b,c)$ and $B(d,e,f)$ be points in $\mathbb{R}^3$. Then the component form of $\vec{v}$ is $\vec{v}=\overrightarrow{AB}=\langle B-A\rangle=\langle d-a,e-b,f-c\rangle$.

The numbers $d-a, e-b$, and $f-c$ are called the components of $\vec{v}$

Let $P=(d-a,e-b,f-c)$ and $\boldsymbol{O}=(0,0,0)$. Then the position vector of $\vec{v}$ is $\vec{v}=\overrightarrow{\boldsymbol{O}P}$

Example 12.2.18: Find and sketch the vector $\overrightarrow{AB}$ where $A(1,-1,-3)$ and $B(3,1,0)$



Solution:

$\overrightarrow{AB}=\langle B-A\rangle=\langle 3-1,1-(-1),0-(-3)\rangle=\langle 2,2,3\rangle$

To sketch $\overrightarrow{AB}$: we sketch it as a position vector

so let $P=(2,2,3)\Rightarrow \overrightarrow{AB}=\overrightarrow{\boldsymbol{O}P}$

Rule 12.2.19:

(1) Let $\vec{v}=\langle a,b,c\rangle$. Then $|\vec{v}|=\sqrt{a^2+b^2+c^2}$ (in 3D)

(2) Let $\vec{v}=\langle a,b\rangle$. Then $|\vec{v}|=\sqrt{a^2+b^2}$ (in 2D)

Example 12.2.20:

(1) Let $\vec{v}=\langle 2,-2,-1\rangle$. Then $|\vec{v}|=\sqrt{2^2+(-2)^2+(-1)^2}=\sqrt{9}=3$

(2) Let $\vec{v}=\langle -5,\sqrt{11}\rangle$. Then $|\vec{v}|=\sqrt{(-5)^2+\sqrt{11}^2}=\sqrt{36}=6$

Rule 12.2.21: Let $\vec{u}=\langle a,b,c\rangle$, $\vec{v}=\langle d,e,f\rangle$ and let α be a scalar.

(1) $\vec{u}+\vec{v}=\langle a+d,b+e,c+f\rangle$ (2) $\vec{u}-\vec{v}=\langle a-d,b-e,c-f\rangle$

(3) $\alpha\,\vec{u}=\langle \alpha a,\alpha b,\alpha c\rangle$ (4) $\vec{u}=\vec{v} \iff a=d, b=e, c=f$

Example 12.2.22: Let $\vec{a}=\langle -1,0,3\rangle$ and $\vec{b}=\langle 2,-1,5\rangle$. Find $\left|2\,\vec{a}-\frac{\vec{b}}{3}\right|$.

Solution: First we find the vector $2\,\vec{a}-\frac{\vec{b}}{3}$:

$$2\,\vec{a}-\frac{\vec{b}}{3}=2\langle -1,0,3\rangle-\frac{\langle 2,-1,5\rangle}{3}=\langle 2(-1)-\frac{2}{3},2(0)-\frac{-1}{3},2(3)-\frac{5}{3}\rangle=\langle -\frac{8}{3},\frac{1}{3},-\frac{13}{3}\rangle$$

$$\left|2\,\vec{a}-\frac{\vec{b}}{3}\right|=\sqrt{\frac{64}{9}+\frac{1}{9}+\frac{169}{9}}=\frac{\sqrt{234}}{3}$$

Standard Basis Vectors 12.2.23:

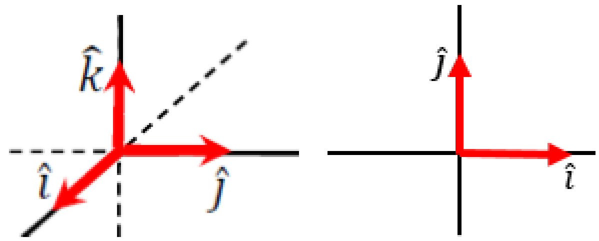
(1) In 3D, let

$\hat{\imath} = \langle 1,0,0\rangle, \hat{\jmath} = \langle 0,1,0\rangle, \hat{k} = \langle 0,0,1\rangle$

$\Rightarrow \langle a,b,c\rangle = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

(2) In 2D, let $\hat{\imath} = \langle 1,0\rangle, \hat{\jmath} = \langle 0,1\rangle$

$\Rightarrow \langle a,b\rangle = a\hat{\imath} + b\hat{\jmath}$



Example 12.2.24:

(1) $5i - j - 7k = \langle 5,-1,-7\rangle$ (2) $\frac{i}{2} + 6k = \langle \frac{1}{2}, 0,6\rangle$

Example 12.2.25: Let $\vec{a} = 5i - j$ and $\vec{b} = \langle 2,4,-1\rangle$. Find $|2\,\vec{a} + 3\vec{b}|$

Solution: First we find the vector $2\,\vec{a} + 3\vec{b}$:

$2\,\vec{a} + 3\vec{b} = 2\langle 5,-1,0\rangle + 3\langle 2,4,-1\rangle = \langle 16,10,-3\rangle \Rightarrow |2\,\vec{a} + 3\vec{b}| = \sqrt{256+100+9} = \sqrt{365}$

Notations 12.2.26:

(1) The set of all vectors in $\mathbb{R}^2$ is written as V_2.
(2) The set of all vectors in $\mathbb{R}^3$ is written as V_3.

Example 12.2.27:

(1) $2i - 5j$ and $\langle -1,0.6\rangle$ are vectors in V_2.
(2) $-3i + 2k$ and $\langle 3,-2,7\rangle$ are vectors in V_3.

Definition 12.2.28: A vector $\vec{v}$ is called a unit vector if $|\vec{v}| = 1$

Example 12.2.29:

(1) $\vec{v} = \langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle \Rightarrow |\vec{v}| = \sqrt{\frac{1}{9} + \frac{4}{9} + \frac{4}{9}} = 1 \Rightarrow \vec{v}$ is a unit vector.

(2) $\vec{u} = 0.5i - 0.2j \Rightarrow |\vec{u}| = \sqrt{0.25 + 0.04} = \sqrt{0.29} \neq 1 \Rightarrow \vec{u}$ is a not unit vector.

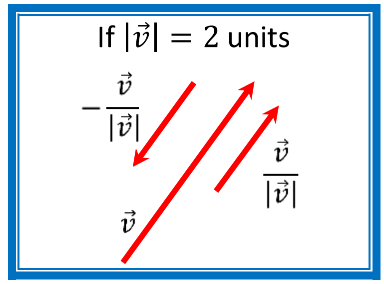
(3) The zero vector $\vec{0}$ is not a unit vector.

Example 12.2.30: Find all values of a that make $\vec{v} = \langle -\frac{1}{2}, \frac{1}{\sqrt{3}}, a\rangle$ a unit vector

u $\vec{v}$ a unit vector $\Rightarrow |\vec{v}| = 1 \Rightarrow \sqrt{\frac{1}{4} + \frac{1}{3} + a^2} = 1 \Rightarrow \frac{7}{12} + a^2 = 1$

$\Rightarrow a^2 = 1 - \frac{7}{12} = \frac{5}{12} \Rightarrow a = \pm\frac{\sqrt{5}}{\sqrt{12}} = \pm\frac{\sqrt{5}}{2\sqrt{3}}$

Rule 12.2.31: If $\vec{v} \neq \vec{0}$, then $\frac{\vec{v}}{|\vec{v}|}$ and $-\frac{\vec{v}}{|\vec{v}|}$ are two unit vectors.



In fact: $\begin{cases} \frac{\vec{v}}{|\vec{v}|} \text{ is a unit vector in the same direction of } \vec{v} \\ -\frac{\vec{v}}{|\vec{v}|} \text{ is a unit vector in the opposit direction of } \vec{v} \end{cases}$

Example 12.2.32: Let $\vec{v} = 2i - 2j + k$.

(1) Find a unit vector in the same direction of $\vec{v}$

(2) Find a vector of length $\frac{3}{2}$ in the same direction of $\vec{v}$

(3) Find a unit vector in the opposite direction of $\vec{v}$

(4) Find a vector of length $\sqrt{\pi}$ in the opposite direction of $\vec{v}$

Solution: $\vec{v} = 2i - 2j + k \Rightarrow |\vec{v}| = 3$

(1) a unit vector in the same direction of $\vec{v}$ is

$$\frac{\vec{v}}{|\vec{v}|} = \frac{2i - 2j + k}{3} = \frac{2}{3}i - \frac{2}{3}j + \frac{1}{3}k$$

(2) a vector of length $\frac{3}{2}$ in the same direction of $\vec{v}$ is

$$\frac{3}{2}\left(\frac{\vec{v}}{|\vec{v}|}\right) = \frac{3}{2}\left(\frac{2}{3}i - \frac{2}{3}j + \frac{1}{3}k\right) = i - j + \frac{1}{2}k$$

(3) a unit vector in the opposite direction of $\vec{v}$ is

$$-\frac{\vec{v}}{|\vec{v}|} = -\frac{2i - 2j + k}{3} = -\frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k$$

(4) a vector of length $\sqrt{\pi}$ in the opposite direction of $\vec{v}$ is

$$\sqrt{\pi}\left(-\frac{\vec{v}}{|\vec{v}|}\right) = \sqrt{\pi}\left(-\frac{2}{3}i + \frac{2}{3}j - \frac{1}{3}k\right) = -\frac{2\sqrt{\pi}}{3}i + \frac{2\sqrt{\pi}}{3}j - \frac{\sqrt{\pi}}{3}k$$

Remark 12.2.33: The standard basis vectors i, j, k are unit vectors since $|i| = |j| = |k| = 1$

Section 12.3: The Dot Product

Definition 12.3.1: Let $\vec{u} = \langle a, b, c\rangle$ and $\vec{v} = \langle d, e, f\rangle$. Then the dot product of $\vec{u}$ and $\vec{v}$ is defined by $\vec{u} \cdot \vec{v} = ad + be + cf$

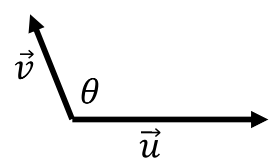
Example 12.3.2:

(1) $\langle 1, -2,3\rangle \cdot \langle 6,3,0\rangle = 1(6) + (-2)(3) + 3(0) = 0$
(2) $\langle 2,6\rangle \cdot \langle -5,2\rangle = 2(-5) + 6(2) = 2$
(3) $(3i - j) \cdot (-2i + 4k) = 3(-2) + (-1)(0) + 0(4) = -6$

Properties of Dot Product 12.3.3: Let $\vec{u}$,$\vec{v}$, and $\vec{w}$ be vectors in V_2 or V_3 and let a, b be a scalar. Then

(1) $\vec{0} \cdot \vec{v} = 0$ (2) $\vec{u} \cdot \vec{u} = |\vec{u}|^2$ (3) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
(4) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (5) $(a\vec{u}) \cdot \vec{v} = \vec{u} \cdot (a\vec{v}) = a(\vec{u} \cdot \vec{v})$
(6) $|\vec{u} + \vec{v}|^2 = |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$ (7) $|\vec{u} - \vec{v}|^2 = |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$
(8) $|a\vec{u} + b\vec{v}|^2 = a^2|\vec{u}|^2 + 2ab\vec{u} \cdot \vec{v} + b^2|\vec{v}|^2$
(9) $|a\vec{u} - b\vec{v}|^2 = a^2|\vec{u}|^2 - 2ab\vec{u} \cdot \vec{v} + b^2|\vec{v}|^2$

Definition 12.3.4: The angle θ between two vectors $\vec{u}$ and $\vec{v}$ is the angle between them when the vectors have the same initial point, where $0 \le \theta \le \pi$.



Rule 12.3.5: $\vec{u} \cdot \vec{v} = |\vec{u}|\, |\vec{v}\,|\cos\theta$

If $\vec{u} \neq \vec{0}$ and $\vec{v} \neq \vec{0} \Rightarrow \cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|} \Rightarrow\ \theta = \cos^{-1}\left(\dfrac{\vec{u} \cdot \vec{v}}{|\vec{u}|\,|\vec{v}|}\right)$

Example 12.3.6: Find the angle between the two vectors $\vec{u} = -i + k$ and $\vec{v} = 3i + j + k$

Solution:

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}\right) = \cos^{-1}\left(\frac{-1(3) + 0(1) + 1(1)}{\sqrt{2}\sqrt{11}}\right)$$

$$= \cos^{-1}\left(\frac{-2}{\sqrt{22}}\right) \cong \underbrace{115.2^\circ}_{\text{in Degrees}} \cong \underbrace{2.01}_{\text{in radian}}$$

Example 12.3.7: Find the value of x that makes the angle between the two vectors $\vec{u} = \langle 2,1, -1\rangle$ and $\vec{v} = \langle 1, x, 0\rangle$ is $\frac{\pi}{4}$

Solution:

$$\vec{u} \cdot \vec{v} = |\vec{u}|\, |\vec{v}\,|\cos\theta \Rightarrow\ 2(1) + 1(x) + (-1)(0) = \sqrt{6}\sqrt{1 + x^2}\cos\frac{\pi}{4}$$

$$\Rightarrow 2 + x = \frac{\sqrt{6}\sqrt{1+x^2}}{\sqrt{2}}\ \Rightarrow\ (2 + x)^2 = 3(1 + x^2) \Rightarrow x^2 + 4x + 4 = 3 + 3x^2$$

$$\Rightarrow 2x^2 - 4x - 1 = 0 \Rightarrow x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-1)}}{2(2)} = \frac{4 \pm \sqrt{24}}{4} = \frac{4 \pm 2\sqrt{6}}{4} = 1 \pm \frac{\sqrt{6}}{2}$$

Example 12.3.8: Find the angle between the two lines in $\mathbb{R}^2$: $y = 2x - 3$ and $y = 7 - 3x$

Solution: Take any 2 pts A, B on $y = 2x - 3$ and 2 pts C, D on $y = 7 - 3x$

$\Rightarrow A(0,-3), B(1,-1)$ and $C(0,7), D(1,4)$

Let $\overrightarrow{AB} = \langle 1,2 \rangle$ parallel to the line $y = 2x - 3$

Let $\overrightarrow{CD} = \langle 1,-3 \rangle$ parallel to the line $y = 7 - 3x$

$\Rightarrow$ The angle θ between the two lines is the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$

ملاحظة: الزاوية بين المستقيمات المتقاطعة تؤخذ زاوية حادة

$$\cos^{-1}\left(\frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{AB}|\,|\overrightarrow{CD}|}\right) = \cos^{-1}\left(\frac{1-6}{\sqrt{5}\sqrt{10}}\right) = \cos^{-1}\left(\frac{-5}{5\sqrt{2}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \underbrace{\frac{3\pi}{4}}_{\text{الزاوية منفرجة}}$$

$$\Rightarrow \quad \theta = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$$

Example 12.3.9: If $\vec{u}$ and $\vec{v}$ are vectors such that $|\vec{u}| = 4$, $|\vec{v}| = 3$ and the angle between $\vec{u}$ and $\vec{v}$ is $\frac{2\pi}{3}$.

(1) Find $\vec{u}\cdot\vec{v}$ (2) Find $|2\vec{u} - 3\vec{v}|$ (3) Find $|3\vec{u} + \vec{v}|$

Solution:

(1) $\vec{u}\cdot\vec{v} = |\vec{u}|\,|\vec{v}|\cos\theta = 4(3)\cos\left(\frac{2\pi}{3}\right) = 12\left(-\cos\frac{\pi}{3}\right) = -12\left(\frac{1}{2}\right) = -6$

(2) $|2\vec{u} - 3\vec{v}|^2 = 4|\vec{u}|^2 - 2(2)(3)\vec{u}\cdot\vec{v} + 9|\vec{v}|^2 = 4(16) - 12(-6) + 9(9) = 217$

$\Rightarrow \quad |2\vec{u} - 3\vec{v}| = \sqrt{217}$

(3) $|3\vec{u} + \vec{v}|^2 = 3^2|\vec{u}|^2 + 2(3)\vec{u}\cdot\vec{v} + |\vec{v}|^2 = 9(16) + 6(-6) + 9 = 117$

$\Rightarrow |3\vec{u} + \vec{v}| = \sqrt{117}$

Example 12.3.10: If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a}| = \sqrt{3}$, $|2\vec{a} - 3\vec{b}| = \sqrt{45}$ and $|\vec{a} + 2\vec{b}| = \sqrt{27}$.

(1) Find $\vec{a}\cdot\vec{b}$ (2) Find the angle between $\vec{a}$ and $\vec{b}$ (3) Find $|\vec{a} + 3\vec{b}|$

Solution:

(1) $|2\vec{a} - 3\vec{b}|^2 = \sqrt{45}^2 \Rightarrow 4|\vec{a}|^2 - 2(2)(3)\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 45$

$\Rightarrow \quad 4\sqrt{3}^2 - 12\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 45 \Rightarrow -12\vec{a}\cdot\vec{b} + 9|\vec{b}|^2 = 33$

$\Rightarrow \quad -4\vec{a}\cdot\vec{b} + 3|\vec{b}|^2 = 11 \; \ldots\ldots ①$

$|\vec{a} + 2\vec{b}|^2 = \sqrt{27}^2 \Rightarrow |\vec{a}|^2 + 2(2)\vec{a}\cdot\vec{b} + (2)^2|\vec{b}|^2 = 27$

$\Rightarrow \quad \sqrt{3}^2 + 4\vec{a}\cdot\vec{b} + 4|\vec{b}|^2 = 27 \Rightarrow 4\vec{a}\cdot\vec{b} + 4|\vec{b}|^2 = 24 \; \ldots\ldots ②$

① + ②: $7|\vec{b}|^2 = 35 \Rightarrow |\vec{b}|^2 = 5 \Rightarrow |\vec{b}| = \sqrt{5}$

②: $4\vec{a}\cdot\vec{b} + 4(5) = 24 \Rightarrow \vec{a}\cdot\vec{b} = 1$

(2) $\theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}\right) = \cos^{-1}\left(\frac{1}{\sqrt{3}(\sqrt{5})}\right) = \cos^{-1}\left(\frac{1}{\sqrt{15}}\right) \cong \underbrace{75.04^\circ}_{\text{in degrees}} \cong \underbrace{1.31}_{\text{in radian}}$

(3) $|\vec{a} + 3\vec{b}|^2 = |\vec{a}|^2 + 2(3)\vec{a}\cdot\vec{b} + (3)^2|\vec{b}|^2 = 3 + 6(1) + 9(5) = 54$

$\Rightarrow |\vec{a} + 3\vec{b}| = \sqrt{54} = 3\sqrt{6}$

Example 12.3.11: Prove that

(1) $\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2=2(|\vec{a}|^2+\left|\vec{b}\right|^2)$ (2) $\left|\vec{a}+\vec{b}\right|^2-\left|\vec{a}-\vec{b}\right|^2=4\vec{a}\cdot\vec{b}$

Proof:

(1) $\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2=\left(|\vec{a}|^2+2\vec{a}\cdot\vec{b}+\left|\vec{b}\right|^2\right)+(|\vec{a}|^2-2\vec{a}\cdot\vec{b}+\left|\vec{b}\right|^2)=2(|\vec{a}|^2+\left|\vec{b}\right|^2)$

(2) $\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2=\left(|\vec{a}|^2+2\vec{a}\cdot\vec{b}+\left|\vec{b}\right|^2\right)-\left(|\vec{a}|^2-2\vec{a}\cdot\vec{b}+\left|\vec{b}\right|^2\right)=4\vec{a}\cdot\vec{b}$

Example 12.3.12: If $|\vec{a}|=3$ and $\left|\vec{b}\right|=4$, find $\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2$

Solution: $\left|\vec{a}+\vec{b}\right|^2+\left|\vec{a}-\vec{b}\right|^2=2\left(|\vec{a}|^2+\left|\vec{b}\right|^2\right)=2(9+16)=50$

Remark 12.3.13:

Two vectors $\vec{u}$ and $\vec{v}$ are perpendicular (or orthogonal) written $\vec{u}\perp\vec{v}\Leftrightarrow\vec{u}\cdot\vec{v}=0$

Example 12.3.14: Show that $2i+2j-k$ is perpendicular to $5i-4j+2k$

Solution:

$$(2i+2j-k)\cdot(5i-4j+2k)=2(5)+2(-4)+(-1)(2)=0$$

$\Rightarrow(2i+2j-k)\perp(5i-4j+2k)$

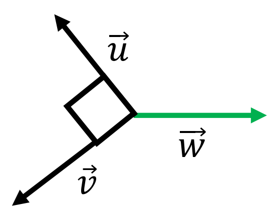
Example 12.3.15: Find all values of a that make $ai-2j+k$ perpendicular to $2i+j+ak$

Solution:

$(ai-2j+k)\cdot(2i+j+ak)=0\Rightarrow a(2)+(-2)(1)+1(a)=0\Rightarrow 3a-2=0$

$\Rightarrow a=\frac{2}{3}$

Example 12.3.16: If $\vec{u}$ and $\vec{v}$ are unit vectors such that $\vec{u}+\vec{v}+\vec{w}=0$, then find $|\vec{w}|$



Solution:

$\vec{u}$ and $\vec{v}$ are unit vectors $\Rightarrow|\vec{u}|=1$ and $|\vec{v}|=1$

$\vec{u}\perp\vec{v}$ (from figure) $\Rightarrow\vec{u}\cdot\vec{v}=0$

$\vec{u}+\vec{v}+\vec{w}=0\Rightarrow\vec{w}=-(\vec{u}+\vec{v})$

$\Rightarrow|\vec{w}|^2=|-(\vec{u}+\vec{v})|^2=|\vec{u}+\vec{v}|^2=|\vec{u}|^2+2\vec{u}\cdot\vec{v}+|\vec{v}|^2=1+0+1=2$

$\Rightarrow|\vec{w}|=\sqrt{2}$

Remark 12.3.17: $\vec{u}\cdot\vec{v}=|\vec{u}|\,|\vec{v}|\cos\theta\Rightarrow\cos\theta=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

(1) $\vec{u}\cdot\vec{v}>0\Rightarrow\theta$ is a cute angle (زاوية حادة)
(2) $\vec{u}\cdot\vec{v}<0\Rightarrow\theta$ is an obtuse angle (زاوية منفرجة)
(3) $\vec{u}\cdot\vec{v}=0\Rightarrow\theta$ is a right angle (زاوية قائمة)

Example 12.3.18: The angle between the vectors $2i - k$ and $j + 2k$ is an obtuse angle since

$$(2i - k) \cdot (j + 2k) = -2 < 0$$

Definition 12.3.19:

(1) The scalar projection of the vector $\vec{u}$ onto the vector $\vec{v}$ is $\operatorname{comp}_{\vec{v}} \vec{u} = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$

(2) The vector projection of the vector $\vec{u}$ onto the vector $\vec{v}$ is $\operatorname{proj}_{\vec{v}} \vec{u} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right) \vec{v}$

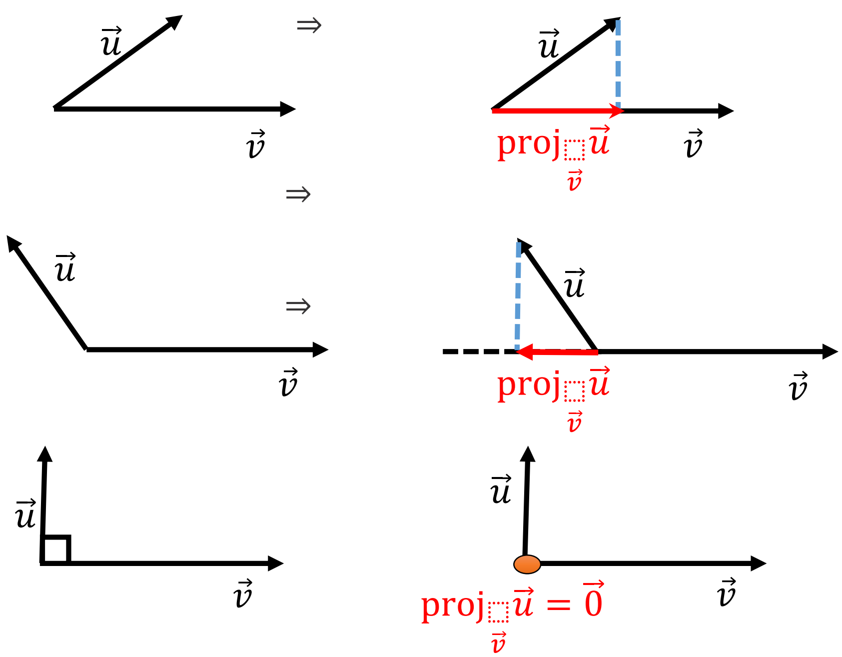
Remark 12.3.20:

1) $\operatorname{proj}_{\vec{v}} \vec{u} = \left(\operatorname{comp}_{\vec{v}} \vec{u}\right) \frac{\vec{v}}{|\vec{v}|}$

2) $\left|\operatorname{proj}_{\vec{v}} \vec{u}\right| = \left|\operatorname{comp}_{\vec{v}} \vec{u}\right|$

2) $\operatorname{comp}_{\vec{v}} \vec{u} = |\vec{u}| \cos\theta$

4)



Example 12.3.21: Find the scalar and vector projections of $\vec{v} = \langle 1,1,2 \rangle$ onto $\vec{u} = -2i + j - k$

Solution: The scalar projection of the vector $\vec{v}$ onto the vector $\vec{u}$ is:

$$\operatorname{comp}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}|} = \frac{1(-2) + 1(1) + 2(-1)}{\sqrt{4 + 1 + 1}} = \frac{-3}{\sqrt{6}}$$

The vector projection of the vector $\vec{v}$ onto the vector $\vec{u}$ is:

$$\operatorname{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}|^2}\right) \vec{u} = \frac{-3}{6} \vec{u} = -\frac{1}{2}(-2i + j - k) = i - \frac{1}{2} j + \frac{1}{2} k$$

Example 12.3.22:

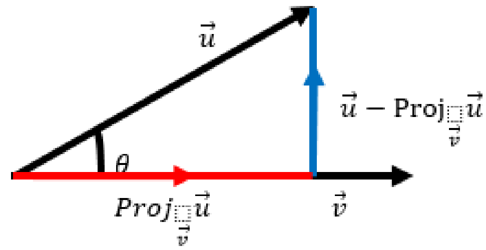
(1) If $|\vec{u}| = 5$ and the angle between $\vec{u}$ and $\vec{v}$ is $\frac{5\pi}{6}$, then find the scalar projection of $\vec{u}$ onto $\vec{v}$

(2) If $\text{comp}_{\vec{v}}\, \vec{u} = -4$ and $\vec{v} = 3j - 4k$, then find the vector projection of $\vec{u}$ onto $\vec{v}$

Solution:

(1) $\text{comp}_{\vec{v}}\, \vec{u} = |\vec{u}|\cos\theta = 5\cos\frac{5\pi}{6} = 5\left(-\cos\frac{\pi}{6}\right) = -\frac{5\sqrt{3}}{2}$

(2) $\text{proj}_{\vec{v}}\, \vec{u} = \left(\text{comp}_{\vec{v}}\, \vec{u}\right)\frac{\vec{v}}{|\vec{v}|} = -4\frac{3j-4k}{\sqrt{9+16}} = \frac{-12j+16k}{5}$



Example 12.3.23: Show that $\vec{u} - \text{proj}_{\vec{v}}\, \vec{u}$ is orthogonal to $\vec{v}$

Proof: $\vec{v} \cdot \left(\vec{u} - \text{proj}_{\vec{v}}\, \vec{u}\right) = \vec{v} \cdot \vec{u} - \vec{v} \cdot \left(\text{proj}_{\vec{v}}\, \vec{u}\right)$

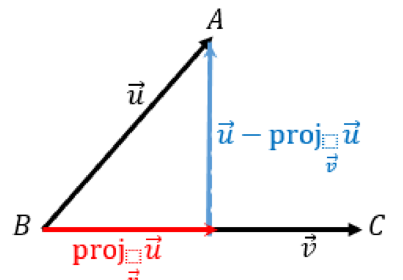
$$= \vec{u} \cdot \vec{v} - \vec{v} \cdot \left(\left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v}\right) = \vec{u} \cdot \vec{v} - \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} \cdot \vec{v}$$

$$= \vec{u} \cdot \vec{v} - \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)|\vec{v}|^2 = \vec{u} \cdot \vec{v} - \vec{u} \cdot \vec{v} = 0 \quad \Rightarrow \left(\vec{u} - \text{proj}_{\vec{v}}\, \vec{u}\right) \perp \vec{v}$$

Remark 12.3.24: Let L a line that pass through the points B and C.

Then the distance from the point A to the line L is:

$\left|\vec{u} - \text{proj}_{\vec{v}}\, \vec{u}\right|$, where $\vec{u} = \overrightarrow{BA}$ and $\vec{v} = \overrightarrow{BC}$



Example 12.3.25: Find the distance from the point $A(1,2,3)$ and the line that pass through the points $B(2,1,3)$ and $C(0,1,0)$

Solution: $\vec{u} = \overrightarrow{BA} = \langle A - B\rangle = \langle -1,1,0\rangle$ and $\vec{v} = \overrightarrow{BC} = \langle C - B\rangle = \langle -2,0,-3\rangle$

$$\vec{u} - \text{proj}_{\vec{v}}\, \vec{u} = \vec{u} - \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2}\right)\vec{v} = \vec{u} - \left(\frac{2}{13}\right)\vec{v}$$

$$= \langle -1 - \frac{2}{13}(-2), 1 - \frac{2}{13}(0), 0 - \frac{2}{13}(-3)\rangle = \langle -\frac{9}{13}, 1, \frac{6}{13}\rangle$$

$$\text{Distance} = \left|\vec{u} - \text{proj}_{\vec{v}}\, \vec{u}\right| = \sqrt{\frac{81}{169} + 1 + \frac{36}{169}} = \sqrt{\frac{286}{169}} = \frac{\sqrt{286}}{13}$$

هناك طريقة اسهل لحل هذا السؤال ستأتي في
Section 12.4: The Cross Product

Section 12.4: The Cross Product

Definition 12.4.1: The Cross product of two vectors $\vec{a} = \langle a_1, a_2, a_3\rangle$ and $\vec{b} = \langle b_1, b_2, b_3\rangle$ is given by:

$$\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)i - (a_1b_3 - a_3b_1)j + (a_1b_2 - a_2b_1)k$$

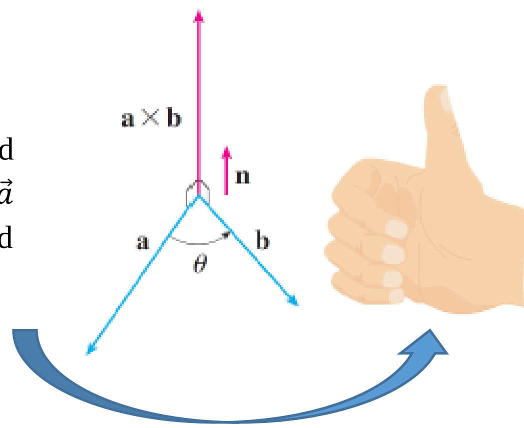
$\Rightarrow$ $\vec{a}\times\vec{b}$ is a vector in V_3.

Remark 12.4.2:

(1) To find $\vec{a}\times\vec{b}$ we must have $\vec{a}$ and $\vec{b}$ in V_3.

(2) To find $\vec{a}\cdot\vec{b}$, the vectors $\vec{a}$ and $\vec{b}$ may be in V_2 or V_3.

(3) $\vec{a}\times\vec{b}$ is a vector orthogonal (يعامد) to the vectors $\vec{a}$ and $\vec{b}$ and so $\vec{a}\times\vec{b}$ is orthogonal to the plane containing both vectors $\vec{a}$ and $\vec{b}$. The direction of $\vec{a}\times\vec{b}$ is determined by the right hand rule.



Example 12.4.3: Let $\vec{a} = \langle 3,2,1\rangle$ and $\vec{b} = \langle -1,1,0\rangle$

(1) Find $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$

(2) Find two vectors perpendicular (orthogonal) to both $\vec{a}$ and $\vec{b}$

(3) Find two unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

(4) Find two unit vectors orthogonal to the plane that pass through the points $A(1,2,3), B(4,4,4)$, and $C(0,3,3)$

Solution:

(1) $\vec{a}\times\vec{b} = \begin{vmatrix} i & j & k \\ 3 & 2 & 1 \\ -1 & 1 & 0 \end{vmatrix} = i\big(2(0) - 1(1)\big) - j\big(3(0) - 1(-1)\big) + k\big(3(1) - 2(-1)\big)$

$= -i - j + 5k$

$\vec{b}\times\vec{a} = \begin{vmatrix} i & j & k \\ -1 & 1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = i\big(1(1) - 2(0)\big) - j\big(-1(1) - 3(0)\big) + k\big(-1(2) - 3(1)\big)$

$= i + j - 5k$

(2) Two vectors orthogonal to both $\vec{a}$ and $\vec{b}$ are $\vec{a}\times\vec{b}$ and $-\vec{a}\times\vec{b}$

$\Rightarrow -i - j + 5k$ and $i + j - 5k$ are orthogonal to both $\vec{a}$ and $\vec{b}$

(3) Two unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$ are $\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ and $-\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$

$\Rightarrow$ $\frac{-i-j+5k}{|-i-j+5k|}$ and $\frac{i+j-5k}{|i+j-5k|}$ are unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

$\Rightarrow$ $\frac{-i-j+5k}{\sqrt{27}}$ and $\frac{i+j-5k}{\sqrt{27}}$ are unit vectors orthogonal to both $\vec{a}$ and $\vec{b}$

(4) Let $\vec{a} = \overrightarrow{AB} = \langle B - A\rangle = \langle 3,2,1\rangle$ and $\vec{b} = \overrightarrow{AC} = \langle C - A\rangle = \langle -1,1,0\rangle$

$\Rightarrow \vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are orthogonal to both $\vec{a}$ and $\vec{b}$

$\Rightarrow \vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow -i-j+5k$ and $i+j-5k$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{|-i-j+5k|}$ and $\frac{i+j-5k}{|i+j-5k|}$ are orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

$\Rightarrow \frac{-i-j+5k}{\sqrt{27}}$ and $\frac{i+j-5k}{\sqrt{27}}$ are unit vectors orthogonal to the plane containing both $\vec{a}$ and $\vec{b}$

Rule 12.4.4: $i\times j=k,\ j\times k=i,\ k\times i=j,\ j\times i=-k,\ k\times j=-i,\ i\times k=-j$

Properties of Cross Product 12.4.5: Let $\vec{u}$,$\vec{v}$, and $\vec{w}$ be vectors V_3 and let a be a scalar. Then

(1) $\vec{u}\times\vec{u}=\vec{0}$
(2) $\vec{u}\times\vec{v}=-(\vec{v}\times\vec{u})$
(3) $\vec{0}\times\vec{v}=\vec{v}\times\vec{0}=\vec{0}$
(4) $(a\vec{u})\times\vec{v}=\vec{u}\times(a\vec{v})=a(\vec{u}\times\vec{v})$
(5) $\vec{u}\times(\vec{v}+\vec{w})=(\vec{u}\times\vec{v})+(\vec{u}\times\vec{w})$
(6) $(\vec{u}+\vec{v})\times\vec{w}=(\vec{u}\times\vec{w})+(\vec{v}\times\vec{w})$

In general

$\vec{a}\times(\vec{b}\times\vec{c})\neq(\vec{a}\times\vec{b})\times\vec{c}$

For Example:

$j\times(j\times k)=j\times(i)=-k$

$(j\times j)\times k=\vec{0}\times k=\vec{0}$

Rule 12.4.6: $\underbrace{\vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}}_{\text{قاعدة مهمة}}$

Example 12.4.7: Let $\vec{a}$ and $\vec{b}$ be orthogonal such that $|\vec{a}|=2$ and $|\vec{b}|=3$.
Find $(\vec{b}\times\vec{a})\times\vec{a}$ and $|(\vec{b}\times\vec{a})\times\vec{a}|$

Solution:

$(\vec{b}\times\vec{a})\times\vec{a}=-\vec{a}\times(\vec{b}\times\vec{a})=-\left((\vec{a}\cdot\vec{a})\vec{b}-(\vec{a}\cdot\vec{b})\vec{a}\right)=-(|\vec{a}|^2\vec{b}-0\vec{a})=-4\vec{b}$

$|(\vec{b}\times\vec{a})\times\vec{a}|=|-4\vec{b}|=4|\vec{b}|=4(3)=12$

Example 12.4.8: Simplify (بسّط) $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$

Solution: $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=\vec{a}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{a}-\vec{b}\times\vec{b}$

$=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}$

$=2\vec{a}\times\vec{b}$

Rule 12.4.9:

(1) The length of $\vec{a}\times\vec{b}$ is given by: $|\vec{a}\times\vec{b}|=|\vec{a}|\,|\vec{b}|\sin\theta$

(2) The length of $\vec{a}\times\vec{b}$ is given by: $|\vec{a}\times\vec{b}|=\sqrt{|\vec{a}|^2|\vec{b}|^2-(\vec{a}\cdot\vec{b})^2}$ (Lagrange identity)

Proof. (2) $\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$ and $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

Now, $\sin^2\theta + \cos^2\theta = 1 \Rightarrow \frac{|\vec{a}\times\vec{b}|^2}{|\vec{a}|^2\,|\vec{b}|^2} + \frac{(\vec{a}\cdot\vec{b})^2}{|\vec{a}|^2\,|\vec{b}|^2} = 1 \Rightarrow |\vec{a}\times\vec{b}|^2 + (\vec{a}\cdot\vec{b})^2 = |\vec{a}|^2\,|\vec{b}|^2$

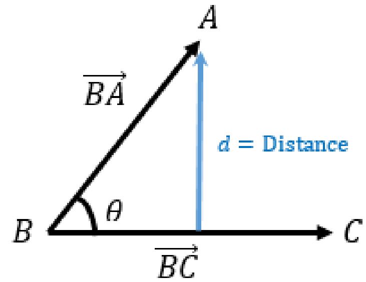
$\Rightarrow |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2 \Rightarrow |\vec{a}\times\vec{b}| = \sqrt{|\vec{a}|^2\,|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2}$

Example 12.4.10: if $\vec{a} = \langle 2,-1,0\rangle$ and $\vec{b} = \langle 3,0,4\rangle$, then $|\vec{a}\times\vec{b}| = \sqrt{5(25)-(6)^2} = \sqrt{89}$

Rule 12.4.11: Let L a line that pass through the points B and C.

Then the distance from the point A to the line L is:

Distance $= \frac{|\overrightarrow{BA}\times\overrightarrow{BC}|}{|\overrightarrow{BC}|}$



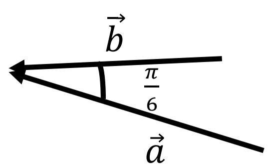
Proof: $\sin\theta = \frac{d}{|\overrightarrow{BA}|} \Rightarrow d = |\overrightarrow{BA}|\sin\theta = \frac{|\overrightarrow{BA}||\overrightarrow{BC}|\sin\theta}{|\overrightarrow{BC}|} = \frac{|\overrightarrow{BA}\times\overrightarrow{BC}|}{|\overrightarrow{BC}|}$

Example 12.4.12: Find the distance from the point $A(1,2,3)$ and the line that pass through the points $B(2,1,3)$ and $C(0,1,0)$

Solution: $\overrightarrow{BA} = \langle A-B\rangle = \langle -1,1,0\rangle$ and $\overrightarrow{BC} = \langle C-B\rangle = \langle -2,0,-3\rangle$

$$\text{Distance} = \frac{|\overrightarrow{BA}\times\overrightarrow{BC}|}{|\overrightarrow{BC}|} = \frac{\sqrt{|\overrightarrow{BA}|^2|\overrightarrow{BC}|^2 - (\overrightarrow{BA}\cdot\overrightarrow{BC})^2}}{|\overrightarrow{BC}|} = \frac{\sqrt{2(13)-(2)^2}}{\sqrt{13}} = \frac{\sqrt{22}}{\sqrt{13}}$$

Example 12.4.13: Find $|\vec{a}\times\vec{b}|$, where $\vec{a}$ and $\vec{b}$ are given in the figure with $|\vec{a}| = 8$, $|\vec{b}| = 6$



Solution: $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta = 8(6)\sin\left(\frac{\pi}{6}\right) = 48\left(\frac{1}{2}\right) = 24$

Example 12.4.14: Find $|\vec{a}\times\vec{b}|$ and $\vec{a}\times\vec{b}$, where $|\vec{a}| = 2$ and $|\vec{b}| = \frac{1}{2}$ and $|\vec{a}+2\vec{b}| = 3$

Solution: $|\vec{a}+2\vec{b}|^2 = 3^2 \;\Rightarrow\; |\vec{a}|^2 + 4\vec{a}\cdot\vec{b} + 4|\vec{b}|^2 = 9 \;\Rightarrow\; 4 + 4\vec{a}\cdot\vec{b} + 1 = 9 \Rightarrow \vec{a}\cdot\vec{b} = 1$

$\Rightarrow |\vec{a}\times\vec{b}| = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a}\cdot\vec{b})^2} = \sqrt{4\left(\frac{1}{4}\right) - (1)^2} = 0 \;\Rightarrow\; \vec{a}\times\vec{b} = \vec{0}$

Rule 12.4.15: Two vectors $\vec{a}$ and $\vec{b}$ are parallel written $\vec{a}//\vec{b}$ if $\vec{a}\times\vec{b}=\vec{0}$.

Observe the following:

(1) in **Example 12.4.12** we have $\vec{a}\times\vec{b}=\vec{0}$ so $\vec{a}//\vec{b}$.
(2) If $\vec{a}$ is any vector then $\vec{a}//\vec{0}$ since $\vec{a}\times\vec{0}=\vec{0}$

Remark 12.4.16: $\vec{a}//\vec{b} \Leftrightarrow \vec{a}=c\vec{b}$ or $\vec{b}=c\vec{a}$ for some scalar c.

Consequently: Let $\vec{a}=\langle a_1,a_2,a_3\rangle$ and $\vec{b}=\langle b_1,b_2,b_3\rangle$, Then $\vec{a}//\vec{b} \Leftrightarrow \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$, where b_1,b_2,b_3 are nonzero scalars.

Example 12.4.17:

(1) $\langle 6,3,15\rangle//\langle 4,2,10\rangle$ since the ratio $\frac{6}{4}:\frac{3}{2}:\frac{15}{10}$ are all equal

(2) $\langle 4,6,-28\rangle$ and $\langle 2,3,14\rangle$ are not parallel since the ratios $\frac{4}{2}:\frac{6}{3}:\frac{-28}{7}$ are not all equal

Example 12.4.18: Find the value of x that makes $\vec{a}=\langle 2,x-1,x\rangle$ and $\vec{b}=\langle x^2-1,0,x+1\rangle$ parallel.

Solution: $\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}$ $\Rightarrow$ We have 2 equations:

$\frac{x^2-1}{2}=\frac{0}{x-1}$ .....① and $\frac{0}{x-1}=\frac{x+1}{x}$.......②

Solving equation ②: $\frac{0}{x-1}=\frac{x+1}{x}\Rightarrow x=-1$

Check using equation ①: $\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}}_{\text{عوّض } x=-1} \Rightarrow \frac{0}{2}=\frac{0}{-2}=\frac{0}{-1}$ (no error)

$\Rightarrow$ the value of x is $x=-1$.

Another solution: Solving equation ①: $\frac{x^2-1}{2}=\frac{0}{x-1} \Rightarrow x^2-1=0 \Rightarrow x=\pm 1$

Check using equation ②:

②: $\underbrace{\frac{0}{x-1}=\frac{x+1}{x}}_{\text{عوّض } x=-1} \Rightarrow \frac{0}{2}=\frac{0}{-2}=\frac{0}{-1}$ (which is true) $\Rightarrow x=-1$ is a correct value

$\underbrace{\frac{x^2-1}{2}=\frac{0}{x-1}=\frac{x+1}{x}}_{\text{عوّض } x=1} \Rightarrow \frac{0}{2}=\frac{0}{0}=\frac{2}{1}$ (which is false) $\Rightarrow x=1$ is a false value $\Rightarrow x\neq 1$

$\Rightarrow x=-1$ is the only value only for x

Exercise 12.4.19: Find the value of x that makes:

$$\vec{a} = \langle 3x^2 - 3, 3, x^2 - x - 3\rangle \text{ and } \vec{b} = \langle 3,1,1\rangle \text{ parallel.}$$

Answer is $x = -2$

Definition 12.4.20: Three points A, B, C are collinear (على استقامة واحدة) $\Leftrightarrow \overrightarrow{AB} // \overrightarrow{AC}$

Example 12.4.21: Determine whether the points $A(2,4,-3), B(3,-1,1), C(4,-6,5)$ are collinear or not.

Solution:

$\overrightarrow{AB} = \langle 1,-5,4\rangle$ and $\overrightarrow{AC} = \langle 2,-10,8\rangle \Rightarrow \frac{2}{1} : \frac{-10}{-5} : \frac{8}{4}$ are all equal $\Rightarrow \overrightarrow{AB} // \overrightarrow{AC}$

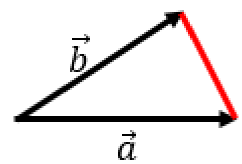
$\Rightarrow$ The points A, B, C are collinear

Another solution: $\overrightarrow{AC} = 2\overrightarrow{AB} \Rightarrow \overrightarrow{AB} // \overrightarrow{AC} \Rightarrow$ The points A, B, C are collinear

Rule 12.4.22:

(1) The area (مساحة) of the parallelogram determined by the vectors $\vec{a}$ and $\vec{b}$ is $A = |\vec{a} \times \vec{b}|$

(2) The area of the triangle determined by the vectors $\vec{a}$ and $\vec{b}$ is $A = \frac{1}{2}|\vec{a} \times \vec{b}|$



Remark 12.4.23: Let A, B, C, D be points.

(1) The area of the parallelogram (متوازي اضلاع) with vertices A, B, C, D is $A = |\overrightarrow{AB} \times \overrightarrow{AC}|$

(2) The area of the triangle (مثلث) with vertices A, B, C is $A = \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$

Example 12.4.24: let $\vec{a} = i + 2j - k$ and $\vec{b} = j + 3k$ and let $A(1,0,1), B(2,2,0), C(1,1,4), D$ be four points.

(1) Find the area of the parallelogram determined by the vectors $\vec{a}$ and $\vec{b}$.
(2) Find the area of the triangle determined by the vectors $\vec{a}$ and $\vec{b}$.
(3) Find the area of the parallelogram with vertices A, B, C, D
(4) Find the area of the triangle with vertices A, B, C

Solution:

(1) Area $= |\vec{a} \times \vec{b}| = \sqrt{|\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2} = \sqrt{(6)(10) - (-1)^2} = \sqrt{59}$

(2) Area $= \frac{\sqrt{59}}{2}$

(3) $\overrightarrow{AB} = \langle 1,2,-1\rangle = \vec{a}$ and $\overrightarrow{AC} = \langle 0,1,3\rangle = \vec{b} \Rightarrow$ Area $= |\vec{a} \times \vec{b}| = \sqrt{59}$

(4) Area $= \frac{\sqrt{59}}{2}$

Definition 12.4.25: Let $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$, and $\vec{c} = \langle c_1, c_2, c_3 \rangle$ be vectors. The scalar triple of the vectors $\vec{a}, \vec{b}, \vec{c}$ written $\vec{a} \cdot (\vec{b} \times \vec{c})$ is defined by

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

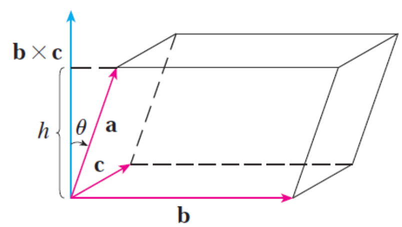
Rule 12.4.26: $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

Example 12.4.27: If $\vec{a} \cdot (\vec{b} \times \vec{c}) = -3$, find $2\vec{c} \cdot (\vec{b} \times 3\vec{a})$

Solution: $2\vec{c} \cdot (\vec{b} \times 3\vec{a}) = 2(3)\vec{c} \cdot (\vec{b} \times \vec{a}) = 6\left(-\vec{a} \cdot (\vec{b} \times \vec{c})\right) = -6(-3) = 18$

Rule 12.4.28: The volume of the parallelepiped (متوازي السطوح) determined by the vectors $\vec{a}, \vec{b}, \vec{c}$ is

$$V = \underbrace{\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|}_{\text{القيمة المطلقة}}$$



Remark 12.4.29: Let A, B, C, D be vertices of a parallelepiped not in the same plane and let $\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{AC}, \vec{c} = \overrightarrow{AD}$. Then the volume of this parallelepiped is

$$V = \underbrace{\left|\vec{a} \cdot (\vec{b} \times \vec{c})\right|}_{\text{القيمة المطلقة}}$$

Example 12.4.30: Find the volume of the parallelepiped:

(1) Determined by the vectors $\vec{a} = \langle 0, -2, 5 \rangle, \vec{b} = \langle 0, 1, 2 \rangle, \vec{c} = \langle 6, 3, -1 \rangle$

(2) With adjacent edges PQ, PR, PS, where $P(-2,1,0), Q(-2,-1,5), R(-2,2,2)$, and $S(4,4,-1)$.

Solution:

(1) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 0 & -2 & 5 \\ 0 & 1 & 2 \\ 6 & 3 & -1 \end{vmatrix}$

$= 0(-1-6) - (-2)(0-12) + 5(0-6) = 0 - 24 - 30 = -54$

Volume $= \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right| = |-54| = 54$

(2) Let $\vec{a} = \overrightarrow{PQ} = \langle 0, -2, 5 \rangle, \vec{b} = \overrightarrow{PR} = \langle 0, 1, 2 \rangle, \vec{c} = \overrightarrow{PS} = \langle 6, 3, -1 \rangle$

$\Rightarrow \vec{a} \cdot (\vec{b} \times \vec{c}) = -54$ (by part (1))$\Rightarrow$ Volume $= \left|\vec{a} \cdot (\vec{b} \times \vec{c})\right| = |-54| = 54$

Rule 12.4.31:

(1) Three vectors $\vec{a}, \vec{b}$, and $\vec{c}$ in V_3 are coplanar (lie in the same plane) if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$.

(2) Four points A, B, C, D in $\mathbb{R}^3$ are coplana if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$, where $\vec{a} = \overrightarrow{AB}, \vec{b} = \overrightarrow{AC}$, and $\vec{c} = \overrightarrow{AD}$

Example 12.4.32:

(1) Find the value of x that makes $\vec{a} = \langle 1, x, 0\rangle, \vec{b} = \langle x, 2,1\rangle, \vec{c} = \langle 0,1,1\rangle$ coplanar

(2) Find the value of x that makes the points $A(1,-1,2), B(2, x-1,2)$, $C(x+1,1,3)$, and $D(1,0,3)$ lie in the same plane.

Solution:

(1) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & x & 0 \\ x & 2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(2-1) - x(x-0) + 0(x-1) = 1 - x^2$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \;\Rightarrow\; 1 - x^2 = 0 \;\Rightarrow\; x = \pm 1$$

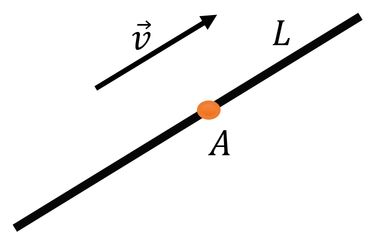
(2) $\vec{a} = \overrightarrow{AB} = \langle 1, x, 0\rangle, \vec{b} = \overrightarrow{AC} = \langle x, 2,1\rangle$, and $\vec{c} = \overrightarrow{AD} = \langle 0,1,1\rangle$

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 1 - x^2$ (by part (1))

$\vec{a}\cdot(\vec{b}\times\vec{c}) = 0 \;\Rightarrow\; 1 - x^2 = 0 \;\Rightarrow\; x = \pm 1$

Section 12.5: Equations of lines and Planes

Definition 12.5.1: Let L be the line that pass through the pt $A(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c\rangle$. Then



(1) The parametric (param.) eqs of L are:
$x = x_0 + at, \;\; y = y_0 + bt, \;\; z = z_0 + ct,$ where $t \in \mathbb{R}$

(2) The vector eq. of L is $\vec{r}(t) = \langle x_0, y_0, z_0\rangle + \langle a, b, c\rangle t, t \in \mathbb{R}$
$\Rightarrow \vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle$

(3) The symmetric (symm) eqs of L are:
$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$, whenever $a \neq 0, b \neq 0, c \neq 0$

- If $a \neq 0, b \neq 0, c = 0$, the symm. Eqs are: $\frac{x-x_0}{a} = \frac{y-y_0}{b}, z = z_0$
- If $a \neq 0, c \neq 0, b = 0$, the symm. Eqs are: $\frac{x-x_0}{a} = \frac{z-z_0}{c}, y = y_0$
- If $b \neq 0, c \neq 0, a = 0$, the symm. Eqs are $\frac{y-y_0}{b} = \frac{z-z_0}{c}, x = x_0$

Example 12.5.2: Find param, symm, and vector equations of the line through the pt $A(1,2,3)$ and parallel to the vector $6\hat{\imath} - 7\hat{k}$. Also:

(1) find two pts on the line other than A.

(2) at what pts the line intersects:

(a) The xy-plane (b) the xz-plane (c) the y-axis

Solution: The parallel $6\hat{\imath} - 7\hat{k} = \langle 6,0,-7\rangle$

The param. Eqs are: $x = 1 + 6t, \;\; y = 2 + 0t, \;\; z = 3 - 7t, \;\; t \in \mathbb{R}$

$$\Rightarrow x = 1 + 6t, \;\; y = 2, \;\; z = 3 - 7t, \;\; t \in \mathbb{R}$$

The vector eq is: $\vec{r}(t) = \langle 1,2,3\rangle + \langle 6,0,-7\rangle t, t \in \mathbb{R} \Rightarrow \vec{r}(t) = \langle 1 + 6t, 2, 3 - 7t\rangle, t \in \mathbb{R}$

The symm. Eqs are: $\frac{x-1}{6} = \frac{z-3}{-7}, y = 2 \Rightarrow \frac{x-1}{6} = \frac{3-z}{7}, y = 2$

(1) To find pts on the line we use the parmm eqs: $x = 1 + 6t,\ \ y = 2,\ \ z = 3 - 7t,\ \ t \in \mathbb{R}$
Taking $t = 0 \Rightarrow x = 1, y = 2, z = 3 \Rightarrow$ a pt on the line is $(1,2,3)$ which is the pt Taking Taking $t = 1 \Rightarrow x = 7, y = 2, z = -4 \Rightarrow$ a pt on the line is $B(7,2,-4)$
Taking $t = -1 \Rightarrow x = -5, y = 2, z = 10 \Rightarrow$ a pt on the line is $C(-5,2,10)$

We can find infinitely many pts on the line

(2)

(a) The intersects the xy-plane when $z = 0 \Rightarrow 3 - 7t = 0 \Rightarrow t = \frac{3}{7}$

In the param. Eqs ($x = 1 + 6t,\ \ y = 2,\ \ z = 3 - 7t$) substitute $t = \frac{3}{7}$:

$$x = 1 + 6\left(\frac{3}{7}\right) = \frac{25}{7}, y = 2, z = x = 3 - 7\left(\frac{3}{7}\right) = 0$$

The intersects the xy-plane at the pt $\left(\frac{25}{7}, 2, 0\right)$

(b) The line intersects the xz-plane when $y = 0$ but in the param. Eqs $y = 2 \neq 0$
$\Rightarrow$ the line does not intersect the xz-plane

(c) The line intersects the y-axis when $x = 0$ and $z = 0$
From the param. Eqs. $x = 1 + 6t,\ \ z = 3 - 7t$

$$\Rightarrow \begin{Bmatrix} 1 + 6t = 0 \ \Rightarrow t = -\frac{1}{6} \\ 3 - 7t = 0 \ \Rightarrow \ t = \frac{3}{7} \end{Bmatrix} \text{ but } -\frac{1}{6} \neq \frac{3}{7} \Rightarrow \text{the does not intersect the } y\text{-axis}$$

Example 12.5.3: Find a vector parallel to the given line:

(1) $x = 2 - t, y = t, z = 2t - 2$ (2) $\vec{r}(t) = \langle -1,1,0\rangle + \langle 0,1,3\rangle t$

(3) $\frac{2-x}{3} = \frac{y+1}{-2}, z = 4$

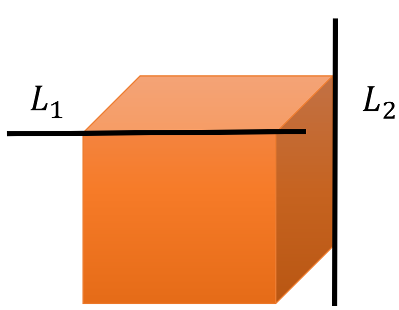
Solution: The vectors are:

(1) $\vec{v} = \langle -1,1,2\rangle$ (2) $\vec{v} = \langle 0,1,3\rangle$ (3) $\vec{v} = \langle -3,-2,0\rangle$

Remark 12.5.4: Let L_1 and L_2 be two lines such that $\vec{u}//L_1$ and $\vec{v}//L_2$

(1) $L_1//L_2 \Leftrightarrow \vec{u}//\vec{v} \Leftrightarrow \vec{u} = c\vec{v}$ or $\vec{v} = c\vec{u}$
$\Leftrightarrow$ the ratios of the components of $\vec{u}$ and $\vec{v}$ remains the same (النسبة تبقى ثابتة)

(2) If L_1 and L_2 are not parallel, the they are interesected (متقاطعين) or skew (متخالفين).



L_1 and L_2 in the figure are skew

Example 12.5.5: Determine whether the given two lines are parallel, skew or intersects. If they are intersecting, find the pt of intersection:

(1) $L_1: x = 2 - 3t, y = 2t, z = 7$
$L_2: x = 5 + 9s, y = 3 - 6s, z = 1$

(2) $L_1: x = t, y = 3 - t, z = 2 + 3t$
$L_2: x = 1 + s, y = 2 + s, z = -3$

(3) $L_1: x = 6 + t, y = -2 + 3t, z = 4 + 2t$ -3,2
$L_2: x = -1 + 2s, y = -1 - 5s, z = 2 - 2s$

Solution:

(1) $L_1: x = 2 - 3t, y = 2t, z = 7 \Rightarrow \vec{u} = \langle -3,2,0 \rangle // L_1$
$L_2: x = 5 + 9s, y = 3 - 6s, z = 1 \Rightarrow \vec{v} = \langle 9,-6,0 \rangle // L_2$
Observe that $\vec{v} = -3\vec{u} \Rightarrow \vec{u} // \vec{v} \Rightarrow L_1 // L_2$.

ندرس التوازي بين المتجهات $\vec{u}, \vec{v}$ لاحظ ان طريقة الدراسة من خلال االنسب تفشل لان الإحداثي z في كلا المتجهين 0

(2) $L_1: x = t, y = 3 - t, z = 2 + 3t \Rightarrow \vec{u} = \langle 1,-1,3 \rangle // L_1$
$L_2: x = 1 + s, y = 2 + s, z = -3 \Rightarrow \vec{v} = \langle 1,1,0 \rangle // L_2$
Ratio method: $\frac{1}{1}, \frac{1}{-1}, \frac{0}{3}$ are not equal $\Rightarrow$ L_1 and L_2 are not parallel
$\Rightarrow$ L_1 and L_2 are interesected or skew
To determine whether they are interesected or skewed:
Assume that the are intersected:
(اذا حصلنا على شيء مستحيل التحقق تكون فرضيتنا خطأ وبالتالي لا تتقاطع المستقيمات ويحدث التخالف)

$L_1: x = t, y = 3 - t, z = 2 + 3t$

$L_2: x = 1 + 2s, y = 2 + s, z = -3$

$$\Rightarrow \left\{ \begin{array}{c} t = 1 + s \ \Rightarrow t - s = 1 \ldots ① \\ 3 - t = 2 + s \ \Rightarrow \ -t - s = -1 \ldots ② \\ 2 + 3t = -3 \ \Rightarrow \ t = -1 \ldots ③ \end{array} \right\}$$

اخترت المعادلات ①،③ للحل و المعادلة ② للتحقق:

③: $t = -1$ and ① $\Rightarrow -1 - s = 1 \Rightarrow s = -2$

$\therefore t = -1$ and $s = -2$

(التحقق) ②: $-t - s = -1$

$\Rightarrow -(-1) - (-2) = 3 \neq -1$

Eq ② does not satisfied $\Rightarrow$ our assumption is false

$\Rightarrow$ the lines are not intersected but they are not parallel

$\therefore$ the lines are skewed

خطة الحل

نساوي بين معدلات L_1 ومعادلات L_2 فنحصل على 3 معادلات بمتغيرين هما t, s نختار معادلتين للحل ومعادلة للتحقق: معادلتي الحل نجد من خلالهما قيم t, s ثم نذهب لمعادلة التحقق ونعوض هذه القيم فيها:

- فإن لم تتحقق المعادلة يكون الفرض بالتقاطع خطأ وبالتالي تكون الخطوط متخالفة
- فإن تحققت المعادلة (أي حدثت المساواة) فتكون الخطوط متقاطعة ولإيجاد نقطة التقاطع نعوض قيمة t في L_1 أو قيمة s في L_2 يجب ان نحصل من كليهما على نفس النقطة

(3) $L_1: x = 6 + t, y = -2 + 3t, z = 4 + 2t \ \Rightarrow \ \vec{u} = \langle 1,3,-1 \rangle // L_1$
$L_2: x = -1 + 2s, y = -1 - 5s, z = 2 - 2s \Rightarrow \vec{v} = \langle 2,-5,3 \rangle // L_2$
Ratio method: $\frac{1}{2}, \frac{3}{-5}, \frac{2}{-2}$ are not equal $\Rightarrow$ L_1 and L_2 are not parallel
$\Rightarrow$ L_1 and L_2 are interesected or skew
To determine whether they are interesected or skewed:

Assume that the are intersected:

$L_1: x = 6 + t, y = -2 + 3t, z = 4 + 2t$

$L_2: x = -1 + 2s, y = -1 - 5s, z = 2 - 2s$

$$\Rightarrow \begin{Bmatrix} 6 + t = -1 + 2s \Rightarrow t - 2s = -7 \dots ① \\ -2 + 3t = -1 - 5s \Rightarrow 3t + 5s = 1 \dots ② \\ 4 + 2t = 2 - 2s \Rightarrow 2t + 2s = -2 \dots ③ \end{Bmatrix}$$

اخترت المعادلات ①،② للحل و المعادلة ③ للتحقق:

$$\begin{Bmatrix} -3 \times ①: -3t + 6s = 21 \\ ②: \quad 3t + 5s = 1 \end{Bmatrix} \Rightarrow 11s = 22 \Rightarrow s = 2$$

$① \Rightarrow -3t + 6(2) = 21 \Rightarrow -3t = 9 \Rightarrow t = -3$

$\therefore t = -3$ and $s = 2$

(**التحقق**) $③: 2t + 2s = -2 \Rightarrow 2(-3) + 2(2) = -6 + 4 = -2$

Eq ② is satisfied

$\Rightarrow$ our assumption is true $\Rightarrow$ the lines are intersected

To find the pt of intersection: substitute $t = -3$ in L_1 (or $s = 2$ in L_2):

$t = -3: x = 6 + t = 6 - 3 = 3$

$y = -2 + 3t = -2 + 3(-3) = -11$

$z = 4 + 2t = 4 + 2(-3) = -2$

The pt of intersection is $(3, -11, -2)$

للتأكد من صحة الحل نحسب نقطة التقاطع من خلال الخط L_2:

$s = 2: x = -1 + 2s = -1 + 2(2) = 3$

$y = -1 - 5s = -1 - 5(2) = -11$

$z = 2 - 2s = 2 - 2(2) = -2$

The pt of intersection is $(3, -11, -2)$ (لاحظ الناتج نفس النقطة)

Example 12.5.6: Find param. Eqs of the line through the points $A(1,2,8)$ and $B(-2,8,-4)$

Solution:

A vector parallel to the line is $\overrightarrow{AB} = \langle B - A \rangle = \langle -2 - 1, 8 - 2, 8 - (-4) \rangle = \langle -3, 6, 12 \rangle$

انتبه لما يلي:

انت غير مضطر انك تتعامل مع المتجه $\langle -3,6,12 \rangle$ بل ممكن تغييره بأي متجه اخر يوازيه: يعني ممكن تضرب أو تقسم المتجه بأي عدد لجعل مركبات المتجه اقل وافضل لذلك سوف اقسم المتجه على العدد -3: المتجه الجديد $\langle 1, -2, -4 \rangle$

- A pt. on the line: $A(1,2,8)$ and a parallel vector: $\langle 1, -2, -4 \rangle$

  $\Rightarrow$ Param. Eqs are: $x = 1 + t, y = 2 - 2t, z = 8 - 4t$

انتبه هناك حلول أخرى للسؤال ولكنها جميعاً متكافئة:

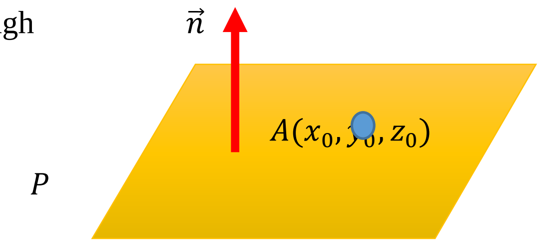
1) اذا بنينا الحل على النقطة $B(-2,8,-4)$ والمتجه $\langle 1,-2,-4\rangle$ فيكون الجواب:
$x=-2+t, y=8-2t, z=-4-4t$

2) اذا بنينا الحل على النقطة $A(-2,8,-4)$ ولكن على متجه آخر بضرب المتجه الأصلي بـ -1 أي المتجه $\langle -1,2,4\rangle$ فيكون الجواب: $x=-2-t, y=8+2t, z=-4+4t$

Definition 12.5.7: The eq. of the plane P that passes through the point $A(x_0,y_0,z_0)$ and with normal (عامودي) vector $\vec{n}=\langle a,b,c\rangle$ is:

$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$

$\Leftrightarrow ax+by+cz=ax_0+by_0+cz_0$



A vector eq of the plane is: $\vec{n}\cdot\vec{r}=\vec{n}\cdot\vec{r}_0$, where $\vec{r}=\langle x,y,z\rangle$ and $\vec{r}_0=\langle x_0,y_0,z_0\rangle$

Example 12.5.8: Find the equation of the plane through the pt $(0,-6,7)$ with normal vector $\langle 2,5,6\rangle$. Find the intercepts (التقاطعات مع المحاور الإحداثية) and sketch the graph of the plane.

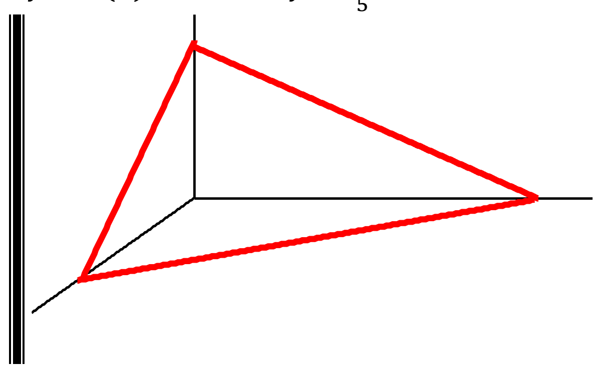
Solution: The eq is $2x+5y+6z=2(0)+5(-6)+6(7)\Rightarrow 2x+5y+6z=12$

The intercepts:

- The $\overbrace{x-\text{intercept}}^{\text{المقطع السيني}}$: when $y=0,z=0\Rightarrow \overbrace{2x+5(0)+6(0)=12}^{\substack{2x+5y+6z=12\\ \Downarrow}}\Rightarrow \overbrace{x=6}^{x-\text{intercept}}$
- The $\overbrace{y-\text{intercept}}^{\text{المقطع الصادي}}$: when $x=0,z=0\Rightarrow \overbrace{2(0)+5y+6(0)=12}^{\substack{2x+5y+6z=12\\ \Downarrow}}\Rightarrow \overbrace{y=\frac{12}{5}}^{y-\text{intercept}}$
- The $\overbrace{z-\text{intercept}}^{\text{المقطع } z}$: when $x=0,y=0$

  $\Rightarrow \overbrace{2(0)+5(0)+6z=12}^{\substack{2x+5y+6z=12\\ \Downarrow}}\Rightarrow \overbrace{z=2}^{z-\text{intercept}}$

  The Graph is:



Example 12.5.9: Find the pt at which the line $x=2+3t, y=-4t, z=5+t$ intersects the plane $4x+5y-2z=18$.

Solution: Substitute (عوض) the eqs of the in the eq of the plane:

$4(2+3t)+5(-4t)-2(5+t)=18\Rightarrow 8+12t-20t-10-2t=18$

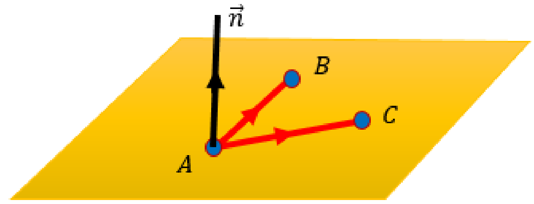
$\Rightarrow -10t=20\Rightarrow t=-2$. Now substitute this value in the eqs of the line:

$$x=2+3(-2)=-4,\qquad y=-4(-2)=8,\quad z=5+(-2)=3$$

The pt of intersection is $(-4,8,3)$

Example 12.5.10: Find the eq. of the plane that pass through the pts $A(1,3,2), B(3,-1,6), C(5,2,0)$

Solution: $\overrightarrow{AB} = \langle 2,-4,4\rangle$ and $\overrightarrow{AC} = \langle 4,-1,-2\rangle$



$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 2 & -4 & 4 \\ 4 & -1 & -2 \end{vmatrix}$$

$$= \oplus\, i\big(-4(-2) - 4(-1)\big)$$
$$\ominus\, j\big(2(-2) - 4(4)\big)$$
$$\oplus\, k\big(2(-1) - 4(-4)\big)$$

$= 12i + 20j + 14k$ is normal on the plane

لاحظ ان المتجه الذي حصلنا عليه مركباته كبيره ويمكن قسمته على 2 فيكون المتجه الناتج موازيا للمتجه الأصلي وبالتالي يكون عاموديا على المستوى

$\Rightarrow$ A vector normal to the plane is: $\frac{12i+20j+14k}{2} = 6i + 10j + 7k$ and a pt is $A(1,3,2)$

The eq of the plane is: $6x + 10y + 7z = 6(1) + 10(3) + 7(2)$

$$\Rightarrow 6x + 10y + 7z = 50$$

يمكن ان نجد معادلة المستوى باستخدام أي من النقاط B أو C فلاحظ أن المعادلة النهائية بعد التبسيط ستكون هي نفس معادلتنا التي اوجدناها، مثلاً:

$$6x + 10y + 7z = 6(3) + 10(-1) + 7(6) = 18 - 10 + 42 = 50$$

Example 12.5.11: Find param. eqs of the line of intersection of the two planes:

$$P_1: x + y - z = 1 \text{ and } P_2: 3x - 3y + z = 3$$

Solution: First we find 2 pts on the line of intersection of P_1 and P_2:

Take $x = 0$:

$$\left.\begin{cases} P_1:: x + y - z = 1 & \Rightarrow \quad y - z = 1 \ldots\ldots \textcircled{1} \\ P_2: 3x - 3y + z = 3 & \Rightarrow -3y + z = 3 \ \ldots.. \textcircled{2} \end{cases}\right\} \Rightarrow \begin{cases} -2y = 4 \Rightarrow y = -2 \\ \textcircled{1}: -2 - z = 1 \\ \Rightarrow z = -3 \end{cases}$$

A pt on the line is $B(0,-2,-3)$

Take $x = 1$:

$$\left.\begin{cases} P_1:: x + y - z = 1 & \Rightarrow \quad y - z = 0 \ldots\ldots \textcircled{3} \\ P_2: 3x - 3y + z = 3 & \Rightarrow -3y + z = 0 \ \ldots.. \textcircled{4} \end{cases}\right\} \Rightarrow \begin{cases} -2y = 0 \Rightarrow y = 0 \\ \textcircled{3}: 0 - z = 0 \\ \Rightarrow z = 0 \end{cases}$$

A pt on the line is $C(1,0,0)$

$\Rightarrow \overrightarrow{BC} = \langle 1,2,3\rangle$ parallel to the line of intersection

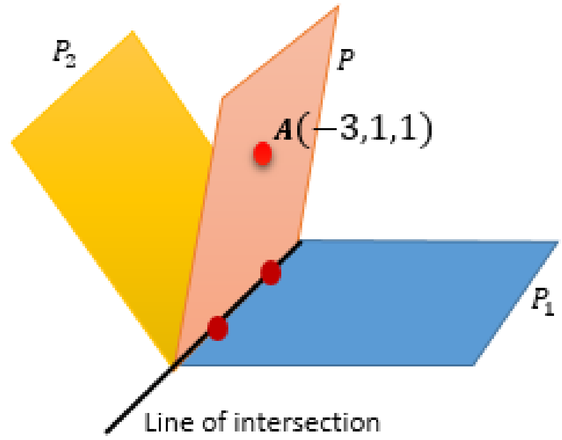
To find param. eqs of the line of intersection we use the point $B(0,-2,-3)$. The eqs are:

$x = 0 + 1t, y = -2 + 2t, z = 3 + (-3)t \Rightarrow x = t, y = -2 + 2t, z = 3 + 3t$

Example 12.5.12: Find the eq of the plane through the pt $A(-3,1,1)$ and contains the line of intersection of the two planes $P_1: z = y$ and $P_2: x + z = -1$.

Solution: First we find 2 pts on the line of intersection of P_1 and P_2:

ملاحظة:
اذا كان لدينا معادلتين بـ 3 متغييرات ونريد نقطة تحقق المعتدلتين فنقوم بما يلي: نأخذ قيمة لأحد المتغيرات ونعوضها في المعادلتين فنحصل على معادلتين بمتغيرين فنحل المعادلتين بالحذف او التعويض لنحصل على قيم المتغيرين الذين بقيا، وإن أردنا نقطة اخرى نعيد العملية السابقة مرة اخرى.



Take $x = 0$:

$$\begin{cases} P_1: z = y & \Rightarrow -y + z = 0 \ \ldots\ldots (1) \\ P_2: x + z = -1 \Rightarrow 0 + z = -1 \Rightarrow z = -1 & \ldots\ldots (2) \end{cases} \Rightarrow \begin{cases} z = -1 \\ -y + (-1) = 0 \\ \Rightarrow y = -1 \end{cases}$$

A pt on the line is $B(0,-1,-1)$

Take $x = 1$:

$$\begin{cases} P_1: z = y & \Rightarrow -y + z = 0 \ \ldots\ldots (3) \\ P_2: x + z = -1 \Rightarrow 1 + z = -1 \Rightarrow z = -2 & \ldots\ldots (4) \end{cases} \Rightarrow \begin{cases} z = -2 \\ -y + (-2) = 0 \\ \Rightarrow y = -2 \end{cases}$$

A pt on the line is $C(0,-2,-2)$

Since the line in the required plane $\Rightarrow$ the pts $B(0,-1,-1), C(1,-2,-2)$ are in the plane

In our plane we get 3 pts: $A(-3,1,1), B(0,-1,-1), C(1,-2,-2)$

اصبح الحل مثل Example 10 لذلك نريد صنع المتجه العامودي على المستوى المطلوب:

$\overrightarrow{AB} = \langle 3,-2,-2 \rangle$ and $\overrightarrow{AC} = \langle 4,-3,-3 \rangle$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} i & j & k \\ 3 & -2 & -2 \\ 4 & -3 & -3 \end{vmatrix}$$

$$= \oplus\, i(6-6) \ominus j(-9-(-8)) \oplus k(-9-(-8))$$

$$= j - k \text{ is normal on the plane}$$

To find the eq. of the plane we use the point $A(-3,1,1)$. The eq is:

$0x + 1y - 1z = 0(-3) + 1(1) - 1(1) \Rightarrow y - z = 0$

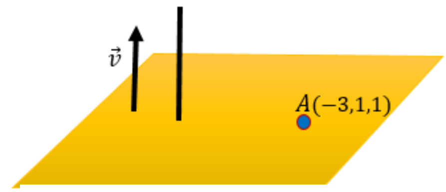
Example 12.5.13:

(1) Find the eq of the plane through the pt $A(-3,1,1)$ and perpendicular to the line:

$$x = t, y = -2 + 2t, z = 3 + 3t$$

(2) Find the eq of the plane through the pt $A(-3,1,1)$ and perpendicular to the line of intersection of the two planes $P_1: z = y$ and $P_2: x + z = -1$.

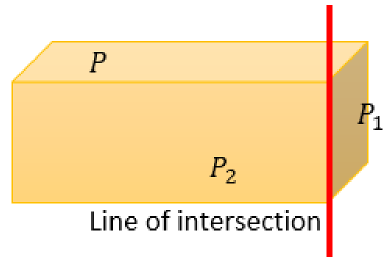
Solution:



(1) A vector parallel to the line is:
$\vec{v} = \langle 1,2,3 \rangle \Rightarrow \vec{v}$ normal to the plane
The eq of the plane is:

$$1x + 2y + 3z = 1(-3) + 2(1) + 3(1)$$

$\Rightarrow x + 2y + 3z = 2$



(2) First we find two on the line of intersection:
See example 12: the pts $B(0,-1,-1), C(1,-2,-2)$ are on the line of intersection
$\overrightarrow{BC} = \langle 1,-1,-1 \rangle //$ line of intersection
$\Rightarrow \overrightarrow{BC} = \langle 1,-1,-1 \rangle$ normal to the required plane
To find the eq. of the plane we use the point $A(-3,1,1)$. The eq is:

$$1x - 1y - 1z = 1(-3) - 1(1) - 1(1) \Rightarrow -y - z = -5 \Rightarrow y + z = 5$$

Definition 12.5.14: Let P_1 and P_2 be two planes and let $\vec{n}_1 \perp P_1$ and $\vec{n}_2 \perp P_2$.

(1) $P_1 // P_2 \Leftrightarrow \vec{n}_1 // \vec{n}_2$
(2) P_1 and P_2 are intersected (at a line) $\Leftrightarrow \vec{n}_1$ and $\vec{n}_2$ are not parallel

Rule 12.5.15: Let P_1 and P_2 be two planes and let $\vec{n}_1 \perp P_1$ and $\vec{n}_2 \perp P_2$. If P_1 and P_2 are intersected, then the angle θ between P_1 and P_2 is the same angle between $\vec{n}_1$ and $\vec{n}_2$, that is $\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} \Leftrightarrow \theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right)$ where $0 \le \theta \le \frac{\pi}{2}$

Example 12.5.16: Find the angle between the planes $P_1: x - y = 3$ and $P_2: x + 2y - z = 1$

Solution: $\vec{n}_1 = \langle 1,-1,0 \rangle$ and $\vec{n}_2 = \langle 1,2,-1 \rangle$

$$\cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|}\right) = \cos^{-1}\left(\frac{1-2}{\sqrt{2}\sqrt{6}}\right) = \cos^{-1}\left(\frac{-1}{2\sqrt{3}}\right) \cong 106.7°$$

$\Rightarrow \theta \cong 180 - 106.7 = 73.3 \Rightarrow \theta \cong 73.3$

Example 12.5.17: Find all values of a (if exist) that make the planes $P_1: 4x + 3ay - 2z = 1$ and $P_2: \frac{3}{2}ax + 9y - 3z = 0$: (1) perpendicular (2) parallel

Solution: $\vec{n}_1 = \langle 4, 3a, -2 \rangle$ and $\vec{n}_2 = \langle \frac{3}{2}a, 9, -3 \rangle$

(1) P_1 perpendicular to P_2 $\Leftrightarrow$ $\vec{n}_1 \perp \vec{n}_2 \Leftrightarrow \vec{n}_1 \cdot \vec{n}_2 = 0 \Rightarrow 4\left(\frac{3}{2}a\right) + 3a(9) - 2(-3) = 0$

$\Rightarrow 6a + 27a + 6 = 0 \Rightarrow 33a = -6 \Rightarrow a = -\frac{6}{33} = -\frac{2}{11}$

(2) P_1 and P_2 parallel $\Rightarrow \vec{n}_1$ and $\vec{n}_2$ are parallel

Ratio Method: $\frac{\frac{3}{2}a}{4} = \frac{9}{3a} = \frac{-3}{-2} \Rightarrow \begin{cases} \frac{\frac{3}{2}a}{4} = \frac{-3}{-2} \Rightarrow \frac{3a}{8} = \frac{3}{2} \Rightarrow a = 4 \\ \frac{9}{3a} = \frac{-3}{-2} \Rightarrow a = 2 \end{cases}$ which is impossible

There is no value of a that make P_1 and P_2 parallel

Rule 12.5.18: The distance from the pt $A(x_0, y_0, z_0)$ to the plane $P: ax + by + cz + d = 0$ is:

$$\text{distance} = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

If the distance equals 0, then the point A is on the plane.

Example 12.5.19: Find the distance from the pt $A(-1,2,3)$ to the plane $2x - 4y + z = 1$

Solution: Eq of the plane $2x - 4y + z = 1 \Rightarrow 2x - 4y + z - 1 = 0$

$$\Rightarrow \text{distance} = \frac{|2(-1) - 4(2) + 3 - 1|}{\sqrt{(2)^2 + (-4)^2 + (1)^2}} = \frac{|-8|}{\sqrt{21}} = \frac{8}{\sqrt{21}}$$

Rule 12.5.20: Let P_1 and P_2 be two planes.

(1) If P_1 and P_2 are intersected then the distance between P_1 and P_2 is zero 0

(2) The distance between two parallel planes:

$ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is

$$\text{distance} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

Example 12.5.21: Find the distance between the given two planes:

(1) $P_1: x - 2y + z = 3$ and $P_2: 2x - y + 2z = 6$

(2) $P_1: 10x + 2y - 2z = 5$ and $P_2: 5x + y - z = 1$

Solution:

(1) $\vec{n}_1 = \langle 1, -2, 1 \rangle$ and $\vec{n}_2 = \langle 2, -1, 2 \rangle \Rightarrow$ To check if $\vec{n}_1$ and $\vec{n}_2$ are parallel or not:

Ratio Method: $\frac{1}{2}, \frac{-2}{-1}, \frac{1}{2}$ are not all equal $\Rightarrow \vec{n}_1$ and $\vec{n}_2$ are not parallel

$\Rightarrow P_1$ and P_2 are intersected $\Rightarrow$ distance between P_1 and P_2 is 0

(2) $\vec{n}_1 = \langle 10, 2, -2 \rangle$ and $\vec{n}_2 = \langle 5, 1, -1 \rangle \Rightarrow$ To check if $\vec{n}_1$ and $\vec{n}_2$ are parallel or not:

Ratio Method: $\frac{10}{5}, \frac{2}{1}, \frac{-2}{-1}$ are all equal to 2 $\Rightarrow \vec{n}_1$ and $\vec{n}_2$ are parallel

نعيد كتابة معالات المستويات لتكون معاملات (x, y, z) متماثلة:

$P_1: 10x + 2y - 2z = 5 \Rightarrow 5x + y - z = \frac{5}{2} \Rightarrow 5x + y - z - \frac{5}{2} = 0$ ....①

$P_2: 5x + y - z - 1 = 0$..........②

The distance between P_1 and P_2 is distance$(A, P_2) = \frac{\left|-\frac{5}{2}-(-1)\right|}{\sqrt{(5)^2+(1)^2+(-1)^2}} = \frac{\left|-\frac{3}{2}\right|}{\sqrt{27}} = \frac{1}{2\sqrt{3}}$

Section 12.6: Cylinders and Quadric Surfaces

Definition 12.6.1: Cylinders are surfaces that results (ينتج) by moving a curve in a direction of a fixed axis (line).

Example 12.6.2: Each of the following are cylinders:

(1) $z = x^2$ (we move the curve along the y-axis)

(2) $x^2 + y^2 = 4$ (we move the curve along the z-axis)

(3) The plane $x - 2y + z = 1$ (we move line L in the plane along a line in the plane perpendicular to L)

(4) All planes are cylinders $\Rightarrow$ $x = 3, 2x - z = 1$, and $2x + 3y - z = 2$ are all cylinders

Example 12.6.3: Each of the following are not cylinders:

(1) $x^2 - 3 + z = 5$ (2) $x + 2y = \cos z$ (3) $z = e^x - 3\ln y$

Definition 12.6.4: A quadric surface is the graph of a second-degree equation in the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0,$$

where $A, B, C, \ldots, J$ are scalars.

Example 12.6.5: Identify (أعط الإسم) the trace of the quadric surface $2x^2 + y^2 - z^2 = 16$ in the:

(1) plane $z = 1$ (2) plane $y = 4$

Solution: (1) Substitute $z = 1$ in the surface $\Rightarrow 2x^2 + y^2 - 1^2 = 16$

$\Rightarrow$ The trace is $2x^2 + y^2 = 17$ which is an ellipse (قطع ناقص) in the plane

(2) Substitute $y = 1$ in the surface $\Rightarrow 2x^2 + 1^2 - z^2 = 16$

$\Rightarrow$ The trace is $2x^2 - z^2 = 15$ which is a hyperbola (قطع زائد) in the plane

Example 12.6.6: Identify the trace of the quadric surface $x^2 + y + z^2 = 2$ in the:

(1) plane $z = 1$ (2) plane $y = 1$

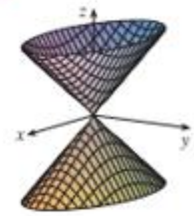
Solution: (1) Substitute $z = 1$ in the surface $\Rightarrow x^2 + y + 1 = 2$

$\Rightarrow$ The trace is $y - 1 = -x^2$ which is a parabola (قطع مكافيء) in the plane

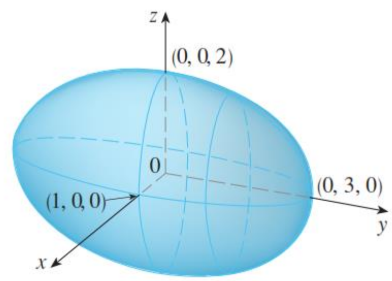
(2) Substitute $y = 1$ in the surface $\Rightarrow x^2 + 1 + z^2 = 2$

$\Rightarrow$ The trace is $x^2 + z^2 = 1$ which is a circle (دائرة) in the plane

Remark 12.6.7:

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.		

Example 12.6.8: Use traces to sketch the surface $x^2+\frac{y^2}{9}+\frac{z^2}{4}=1$. Show the intercepts and give the name.



Solution: The surface name is **Ellipsoid**

Intercepts:

***x*-intercept:** $y=0, z=0$: $x^2+\frac{0^2}{9}+\frac{0^2}{4}=1 \Rightarrow x^2=1 \Rightarrow x=1,-1$

***y*-intercept:** $x=0, z=0$: $0^2+\frac{y^2}{9}+\frac{0^2}{4}=1 \Rightarrow y^2=9 \Rightarrow y=3,-3$

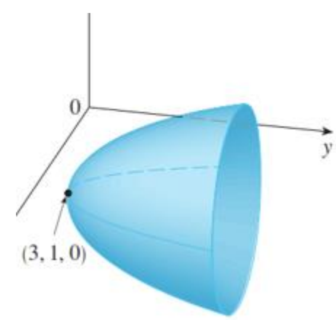
***z*-intercept:** $x=0, y=0$: $0^2+\frac{0^2}{9}+\frac{z^2}{4}=1 \Rightarrow z^2=4 \Rightarrow z=2,-2$

Example 12.6.9: Classify and sketch each of the following surfaces:

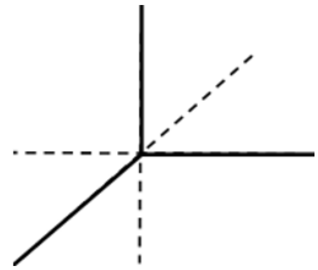
(1) $x^2+2z^2-6x-y+10=0$ (2) $x^2+2z^2-6x+y+10=0$.

Solution:

(1)

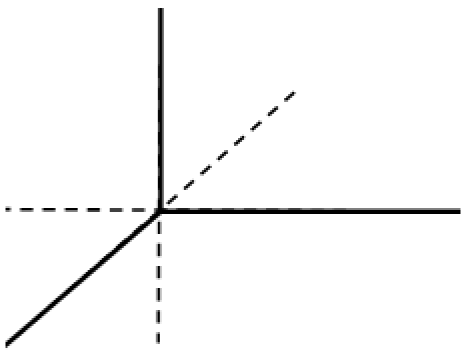


(2)

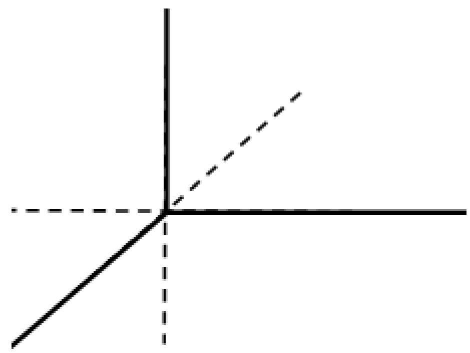


Example 12.6.10: Identify and sketch the surfaces:

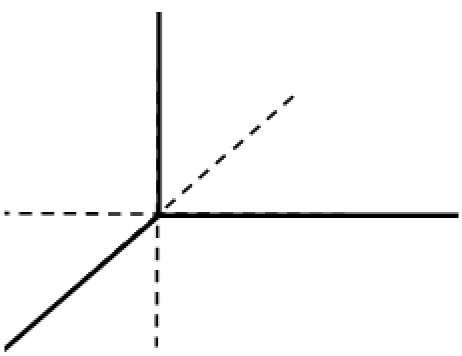
(1) $z = 4x^2 + y^2$



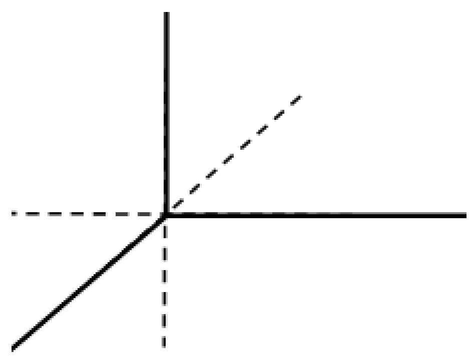
(2) $\frac{x^2}{4} + y^2 - 6y + \frac{z^2}{3} = 0.$



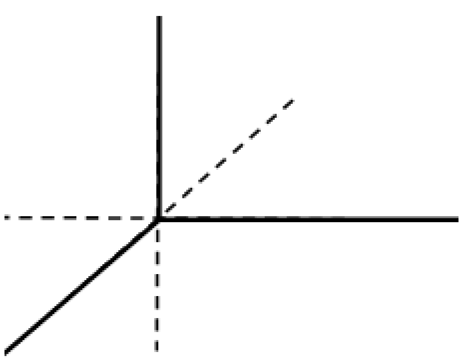
(3) $\frac{x^2}{4} - y + \frac{z^2}{4} + 2z = 1.$



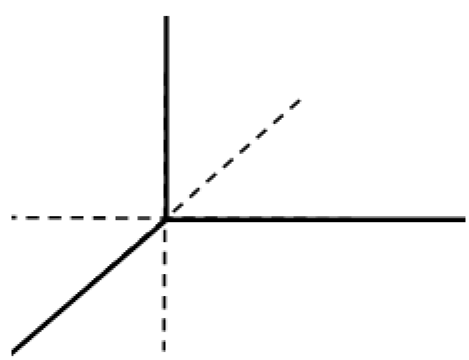
(4) $z^2 = 2x^2 + y^2$



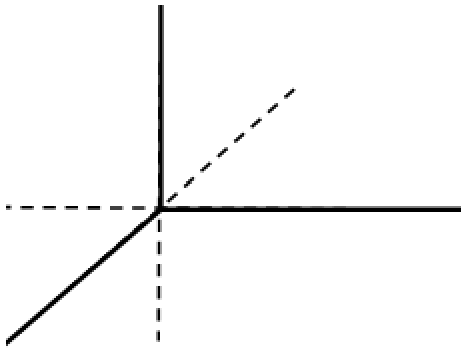
(5) $z^2 = 2x^2 + y^2$



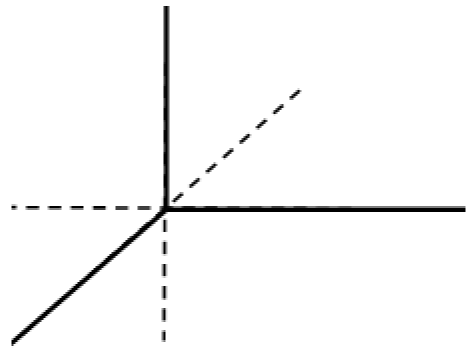
(6) $y^2 = x^2 + 4z^2$



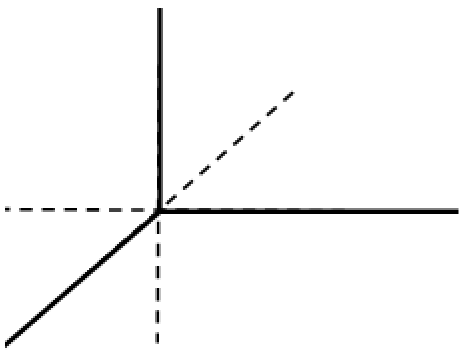
(7) $z = \sqrt{2x^2 + y^2}$



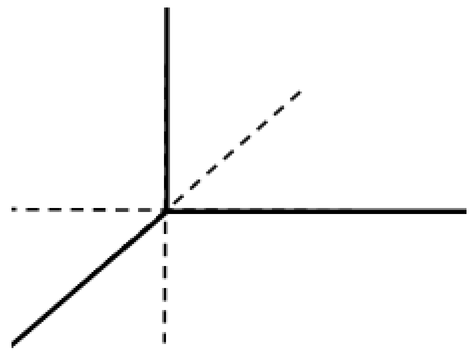
(8) $z = -\sqrt{2x^2 + y^2}$



(9) $2 - y = \sqrt{2x^2 + z^2}$



(10) $x - 1 = \sqrt{(y-1)^2 + z^2}$



End of Chapter 12

Study Hard - Good Luck

Chapter 13: Vector Functions

Section 13.1: Vector Functions and vector curves

Definition 13.1.1: A vector function, denoted by $\mathbf{r}(t)$, is a function in the variable t with domain $A \subseteq \mathbb{R}$ and its range is the set of vectors:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)i + g(t)j + h(t)k, \ \ t \in A$$

that is $A = \text{dom}\,\mathbf{r}(t) = \text{dom}(f) \cap \text{dom}(g) \cap \text{dom}(h)$

Example 13.1.2: Find the domain of $\mathbf{r}(t) = \langle \frac{t}{t^2-1}, \ln(3-t), \sqrt{t} \rangle$.

Solution:

- $\frac{t}{t^2-1}$: $\Rightarrow t^2 - 1 \neq 0 \Rightarrow t \neq \pm 1 \Rightarrow \text{dom}\left(\frac{t}{t^2-1}\right) = (-\infty, \infty)\backslash\{\pm 1\}$.
- $\ln(3-t)$: $\Rightarrow 3 - t > 0 \Rightarrow t < 3 \Rightarrow \text{dom}(\ln(3-t)) = (-\infty, 3)$
- $\sqrt{t}$: $\Rightarrow t \geq 0 \Rightarrow \text{dom}(\sqrt{t}) = [0, \infty)$

$\Rightarrow \text{dom}\,\mathbf{r}(t) = \text{dom}\left(\frac{t}{t^2-1}\right) \cap \text{dom}(\ln(3-t)) \cap \text{dom}(\sqrt{t}) \Rightarrow \text{dom}\,\mathbf{r}(t) = [0,1) \cup (1,3)$.

Geometrically 13.1.3: The vector function:

$$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$$

Defines a vector curve C traced out by the tip of the moving vector $\mathbf{r}(t)$. The direction of C is as the direction of the moving tip when t increases as shown in the figure. The vector $\mathbf{r}(t)$ is called the position vector. The parametric eqs of C are: $x = f(t), y = g(t), z = h(t)$

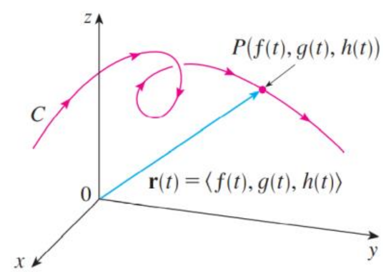


FIGURE 1
C is traced out by the tip of a moving position vector $\mathbf{r}(t)$.

Example 13.1.4: Sketch the curve and show the direction for each of the following vector functions:

(1) $\mathbf{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t \rangle$ (2) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k}$

(3) $\mathbf{r}(t) = \langle 1 + 4\sin t, 3\cos t \rangle$ (4) $\mathbf{r}(t) = \langle t, \sin t \rangle$

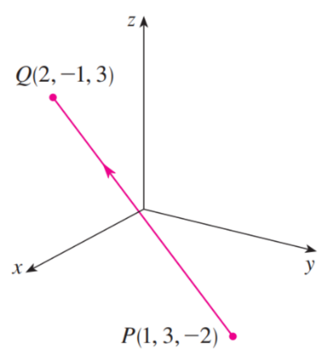
Solution:

(1) $\mathbf{r}(t) = \langle 1 + t, 3 - 4t, -2 + 5t \rangle \Rightarrow x = 1 + t, \ y = 3 - 4t, z = -2 + 5t$

$\Rightarrow$ The curve is a line in 3D

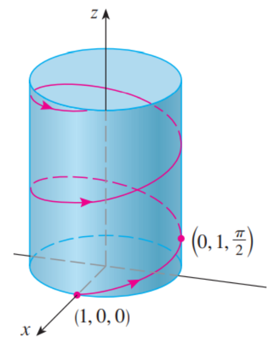
To determine direction: Find 2 pts on the curve

- $t = 0 \Rightarrow x = 1, y = 3, z = -2 \Rightarrow P(1,3,-2)$
- $t = 1 \Rightarrow x = 2, y = -1, z = 3 \Rightarrow Q(2,-1,3)$



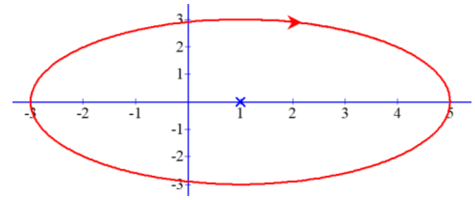
(2) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\mathbf{k} \Rightarrow x = \cos t\,, y = \sin t\,, z = t$
$\Rightarrow x^2 + y^2 = \cos^2 t + \sin^2 t = 1$
the curve lies on the circular cylinder $x^2 + y^2 = 1$
Since $z = t$, the curve spirals upward around the cylinder as t increases.
This curve is called a **helix.**
To determine direction: Find 2 pts on the curve

- $t = 0 \Rightarrow x = \cos 0 = 1, y = \sin 0 = 0, z = 0 \Rightarrow P(1,0,0)$
- $t = \frac{\pi}{2} \Rightarrow x = \cos\frac{\pi}{2} = 0, y = \sin\frac{\pi}{2} = 1, z = \frac{\pi}{2} \Rightarrow Q\left(0,1,\frac{\pi}{2}\right)$



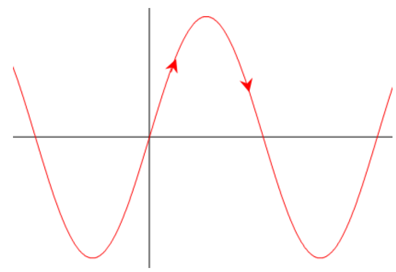
(3) $\mathbf{r}(t) = \langle 1 + 4\sin t\,, 3\cos t\rangle \Rightarrow\ x = 1 + 4\sin t\,, y = 3\cos t$
$\Rightarrow \sin t = \frac{x-1}{4}$ and $\cos t = \frac{y}{3}$
$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{(x-1)^2}{16} + \frac{y^2}{9} = 1$ (Ellipse)
To determine direction: Find 2 pts on the curve

- $t = 0 \Rightarrow x = 1 + 4\sin 0 = 1, y = 3\cos 0 = 3 \Rightarrow P(1,3)$
- $t = \frac{\pi}{2} \Rightarrow x = 1 + 4\sin\frac{\pi}{2} = 5, y = 3\cos\frac{\pi}{2} = 0 \Rightarrow Q(5,0)$



(4) $\mathbf{r}(t) = \langle t, \sin t\rangle \Rightarrow\ x = t, y = \sin t \Rightarrow y = \sin x$
To determine direction: Find 2 pts on the curve

- $t = 0 \Rightarrow x = 0, y = \sin 0 = 0 \Rightarrow P(0,0)$
- $t = \frac{\pi}{2} \Rightarrow x = \frac{\pi}{2}, y = \sin\frac{\pi}{2} = 1 \Rightarrow Q\left(\frac{\pi}{2}, 1\right)$



Exercise 13.1.5: Sketch the curve defined by the vector functions

(1) $\mathbf{r}(t) = \langle 1 - \frac{1}{2}\cos t\,, 4t, 3 + 2\sin t\rangle$.
(2) $\mathbf{r}(t) = \langle t, 5 - t^2\rangle$.

Remark 13.1.6: In general, the curve defined by the vector function

$$\mathbf{r}(t) = \langle a + b\cos t\,, c + d\sin t\,, et\rangle$$

is called a Helix. When $b = d$ it is called a circular Helix.

Example 13.1.7: Find a vector function that represents the curve of intersection of the two surfaces:

(1) $9(x-1)^2 + 2y^2 = 36, z = xy$
(2) $x^2 + z^2 = 1, y = x^2 - z^2$

Solution:

(1) $9(x-1)^2 + 2y^2 = 36 \Rightarrow \frac{(x-1)^2}{4} + \frac{y^2}{18} = 1 \Rightarrow \left(\frac{x-1}{2}\right)^2 + \left(\frac{y}{\sqrt{18}}\right)^2 = 1$

Let $\frac{x-1}{2} = \cos t\,, \frac{y}{\sqrt{18}} = \sin t \Rightarrow\ x = 1 + 2\cos t\,, y = \sqrt{18}\sin t\ \Rightarrow y = 3\sqrt{2}\sin t$

$z = xy = (1 + 2\cos t)(3\sqrt{2}\sin t) = 3\sqrt{2}\sin t + 6\sqrt{2}\cos t\sin t = 3\sqrt{2}\sin t + 2\sqrt{2}\sin(2t)$

The vector function is $\mathbf{r}(t) = \langle 1 + 2\cos t\,, 3\sqrt{2}\sin t\,, 3\sqrt{2}\sin t + 2\sqrt{2}\sin(2t)\rangle$

(2) $x^2 + z^2 = 1 \Rightarrow x = \sin t\,, z = \cos t$

$y = x^2 - z^2 = \sin^2 t - \cos^2 t = -(\cos^2 t - \sin^2 t) = -\cos(2t)$

The vector function is $\mathbf{r}(t) = \sin t\,\mathbf{i} - \cos(2t)\mathbf{j} + \cos t\,\mathbf{k}$

Example 13.1.8: Find a surface on which the vector curve lies on:

(1) $\mathbf{r}(t) = \langle \cos t\,, \sin t\,, t\rangle$

(2) $\mathbf{r}(t) = \langle t\cos t\,, t\sin t\,, t\rangle$

(3) $\mathbf{r}(t) = \langle e^t, t^2 + 2e^{2t}, t\rangle$

(4) $\mathbf{r}(t) = \langle \cos t\,, \sin t\,, -\cos t\rangle$

Solution:

(1) $x = \cos t\,, y = \sin t\,, z = t$

$\Rightarrow \cos^2 t + \sin^2 t = 1 \Rightarrow x^2 + y^2 = 1 \Rightarrow$ The curve lies on a cylinder

(2) $x = t\cos t\,, y = t\sin t\,, z = t$

$\cos^2 t + \sin^2 t = 1 \Rightarrow \frac{x^2}{t^2} + \frac{y^2}{t^2} = 1 \Rightarrow x^2 + y^2 = t^2 \Rightarrow x^2 + y^2 = z^2$

$\Rightarrow z^2 = x^2 + y^2 \Rightarrow$ The curve lies on a cone

(3) $x = e^t, y = t^2 + 2e^{2t}, z = t \Rightarrow y = z^2 + 2x^2 \Rightarrow$ The curve lies on a paraboloid

(4) $x = \cos t\,, y = \sin t\,, z = -\cos t \Rightarrow z = -x \Rightarrow$ The curve lies on a plane

Also, since $\cos^2 t + \sin^2 t = 1 \Rightarrow x^2 + y^2 = 1 \Rightarrow$ The curve lies on a cylinder

Remark 13.1.9: A vector function for the line segment (القطعة المستقيمة) from the pt P to the pt Q is

$$\mathbf{r}(t) = \langle (1-t)P + tQ\rangle, \text{ where } 0 \le t \le 1.$$

Example 13.1.10: Find a vector function for the line segment from the pt $(1,0,2)$ to the pt $(2,3,1)$.

Solution: $\mathbf{r}(t) = \langle (1-t)P + tQ\rangle$, where $0 \le t \le 1$

$\Rightarrow \mathbf{r}(t) = \langle (1-t)(1,0,2) + t(2,3,1)\rangle = \langle 1+t, 3t, 2-t\rangle$, where $0 \le t \le 1$

Rule 13.1.11: The vector function $\mathbf{r}(t)$ is continuous at $t = a \Leftrightarrow \begin{Bmatrix} ①: a \in \operatorname{dom}(\mathbf{r}(t)) \\ ②: \lim_{t\to a} \mathbf{r}(t) = \mathbf{r}(a) \end{Bmatrix}$

Example 13.1.12: Find where the vector function $\mathbf{r}(t) = \langle \frac{t}{\tan(t)}, \frac{1}{t-1}, \ln(t)\rangle$ is continuous.

Solution: $\vec{r}(t)$ is continuous on its domain

- $\frac{t}{\tan(t)} = \frac{t}{\tan(t)}$ is continuous on: $\mathbb{R}\backslash\left(\{0, \pm\pi, \pm 2\pi, \ldots\} \cup \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}, \ldots\right\}\right)$
- $\frac{1}{t-1}$ is continuous on: $\mathbb{R}\backslash\{1\}$
- $\ln(t)$ is continuous on: $(0,\infty)$.

$\Rightarrow \mathbf{r}(t)$ is continuous on its $\operatorname{dom}\big(\mathbf{r}(t)\big) = (0,\infty)\backslash\left\{1, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, \frac{5\pi}{2}, 3\pi, \ldots\right\}$

Section 13.2: Derivative and Integral of Vector Functions

Definition 13.2.1: Let $\mathbf{r}(t)$ be a smooth curve. Then the derivative of $\mathbf{r}(t)$ is defined by:

$$\frac{d\mathbf{r}}{dt} = \lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

If the limit exists. The derivative $\frac{d\mathbf{r}}{dt}$ is written as $\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t)$.

Moreover, if $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$ is a smooth curve, then $\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t)\rangle$

Geometrically 13.2.2: Let C be the curve defined by the vector function $\mathbf{r}(t)$ at a pt P on the curve C.

(1) $\mathbf{r}'(t)$ is a tangent vector to the curve C at P points in the direction of increasing t.

(2) The vector $\mathbf{r}'(t)$ is called the ***tangent vector*** to $\mathbf{r}(t)$ at the pt P.

(3) The ***unit tangent vector*** to the curve is given by $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$

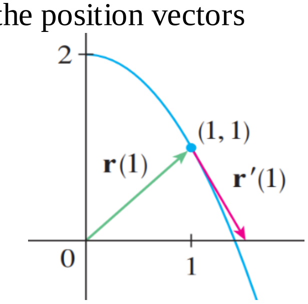
The tangent line to the curve C at the pt P is the line parallel to the tangent vector $\mathbf{r}(t)$.

Example 13.2.3: For the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2-t)\mathbf{j}$, find $\mathbf{r}'(t)$ and sketch the position vectors $\mathbf{r}(1)$ and $\mathbf{r}'(1)$

Solution: $\mathbf{r}'(t) = \frac{\mathbf{i}}{2\sqrt{t}} - \mathbf{j}$. To sketch $\mathbf{r}(1)$ and $\mathbf{r}'(1)$: First we need to sketch the curve $\mathbf{r}(t) = \sqrt{t}\mathbf{i} + (2-t)\mathbf{j}$ (We are in 2D)

$\Rightarrow x = \sqrt{t}, y = 2 - t \Rightarrow t = x^2 \Rightarrow y = 2 - x^2$ is a parabola

$\mathbf{r}(1) = \mathbf{i} + \mathbf{j} = \langle 1,1\rangle$ and $\mathbf{r}'(1) = \frac{\mathbf{i}}{2} - \mathbf{j} = \langle \frac{1}{2}, -1\rangle$



Example 13.2.4: Find a tangent vector and a unit tangent vector to the curve:

$\mathbf{r}(t) = \langle 1 + t^3, \ln t, \sin(\pi t)\rangle$

(1) at $t = 1$ (2) at the point $A(9, \ln 2, 0)$

Solution: (1) $\mathbf{r}'(t) = \langle 3t^2, \frac{1}{t}, \pi\cos(\pi t)\rangle \Rightarrow \mathbf{r}'(1) = \langle 3,1,-\pi\rangle$ is a tangent vector to $\mathbf{r}(t)$

The unit tangent vector is $\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\langle 3,1,-\pi\rangle}{\sqrt{10+\pi^2}}$

(2) At $A(9, \ln 2, 0)$ we have to find the value of t:

$A(9, \ln 2, 0)$: $x = 9, y = \ln 2, z = 0 \Rightarrow 1 + t^3 = 9, \ln t = \ln 2, \sin(\pi t) = 0 \Rightarrow t = 2$

$\mathbf{r}'(2) = \langle 12, \frac{1}{2}, \pi\rangle$ is a tangent line $\Rightarrow$ The unit tangent vector is $\mathbf{T}(2) = \frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \frac{\langle 12, \frac{1}{2}, \pi\rangle}{\sqrt{144.25+\pi^2}}$

Example 13.2.5: Find all unit vectors parallel to the tangent line to the parabola $y = x^2$ at the point $(2,4)$.

Solution:

Step 1: We write the curve $y = x^2$ in parametric form: Let $x = t \Rightarrow y = t^2$

$\Rightarrow$ the curve is $\mathbf{r}(t) = \langle t, t^2\rangle \Rightarrow \mathbf{r}'(t) = \langle 1, 2t\rangle$

Step 2: We find the value of t at the pt $(2,4)$: $x = 2, y = 4$ but $x = t, y = t^2 \Rightarrow t = 2$

Step 3: A vector parallel to the curve at $(2,4)$ is $\mathbf{r}'(2) = \langle 1,4\rangle$

The unit vectors are $\pm\frac{\mathbf{r}'(2)}{|\mathbf{r}'(2)|} = \pm\frac{\langle 1,4\rangle}{\sqrt{17}} = \pm\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}\rangle$

Example 13.2.6: Find parametric equations of the tangent line to the curve:

$$x = 2\cos t\,, y = \sin t\,, z = t \text{ at the pt } A\left(0,1,\frac{\pi}{2}\right).$$

Solution:

- First we write the vector function of the curve: $\mathbf{r}(t) = \langle 2\cos t\,, \sin t\,, t\rangle$
  $\Rightarrow \mathbf{r}'(t) = \langle -2\sin t\,, \cos t\,, 1\rangle$
- Second we have to find the value of t at the pt $A\left(0,1,\frac{\pi}{2}\right)$:
  $x = 0, y = 1, z = \frac{\pi}{2}$ but $x = 2\cos t\,, y = \sin t\,, z = t \Rightarrow t = \frac{\pi}{2}$
  A vector parallel to the line is: $\mathbf{r}'\left(\frac{\pi}{2}\right) = \langle -2\sin\left(\frac{\pi}{2}\right), \cos\left(\frac{\pi}{2}\right), 1\rangle = \langle -2,0,1\rangle$
  Parametric equations of the tangent line are:
  $x = 0 + (-2)t, y = 1 + 0t, z = \frac{\pi}{2} + 1t \Rightarrow$ $x = -2t, y = 1, z = \frac{\pi}{2} + t$

Properties 13.2.7: Let $\mathbf{u}(t), \mathbf{v}(t)$ be vector functions, $f(t)$ is a function and a, b are scalars. Then

(1) $\frac{d}{dt}(a\mathbf{u}(t) + b\mathbf{v}(t)) = a\mathbf{u}'(t) + b\mathbf{v}'(t)$

(2) $\frac{d}{dt}(\mathbf{u}(t)\cdot\boldsymbol{v}(t)) = \boldsymbol{u}(t)\cdot\boldsymbol{v}'(t) + \boldsymbol{u}'(t)\cdot\boldsymbol{v}(t)$

(3) $\frac{d}{dt}(\boldsymbol{u}(t)\times\boldsymbol{v}(t)) = \boldsymbol{u}(t)\times\boldsymbol{v}'(t) + \boldsymbol{u}'(t)\times\boldsymbol{v}(t)$

(4) (Chain Rule): $\frac{d}{dt}(\boldsymbol{u}(f(t))) = f'(t)\boldsymbol{u}'(f(t))$

Example 13.2.8:

(1) Show that $\frac{d}{dt}(\mathbf{r}(t)\times\mathbf{r}'(t)) = \mathbf{r}(t)\times\mathbf{r}''(t)$.

(2) Let $\mathbf{r}(t)$ be a smooth curve such that $|\mathbf{r}(t)| = C$ (constant for all t). Show that $\mathbf{r}(t)$ is orthogonal to $\mathbf{r}'(t)$ for all t

Proof:

(1) $\frac{d}{dt}(\mathbf{r}(t)\times\mathbf{r}'(t)) = \mathbf{r}(t)\times\mathbf{r}''(t) + \underbrace{\mathbf{r}'(t)\times\mathbf{r}'(t)}_{=\mathbf{0}} = \mathbf{r}(t)\times\mathbf{r}''(t)$

(2) $|\mathbf{r}(t)| = C \Rightarrow |\mathbf{r}(t)|^2 = C^2 \Rightarrow \mathbf{r}(t)\cdot\mathbf{r}(t) = C^2 \Rightarrow \frac{d}{dt}\big(\mathbf{r}(t)\cdot\mathbf{r}(t)\big) = \frac{d}{dt}(C^2)$
$\Rightarrow \mathbf{r}(t)\cdot\mathbf{r}'(t) + \mathbf{r}'(t)\cdot\mathbf{r}(t) = 0 \Rightarrow 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0 \Rightarrow \mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$
$\Rightarrow \mathbf{r}(t)\perp\mathbf{r}'(t)$

Rule 13.2.9: Let $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. Then $\int_a^b \mathbf{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt\rangle$

Example 13.2.10: Let $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 2t\mathbf{k}$. Then

(1) $\int \mathbf{r}(t)\,dt = (\int 2\cos t\,dt)\mathbf{i} + (\int \sin t\,dt)\mathbf{j} + (\int 2t\,dt)\mathbf{k}$

$$= (2\sin t + c_1)\mathbf{i} + (-\cos t + c_2)\mathbf{j} + (t^2 + c_3)\mathbf{k}$$

$= 2\sin t\,\mathbf{i} - \cos t\,\mathbf{j} + t^2\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a vector constant of integration

(2) $\int_0^{\pi/2} \mathbf{r}(t)\,dt = \left(\int_0^{\frac{\pi}{2}} 2\cos t\,dt\right)\mathbf{i} + \left(\int_0^{\frac{\pi}{2}} \sin t\,dt\right)\mathbf{j} + \left(\int_0^{\frac{\pi}{2}} 2t\,dt\right)\mathbf{k}$

$$= 2\sin t\rrbracket_0^{\pi/2}\mathbf{i} - \cos t\rrbracket_0^{\pi/2}\mathbf{j} + t^2\rrbracket_0^{\pi/2}\mathbf{k}$$

$$= 2(\sin\left(\frac{\pi}{2}\right) - \sin 0)\mathbf{i} - (\cos\left(\frac{\pi}{2}\right) - \cos 0)\mathbf{j} + (\left(\frac{\pi}{2}\right)^2 - 0)\mathbf{k} = 2\mathbf{i} + \mathbf{j} + \frac{\pi^2}{4}\mathbf{k}$$

Section 13.3: Arc length and Curvature (طول القوس والإنحناء)

Definition 13.3.1: The ***arc length*** (طول القوس) of the curve $\mathbf{r}(t), a \le t \le b$ is $L = \int_a^b |\mathbf{r}'(t)|\, dt$

Example 13.3.2: Find the arc length of the helix $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + t\mathbf{j} + 4\sin t\,\mathbf{k}$ from the pt $A(4,0,0)$ to the pt $B\left(0, \frac{\pi}{2}, 4\right)$

Solution:

Step 1: $\mathbf{r}'(t) = \langle -4\sin t, 1, 4\cos t\rangle$

$\Rightarrow |\mathbf{r}'(t)| = \sqrt{16\sin^2 t + 1 + 16\cos^2 t} = \sqrt{1 + 16(\sin^2 t + \cos^2 t)} = \sqrt{17}$

Step 2: We find the values of t at the pts A and B:

- $A(4,0,0) \Rightarrow x = 4, y = 0, z = 0$ but $x = 4\cos t,\ y = t, z = 4\sin t \Rightarrow t = 0$
- $B\left(0, \frac{\pi}{2}, 4\right) \Rightarrow x = 0, y = \frac{\pi}{2}, z = 4$ but $x = 4\cos t,\ y = t, z = 4\sin t \Rightarrow t = \frac{\pi}{2}$

$\Rightarrow 0 \le t \le \frac{\pi}{2}$

Step 3: $L = \int_0^{\frac{\pi}{2}} |\mathbf{r}'(t)|\, dt = \int_0^{\frac{\pi}{2}} \sqrt{17}\, dt = \frac{\pi\sqrt{17}}{2}$

Example 13.3.3: Find the arc length of the following curves:

(1) $\mathbf{r}(t) = \cos t\, i + \sin t\, j + \ln(\cos t), \frac{3\pi}{4} \le t \le \pi$

(2) $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + t\mathbf{j} + 4\sin t\,\mathbf{k}$ from the pt $A(4,0,0)$ to the pt $B\left(0, \frac{\pi}{2}, 4\right)$

(3) $\mathbf{r}(t) = \langle 2t, t^2, \frac{t^3}{3}\rangle, 0 \le t \le 1$

(4) $\mathbf{r}(t) = \langle \sqrt{2}\, t, e^t, e^{-t}\rangle, 0 \le t \le 2$

Solution:

(1) **Step 1:** $\mathbf{r}'(t) = \langle -\sin t, \cos t, \frac{-\sin t}{\cos t}\rangle = \langle -\sin t, \cos t, -\tan t\rangle$

$\Rightarrow |\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t} = \sqrt{1 + \tan^2 t} = \sqrt{\sec^2 t} = |\sec t|$

Step 2: $L = \int_{\frac{3\pi}{4}}^{\pi} |\mathbf{r}'(t)|\, dt = \int_{\frac{3\pi}{4}}^{\pi} |\sec t|\, dt = -\int_{\frac{3\pi}{4}}^{\pi} \sec t\, dt = -\ln|\sec t + \tan t|\Big]_{\frac{3\pi}{4}}^{\pi}$

$= -\left[\ln|-1 + 0| - \ln\left|-\sqrt{2} + (-1)\right|\right] = \ln\left(\sqrt{2} + 1\right)$

(2) **Step 1:** $\mathbf{r}'(t) = \langle -4\sin t, 1, 4\cos t\rangle$

$\Rightarrow |\mathbf{r}'(t)| = \sqrt{16\sin^2 t + 1 + 16\cos^2 t} = \sqrt{1 + 16(\sin^2 t + \cos^2 t)} = \sqrt{17}$

Step 2: Find the values of t at the pts A and B:

- $A(4,0,0) \Rightarrow x = 4, y = 0, z = 0$ but $x = 4\cos t,\ y = t, z = 4\sin t \Rightarrow t = 0$
- $B\left(0, \frac{\pi}{2}, 4\right) \Rightarrow x = 0, y = \frac{\pi}{2}, z = 4$ but $x = 4\cos t,\ y = t, z = 4\sin t \Rightarrow t = \frac{\pi}{2}$

$\Rightarrow 0 \le t \le \frac{\pi}{2}$

Step 3: $L = \int_0^{\frac{\pi}{2}} |\mathbf{r}'(t)|\, dt = \int_0^{\frac{\pi}{2}} \sqrt{17}\, dt = \frac{\pi\sqrt{17}}{2}$

(3) **Step 1:** $\mathbf{r}'(t) = \langle 2, 2t, t^2\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{4+4t^2+t^4} = \sqrt{(t^2+2)^2} = t^2+2$

Step 2: $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (t^2+2)\,dt = \frac{t^3}{3} + 2t\Big]_0^1 = \left(\frac{1}{3}+2\right) - (0) = \frac{7}{2}$

(4) **Step1:** $\mathbf{r}'(t) = \langle 2, 2t, t^2\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{4+4t^2+t^4} = \sqrt{(t^2+2)^2} = t^2+2$

Step 2: $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (t^2+2)\,dt = \frac{t^3}{3} + 2t\Big]_0^1 = \left(\frac{1}{3}+2\right) - (0) = \frac{7}{3}$

(5) **Step 1:** $\mathbf{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t}\rangle$

$$\Rightarrow |\mathbf{r}'(t)| = \sqrt{2+(e^t)^2+(-e^{-t})^2} = \sqrt{2+e^{2t}+e^{-2t}}$$

$$= \sqrt{e^{-2t}(2e^{2t}+e^{4t}+1)} = \sqrt{e^{-2t}}\sqrt{e^{4t}+2e^{2t}+1}$$

$$= e^{-t}\sqrt{(e^{2t}+1)^2} = e^{-t}(e^{2t}+1) = e^t + e^{-t}$$

Step 2: $L = \int_0^2 |\mathbf{r}'(t)|\,dt = \int_0^2 (e^t+e^{-t})\,dt = e^t + \frac{e^{-t}}{-1}\Big]_0^2 = (e^2 - e^{-2}) - (1-1) = e^2 - e^{-2}$

Exercise 13.3.4: Find the arc length (or the length of the arc) of the following curves:

(1) $\mathbf{r}(t) = \boldsymbol{i} + t^2\mathbf{j} + t^3\mathbf{k}, 0 \le t \le 1$

(2) $\mathbf{r}(t) = 12t\mathbf{i} + 8t^{3/2}\mathbf{j} + 3t^2\mathbf{k}, 0 \le t \le 4$

Example 13.3.5: Find the arc length of the curve of intersection of the parabolic cylinder $x^2 = 2y$ and the surface $3z = xy$ from the origin to the pt $A(6,18,36)$

Solution: First we parametrize the curve: $y = \frac{x^2}{2}$ and $z = \frac{xy}{3}$

Let $x = t \Rightarrow y = \frac{t^2}{2}$ and $z = \frac{t(t^2)}{6} = \frac{t^3}{6} \Rightarrow \mathbf{r}(t) = \langle t, \frac{t^2}{2}, \frac{t^3}{6}\rangle \Rightarrow \mathbf{r}'(t) = \langle 1, t, \frac{t^2}{2}\rangle$

$$L = \int_0^6 |\mathbf{r}'(t)|\,dt = \int_0^6 \left|\langle 1, t, \tfrac{1}{2}t^2\rangle\right|\,dt = \int_0^6 \sqrt{1^2+t^2+\left(\tfrac{1}{2}t^2\right)^2}\,dt = \int_0^6 \sqrt{1+t^2+\tfrac{1}{4}t^4}\,dt$$

$$= \int_0^6 \sqrt{(1+\tfrac{1}{2}t^2)^2}\,dt = \int_0^6 (1+\tfrac{1}{2}t^2)\,dt = \left[t+\tfrac{1}{6}t^3\right]_0^6 = 6+36 = 42$$

Definition 13.3.6:

(1) ***The curvature*** (الإنحناء) of a curve $\mathbf{r}(t)$ is defined by $\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$

(2) ***The curvature*** of a curve $y = f(x)$ (a curve in the plane) is defined by $\kappa(x) = \frac{|f''(x)|}{\left[1+(f'(x))^2\right]^{3/2}}$

Example 13.3.7:

(1) Find the curvature of the curve $\mathbf{r}(t) = \langle t, t^2, t^3\rangle$

(2) Find the curvature of the curve $\mathbf{r}(t) = \langle t^2 - t + 1, \ln(1-t), (1-t)\ln(1-t)\rangle$ at the point $(1,0,0)$

(3) Find the curvature of the curve with parametric equations $x = \sin t, y = \cos t, z = \sin 5t$ at the point $A(0,1,0)$.

(4) Find the curvature of the parabola $y = 1 - x^2$ at the point $(2,-3)$.

Solution:

(1) $\mathbf{r}'(t) = \langle 1,2t,3t^2\rangle \Rightarrow \mathbf{r}''(t) = \langle 0,2,6t\rangle$

$|\mathbf{r}'(t)| = \sqrt{1+4t^2+9t^4}$ and $|\mathbf{r}''(t)| = \sqrt{4+36t^2}$

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{|\mathbf{r}'(t)|^2|\mathbf{r}''(t)|^2 - \left(\mathbf{r}'(t)\cdot\mathbf{r}''(t)\right)^2}}{|\mathbf{r}'(t)|^3}$$
$$= \frac{\sqrt{(1+4t^2+9t^4)(4+36t^2)-(4t+18t^3)^2}}{(1+4t^2+9t^4)^{3/2}}$$

(2) $\mathbf{r}(t) = \langle t^2-t+1, \ln(1-t), (1-t)\ln(1-t)\rangle$ at the point $(1,0,0)$

- $t^2-t+1 = 1 \Rightarrow t^2-t=0 \Rightarrow t(t-1)=0 \Rightarrow t=0$ or $t=1$.

but $\ln(1-t)$ is undefined at $t=1$. So, we must have $t=0$.

- $\mathbf{r}'(t) = \langle 2t, -\frac{1}{1-t}, -1-\ln(1-t)\rangle \Rightarrow \mathbf{r}'(0) = \langle 0,-1,-1\rangle$
- $\mathbf{r}''(t) = \langle 2, \frac{1}{(1-t)^2}, \frac{1}{1-t}\rangle \Rightarrow \mathbf{r}''(0) = \langle 2,1,1\rangle$
- $|\mathbf{r}'(0)| = \sqrt{2}$, $|\mathbf{r}''(0)| = \sqrt{6}$ and $\mathbf{r}'(0)\cdot\mathbf{r}''(0) = -2$
- $\kappa(0) = \frac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{\sqrt{|\mathbf{r}'(0)|^2|\mathbf{r}''(0)|^2-\left(\mathbf{r}'(0)\cdot\mathbf{r}''(0)\right)^2}}{|\mathbf{r}'(0)|^3}$

$$= \frac{\sqrt{2(6)-(-2)^2}}{(2)^{3/2}} = \frac{\sqrt{8}}{2\sqrt{2}} = \frac{2\sqrt{2}}{2\sqrt{2}} = 1$$

(3) $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 5t\rangle$

- At the pt $A(0,1,0) \Rightarrow \sin t = 0, \cos t = 1, \sin 5t = 0 \Rightarrow t = 0$
- $\mathbf{r}'(t) = \langle \cos t, -\sin t, 5\cos t\rangle \Rightarrow \mathbf{r}'(0) = \langle 1,0,5\rangle$
- $\mathbf{r}''(t) = \langle -\sin t, -\cos t, -25\sin t\rangle \Rightarrow \mathbf{r}''(0) = \langle 0,-1,0\rangle$
- $|\mathbf{r}'(0)| = \sqrt{26}$, $|\mathbf{r}''(0)| = \sqrt{1}$ and $\mathbf{r}'(0)\cdot\mathbf{r}''(0) = 0$
- $\kappa(0) = \frac{|\mathbf{r}'(0)\times\mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \frac{\sqrt{|\mathbf{r}'(0)|^2|\mathbf{r}''(0)|^2-\left(\mathbf{r}'(0)\cdot\mathbf{r}''(0)\right)^2}}{|\mathbf{r}'(0)|^3} = \frac{\sqrt{26}}{(26)^{3/2}} = \frac{1}{26}$

(4) $f(x) = 1-x^2$ at the pt $(2,-3) \Rightarrow x = 2$, $f'(x) = -2x$ and $f''(x) = -2$

$\Rightarrow f'(2) = -4$ and $f''(2) = -2$.

$\Rightarrow \kappa(2) = \frac{|f''(2)|}{\left[1+\left(f'(2)\right)^2\right]^{3/2}} = \frac{|-2|}{[1+(-4)^2]^{3/2}} = \frac{2}{(17)^{3/2}} = \frac{2}{17\sqrt{17}}$

Example 13.3.8:

(1) Find the curvature of the curve with parametric equations $x = \sin t$, $y = \cos t$, $z = \sin 5t$ at the point $A(0,1,0)$.
(2) Find the curvature of each of the following curves: (a) $y = \tan x$ (b) $y = xe^x$
(3) Find the curvature of the parabola $y = 1-x^2$ at the point $(2,-3)$.

Example 13.3.9:

(1) Show that the curvature of any line is zero.

(2) Show that the curvature of any circle of radius a is $\frac{1}{a}$.

Proof:

(1) First we give a vector function of the line in 3D:

$$x = x_0 + at, y = y_0 + bt, z = z_0 + ct, t_0 \le t \le t_1$$

Let $\mathbf{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct\rangle \Rightarrow \mathbf{r}'(t) = \langle a, b, c\rangle \Rightarrow \mathbf{r}''(t) = \langle 0,0,0\rangle$

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{|\mathbf{r}'(t)|^2|\mathbf{r}''(t)|^2 - \left(\mathbf{r}'(t) \cdot \mathbf{r}''(t)\right)^2}}{|\mathbf{r}'(t)|^3}$$

$$= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{|\mathbf{r}'(t)|^2|\mathbf{r}''(t)|^2 - \left(\mathbf{r}'(t) \cdot \mathbf{r}''(t)\right)^2}}{|\mathbf{r}'(t)|^3}$$

$$= \frac{\sqrt{|\mathbf{r}'(t)|^2 0 - (0)^2}}{|\mathbf{r}'(t)|^3} = 0$$

(2) First we give a vector function of the circle in 3D: $x^2 + y^2 = a^2, z = 0$

Let $x = a\cos t$ and $y = a\sin t, z = 0 \Rightarrow \mathbf{r}(t) = \langle a\cos t, a\sin t, 0\rangle$

$\Rightarrow \mathbf{r}'(t) = \langle -a\sin t, a\cos t, 0\rangle$ and $\mathbf{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle$

$|\mathbf{r}'(t)| = \sqrt{a^2\sin^2 t + a^2\cos^2 t} = \sqrt{a^2(\sin^2 t + \cos^2 t)} = \sqrt{a^2} = a$

$|\mathbf{r}''(t)| = \sqrt{a^2\cos^2 t + a^2\sin^2 t} = \sqrt{a^2(\cos^2 t + \sin^2 t)} = \sqrt{a^2} = a$

$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = -a\sin t\,(-a\cos t) + a\cos t\,(-a\sin t) = 0$

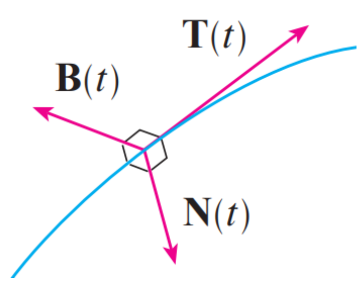
$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{|\mathbf{r}'(t)|^2|\mathbf{r}''(t)|^2 - \left(\mathbf{r}'(t) \cdot \mathbf{r}''(t)\right)^2}}{|\mathbf{r}'(t)|^3}$$

$$= \frac{\sqrt{a^2(a^2) - (0)^2}}{(a)^3} = \frac{a^2}{a^3} = \frac{1}{a}$$

Definition 13.3.10: Let $\mathbf{r}(t)$ be a vector function.

(1) The ***unit normal vector*** is defined by $\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$.

(2) The ***Binormal vector*** is defined by $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.



Example 13.3.11:

(1) Find the unit normal and binormal vectors of the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$$

(2) Find the unit normal and binormal vectors of the circular helix

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \text{ at the point } (1,0,2\pi)$$

Solution:

(1) $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{1}{\sqrt{2}}(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+\ \mathbf{k})=\langle-\frac{\sin t}{\sqrt{2}},\frac{\cos t}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle$$

$$\mathbf{T}'(t)=\frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i}-\sin t\,\mathbf{j})\ \Rightarrow\ |\mathbf{T}'(t)|=\sqrt{\frac{\cos^2 t}{2}+\frac{\sin^2 t}{2}}=\frac{1}{\sqrt{2}}$$

The unit normal is:

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\frac{\frac{1}{\sqrt{2}}(-\cos t\,\mathbf{i}-\sin t\,\mathbf{j})}{\frac{1}{\sqrt{2}}}=-\cos t\,\mathbf{i}-\sin t\,\mathbf{j}=\langle-\cos t,-\sin t,0\rangle$$

The Binormal vector is:

$$\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -\frac{\sin t}{\sqrt{2}} & \frac{\cos t}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\cos t & -\sin t & 0\end{vmatrix}=\mathbf{i}\left(0+\frac{\sin t}{\sqrt{2}}\right)-\mathbf{j}\left(0+\frac{\cos t}{\sqrt{2}}\right)+\mathbf{k}\left(\frac{\sin^2 t}{\sqrt{2}}+\frac{\cos^2 t}{\sqrt{2}}\right)$$

$$=\frac{\sin t}{\sqrt{2}}\mathbf{i}-\frac{\cos t}{\sqrt{2}}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}$$

(2) $\mathbf{r}(t)=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}+t\,\mathbf{k}\Rightarrow$ pt on carve is $(\cos t,\sin t,t)$

At the pt $(1,0,2\pi)$ we have $(1,0,2\pi)=(\cos t,\sin t,t)\Rightarrow t=2\pi$

$\mathbf{N}(t)=\langle-\cos t,-\sin t,0\rangle\Rightarrow\ \mathbf{N}(2\pi)=\langle-\cos 2\pi,-\sin 2\pi,0\rangle=\langle-1,0,0\rangle$

$\mathbf{B}(t)=\langle\frac{\sin t}{\sqrt{2}},-\frac{\cos t}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle\Rightarrow\mathbf{B}(2\pi)=\langle\frac{\sin 2\pi}{\sqrt{2}},-\frac{\cos 2\pi}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle=\langle 0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\rangle$

Example 13.3.12: Let $\mathbf{r}(t)$ be a smooth curve. Show that the unit tangent vector $\mathbf{T}(t)$ is orthogonal to unit normal vector $\mathbf{N}(t)$

Proof: $\mathbf{T}(t)$ is a unit vector $\Rightarrow|\mathbf{T}(t)|^2=1\Rightarrow\mathbf{T}(t)\cdot\mathbf{T}(t)=1$, for all t

$$\frac{d}{dt}\big(\mathbf{T}(t)\cdot\mathbf{T}(t)\big)=\frac{d}{dt}(1)\Rightarrow\mathbf{T}(t)\cdot\mathbf{T}'(t)+\mathbf{T}'(t)\cdot\mathbf{T}(t)=0$$

$$2\big(\mathbf{T}(t)\cdot\mathbf{T}'(t)\big)=0\Rightarrow\ \mathbf{T}(t)\cdot\mathbf{T}'(t)=0\ \ldots\ldots\ldots\text{①}$$

Divide ① by $|\mathbf{T}'(t)|$ we have: $\frac{1}{|\mathbf{T}'(t)|}\big(\mathbf{T}(t)\cdot\mathbf{T}'(t)\big)=0$

$$\Rightarrow\ \mathbf{T}(t)\cdot\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=0\ \Rightarrow\ \mathbf{T}(t)\cdot\mathbf{N}(t)=0\Rightarrow\ \mathbf{T}(t)\perp\mathbf{N}(t).$$

Exercise 13.3.13: Find the vectors $\mathbf{T},\mathbf{N},\mathbf{B}$ at the given point:

(1) $\mathbf{r}(t)=\langle t^2,\frac{2}{3}t^3,t\rangle,\ \left(1,\frac{2}{3},1\right)$

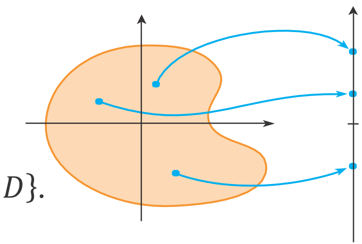
(2) $\mathbf{r}(t)=\langle\cos t,\sin t,\ln(\cos t)\rangle,\ (1,1,0).$

End of Chapter 13

Study Hard - Good Luck

Chapter 14: Partial Derivatives

Section 14.1: Functions of Several Variables



Definition 14.1.1: A **function f of two variables** is a rule that assigns to each ordered pair of real numbers x, y in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is,
$D = \{(x, y) \in \mathbb{R}^2: f(x, y) \in \mathbb{R}\}$ and Range $= \{z \in \mathbb{R}: z = f(x, y), (x, y) \in D\}$.

Example 14.1.2: Let $f(x, y) = x + \ln(y^2 - x)$. Then $f(3,2) = 3 + \ln(2^2 - 3) = 3 + \ln 1 = 3$

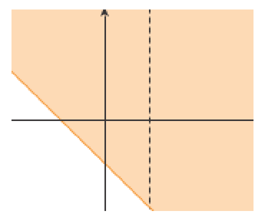
Example 14.1.3: Find and sketch the domain of the functions:

(1) $f(x, y) = \frac{\sqrt{x+y+1}}{x-1}$ (2) $f(x, y) = \ln(y^2 - x)$ (3) $f(x, y) = \sqrt{x^2 + y^2 - 9}$

(4) $f(x, y) = \sqrt{xy}$ (5) $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$

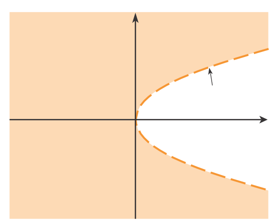
Solution:

(1) $\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2: x + y + 1 \geq 0, x \neq 1\}$



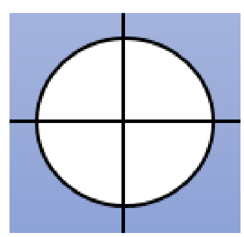
$x + y + 1 \geq 0, x \neq 1: x + y + 1 = 0$
$\Rightarrow x + y = -1$

(2) $\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2: y^2 - x > 0\}$
$= \{(x, y) \in \mathbb{R}^2: y^2 > x\}$

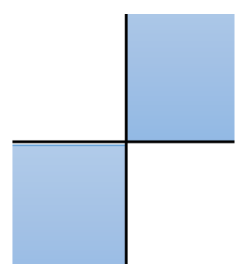


$y^2 > x \Rightarrow y^2 = x$

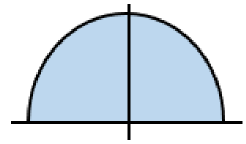
(3) $\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2: x^2 + y^2 - 9 \geq 0\}$
$= \{(x, y) \in \mathbb{R}^2: x^2 + y^2 \geq 9\}$



(4) $\text{Dom}(f) = \{(x, y) \in \mathbb{R}^2: xy \geq 0\}$
$xy \geq 0$:
$\Rightarrow$ $x \geq 0$ and $y \geq 0$ (first quadrant)
or $x \leq 0$ and $y \leq 0$ (third quadrant)



(5) $\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2 : y \geq 0, 25 - x^2 - y^2 \geq 0\}$
$= \{(x,y) \in \mathbb{R}^2 : y \geq 0, x^2 + y^2 \leq 25\}$



Example 14.1.4: Find the domain and range of the function $f(x,y) = 2 - 3\sqrt{9 - x^2 - y^2}$

Solution: $\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2 : 9 - x^2 - y^2 \geq 0\} = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$.

For range f: Let $z = f(x,y) \implies z = 2 - 3\sqrt{9 - x^2 - y^2}$. So,

$\sqrt{9 - x^2 - y^2} \geq 0 \implies 3\sqrt{9 - x^2 - y^2} \geq 0 \implies -3\sqrt{9 - x^2 - y^2} \leq 0$

$\implies 2 - 3\sqrt{9 - x^2 - y^2} \leq 2 \implies z \leq 2$

$x^2 + y^2 \geq 0 \implies -x^2 - y^2 \leq 0 \implies 9 - x^2 - y^2 \leq 9 \implies \sqrt{9 - x^2 - y^2} \leq \sqrt{9} = 3$

$\implies -3\sqrt{9 - x^2 - y^2} \geq -9 \implies 2 - 3\sqrt{9 - x^2 - y^2} \geq 2 - 9 \implies z \geq -7$

So, $-7 \leq z \leq 2 \implies \text{Range} = [-7,2]$

Example 14.1.5: Find the domain and range of the function:

$$f(x,y) = x^2 + 2y^2$$

Solution: $Dom(f) = \{(x,y) \in \mathbb{R}^2\} = \mathbb{R}^2$.

For range f: Let $z = f(x,y) \implies z = 2 + x^2 + 2y^2$. So,

$$x^2 + 2y^2 \geq 0 \implies 2 + x^2 + 2y^2 \geq 2 \implies z \geq 2 \implies \text{range}(f) = [2,\infty)$$

Definition 14.1.6: If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x,y,z) in $\mathbb{R}^3$ such that $z = f(x,y)$ and $(x,y) \in D$.

Example 14.1.7: Sketch the graph of the functions:

(1) $f(x,y) = 6 - 3x - 2y$ (2) $f(x,y) = \sqrt{9 - x^2 - y^2}$

(3) $f(x,y) = 6 - \sqrt{x^2 + 2y^2}$ (4) $f(x,y) = x^2 + 2y^2$

Solution:

(1) $z = f(x,y) \implies z = 6 - 3x - 2y \implies 3x + 2y + z = 6$ (is a plane)

Intercepts:

x-intercept: $y = z = 0$

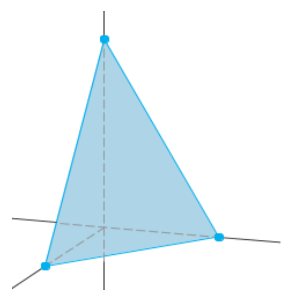
$\implies 3x + 2(0) + 0 = 6 \implies 3x = 6 \Rightarrow x = 2$

y-intercept: $x = z = 0$

$\implies 3(0) + 2y + 0 = 6 \implies 2y = 6 \Rightarrow y = 3$

z-intercept: $x = y = 0$

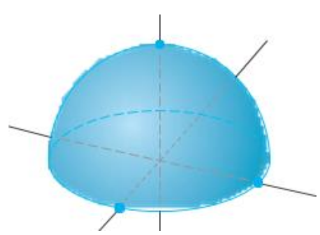
$\implies 3(0) + 2(0) + z = 6 \implies z = 6$



(2) $z = f(x,y) \implies z = \sqrt{9 - x^2 - y^2}$

$\implies z^2 = 9 - x^2 - y^2$ with $z \geq 0$

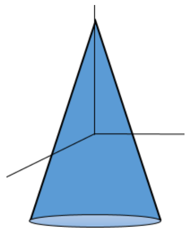
$\implies x^2 + y^2 + z^2 = 9$ with $z \geq 0$ (is a hemisphere)



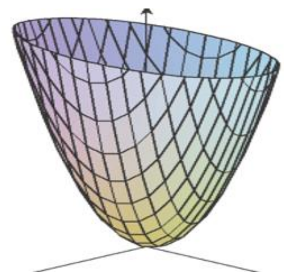
(3) $z = f(x, y) \Longrightarrow \quad z = 6 - \sqrt{x^2 + 2y^2}$

$\Longrightarrow \quad z - 6 = -\sqrt{x^2 + 2y^2}$

$\Longrightarrow \quad (z-6)^2 = x^2 + 2y^2$ (cone)

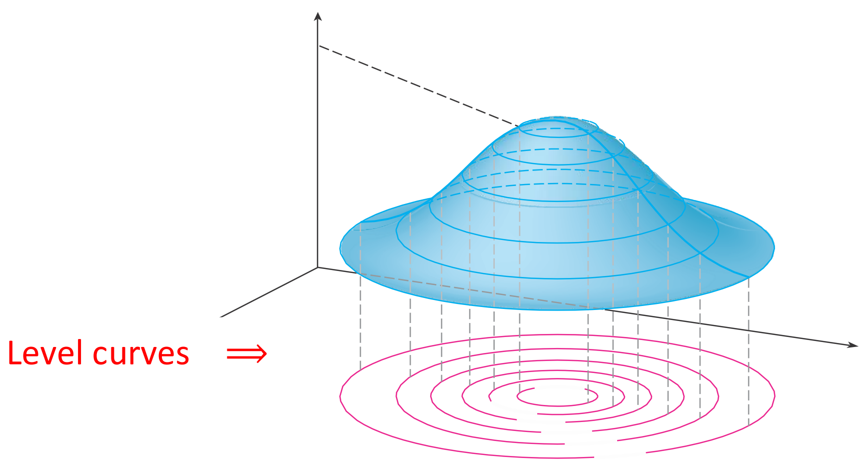


(4) $z = f(x, y) \Longrightarrow \quad z = x^2 + 2y^2$ (paraboloid)



Definition 14.1.8: The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is a constant ($k \in range(f)$).

- ❖ The level curves $f(x, y) = k$ are just the traces of the graph of f in the horizontal plane $z = k$ projected down to the xy-plane.



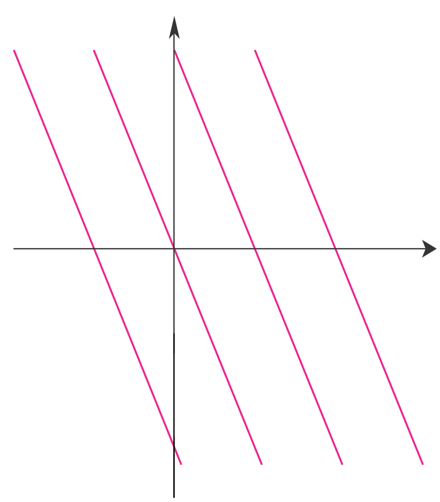
- ❖ The graph of several level curves in the plane is called a contour map of the function f

Example 14.1.9: Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for the values $k = -6, 0, 6,$

Solution: The level curves are:

$$6 - 3x - 2y = k \quad \Longrightarrow \quad 3x + 2y = 6 - k$$

- ➢ $k = -6 \Longrightarrow 3x + 2y = 12$ (line with slope $-\frac{3}{2}$)
- ➢ $k = 0 \Longrightarrow 3x + 2y = 6$ $\left(\text{line with slope } -\frac{3}{2}\right)$
- ➢ $k = 6 \Longrightarrow 3x + 2y = 0$ $\left(\text{line with slope } -\frac{3}{2}\right)$
- ➢ $k = 12 \Longrightarrow 3x + 2y = -6$ (line with slope $-\frac{3}{2}$)



Example 14.1.10: Sketch the level curves of the function $f(x,y)=\sqrt{9-x^2-y^2}$ for the values $k=0,1,3$

Solution: The level curves are: $f(x,y)=k$

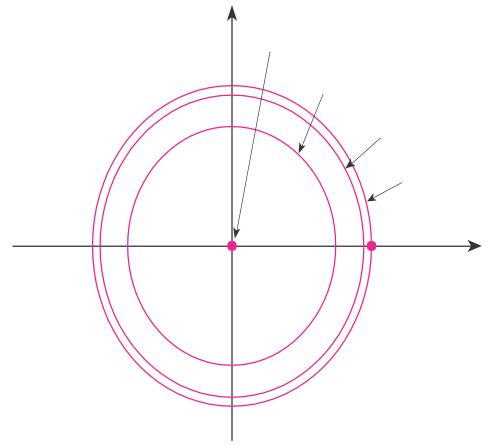
$\Rightarrow \sqrt{9-x^2-y^2}=k \quad \text{for } k=0,1,2,3$

$\Rightarrow 9-x^2-y^2=k^2 \quad \text{for } k=0,1,2,3$

$\Rightarrow x^2+y^2=9-k^2 \quad \text{for } k=0,1,2,3$

- $k=0 \implies x^2+y^2=9 \text{ (circle)}$
- $k=1 \implies x^2+y^2=8 \text{ (circle)}$
- $k=3 \implies x^2+y^2=0$

$\implies x=0, y=0 \implies$ A point $(0,0)$



Functions of Three or More Variables 14.1.11:

A **function** f **of three variables** is a rule that assigns to each ordered triple (x,y,z) in a domain D in $\mathbb{R}^3$ a unique real number denoted by $f(x,y,z)$.

Example 14.1.12: Find and sketch the domain of the function:

(a) $f(x,y,z)=\ln(z-y)+xy\sin z$.

(b) $f(x,y,z)=\sqrt{z-x^2-2y^2}$

Solution:

(a) $\text{Dom}(f)=\{(x,y,z)\in\mathbb{R}^3: z-y>0\}$

$=\{(x,y,z)\in\mathbb{R}^3: z>y\}$

To sketch $\text{Dom}(f)$:

$z>y$: $\Rightarrow$ $z=y$ (plane)

So, Dom(f) is a **half-space** consisting of all points that lie above the plane $z=y$.

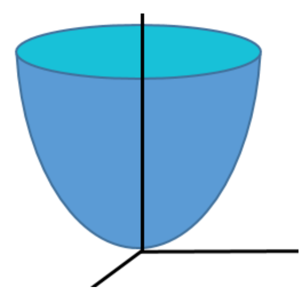
(b) $\text{Dom}(f)=\{(x,y,z)\in\mathbb{R}^3: z-x^2-2y^2\ge 0\}$

$=\{(x,y,z)\in\mathbb{R}^3: z\ge x^2+2y^2\}$

To sketch $\text{Dom}(f)$:

$z\ge x^2+2y^2$: $\Rightarrow$ $z=x^2+2y^2$(paraboloid)

So, $\text{Dom}(f)$ is the region inside and on the paraboloid $z=x^2+2y^2$



Example 14.1.13:

(a) $f(x,y,z)=\frac{1}{x} \Rightarrow \text{Dom}(f)=\{(x,y,z)\in\mathbb{R}^3: x\neq 0\}$

(b) $f(x,y)=\frac{1}{x} \Rightarrow \text{Dom}(f)=\{(x,y)\in\mathbb{R}^2: x\neq 0\}$

(c) $f(x,y,z)=2e^{xyz} \Rightarrow \text{Dom}(f)=\{(x,y,z)\in\mathbb{R}^3\}=\mathbb{R}^3$

(d) $f(x,y,z)=x+y \Rightarrow \text{Dom}(f)=\{(x,y,z)\in\mathbb{R}^3\}=\mathbb{R}^3$

(e) $f(x,y,z,w)=\sqrt{w-z} \Rightarrow \text{Dom}(f)=\{(x,y,z,w)\in\mathbb{R}^4: w-z\ge 0\}=\{(x,y,z,w)\in\mathbb{R}^4: w\ge z\}$

Example 14.1.14: Find the domain and range of the function $f(x,y,z) = 2+\sqrt{x^2+3}$

Solution:

$f(x,y,z) = 2+\sqrt{x^2+3} \quad \Rightarrow \text{ Dom}(f) = \{(x,y,z) \in \mathbb{R}^3\} = \mathbb{R}^3$

To find range(f): Let $w = f(x,y,z)$

$\Rightarrow \text{range}(f) = \{w \in \mathbb{R}: w = f(x,y,z), (x,y,z) \in \text{Dom}(f)\}$

- $\sqrt{x^2+3} \geq 0 \Rightarrow 2+\sqrt{x^2+3} \geq 2 \Rightarrow w \geq 2$ …….①
- $x^2 \geq 0 \Rightarrow x^2+3 \geq 3 \Rightarrow \sqrt{x^2+3} \geq \sqrt{3} \Rightarrow 2+\sqrt{x^2+3} \geq 2+\sqrt{3} \Rightarrow w \geq 2+\sqrt{3}$…….②
- Intersection of ① and ② $\Rightarrow w \in [2+\sqrt{3}, \infty) \Rightarrow \text{ range}(f) = [2+\sqrt{3}, \infty)$

Definition 14.1.15: The **level surfaces** of a function $f(x,y,z)$ for the value k are the surfaces given by the equation $f(x,y,z) = k$, where k is a constant, that is if the point (x,y,z) moves along a level surface, the value of $f(x,y,z)$ remains fixed.

Example 14.1.16: Find and sketch the level surfaces of the function $f(x,y,z) = x^2+y^2+z^2$.

Solution: Observe that $f(x,y,z) = x^2+y^2+z^2 \geq 0$

$\Rightarrow$ the values of k are $k \geq 0$ since for the level surfaces we have $f(x,y,z) = k$

For $k = 0$: $f(x,y,z) = 0 \Rightarrow x^2+y^2+z^2 = 0$

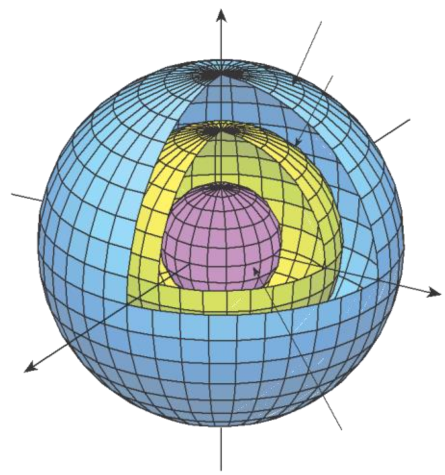
$\Rightarrow x = 0, y = 0, z = 0 \Rightarrow$ we have a point $(0,0,0)$

For $k = 1$: $f(x,y,z) = 1 \Rightarrow x^2+y^2+z^2 = 1$

$\Rightarrow$ A sphere of radius 1 centered at $(0,0,0)$

For $k > 0$: : $f(x,y,z) = k \Rightarrow x^2+y^2+z^2 = k$

$\Rightarrow$ A sphere of radius $\sqrt{k}$ centered at $(0,0,0)$



Section 14.2: Limits and Continuity

Definition 14.2.1: Let $C: x = f(t),\ y = g(t)$ be a path (curve) in the xy-plane. Then

C passes through the point $P_0(a,b)$ in $\mathbb{R}^2$	$\Leftrightarrow$ (يكافيء)	there exists $t_0 \in \mathbb{R}$ such $f(t_0) = a$ and $g(t_0) = b$.

Definition 14.2.2: Let $P_0(a,b)$ in $\mathbb{R}^2$ and let C be a path that pass through the point $P_0(a,b)$ when $t = t_0$. Then $\lim\limits_{\substack{(x,y)\to P_0 \\ along\ C}} F(x,y) = \lim\limits_{t\to t_0} F(f(t), g(t))$

Definition 14.2.3: Let $P_0(a,b)$ be a point in $\mathbb{R}^2$ and let $L \in \mathbb{R}$.

(1) $\lim\limits_{(x,y)\to P_0} F(x,y) = L$ (exists) $\Leftrightarrow \lim\limits_{\substack{(x,y)\to P_0 \\ along\ C}} F(x,y) = L$ for **all** paths C in Dom$(F(x,y))$ that pass through the point P_0.

لاثبات ان النهاية موجودة: علينا ان نأخذ **كل** المسارات paths (المنحنيات curves) المارة بالنقطة P_0 وحساب النهاية من خلالها، **وهذا مستحيل وذلك لأن عدد المسارات لانهائي**. لذلك إن كانت النهاية موجودة واردنا حسابها فإننا لا نستخدم المسارات في حسابها **بل نلجأ الى طرق أخرى مثل:**

التعويض المباشر	أو	التحليل والإختصار	أو	الضرب بالمرافق	أو	استخدام تعويضات خاصة

(2) Let C_1 and C_2 be two paths in $\text{Dom}(F(x,y))$ that pass through a point P_0. If

$$\lim_{\substack{(x,y)\to P_0 \\ along\ C_1}} F(x,y) \neq \lim_{\substack{(x,y)\to P_0 \\ along\ C_2}} F(x,y), \text{ then } \lim_{(x,y)\to P_0} F(x,y) \text{ dose not exist (DNE)}$$

لاثبات ان النهاية غير موجودة: علينا ان نجد **مسارين** كليهما مار بالنقطة P_0 وحساب النهاية من خلال كل واحد من المسارين بحيث يكون جوابا النهايتين **مختلفين**

Example 14.2.4: Find each of the following limit, if it exists:

(1) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4+x^2y^2-6y^4}{x^2+3y^2}$ (2) $\lim\limits_{(x,y)\to(-1,1)} \frac{y^6-x^2}{y^3+x}$

(3) $\lim\limits_{(x,y)\to(4,2)} \frac{x^2-5xy^2+4y^4}{\sqrt{x}-2y}$ (4) $\lim\limits_{(x,y)\to(4,1)} \frac{x^2-3xy^2-4y^4}{\sqrt{x}-2y}$

Solution:

(1) $\lim\limits_{(x,y)\to(0,0)} \frac{x^4+x^2y^2-6y}{x^2+3y^2} = \lim\limits_{(x,y)\to(0,0)} \frac{(x^2+3y^2)(x^2-2y^2)}{x^2+3y^2} = \lim\limits_{(x,y)\to(0,0)} (x^2-2y^2) = 0$

(2) $\lim\limits_{(x,y)\to(-1,1)} \frac{y^6-x^2}{y^3+x} = \lim\limits_{(x,y)\to(-1,1)} \frac{(y^3+x)(y^3-x)}{y^3+x} = \lim\limits_{(x,y)\to(-1,1)} (y^3-x) = 2$

(3) $\lim\limits_{(x,y)\to(4,2)} \frac{x^2-5xy^2+4y^4}{\sqrt{x}-2y} = \frac{(4)^2-5(4)(2)^2+4(2)^4}{\sqrt{4}-2(2)} = -24$

(4) $\lim\limits_{(x,y)\to(4,1)} \frac{x^2-3xy^2-4y^4}{\sqrt{x}-2y} = \lim\limits_{(x,y)\to(4,1)} \frac{x^2-3xy^2-4y^4}{\sqrt{x}-2y} \times \frac{\sqrt{x}+2y}{\sqrt{x}+2y} = \lim\limits_{(x,y)\to(4,1)} \frac{x^2-3xy^2-4y^4}{x-4y^2} \times 4$

$$= 4 \lim_{(x,y)\to(4,1)} \frac{(x-4y^2)(x+y^2)}{x-4y^2} = 4 \lim_{(x,y)\to(4,1)} (x+y^2) = 4(5) = 20$$

Remark 14.2.5: Recall that (1) $\lim\limits_{\theta\to0} \frac{\sin(\theta)}{\theta} = 1$ (2) $\lim\limits_{\theta\to0} \frac{\theta}{\sin(\theta)} = 1$ (3) $\lim\limits_{\theta\to0} \frac{\tan(\theta)}{\theta} = 1$ (4) $\lim\limits_{\theta\to0} \frac{\theta}{\tan(\theta)} = 1$

Example 14.2.6: Find each of the following limit, if it exists:

(1) $\lim\limits_{(x,y)\to(0,-3)} \frac{\sin(2xy^2)}{x}$ (2) $\lim\limits_{(x,y)\to(1,2)} \frac{4x^2-y^2}{\tan(4x-2y)}$

Solution:

(1) Let $\theta = 2xy^2$. When $(x,y)\to(0,-3)$ we have $x\to0$ and $y\to-3$.

Then $\theta\to0$. So, $\lim\limits_{(x,y)\to(0,-3)} \frac{\sin(2xy^2)}{x} = \lim\limits_{(x,y)\to(0,-3)} \frac{\sin(2xy^2)}{1} \times \frac{1}{x} = \lim\limits_{(x,y)\to(0,-3)} \frac{\sin(2xy^2)}{2xy^2} \times \frac{2xy^2}{x}$

$$= \lim_{(x,y)\to(0,-3)} \frac{\sin(2xy^2)}{2xy^2} \times \lim_{(x,y)\to(0,-3)} \frac{2xy^2}{x}$$

$$= \lim_{\theta\to0} \frac{\sin(\theta)}{\theta} \times \lim_{(x,y)\to(0,-3)} 2y^2 = 1 \times 2(-3)^2 = 18$$

(2) Let $\theta = 4x - 2y$. When $(x,y) \to (1,2)$ we have $x \to 1$ and $y \to 2 \Rightarrow \theta \to 0$. So,

$$\lim_{(x,y)\to(1,2)} \frac{4x^2 - y^2}{\tan(4x-2y)} = \lim_{(x,y)\to(1,2)} \frac{1}{\tan(4x-2y)} \times \frac{4x^2-y^2}{1}$$
$$= \lim_{(x,y)\to(1,2)} \frac{4x-2y}{\tan(4x-2y)} \times \frac{4x^2-y^2}{4x-2y}$$
$$= \lim_{(x,y)\to(1,2)} \frac{4x-2y}{\tan(4x-2y)} \times \lim_{(x,y)\to(1,2)} \frac{4x^2-y^2}{4x-2y}$$
$$= \lim_{(x,y)\to(1,2)} \frac{\theta}{\tan(\theta)} \times \lim_{(x,y)\to(1,2)} \frac{(2x-y)(2x+y)}{2(2x-y)}$$
$$= \lim_{\theta\to 0} \frac{\theta}{\tan(\theta)} \times \lim_{(x,y)\to(1,2)} \frac{(2x+y)}{2} = 1 \times \frac{4}{2} = 2$$

Remark 14.2.7: When $(x,y) \to (0,0)$ and we have the terms x^2+y^2 in the limit, we must think of using of the substitutions: $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then $x^2+y^2 = r^2$ and when $(x,y)\to(0,0)$ we have $x \to 0$ and $y \to 0$. So, $r \to 0^+$ and we have

$$\lim_{(x,y)\to(0,0)} F(x,y) = \lim_{r\to 0^+} F(r\cos(\theta), r\sin(\theta)), \text{ where } 0 \le \theta < 2\pi.$$

Example 14.2.8: Find the following limit, if it exists:

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{\sqrt{x^2+y^2+1}-1}$$

Solution: Let $x = r\cos(\theta)$ and $y = r\sin(\theta) \Rightarrow x^2+y^2 = r^2$ and $(x,y)\to(0,0)$ $\Rightarrow r \to 0^+$. So,

$$\lim_{(x,y)\to(0,0)} \frac{x^2y}{\sqrt{x^2+y^2+1}-1} = \lim_{r\to0^+} \frac{(r\cos(\theta))^2(r\sin(\theta))}{\sqrt{r^2+1}-1} = \lim_{r\to0^+} \frac{r^3\cos^2(\theta)\sin(\theta)}{\sqrt{r^2+1}-1} = \lim_{r\to0^+}\left(\frac{r^3}{\sqrt{r^2+1}-1} \times \cos^2(\theta)\sin(\theta)\right)$$
$$= \lim_{r\to0^+}\left(\frac{r^3}{\sqrt{r^2+1}-1} \times \frac{\sqrt{r^2+1}+1}{\sqrt{r^2+1}+1} \times \cos^2(\theta)\sin(\theta)\right)$$
$$= \lim_{r\to0^+}\left(\frac{r^3}{\sqrt{r^2+1}-1} \times \frac{\sqrt{r^2+1}+1}{\sqrt{r^2+1}+1} \times \cos^2(\theta)\sin(\theta)\right)$$
$$= \lim_{r\to0^+}\left(\frac{r^3}{(r^2+1)-1} \times \frac{\sqrt{r^2+1}+1}{1} \times \cos^2(\theta)\sin(\theta)\right)$$
$$= \lim_{r\to0^+}\left(\frac{r^3}{r^2} \times \cos^2(\theta)\sin(\theta)\right) \times \lim_{r\to0^+}\left(\left(\sqrt{r^2+1}+1\right)\right)$$
$$= \lim_{r\to0^+}\left(r\cos^2(\theta)\sin(\theta)\right) \times 2$$
$$= 0 \times 2 = 0$$

Observe that:
$-1 \le \cos^2(\theta)\sin(\theta) \le 1$
$\Rightarrow -r \le r\cos^2(\theta)\sin(\theta) \le r$
$\Rightarrow \lim_{r\to0^+}(-r) = 0$ and $\lim_{r\to0^+} r = 0$
So, by the squeeze theorem:
$\lim_{r\to0^+} r\cos^2(\theta)\sin(\theta)) = 0$

Example 14.2.9: Find the following limit, if it exists $\lim\limits_{(x,y)\to(1,1)} \dfrac{(6y-4x-1)^5-1}{(2x-3y)^8-1}$

Solution: Observe that $\lim\limits_{(x,y)\to(1,1)} \frac{(6y-4x-1)^5-1}{(2x-3y)^8-1} = \lim\limits_{(x,y)\to(1,1)} \frac{(-2(2x-3y)-1)^5-1}{(2x-3y)^8-1}$

So, let $\theta = 2x-3y$. When $(x,y)\to(1,1)$. Then $x\to1$ and $y\to1$.

So, $\theta\to-1$. Then $\lim\limits_{(x,y)\to(1,1)} \frac{(6y-4x-1)^5-1}{(2x-3y)^8-1} = \lim\limits_{\theta\to-1} \frac{(-2\theta-1)^5-1}{\theta^8-1} = \lim\limits_{\theta\to-1} \frac{5(-2\theta-1)^4(-2)}{8\theta^7} = \frac{5(1)^4(-2)}{8(-1)^7} = \frac{5}{4}$

Remark 14.2.10: Recall the following:

- Let $C: x=f(t),\ y=g(t)$ be a path (curve) in the xy-plane. Then

C passes through the point $P_0(a,b)$ in $\mathbb{R}^2$	$\Leftrightarrow$ (يكافيء)	there exists $t_0\in\mathbb{R}$ such $f(t_0)=a$ and $g(t_0)=b$.

- Let $P_0(a,b)$ in $\mathbb{R}^2$ and let C be a path that pass through the point $P_0(a,b)$ when $t=t_0$. Then $\lim\limits_{\substack{(x,y)\to P_0 \\ along\ C}} F(x,y) = \lim\limits_{t\to t_0} F(f(t),g(t))$

- Let C_1 and C_2 be two curves that pass through a point $P_0(a,b)$. If $\lim\limits_{\substack{(x,y)\to P_0 \\ along\ C_1}} F(x,y) \neq \lim\limits_{\substack{(x,y)\to P_0 \\ along\ C_2}} F(x,y)$, then $\lim\limits_{(x,y)\to P_0} F(x,y)$ dose not exist (DNE)

لاثبات ان النهاية غير موجودة: علينا ان نجد **مسارين** (C_1) و (C_2) كليهما مار بالنقطة P_0 وحساب النهاية من خلال كل واحد من المسارين بحيث يكون جوابا النهايتين **مختلفين**.

$$\lim\limits_{\substack{(x,y)\to P_0 \\ \text{along } C_1}} F(x,y) \neq \lim\limits_{\substack{(x,y)\to P_0 \\ \text{along } C_2}} F(x,y)$$

Example 14.2.11:

(1) Find $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ along the path $C_1: y=2x-4$

(2) Find $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ along the path $C_2: x=3t, y=1-9t$.

(3) Is $\lim\limits_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ exists? Justify.

Solution:

(1) $\lim\limits_{\substack{(x,y)\to(1,-2) \\ \text{along } C_1}} \frac{x^2+3y+5}{2x+y} = \lim\limits_{x\to1} \frac{x^2+3(2x-4)+5}{2x+(2x-4)} = \lim\limits_{x\to1} \frac{x^2+6x-7}{4x-4} = \lim\limits_{x\to1} \frac{(x-1)(x+7)}{4(x-1)} = \lim\limits_{x\to1} \frac{(x+7)}{4} = 2$

(2) We must find the value of t_0 when the point $(1,-2)$ is on C_2:

The point $(1,-2)$: $x=1,\quad y=-2$

On C_2: $x=3t_0,\ y=1-9t_0$

x-coordinate on path $=$ x-coordinate in point $\Rightarrow$ $3t_0=1 \ \Rightarrow\ t_0=\frac{1}{3}$

Observe that we can find t_0 from y component: y (on path) $= y$ (in point) $\Rightarrow 1-9t_0=-2 \Rightarrow t_0=\frac{1}{3}$	We have the same value for t_0

$$\lim_{\substack{(x,y)\to(1,-2)\\ \text{along } C_2}} \frac{x^2+3y+5}{2x+y} = \lim_{t\to\frac{1}{3}} \frac{(3t)^2+3(1-9t)+5}{2(3t)+(1-9t)} = \lim_{t\to\frac{1}{3}} \frac{9t^2-27t+8}{1-3t} == \lim_{t\to\frac{1}{3}} \frac{18t-27}{-3} = 7$$

(3) $\lim_{(x,y)\to(1,-2)} \frac{x^2+3y+5}{2x+y}$ DNE (does not exist), because: $\lim_{\substack{(x,y)\to(1,-2)\\ along\ C_1}} \frac{x^2+3y+5}{2x+y} \neq \lim_{\substack{(x,y)\to(1,-2)\\ along\ C_2}} \frac{x^2+3y+5}{2x+y}$

Example 14.2.12: Find the limit, if it exists $\lim_{\substack{(x,y)\to(1,0)\\ \text{along } y=\ln(x)}} \frac{x\sin(e^y-1)}{x^2+y^2-x}$

Solution:

$$\lim_{\substack{(x,y)\to(1,0)\\ \text{along } y=\ln(x)}} \frac{x\sin(e^y-1)}{x^2+y^2-x} = \lim_{x\to1} \frac{x\sin(e^{\ln(x)}-1)}{x^2+(\ln(x))^2-x} = \lim_{x\to1} \frac{x\sin(x-1)}{x^2+(\ln(x))^2-x} = \frac{0}{0}$$

$$= \lim_{x\to1} \frac{x\cos(x-1)+\sin(x-1)}{2x+\frac{2\ln(x)}{x}-1} \frac{1\cos0+\sin0}{2+\frac{2\ln(1)}{1}-1} = 1$$

Example 14.2.13: Find the limit, if it exists: $\lim_{(x,y)\to(2,-1)} \frac{x^2-2x-y^2-2y-1}{(x-2)^2+(y+1)^2}$

Solution:

- $C_1: x=t+2, y=0+(-1) \Rightarrow \quad C_1:\ x=t+2,\ y=-1$. So, $(x,y)\to(2,-1) \ \Rightarrow\ t\to0$

$$\lim_{\substack{(x,y)\to(2,-1)\\ \text{along } C_1}} \frac{x^2-2x-y^2-2y-1}{(x-2)^2+(y+1)^2} = \lim_{t\to0} \frac{(t+2)^2-2(t+2)-(-1)^2-2(-1)-1}{(t+2-2)^2+(-1+1)^2} = \lim_{t\to0} \frac{t^2+4t+4-2t-4-1+2-1}{t^2}$$

$$= \lim_{t\to0} \frac{t^2+2t}{t^2} = \lim_{t\to0} \frac{t(t+2)}{t^2} = \lim_{t\to0} \frac{t+2}{t} = \frac{2}{0} \ldots\ldots (1)$$

So, $\lim_{(x,y)\to(2,-1)} \frac{x^2-2x-y^2-2y-1}{(x-2)^2+(y+1)^2}$ Does not exist

Example 14.2.14: Find the following limit, if it exists $\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$

Solution:

- $C_1: x=0+0, y=t+0 \Rightarrow \quad C_1:\ x=0,\ y=t$. So, $(x,y)\to(0,0) \ \Rightarrow\ t\to0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \frac{x^3y^2}{x^6+y^4} == \lim_{t\to0} \frac{0^3t^2}{0^6+t^4} = \lim_{t\to0} \frac{0}{t^4} = \lim_{t\to0} 0 = 0 \ldots\ldots (1)$$

- $C_2:\ x=t^{\frac{12}{6}}+0,\ \ y=t^{\frac{12}{4}}+0 \Rightarrow \quad C_2:\ x=t^2,\ y=t^3$. So, $(x,y)\to(0,0) \ \Rightarrow\ t\to0$:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} \frac{x^3y^2}{x^6+y^4} = \lim_{t\to0} \frac{(t^2)^3(t^3)^2}{(t^2)^6+(t^3)^4} = \lim_{t\to0} \frac{t^6t^6}{t^{12}+t^{12}} = \lim_{t\to0} \frac{t^{12}}{2t^{12}} = \frac{1}{2} \ldots\ldots (2)$$

- (1)&(2) $\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \frac{x^3y^2}{x^6+y^4} \neq \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} \frac{x^3y^2}{x^6+y^4}$. So, $\lim_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$ Does not exist

Example 14.2.15: Find the following limit, if it exists: $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{x^2y^2}{x^6-y^3+2z^4}$

Solution:

- $C_1: x = t + 0, y = t + 0, z = 0 + 0 \Rightarrow \quad C_1: \; x = t, \; y = 0, z = 0.$
  So, $(x,y,z) \to (0,0,0) \;\Rightarrow\; t \to 0$:
  $$\lim_{\substack{(x,y,z)\to(0,0,0)\\ \text{along } C_1}} \frac{x^2y^2}{x^6-y^3+2z^4} = \lim_{t\to 0}\frac{t^2 0^2}{t^6-0^3+2(0^4)} = = \lim_{t\to 0}\frac{0}{t^6} = \lim_{t\to 0} 0 = 0 \ldots\ldots (1)$$

- $C_2: \; x = t^{\frac{12}{6}} + 0, \;\; y = t^{\frac{12}{3}} + 0, x = t^{\frac{12}{4}} + 0 \Rightarrow \quad C_2: \; x = t^2, \; y = t^4, z = t^3.$
  So, $(x,y,z) \to (0,0,0) \;\Rightarrow\; t \to 0$:
  $$\lim_{\substack{(x,y,z)\to(0,0,0)\\ along\ C_2}} \frac{x^2y^2}{x^6-y^3+2z^4} = = \lim_{t\to 0}\frac{(t^2)^2(t^4)^2}{(t^2)^6-(t^4)^3+2(t^3)^4} = \lim_{t\to 0}\frac{t^4t^8}{t^{12}-t^{12}+2t^{12}} = \lim_{t\to 0}\frac{t^{12}}{2t^{12}} = \lim_{t\to 0}\frac{1}{2} = \frac{1}{2} \ldots. (2)$$

- $(1) \& (2) \Rightarrow \lim\limits_{\substack{(x,y,z)\to(0,0,0)\\ \text{along } C_1}} \frac{x^2y^2}{x^6-y^3+2z^4} \neq \lim\limits_{\substack{(x,y,z)\to(0,0,0)\\ \text{along } C_2}} \frac{x^2y^2}{x^6-y^3+2z^4}.$

  So, $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y^2}{x^6+y^4}$ Does not exist

Definition 14.2.16:

(1) A function $f(x,y)$ is said to be continuous at a point (a,b) in $\text{Dom}(f)$ if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

(2) A function $f(x,y)$ is said to be continuous on a set $S \subseteq Dom(f)$ if $f(x,y)$ is continuous at every point in S.

(3) A function $f(x,y)$ is said to be continuous everywhere if it is continuous on $\mathbb{R}^2$

(4) A function $f(x,y,y)$ is said to be continuous everywhere if it is continuous on $\mathbb{R}^3$.

Example 14.2.17:

(1) $f(x,y) = \frac{x^4+x^2y-5}{y^2+1}$ is continuous on $\text{Dom}(f) = \mathbb{R}^2 \;\Rightarrow\; f$ is continuous everywhere.

(2) $f(x,y) = \frac{ye^x-5}{x^2+y^2}$ is continuous on $\text{Dom}(f) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \neq 0\}$

$\Rightarrow f$ is continuous on $\text{Dom}(f) = \mathbb{R}^2 \backslash \{(0,0)\}$.

(3) $f(x,y,z) = \frac{z\ln(y)-5x}{x-2y-z}$ is continuous on $\text{Dom}(f) = \{(x,y,z) \in \mathbb{R}^3 : x - 2y - z \neq 0, \; y > 0\}$

(4) $f(x,y,z) = 2$ is continuous on $\text{Dom}(f) = \mathbb{R}^3 \;\Rightarrow\; f$ is continuous everywhere.

Example 14.2.18: Find the region on which the function f is continuous.

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x^2+y^2} & , \quad (x,y)\neq(0,0)\\ 0 & , \quad (x,y)=(0,0)\end{cases}$$

Solution: When $(x,y)\neq(0,0)$ the function f is continuous since $f(x,y)=\frac{x^2-y^2}{x^2+y^2}$

To check whether the function f is continuous at $(0,0)$ or not we must study:

(1) Is $\lim_{(x,y)\to(0,0)} f(x,y)$ exists or not?

(2) If $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, is $\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)$?

- $C_1: x=t+0, y=0+0 \Rightarrow x=t, y=0.\ (x,y)\to(0,0) \ \Rightarrow \ t\to 0$

  $$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} f(x,y)=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_1}} \frac{x^2-y^2}{x^2+y^2}=\lim_{t\to 0}\frac{t^2-0^2}{t^2+0^2}=1 \ \ldots\ldots ①$$

- $C_2: x=0+0, y=t+0 \Rightarrow x=t, y=t.\ (x,y)\to(0,0) \ \Rightarrow \ t\to 0$

  $$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} f(x,y)=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C_2}} \frac{x^2-y^2}{x^2+y^2}=\lim_{t\to 0}\frac{0^2-t^2}{0^2+t^2}=-1 \ \ldots\ldots ②$$

- So, $\lim_{\substack{(x,y)\to(0,0)\\ along\ C_1}} f(x,y)\neq \lim_{\substack{(x,y)\to(0,0)\\ along\ C_2}} f(x,y) \quad \Rightarrow \quad \lim_{(x,y)\to(0,0)} f(x,y)$ Does not exist

$\Rightarrow f(x,y)$ is discontinuous at $(0,0)$

$\Rightarrow f(x,y)$ is continuous only on $\mathbb{R}^2\backslash\{(0,0)\}$

Remark 14.2.19:

(1) Recall that:

$$\boxed{\lim_{(x,y)\to(a,b)} f(x,y)=L \text{ (exists)}} \Leftrightarrow \boxed{\left.\lim_{\substack{(x,y)\to(a,b)\\ \text{along } C}} f(x,y)=L\right\} \text{ for **all** paths } C \text{ that pass through the point } (a,b)}$$

(2) If a function $f(x,y)$ is continuous at a point (a,b), then

$$\lim_{(x,y)\to(a,b)} f(x,y)=f(a,b) \text{ exists}$$

$\Rightarrow$ If C is a given path that passes through the point (a,b), then

$$\lim_{\substack{(x,y)\to(a,b)\\ \text{along } C}} F(x,y)=f(a,b)$$

Example 14.2.20: Find the value of k such that the function f is continuous at the origin, where

$$f(x,y)=\begin{cases}\dfrac{1-\cos\left(\sqrt{x^2+y^2}\right)}{x^2+y^2} & , \quad (x,y)\neq(0,0)\\ k & , \quad (x,y)=(0,0)\end{cases}$$

Solution: f is continuous at the origin $\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)=f(0,0)=k$ (the limit exists)

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} f(x,y)=k\ldots\ldots①$$

where C is any path in $\mathrm{Dom}(f)$ passing through $(0,0)$.

So, take $C: x=t+0, y=0+0$ with $t>0$ we have $(x,y)\to(0,0) \Rightarrow t\to 0$:

$$①\Rightarrow k=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} f(x,y)=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}}\frac{1-\cos\left(\sqrt{x^2+y^2}\right)}{x^2+y^2}=\lim_{t\to0}\frac{1-\cos\left(\sqrt{t^2+0^2}\right)}{t^2+0^2}=\lim_{t\to0}\frac{1-\cos(t)}{t^2}$$

$$=\lim_{t\to0}\frac{\sin(t)}{2t}=\frac{1}{2} \quad \Rightarrow k=\frac{1}{2}.$$

Example 14.2.21: Find the value of k such that the function f is continuous everywhere, where

$$f(x,y)=\begin{cases}\frac{kx^2-2y^2}{x^2+y^2} & , \quad (x,y)\neq(0,0)\\ -2 & , \quad (x,y)=(0,0)\end{cases}$$

Solution: f is continuous everywhere $\Rightarrow f$ is continuous at the origin

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)=-2 \text{ (the limit exists)}$$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} f(x,y)=-2\ldots\ldots①$$

where C is any path in $\mathrm{Dom}(f)$ passing through $(0,0)$.

So, take $C: x=t+0, y=0+0$ with $t>0 \Rightarrow x=t, y=0$. We have $(x,y)\to(0,0) \Rightarrow t\to 0$

$$①\Rightarrow -2=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} f(x,y)=\lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}}\frac{kx^2-2y^2}{x^2+y^2}=\lim_{t\to0}\frac{kt^2-2(0^2)}{t^2+0^2}=k\Rightarrow k=-2.$$

Example 14.2.22: Find the value of a such that the function f is continuous at the point $(0,2)$, where

$$f(x,y)=\begin{cases}\dfrac{\sin(xy)}{x} & , \quad x\neq0\\ a & , \quad x=0\end{cases}$$

Solution: f is continuous at the point $(0,2) \Rightarrow \lim\limits_{(x,y)\to(0,2)} f(x,y)=f(0,2)=a$

$$\Rightarrow \lim_{\substack{(x,y)\to(0,0)\\ \text{along } C}} f(x,y)=a\ldots\ldots①$$

where C is any path in $\mathrm{Dom}(f)$ passing through the point $(0,2)$.

So, take $C: x=t+0, y=0+0$ with $t>0 \Rightarrow x=t, y=0$. We have $(x,y)\to(0,0) \Rightarrow t\to 0$

$$①\Rightarrow a=\lim_{\substack{(x,y)\to(0,2)\\ \text{along } C}} f(x,y)=\lim_{\substack{(x,y)\to(0,2)\\ \text{along } C}}\frac{\sin(xy)}{x}=\lim_{t\to0}\frac{\sin(2t)}{t}=2\Rightarrow a=2.$$

Example 14.2.23: Find the value of k such that the function f is continuous at the point $(1,1)$, where

$$f(x,y)=\begin{cases}\dfrac{\sqrt{xy+8}-3}{xy-1} & , \quad xy\neq 1\\ k & , \quad xy=1\end{cases}$$

Solution: f is continuous at the point $(1,1)$ $\Rightarrow \lim_{(x,y)\to(1,1)} f(x,y)=f(1,1)=k$

$$\Rightarrow \lim_{\substack{(x,y)\to(1,1)\\ \text{along } C}} f(x,y)=k\ldots\ldots ①$$

where C is any path in $\mathrm{Dom}(f)$ passing through the point $(1,1)$.

So, take $C: x=t+1, y=0+1$ with $t>0 \Rightarrow x=t+1, y=01$.

We have $(x,y)\to(1,1) \;\Rightarrow\; t\to 0$

$$① \Rightarrow k=\lim_{\substack{(x,y)\to(1,1)\\ \text{along } C}} f(x,y)=\lim_{\substack{(x,y)\to(1,1)\\ \text{along } C}} \frac{\sqrt{xy+8}-3}{xy-1}=\lim_{t\to0}\frac{\sqrt{(t+1)+8}-3}{(t+1)-1}==\lim_{t\to0}\frac{\sqrt{t+9}-3}{t}\times\frac{\sqrt{t+9}+3}{\sqrt{t+9}+3}$$

$$=\lim_{t\to0}\frac{(t+9)-9}{t}\times\frac{1}{\sqrt{t+9}+3}=\lim_{t\to0}\frac{1}{\sqrt{t+9}+3}=\frac{1}{6}\Rightarrow k=\frac{1}{6}.$$

Section 14.3: Partial Derivative

Definition 14.3.1: The **partial derivative of f:**

(a) **with respect to x** at a point (a,b) written as $f_x(a,b)$ is defined by

$$f_x(a,b)=\lim_{x\to a}\frac{f(x,b)-f(a,b)}{x-a}=\lim_{h\to0}\frac{f(a+h,b)-f(a,b)}{h}.$$

(b) **with respect to y** at a point (a,b) written as $f_y(a,b)$ is defined by

$$f_y(a,b)=\lim_{y\to b}\frac{f(a,y)-f(a,b)}{y-b}=\lim_{h\to0}\frac{f(a,b+h)-f(a,b)}{h}$$

Remark 14.3.2: (a) $f_x(a,b)=g'(a)$, where $g(x)=f(x,b)$.
(b) $f_y(a,b)=h'(b)$, where $h(y)=f(a,y)$.

Example 14.3.3: Find $f_x(1,0)$ and $f_y(1,0)$, where $f(x,y)=\sqrt{x^4+y^3+3}$

Solution:

- $f_x(1,0)$: Let $g(x)=f(x,0)=\sqrt{x^4+0^3+3}=\sqrt{x^4+3}$

  $$g'(x)=\frac{4x^3}{2\sqrt{x^4+3}}\Rightarrow g'(1)=1\Rightarrow f_x(1,0)=g'(1)=1$$

- $f_y(1,0)$: Let $h(y)=f(1,y)=\sqrt{1^4+y^3+3}=\sqrt{y^3+4}$

  $$h'(y)=\frac{3y^2}{2\sqrt{y^3+4}}\Rightarrow h'(0)=0\Rightarrow f_y(1,0)=h'(0)=0$$

Example 14.3.4: Find $f_x(0,0)$, where $f(x,y) = 3x + \sqrt[3]{8x^3 + 27y^6}$

Solution: Let $g(x) = f(x,0) = 3x + \sqrt[3]{8x^3} = 5x \Rightarrow g'(x) = 5 \Rightarrow g'(0) = 5$

$\Rightarrow f_x(0,0) = g'(0) = 5$

Example 14.3.5: Find $f_x(0,0)$ if it exists, where $f(x,y) = \sqrt{x^2 + y^2}$

Solution:

$$f(x,0) = \sqrt{x^2 + (0)^2} = \sqrt{x^2} = |x|$$

Since $|x|$ is not differentiable at $x = 0$, then $f_x(0,0)$ does not exists

That is: the partial derivative of f with respect to x does not exists at $(0,0)$.

Example 14.3.6: Find $f_y(0,0)$, where $f(x,y) = \begin{cases} \frac{3x^2+xy+y^3}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 0 & , \quad (x,y) = (0,0) \end{cases}$

Solution:

- **First Method:** $f(0,y) = \left\{ \begin{matrix} \frac{3(0)^2+(0)y+y^3}{(0)^2+y^2} & , & y \neq 0 \\ 0 & , & y = 0 \end{matrix} \right\} = \left\{ \begin{matrix} y & , & y \neq 0 \\ 0 & , & y = 0 \end{matrix} \right\} = y$

  $\Rightarrow f(0,y) = y \Rightarrow h(y) = f(0,y) \Rightarrow h(y) = y \Rightarrow f_y(0,0) = h'(0) = 1$

- **Second Method:** By definition:

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{\frac{3(0)^2 + (0)y + y^3}{(0)^2 + y^2} - 0}{y - 0}$$

$$= \lim_{y\to 0} \frac{\left(\frac{y^3}{y^2}\right)}{y} = \lim_{y\to 0} \frac{y^3}{y^3} = 1 \Rightarrow f_y(0,0) = 1$$

Example 14.3.7: Find $f_y(0,0)$, where $f(x,y) = \begin{cases} \frac{3x^2+xy+y^3}{x^2+y^2} & , \quad (x,y) \neq (0,0) \\ 1 & , \quad (x,y) = (0,0) \end{cases}$

Solution:

- **First Method:** $f(0,y) = \left\{ \begin{matrix} \frac{3(0)^2+(0)y+y^3}{(0)^2+y^2} & , & y \neq 0 \\ 1 & , & y = 0 \end{matrix} \right\} = \left\{ \begin{matrix} y & , & y \neq 0 \\ 1 & , & y = 0 \end{matrix} \right\}$

  Observe that $f(0,y)$ is discontinuous at $y = 0 \Rightarrow f_y(0,0)$ does not exist

- **Second Method:** By definition:

$$f_y(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{\frac{3(0)^2 + (0)y + y^3}{(0)^2 + y^2} - 1}{y - 0}$$

$$= \lim_{y\to 0} \frac{\left(\frac{y^3}{y^2}\right) - 1}{y} = \lim_{y\to 0} \frac{y-1}{y} = \frac{-1}{0} \Rightarrow f_y(0,0) \text{ does not exist}$$

Example 14.3.8: Find $f_x(x,y)$ and $f_y(x,y)$, where $f(x,y) = xy^4e^{3x} + \cos(2y)$

Solution:

- $f_x(x,y) = xy^4(3e^{3x}) + e^{3x}(y^4) + 0 = 3xy^4e^{3x} + y^4e^{3x}$
- $f_y(x,y) = xe^{3x}(4y^3) + \left(-\sin(2y)\,(2)\right) = 4xy^3e^{3x} - 2\sin(2y)$

Example 14.3.9: Find $f_x(1,0)$ and $f_y(1,-1)$, where $f(x,y)=\begin{cases}\frac{3x^3}{x^2+y^2} & , \quad (x,y)\neq(0,0)\\ 0 & , \quad (x,y)=(0,0)\end{cases}$

Solution: At $(1,0)$ and $(1,-1)$, the function $f(x,y)=\frac{3x^3}{x^2+y^2}$. So,

$$f_x(x,y)=\frac{(x^2+y^2)9x^2-3x^3(2x)}{(x^2+y^2)^2}=\frac{3x^4+9x^2y^2}{(x^2+y^2)^2} \quad \Rightarrow f_x(1,0)=3$$

$$f_y(x,y)=\frac{(x^2+y^2)(0)-3x^3(2y)}{(x^2+y^2)^2}=\frac{-6x^3y}{(x^2+y^2)^2} \quad \Rightarrow f_y(1,-1)=\frac{6}{4}$$

Example 14.3.10: Find $f_x(x,y)$, where $f(x,y)=\begin{cases}\frac{3x^3+xy}{x^2+y^2} & , \quad (x,y)\neq(0,0)\\ 0 & , \quad (x,y)=(0,0)\end{cases}$

Solution:

- If $(x,y)\neq(0,0)$: $\Rightarrow f(x,y)=\frac{3x^3+xy}{x^2+y^2} \Rightarrow f_x=\frac{(x^2+y^2)(9x^2+y)-(3x^3+xy)(2x)}{(x^2+y^2)^2}$

  $\Rightarrow f_x=\frac{3x^4+9x^2y^2-2x^2y}{(x^2+y^2)^2}$

- Finding $f_x(0,0)$: Let $g(x)=f(x,0)=\left\{\begin{matrix}\frac{3x^3+x(0)}{x^2+(0)^2} & , & x\neq 0\\ 0 & , & x=0\end{matrix}\right\}=\left\{\begin{matrix}3x & , & x\neq 0\\ 0 & , & x=0\end{matrix}\right\}=3x$

  $\Rightarrow f_x(0,0)=g'(0)=3$

- So, $f_x(x,y)=\begin{cases}\frac{3x^4+9x^2y^2-2x^2y}{(x^2+y^2)^2} & , \quad (x,y)\neq(0,0)\\ 3 & , \quad (x,y)=(0,0)\end{cases}$

Example 14.3.11: Find $\lim\limits_{h\to 0}\frac{f(1+h,-1)-f(1,-1)}{h}$, where $f(x,y)=\sqrt[5]{x^7-y^2+1}$

Solution: By the definition of the partial derivatives, we have

$\lim\limits_{h\to 0}\frac{f(1+h,-1)-f(1,-1)}{h}=f_x(1,-1)$ But $f_x=\frac{1}{5}(x^7-y^2+1)^{-\frac{4}{5}}(7x^6)$

Example 14.3.12: Let $f(x,y)$ be a function such that $f_x=2xy$, $f_y=x^2+2y$, and $f(1,1)=8$. Find $f(x,y)$ and $f(0,2)$.

Solution: $f_x=2xy \Rightarrow f(x,y)=\int 2xy\,dx+G(y)\Rightarrow f(x,y)=x^2y+G(y)$.....①

Now, we find $G(y)$:

Differentiating equation ① with respect to y:

$f_y=x^2+G'(y)$ but $f_y=x^2+2y \Rightarrow x^2+G'(y)=x^2+2y \Rightarrow G'(y)=2y$

$G(y)=\int 2y dy=y^2+C$, where C is a constant.

① $\Rightarrow f(x,y)=x^2y+G(y)=x^2y+y^2+C \Rightarrow f(x,y)=x^2y+y^2+C$.......②

Now, we find C:

$f(1,1)=8 \Rightarrow 1^2(1)+1^2+C=8 \Rightarrow C=6.$

② $\Rightarrow f(x,y)=x^2y+y^2+6$ and so $f(0,2)=(0)^2(2)+(2)^2+6=10$

Example 14.3.13: Find $f_z(-1,1,e^2)$, where $f(x,y,z) = e^{2xy}\ln(z)$.

Solution: $f_z = \frac{e^{2xy}}{z} \Rightarrow f_z(-1,1,e^2) = \frac{e^{2(-1)(1)}}{e^2} = \frac{e^{-2}}{e^2} = e^{-4}$

Notations 14.3.14: There are several forms of partial derivatives of $z = f(x,y)$:

$$f_x(x,y) = f_x = z_x = \frac{\partial}{\partial x}f(x,y) = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = D_x f = D_1 f$$

and

$$f_y(x,y) = f_y = z_y = \frac{\partial}{\partial y}f(x,y) = \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = D_y f = D_2 f$$

Example 14.3.15: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(\pi, \sqrt{3})$, where $f(x,y) = \sin\left(\frac{x}{y^2+1}\right)$

Solution:

$$\frac{\partial f}{\partial x} = \cos\left(\frac{x}{y^2+1}\right)\left(\frac{1}{y^2+1}\right) \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(\pi,\sqrt{3})} = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\pi \\ y=\sqrt{3}}} = \cos\left(\frac{\pi}{\sqrt{3}^2+1}\right)\left(\frac{1}{\sqrt{3}^2+1}\right) = \frac{1}{4\sqrt{2}}$$

$$\frac{\partial f}{\partial y} = \cos\left(\frac{x}{y^2+1}\right)\left(\frac{-2xy}{(y^2+1)^2}\right) \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(\pi,\sqrt{3})} = \left.\frac{\partial f}{\partial y}\right|_{\substack{x=\pi \\ y=\sqrt{3}}} = \cos\left(\frac{\pi}{\sqrt{3}^2+1}\right)\left(\frac{-2\pi\sqrt{3}}{\left(\sqrt{3}^2+1\right)^2}\right) = -\frac{\pi\sqrt{3}}{8\sqrt{2}}$$

Higher Derivatives 14.3.16: Let $z = f(x,y)$. Then the second partial derivatives of f are:

$$z_{xx} = f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 f}{\partial x^2} \qquad z_{yy} = f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 f}{\partial y^2}$$

$$z_{xy} = f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 f}{\partial y \partial x} \qquad z_{yx} = f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 f}{\partial x \partial y}$$

where $f_{xy} = (f_x)_y$, $f_{yx} = (f_y)_x$, $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$, and $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

Example 14.3.17: Find the second order partial derivatives of $f(x,y) = x^3 + x^2y^3 - 2y^2$

Solution: $f_x = 3x^2 + 2xy^3$ and $f_y = 3x^2y^2 - 4y$. So,

$f_{xx} = 6x + 2y^3$ $\qquad$ $f_{yy} = 6x^2y - 4$

$f_{xy} = (f_x)_y = (3x^2 + 2xy^3)_y = 6xy^2$ $\qquad$ $f_{yx} = (f_y)_x = (3x^2y^2 - 4y)_x = 6xy^2$

Example 14.3.18: Find $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ at the origin, where $f(x,y) = 2x\sqrt[3]{x^3 - 27y^3}$

Solution: Observe that $\frac{\partial^2 f}{\partial y \partial x} = f_{xy}$ and $\frac{\partial^2 f}{\partial x \partial y} = f_{yx}$

$$f_x = 2x\frac{1}{3}(x^3 - 27y^3)^{-\frac{2}{3}}(3x^2) + 2\sqrt[3]{x^3 - 27y^3}$$

$$\Rightarrow \quad f_x = \frac{2x^3}{(x^3 - 27y^3)^{\frac{2}{3}}} + 2\sqrt[3]{x^3 - 27y^3} \Rightarrow f_x(0,y) = \frac{2(0^3)}{((0)^3 - 27y^3)^{\frac{2}{3}}} + 2\sqrt[3]{(0)^3 - 27y^3}$$

$\Rightarrow\ f_x(0,y) = -6y \Rightarrow\ f_{xy}(0,y) = -6 \quad \Rightarrow\ f_{xy}(0,0) = -6 \Rightarrow \left.\frac{\partial^2 f}{\partial y \partial x}\right|_{(0,0)} = f_{xy}(0,0) = -6$

Also, $f_y = 2x\frac{1}{3}(x^3 - 27y^3)^{-\frac{2}{3}}(-27(3)y^2) = \frac{-54xy^2}{(x^3-27y^3)^{\frac{2}{3}}} \Rightarrow\ f_y(x,0) = \frac{-54x(0^2)}{(x^3-27(0^3))^{\frac{2}{3}}}$

$\Rightarrow\ f_y(x,0) = 0 \Rightarrow\ f_{yx}(x,0) = 0 \quad \Rightarrow\ f_{yx}(0,0) = 0 \Rightarrow \left.\frac{\partial^2 f}{\partial x \partial y}\right|_{(0,0)} = f_{yx}(0,0) = 0$

Observe that in this example: $f_{xy}(0,0) \neq f_{yx}(0,0)$

Clairaut's Theorem 14.3.19: Suppose $f(x,y)$ is defined on a disk D that contains the point (a,b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Example 14.3.20: If $f(x,y,z) = \frac{y^2 e^{3xyz}}{8x^2}$, find $f_{yxxzx}(1,2,0)$.

Solution: Since all partial derivatives of f of all orders are continuous near the point $(1,2,0)$, then Clairaut's Theorem implies that $f_{yxxzx} = f_{zyxxx}$. So,

$f_z = \frac{y^2(3xy)e^{3xyz}}{8x^2} \Rightarrow\ f_z(x,y,0) = \frac{y^2(3xy)e^{3xy(0)}}{8x^2} = \frac{3xy^3}{8x^2} = \frac{3y^3}{8x} \Rightarrow\ f_{zy}(x,y,0) = \frac{9y^2}{8x}$

$\Rightarrow f_{zy}(x,2,0) = \frac{9(2)^2}{8x} = \frac{9}{2x} \Rightarrow\ f_{zy}(x,2,0) = \frac{9}{2}x^{-1} \Rightarrow\ f_{zyx}(x,2,0) = -\frac{9}{2}x^{-2}$

$\Rightarrow\ f_{zyxx}(x,2,0) = 9x^{-3} \Rightarrow\ f_{zyxxx}(x,2,0) = -27x^{-4}$

$\Rightarrow\ f_{yxxzx}(1,2,0) = f_{zyxxx}(1,2,0) = -27(1)^{-4} \Rightarrow\ f_{yxxzx}(1,2,0) = -27$

Example 14.3.21: If $f(x,y) = x^3y^5 - \frac{xy^2}{x+\ln(x)}$, find $\left.\frac{\partial^6 f}{\partial y^4 \partial x^2}\right|_{(e,2)}$

Solution: Observe that $\left.\frac{\partial^6 f}{\partial y^4 \partial x^2}\right|_{(e,2)} = f_{xxyyyy}(e,2)$

Since all partial derivatives of f of all orders are continuous near the point $(e,2)$, then Clairaut's Theorem implies that $f_{xxyyyy}(e,2) = f_{yyyyxx}(e,2)$. So,

$f_y = 5x^3y^4 - \frac{2xy}{x+\ln(x)} \quad \Rightarrow\ f_{yy} = 20x^3y^3 - \frac{2x}{x+\ln(x)} \Rightarrow\ f_{yyy} = 60x^3y^2$

$\Rightarrow\ f_{yyyy} = 120x^3y \quad \Rightarrow\ f_{yyyy}(x,2) = 240x^3 \Rightarrow\ f_{yyyyx}(x,2) = 720x^2$

$\Rightarrow\ f_{yyyyxx}(x,2) = 1440x \quad \Rightarrow\ \left.\frac{\partial^6 f}{\partial y^4 \partial x^2}\right|_{(e,2)} = 1440e$

Example 14.3.22: Find $\frac{\partial^{103}}{\partial y^{63} \partial x^{40}}(x^{10}\sin(xy) + x^{50})$ at the point $(-1,0)$.

Solution: Let $f(x,y) = x^{10}\sin(xy) + x^{50}$. Then

$$\frac{\partial^{103}}{\partial y^{63} \partial x^{40}}(x^{10}\sin(xy) + x^{50}) = f_{\underbrace{x\ldots\ldots x}_{40-times}\underbrace{y\ldots\ldots y}_{63-times}}$$

Since all partial derivatives of f of all orders are continuous near the point $(-1,0)$, then Clairaut's Theorem implies that $f_{\underbrace{x\ldots\ldots x}_{40-times}\underbrace{y\ldots\ldots y}_{63-times}}(-1,0) = f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(-1,0)$. So,

$$\begin{Bmatrix} \Rightarrow f_y = x^{10}[x\cos(xy)] + 0 \\ \Rightarrow f_{yy} = x^{10}[-x^2\sin(xy)] \\ \Rightarrow f_{yyy} = x^{10}[-x^3\cos(xy)] \\ \Rightarrow f_{yyyy} = x^{10}[x^4\sin(xy)] \end{Bmatrix} \Rightarrow \begin{Bmatrix} \Rightarrow f_{\underbrace{y\ldots\ldots y}_{60-times}} = x^{10}[x^{60}\sin(xy)] \\ \Rightarrow f_{\underbrace{y\ldots\ldots y}_{61-times}} = x^{10}[x^{61}\cos(xy)] \\ \Rightarrow f_{\underbrace{y\ldots\ldots y}_{62-times}} = x^{10}[-x^{62}\sin(xy)] \\ \Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}} = x^{10}[-x^{63}\cos(xy)] \end{Bmatrix}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}}(x,0) = x^{10}\left[-x^{63}\cos\left(x(0)\right)\right] = -x^{73}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(x,0) = -(73)(72)(71)\ldots(73-39)x^{73-40}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(x,0) = -(73)(72)(71)\ldots(34)x^{33} = -\frac{73!}{33!}x^{33}$$

$$\Rightarrow f_{\underbrace{y\ldots\ldots y}_{63-times}\underbrace{x\ldots\ldots x}_{40-times}}(-1,0) = -\frac{73!}{33!}(-1)^{33} = \frac{73!}{33!}$$

Remark 14.3.23: Recall that $\frac{d}{dx}\int_{g(x)}^{h(x)} F(t)dt = F\left(h(x)\right)h'(x) - F\left(g(x)\right)g'(x)$

Example 14.3.24: If $f(x,y) = \int_y^{xy} \cos(e^t)dt$, find $f_{xy}(0,0)$

Solution: $f_x = \cos(e^{xy})\frac{\partial}{\partial x}(xy) - \cos(e^y)\frac{\partial}{\partial x}(y) = y\cos(e^{xy}) - \cos(e^y)(0)$

$$\Rightarrow \quad f_x = y\cos(e^{xy}) \quad \Rightarrow \quad f_x(0,y) = y\cos\left(e^{(0)y}\right) = y\cos(1)$$

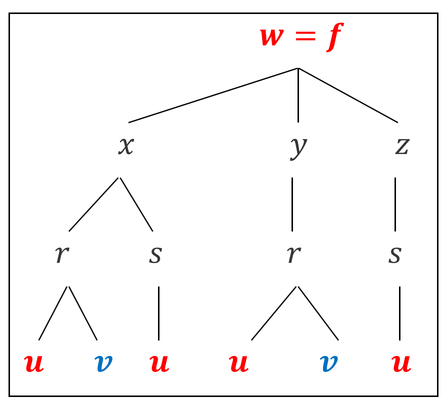
$$\Rightarrow \quad f_{xy} = \cos(1) \quad \Rightarrow \quad f_{xy}(0,0) = \cos(1)$$

Section 14.5: The Chain Rule

Rule 14.5.1: Let $w = f(x,y,z), x = x(r,s), y = y(r), z = z(s), r = r(u,v), and\ s = s(u).$

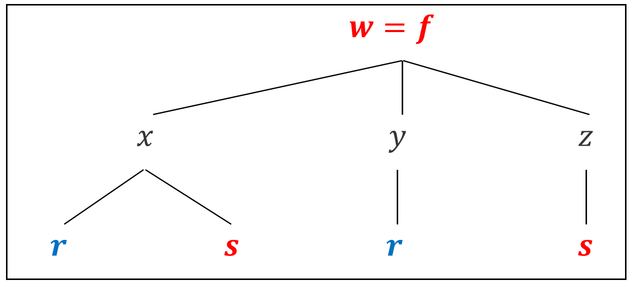
- $\dfrac{\partial w}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r}\dfrac{\partial r}{\partial u} + \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s}\dfrac{ds}{du} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dr}\dfrac{\partial r}{\partial u} + \dfrac{\partial f}{\partial z}\dfrac{dz}{ds}\dfrac{ds}{du}$

- $\dfrac{\partial w}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r}\dfrac{\partial r}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dr}\dfrac{\partial r}{\partial v}$

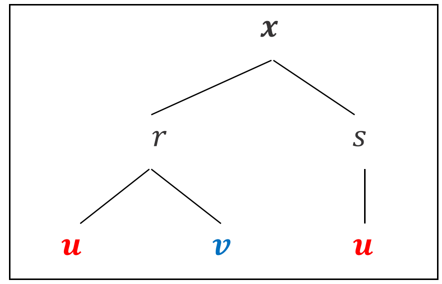


Tree Diagram

- $\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial z}\frac{dz}{ds}$



- $\frac{\partial x}{\partial u} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial u} + \frac{\partial x}{\partial s}\frac{ds}{du}$



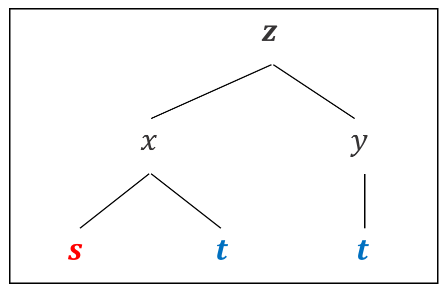
- $\frac{\partial x}{\partial v} = \frac{\partial x}{\partial r}\frac{\partial r}{\partial v}$

Example 14.5.2: Let $z = e^{2x}\sin(y), x = st^2, y = t^3$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Solution:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s}$$

$$= \left(2e^{2x}\sin(y)\right)t^2 = 2t^2e^{2x}\sin(y)$$



$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \left(2e^{2x}\sin(y)\right)(2st) + \left(e^{2x}\cos(y)\right)(3t^2)$$

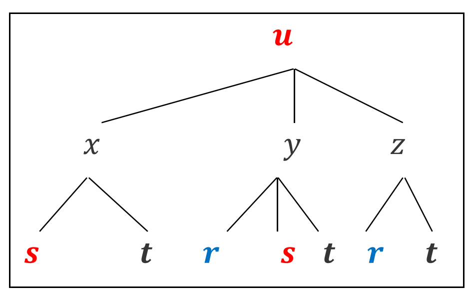
$$= 4ste^{2x}\sin(y) + 3t^2e^{2x}\cos(y) = 4stz + 3t^2e^{2x}\cos(y)$$

Example 14.5.3: Let $u = x^4y + y^2z^3, x = se^{2t}, y = r^2se^{-t}, z = r\cos(t)$.

Find $\frac{\partial u}{\partial s}$ and $\frac{\partial u}{\partial t}$ when $r = 2, s = 1, t = 0$.

Solution: First we have to find x, y, z when $r = 2, s = 1, t = 0$

$$\left.\begin{array}{c} x = 1e^{2(0)} = 1 \\ y = (2)^2(1)e^{-(0)} = 4 \\ z = 2\cos(0) = 2 \end{array}\right\} \Rightarrow \begin{cases} x = 1, y = 4, z = 2 \\ \text{when } r = 2, s = 1, t = 0 \end{cases}$$



- $\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} = (4x^3y)(e^{2t}) + (x^4 + 2yz^3)(r^2e^{-t})$

$$\Rightarrow \left.\frac{\partial u}{\partial s}\right|_{\substack{r=2,s=1,t=0 \\ x=1,y=4,z=2}} = (16)(1) + 65(4) = 276$$

- $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial t} = (4x^3y)(2se^{2t}) + (x^4 + 2yz^3)(-r^2se^{-t}) + (3y^2z^2)\left(-r\sin(t)\right)$

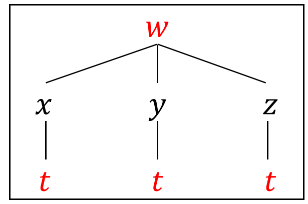
$$\Rightarrow \left.\frac{\partial u}{\partial s}\right|_{\substack{r=2,s=1,t=0 \\ x=1,y=4,z=2}} = 16(2) - 65(4) + 48(2) = -132$$

Example 14.5.4: Let $w=\ln\sqrt{x^2+y^2+z^2}$, $x=\sin(t), y=\cos(t), z=\tan(t)$. Find $\frac{dw}{dt}$.

Solution: Observe that:

$$w=\ln\sqrt{x^2+y^2+z^2}=\frac{1}{2}\ln(x^2+y^2+z^2)$$

$$\Rightarrow \frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$



$$\frac{dw}{dt}=\frac{1}{2}\frac{2x}{x^2+y^2+z^2}\cos(t)+\frac{1}{2}\frac{2y}{x^2+y^2+z^2}(-\sin(t))+\frac{1}{2}\frac{2z}{x^2+y^2+z^2}\sec^2(t)$$

$$=\frac{x\cos(t)-y\sin(t)+z\sec^2(t)}{x^2+y^2+z^2}=\frac{\sin(t)\cos(t)-\sin(t)\cos(t)+\tan(t)\sec^2(t)}{\sin^2(t)+\cos^2(t)+\tan^2(t)}$$

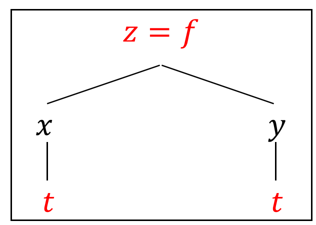
$$=\frac{\tan(t)\sec^2(t)}{1+\tan^2(t)}=\frac{\tan(t)\sec^2(t)}{\sec^2(t)}=\tan(t)$$

Example 14.5.5: Let $z=f(x,y), x=g(t), y=h(t), g(3)=2, h(3)=7, g'(3)=5,$

$h'(3)=-4, f_x(2,7)=6$ and $f_y(2,7)=-8$. Find $\frac{dz}{dt}$ when $t=3$.

Solution: First we have to find x, y when $t=3$:

$$\left.\begin{matrix} x=g(3)=2 \\ y=h(3)=7 \end{matrix}\right\}\Rightarrow\left\{\begin{matrix} x=2, y=7 \\ \text{when } t=3 \end{matrix}\right.$$



$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\frac{dx}{dt}+\frac{\partial z}{\partial y}\frac{dy}{dt}=f_x(x,y)g'(t)+f_y(x,y)h'(t)$$

$$\Rightarrow \left.\frac{dz}{dt}\right|_{t=3,x=2,y=7}=f_x(2,7)g'(3)+f_y(2,7)h'(3)=6(5)+(-8)(-4)=62$$

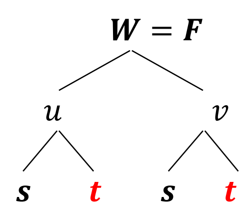
Example 14.5.6: Let $W(s,t)=F\big(u(s,t),v(s,t)\big), F_u(2,3)=-1, F_v(2,3)=10, u(1,0)=2, v(1,0)=3,$

$u_s(1,0)=-2, v_s(1,0)=5, u_t(1,0)=6, v_t(1,0)=4$. Find $W_t(1,0)$ and $W_s(1,0)$.

Solution: Observe that $W(s,t)=F(u,v)$ with $u=u(s,t), v=v(s,t)$.

Also, observe that to find $W_t(1,0)$ we have to find W_t when $s=1, t=0$:

Also, we need to find u, v when $s=1, t=0$:



$$\left.\begin{matrix} u=u(1,0)=2 \\ v=v(1,0)=3 \end{matrix}\right\}\Rightarrow\left\{\begin{matrix} u=2, v=3 \\ \text{when } s=1, t=0 \end{matrix}\right.$$

Now, $W_t=F_u u_t+F_v v_t$

$$W_t(1,0)=F_u(2,3)u_t(1,0)+F_v(2,3)v_t(1,0)=-1(6)+10(4)=34$$

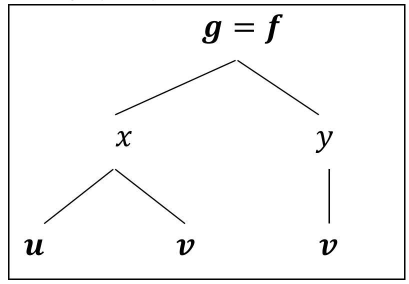
Finding $W_s(1,0)$ is an exercise.

Example 14.5.7: Suppose that $f(x,y)$ is differentiable. Find $g_u(0,0)$ and $g_v(0,0)$, where

$$g(u,v)=f(e^u+\cos(v), 1+\sin(v))$$

	f	g	f_x	f_y
(0,0)	3	6	5	8
(2,1)	6	3	2	7

Solution: Let $x = e^u + \cos(v)$, $y = 1 + \sin(v)$. So, the function is $g(u,v) = f(x,y)$



To find $g_u(0,0)$ means: to find g_u when $u = 0, v = 0$.
So, first we have to find x, y when $u = 0, v = 0$:

$$\left.\begin{matrix} x = e^0 + \cos(0) = 2 \\ y = 1 + \sin(0) = 1 \end{matrix}\right\} \Rightarrow \begin{cases} x = 2, y = 1 \\ \text{when } u = 0, v = 0 \end{cases}$$

Now,

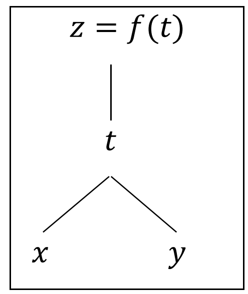
$$g_u = f_x x_u = f_x(x,y)(e^u) \Rightarrow g_u(0,0) = f_x(2,1)(e^0) = 2$$

$$g_v = f_x x_v + f_y \frac{dy}{dv} = f_x(x,y)(-\sin(v)) + f_y(x,y)(\cos(v))$$

$$\Rightarrow g_v(0,0) = f_x(2,1)(-\sin(0)) + f_y(2,1)\cos(0) = 2(0) + 7(1) = 7$$

Example 14.5.8: Let $z = f(x - y)$. Show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$.

Solution: Observe that $f(\ldots)$ is a function in 1-variable, so, let $z = f(t), t = x - y$



$$\frac{\partial z}{\partial x} = f'(t)t_x = f'(t) \text{ since } t_x = 1$$

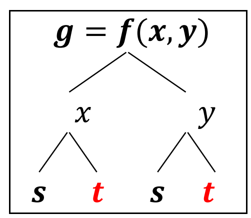
and $\frac{\partial z}{\partial y} = f'(t)t_y = -f'(t)$ since $t_y = -1$

$$\Rightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = f'(t) + \left(-f'(t)\right) = 0 \quad \Rightarrow \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Example 14.5.9: Let $g(s,t) = f(s^2 - t^2, t^2 - s^2)$. Show that $t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$.

Solution: Observe that $f(\ldots,\ldots)$ is a function in 2-variable:

Let $g(s,t) = f(x,y), x = s^2 - t^2, y = t^2 - s^2$



$$\frac{\partial g}{\partial s} = f_x x_s + f_y y_s = 2sf_x + (-2s)f_y$$

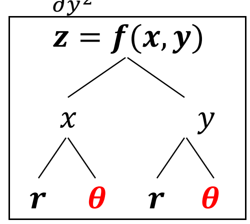
$$\frac{\partial g}{\partial t} = f_x x_t + f_y y_t = -2tf_x + 2tf_y$$

$$\Rightarrow t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = t\left(2sf_x - 2sf_y\right) + s\left(-2tf_x + 2tf_y\right) = 2stf_x - 2stf_y - 2stf_x + 2stf_y = 0$$

$$\Rightarrow t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0$$

Example 14.5.10: Let $z = f(x,y)$ be with continuous second order partial derivatives such that $x = r\cos(\theta), y = r\sin\theta$. Show that $\frac{\partial^2 z}{\partial r^2} = \cos^2\theta\frac{\partial^2 z}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 z}{\partial y \partial x} + \sin^2\theta\frac{\partial^2 z}{\partial y^2}$

Solution:



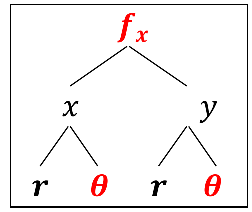
$$\frac{\partial z}{\partial r} = f_x x_r + f_y y_r = \cos\theta f_x + \sin\theta f_y$$

$$\frac{\partial z}{\partial \theta} = f_x x_\theta + f_y y_\theta = -r\sin\theta f_x + r\cos\theta f_y$$

$$= -r(\sin\theta f_x - \cos\theta f_y)$$

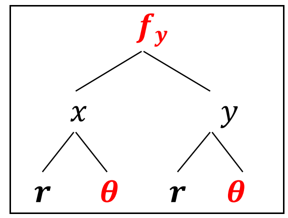
$\Rightarrow \frac{\partial z}{\partial r} = \cos\theta f_x + \sin\theta f_y \;\Rightarrow\; \frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\cos\theta f_x + \sin\theta f_y\right)$

$\Rightarrow \frac{\partial^2 z}{\partial r^2} = \cos\theta \frac{\partial f_x}{\partial r} + \sin\theta \frac{\partial f_y}{\partial r} \ldots\ldots (1)$



So, we have to find: $\frac{\partial f_x}{\partial r}$ and $\frac{\partial f_y}{\partial r}$:

$\frac{\partial f_x}{\partial r} = f_{xx} x_r + f_{xy} y_r = \cos\theta f_{xx} + \sin\theta f_{xy} \ldots\ldots\ldots (2)$



$\frac{\partial f_y}{\partial r} = f_{yx} x_r + f_{yy} y_r = \cos\theta f_{yx} + \sin\theta f_{yy} = \cos\theta f_{yx} + \sin\theta f_{yy}$

$\Rightarrow \frac{\partial f_y}{\partial r} = \cos\theta f_{xy} + \sin\theta f_{yy} \ldots\ldots (3)$ (since $f_{yx} = f_{xy}$)

$\Rightarrow \frac{\partial^2 z}{\partial r^2} = \cos\theta \frac{\partial f_x}{\partial r} + \sin\theta \frac{\partial f_y}{\partial r}$ (by (1))

$= \cos\theta(\cos\theta f_{xx} + \sin\theta f_{xy}) + \sin\theta\left(\cos\theta f_{xy} + \sin\theta f_{yy}\right)$ (by (2) and (3))

$= \cos^2\theta f_{xx} + 2\sin\theta\cos\theta f_{xy} + \sin^2\theta f_{yy}$

Implicit Differentation:

Implicit Function Theorem 14.5.11:

(1) Let $y = f(x)$ is a function defined implicitly by the relation $F(x, y) = 0$, where F is a differentiable function with F_y is nonzero. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

(2) Let $z = f(x, y)$ is a function defined implicitly by the relation $F(x, y, z) = 0$, where F is a differentiable function with F_z is nonzero. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Example 14.5.12: Find y' at $x = 0$ if $\frac{x^3+y^3+1}{6y} = x$

Solution: First, we have to find the value of y when $x = 0$:

$x = 0$: $\frac{x^3 + y^3 + 1}{6y} = x \;\Rightarrow\; \frac{(0)^3 + y^3 + 1}{6y} = 0 \;\Rightarrow\; y^3 = -1 \;\Rightarrow\; y = -1$

So, $x = 0 \Rightarrow\; y = -1$.

Second, we simplify (نبسط) the equation $\frac{x^3+y^3+1}{6y} = x$ (if possible)

Equation: $\frac{x^3 + y^3 + 1}{6y} = x \;\Rightarrow\; x^3 + y^3 + 1 = 6xy \;\Rightarrow\; x^3 + y^3 - 6xy + 1 = 0$

$\Rightarrow$ Let $F(x, y) = x^3 + y^3 - 6xy + 1$: $\Rightarrow y' = -\frac{F_x}{F_y} = -\frac{3x^2 - 6y}{3y^2 - 6x}$

$$y'|_{x=0,y=-1} = -\frac{3(0)^2 - 6(-1)}{3(-1)^2 - 6(0)} = -\frac{6}{3} = -2$$

Example 14.5.13: If $x^3+y^3+z^3+6xyz=9$, find

$$(1)\ \frac{\partial z}{\partial x}\ \text{and}\ \frac{\partial z}{\partial y} \qquad\qquad (2)\ \frac{\partial z}{\partial x}\ \text{and}\ \frac{\partial z}{\partial y}\ \text{at the point}\ (0,1)$$

Solution: $x^3+y^3+z^3+6xy=9 \Rightarrow x^3+y^3+z^3+6xyz-9=0$

Let: $F(x,y,z)=x^3+y^3+z^3+6xyz-9$

(1) $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=-\frac{3x^2+6yz}{3z^2+6xy}$ and $\frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=-\frac{3y^2+6xz}{3z^2+6xy}$

(2) At the point $(0,1) \Rightarrow x=0, y=1$. So, we have to find the value of z:

When $x=0, y=1$:

$x^3+y^3+z^3+6xyz-9=0 \Rightarrow (0)^3+(1)^3+z^3+6(0)(1)z-9=0$

$\Rightarrow z^3=8 \Rightarrow z=2$

$\Rightarrow x=0, y=1, z=2$.

From part (1):

$$\frac{\partial z}{\partial x}=-\frac{3x^2+6yz}{3z^2+6xy} \Rightarrow \left.\frac{\partial z}{\partial x}\right|_{x=0,y=1,z=2}=-\frac{3(0)^2+6(1)(2)}{3(2)^2+6(0)(1)}=-1$$

$$\frac{\partial z}{\partial y}=-\frac{3y^2+6xz}{3z^2+6xy} \Rightarrow \left.\frac{\partial z}{\partial y}\right|_{x=0,y=1,z=2}=-\frac{3(1)^2+6(0)(2)}{3(2)^2+6(0)(1)}=-\frac{1}{4}$$

Example 14.5.14: Suppose that the equation $F(x,y,z)=0$ implicitly defines each of the three variables x,y,z as a function of the other two. If F_x, F_y, F_z are nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}=-1$$

Solution: By the Implicit Function Theorem we have

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \qquad \frac{\partial x}{\partial y}=-\frac{F_y}{F_x} \qquad \frac{\partial y}{\partial z}=-\frac{F_z}{F_y}$$

$$\Rightarrow \frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}=\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)=-1$$

Example 14.5.15: Suppose that the equation $F(x,y)=0$ implicitly defines y as a function of x and defines x as a function of y. If F_x, F_y are nonzero, show that

$$\frac{dy}{dx}\frac{dx}{dy}=1$$

Solution: By the Implicit Function Theorem we have

$$\frac{dy}{dx}=-\frac{F_x}{F_y}\ \text{and}\ \frac{dx}{dy}=-\frac{F_y}{F_x} \Rightarrow \frac{dy}{dx}\frac{dx}{dy}=\left(-\frac{F_x}{F_y}\right)\left(-\frac{F_y}{F_x}\right)=1$$

Example 14.5.16: Suppose that the equation $F(x,y,z)=0$ implicitly defines each of the three variables x,y,z as a function of the other two. If F_x and F_z are nonzero, write $\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}$ in simplicit form.

Solution: By the Implicit Function Theorem we have: $\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}, \frac{\partial x}{\partial y}=-\frac{F_y}{F_x}, \frac{\partial y}{\partial z}=-\frac{F_z}{F_y}$

$$\Rightarrow \frac{\partial z}{\partial x}\frac{\partial x}{\partial y}=\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)=-\left(-\frac{F_y}{F_z}\right)=-\frac{\partial z}{\partial y}$$

Section 14.6 The Directional Derivative and the Gradiant Vector

Definition 14.6.1: The gradient vector of the function $f(x,y)$ at the point (x_0,y_0) is defined by

$$\nabla f(x_0,y_0) = \langle f_x(x_0,y_0), f_y(x_0,y_0)\rangle$$

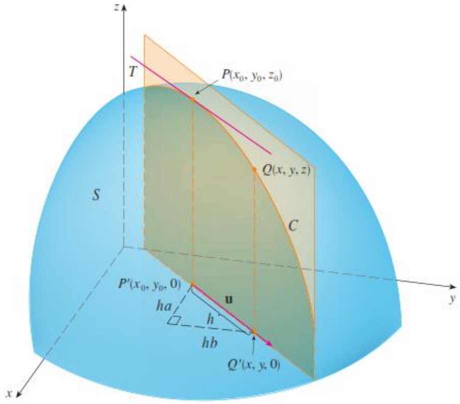
- Observe that if $f(x,y,z)$ is a function in 3-variables, then

$$\nabla f(x_0,y_0,,z_0) = \langle f_x(x_0,y_0,z_0), f_y(x_0,y_0,z_0), f_z(x_0,y_0,z_0)\rangle$$

Definition 14.6.2: The Directional derivative (or the rate of change) of the function $f(x,y)$ at the point (x_0,y_0) in the direction of the unit vector $\hat{v} = \langle a,b\rangle$ is:

$$D_{\hat{v}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

Interpolation of the Directional Derivative 14.6.3:



Suppose that we now wish to find the directional derivative (the rate of change) of the function $f(x,y)$ at a point $P'(x_0,y_0)$ in the direction of a unit vector u To do this we consider the surface S with the equation $z = f(x,y)$ (the graph of f) and we let $z_0 = f(x_0,y_0)$. Then the point $P(x_0,y_0,z_0)$ lies on S. The vertical plane that passes through P in the direction of $\boldsymbol{u}$ intersects S in a curve C. The slope of the tangent line T to C at the point P is the directional derivative (rate of change) of f in the direction of u.

Theorem 14.6.4: The Directional derivative (or the rate of change) of the function $f(x,y)$ at the point (x_0,y_0) in the direction of the unit vector $\hat{v} = \langle a,b\rangle$ is:

$$D_{\hat{v}} f(x_0,y_0) = \nabla f(x_0,y_0)\cdot\hat{v} \text{ (dot product)}$$

Example 5: Find the directional derivative of the function $f(x,y) = x^2y^3$ at the point $(-2,3)$ in the direction of the vector $\vec{v} = 2i - 5j$.

Solution:

- Gradient: $\nabla f = \langle f_x, f_y\rangle = \langle 2xy^3, 3x^2y^2\rangle \Rightarrow \nabla f(-2,3) = \langle 2(-2)(3)^3, 3(-2)^2(3)^2\rangle$

  $\Rightarrow \nabla f(-2,3) = \langle -108,108\rangle$

- Unit vector: $\vec{v} = 2i - 5j \Rightarrow \hat{v} = \frac{\vec{v}}{|\vec{v}|} = \frac{2i-5j}{\sqrt{2^2+5^2}} = \frac{2i-5j}{\sqrt{29}} = \langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\rangle$
- $D_{\hat{v}} f(-2,3) = \nabla f(-2,3)\cdot\hat{v} = \langle -108,108\rangle \cdot \langle \frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}\rangle = (-108)\frac{2}{\sqrt{29}} + 108\left(-\frac{5}{\sqrt{29}}\right) = -\frac{756}{\sqrt{29}}$

Example 14.6.6: Find the rate of change of the function $f(x,y)=\frac{x^2-y}{y^2}$ at the point $(2,1)$ in the direction indicated by the angle $\theta=\frac{\pi}{3}$ (that is in the direction that makes the angle $\theta=\frac{\pi}{3}$ with the positive direction of the x-axis.

Solution:

- Gradient: $\nabla f=\langle f_x,f_y\rangle=\langle\frac{2x}{y^2},\frac{y^2(-1)-(x^2-y)(2y)}{y^4}\rangle \Rightarrow \nabla f(2,1)=\langle 4,-7\rangle$
- Unit vector: $\hat{v}=\langle\cos(\theta),\sin(\theta)\rangle=\langle\frac{1}{2},\frac{\sqrt{3}}{2}\rangle$
- $D_{\hat{v}}f(2,1)=\nabla f(2,1)\cdot\hat{v}=\langle 4,-7\rangle\cdot\langle\frac{1}{2},\frac{\sqrt{3}}{2}\rangle=(4)\frac{1}{2}+(-7)\left(\frac{\sqrt{3}}{2}\right)=\frac{4-7\sqrt{3}}{2}$

Example 14.6.7: Find the rate of change of the function $f(x,y,z)=x^2-3yz^3$ at the point $P(2,-1,1)$ in the direction from P to the point $Q\left(3,1,\frac{1}{2}\right)$.

Solution:

- Gradient: $\nabla f=\langle f_x,f_y,f_z\rangle=\langle 2x,-3z^3,-9yz^2\rangle \Rightarrow \nabla f(2,-1,1)=\langle 4,-3,9\rangle$
- Unit vector: $\vec{v}=\overrightarrow{PQ}=\langle Q-P\rangle=\langle 1,2,-\frac{1}{2}\rangle \Rightarrow |\vec{v}|=\frac{\sqrt{21}}{2} \Rightarrow \vec{v}$ not a unit vector

$$\hat{v}=\frac{\vec{v}}{|\vec{v}|}=\frac{\langle 1,2,-\frac{1}{2}\rangle}{\sqrt{21}/2}=\frac{2\langle 1,2,-\frac{1}{2}\rangle}{\sqrt{21}}=\langle\frac{2}{\sqrt{21}},\frac{4}{\sqrt{21}},-\frac{1}{\sqrt{21}}\rangle$$

- $D_{\hat{v}}f(2,-1,1)=\nabla f(2,,-1,1)\cdot\hat{v}=\langle 4,-3,9\rangle\cdot\langle\frac{2}{\sqrt{21}},\frac{4}{\sqrt{21}},-\frac{1}{\sqrt{21}}\rangle=\frac{8}{\sqrt{21}}-\frac{12}{\sqrt{21}}-\frac{9}{\sqrt{21}}=-\frac{13}{\sqrt{21}}$

Remark 14.6.8: Recall that: The definition of the directional derivative (or the rate of change) of the function $f(x,y)$ at the point (x_0,y_0) in the direction of the unit vector $\hat{v}=\langle a,b\rangle$ is:

$$D_{\hat{v}}f(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+ah,y_0+bh)-f(x_0,y_0)}{h} \text{ and } D_{\hat{v}}f(x_0,y_0)=\nabla f(x_0,y_0)\cdot\hat{v}$$

Example 14.6.9: Let $f(x,y)=\ln(x^2+2y)-2\sqrt{x}$. Find $\lim_{h\to 0}\frac{f\left(4-\frac{h}{3},\frac{\sqrt{8}h}{3}\right)-f(4,0)}{h}$

Solution: By the definition of the directional derivative we have $\lim_{h\to 0}\frac{f\left(4-\frac{h}{3},\frac{\sqrt{8}h}{3}\right)-f(4,0)}{h}=D_{\hat{v}}f(4,0)$, where $\hat{v}=\langle-\frac{1}{3},\frac{\sqrt{8}}{3}\rangle$. So, $\lim_{h\to 0}\frac{f\left(4-\frac{h}{3},\frac{\sqrt{8}h}{3}\right)-f(4,0)}{h}=\nabla f(4,0)\cdot\langle-\frac{1}{3},\frac{\sqrt{8}}{3}\rangle$........①

We have to find $\nabla f(4,0)$:

- $\nabla f=\langle\frac{2x}{x^2+2y}-\frac{1}{\sqrt{x}},\frac{2}{x^2+2y}\rangle \Rightarrow \nabla f(4,0)=\langle 0,\frac{1}{8}\rangle$
- ① $\Rightarrow \lim_{h\to 0}\frac{f\left(4-\frac{h}{3},\frac{\sqrt{8}h}{3}\right)-f(4,0)}{h}=\nabla f(4,0)\cdot\langle-\frac{1}{3},\frac{\sqrt{8}}{3}\rangle=\langle 0,\frac{1}{8}\rangle\cdot\langle-\frac{1}{3},\frac{\sqrt{8}}{3}\rangle=\frac{\sqrt{8}}{24}$

Example 14.6.10: Let $\hat{u}=\langle\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\rangle$, $\hat{v}=\langle\frac{2\sqrt{2}}{3},\frac{1}{3}\rangle$, $D_{\hat{u}}f(2,1)=2$ and $D_{\hat{v}}f(2,1)=\frac{1}{3}$.

(a) Find the gradient vector of f at the point $(2,1)$.
(b) Find the directional derivative of f at the point $(2,1)$ in the direction of $i-2j$.

Solution:
(a) Let $\nabla f(2,1)=\langle a,b\rangle$. Then

- $D_{\hat{u}}f(2,1)=2 \Rightarrow \nabla f(2,1)\cdot\hat{u}=2 \Rightarrow \langle a,b\rangle\cdot\langle\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\rangle=2 \Rightarrow \frac{a-b}{\sqrt{2}}=2$. So,
  $$a-b=2\sqrt{2}\ldots\ldots\ldots (1)$$
- $D_{\hat{v}}f(2,1)=\frac{1}{3} \Rightarrow \nabla f(2,1)\cdot\hat{v}=\frac{1}{3} \Rightarrow \langle a,b\rangle\cdot\langle\frac{2\sqrt{2}}{3},\frac{1}{3}\rangle=\frac{1}{3} \Rightarrow \frac{2\sqrt{2}a+b}{3}=\frac{1}{3}$. So,
  $$2\sqrt{2}a+b=1\ldots\ldots\ldots (2)$$
- (1)+(2): $\Rightarrow (1+2\sqrt{2})a=2\sqrt{2}+1 \Rightarrow a=1$
  (1) $\Rightarrow 1-b=2\sqrt{2} \Rightarrow b=1-2\sqrt{2}$
  $$\nabla f(2,1)=\langle a,b\rangle=\langle 1,1-2\sqrt{2}\rangle$$

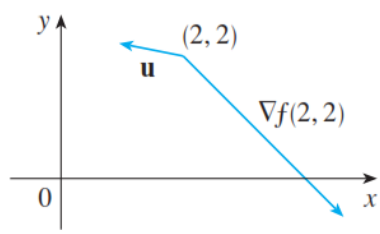
(b)

- $\nabla f(2,1)=\langle 1,1-2\sqrt{2}\rangle$ (by part (a))
- Unit vector: $\vec{w}=i-2j \Rightarrow |\vec{w}|=\sqrt{5} \Rightarrow \hat{w}=\langle\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\rangle$ unit vector.
- $D_{\hat{w}}f(2,1)=\nabla f(2,1)\cdot\hat{w}=\langle 1,1-2\sqrt{2}\rangle\cdot\langle\frac{1}{\sqrt{5}},-\frac{2}{\sqrt{5}}\rangle=\frac{4\sqrt{2}-1}{\sqrt{5}}$

Remark 14.6.11: Since i,j, and k are unit vectors, then:

- $D_i f=\nabla f\cdot i=\langle f_x,f_y,f_z\rangle\cdot\langle 1,0,0\rangle=f_x \quad \Rightarrow \quad D_i f=f_x$
- $D_j f=\nabla f\cdot j=\langle f_x,f_y,f_z\rangle\cdot\langle 0,1,0\rangle=f_y \quad \Rightarrow \quad D_j f=f_y$
- $D_k f=\nabla f\cdot k=\langle f_x,f_y,f_z\rangle\cdot\langle 0,0,1\rangle=f_z \quad \Rightarrow \quad D_k f=f_z$

Example 14.6.12: Use the figure to estimate $D_u f(2,2)$.



Solution: If we take $|u|=1$ unit, then $|\nabla f(2,2)|\cong 3.7$, $\theta\cong 150$. So,
$D_u f(2,-1,1)=\nabla f(x_0,y_0)\cdot u=|\nabla f(2,2)|\,|u|\,\cos\theta=|\nabla f(2,2)|\cos\theta$
$$\cong 3.7\cos(150)=-\frac{3.7\sqrt{3}}{2}\cong -3.2$$

Remark 14.6.13: Let $\hat{v}=\langle a,b\rangle$ be a unit vector. Then
$$D_{\hat{v}}f(x_0,y_0)=\nabla f(x_0,y_0)\cdot\hat{v}$$
$$=|\nabla f(x_0,y_0)|\cdot|\hat{v}|\,\cos\theta$$
$$=|\nabla f(x_0,y_0)|\cos\theta \quad \text{(since } \hat{v} \text{ is a unit vector)}$$
$-1\le\cos\theta\le 1 \Rightarrow -|\nabla f(x_0,y_0)|\le|\nabla f(x_0,y_0)|\cos\theta\le|\nabla f(x_0,y_0)|$
$$\Rightarrow -|\nabla f(x_0,y_0)|\le D_{\hat{v}}f(x_0,y_0)\le|\nabla f(x_0,y_0)|$$

Theorem 14.6.14: Suppose that f is a differentiable function of two (or three variables).

- The maximum value of the directional derivative $D_{\hat{v}} f(x_0, y_0)$ is $|\nabla f(x_0, y_0)|$ which occurs in the direction of $\nabla f(x_0, y_0)$.

$D_{\hat{v}} f = \|\nabla f\|$	$\Leftrightarrow$	$\hat{v}$ and ∇f are in the same direction	$\Leftrightarrow$	$\hat{v} = \frac{\nabla f}{\|\nabla f\|}$

- The minimum value of the directional derivative $D_{\hat{v}} f(x_0, y_0)$ is $-|\nabla f(x_0, y_0)|$ which occurs in the direction of $-\nabla f(x_0, y_0)$.

$D_{\hat{v}} f = -\|\nabla f\|$	$\Leftrightarrow$	$\hat{v}$ and $-\nabla f$ are in the same direction	$\Leftrightarrow$	$\hat{v} = -\frac{\nabla f}{\|\nabla f\|}$

Remark 14.6.15: Observe that:

The maximum value of the directional derivative of f	$\Leftrightarrow$	The maximum rate of change of f	$\Leftrightarrow$	The function f increases fastest.
The minimum value of the directional derivative of f	$\Leftrightarrow$	The minimum rate of change of f	$\Leftrightarrow$	The function f decreases fastest.

Example 14.6.16: Find the maximum directional derivative (or maximum rate of change) of the function $f(x, y) = 2y^2\sqrt{x}$ at the point $(9, -3)$ and find the direction in which it occurs.

Solution:

- $\nabla f = \langle \frac{y^2}{\sqrt{x}}, 4y\sqrt{x} \rangle \;\Rightarrow\; \nabla f(9, -3) = \langle 3, -36 \rangle$
- The maximum directional derivative $= |\nabla f(9, -3)| = \sqrt{3^2 + (36)^2} = \sqrt{1305}$
- The direction in which the maximum directional derivative occurs is in the direction of the vector $\nabla f(9, -3) = \langle 3, -36 \rangle$.

Example 14.6.17: Find the direction in which the function $f(x, y, z) = xe^{x-yz}$

(1) decreases fastest at the point $(2,1,2)$.
(2) increases fastest at the point $(2,1,2)$.

Solution: $\nabla f = \langle xe^{x-yz} + e^{x-yz}, -xze^{x-yz}, -xye^{x-yz} \rangle \Rightarrow \nabla f(2,1,2) = \langle 3, -4, -2 \rangle$

(1) The direction in which the function f decreases fastest is $-\nabla f(2,1,2) = \langle -3, 4, 2 \rangle$

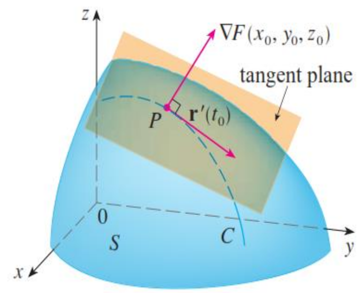
(2) The direction in which the function f increases fastest is $\nabla f(2,1,2) = \langle 3, -4, -2 \rangle$

Example 14.6.18: Find the unit vector $\hat{v}$, if $\nabla f(1,2) = \langle 3, -4 \rangle$ and $D_{\hat{v}} f(1,2) = 5$.

Solution: Since $|\nabla f(1,2)| = \sqrt{9 + 16} = \sqrt{25} = 5$ and $D_{\hat{v}} f(1,2) = 5$ we have $D_{\hat{v}} f(1,2) = |\nabla f(1,2)|$

$\Rightarrow$ $D_{\hat{v}} f(1,2)$ has its maximum value $\Rightarrow$ $\hat{v}$ and $\nabla f(1,2)$ are in the same direction:

$\Rightarrow$ $\hat{v} = \frac{\nabla f(1,2)}{|\nabla f(1,2)|}$ (since $\hat{v}$ is a unit vector) $\Rightarrow$ $\hat{v} = \frac{\langle 3, -4 \rangle}{5} = \langle \frac{3}{5}, -\frac{4}{5} \rangle$.



Theorem 14.6.19: Let $S: F(x, y, z) = k$ be a surface and $P(x_0, y_0, z_0)$ be a point on S. Let $C: \vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a curve on S that passes through P. Prove that ∇F is perpendicular to the tangent vector $\vec{r}'(t)$ of C at the point P.

Proof: The curve C is on $S \Rightarrow C$ satisfies the equation of S

$$\Rightarrow \quad F\big(x(t), y(t), z(t)\big) = k \ldots\ldots\ldots (1)$$

Differentiating both sides of the equation (1) with respect to t:

$$\Rightarrow \quad \frac{dF}{dt} = 0 \quad \Rightarrow \quad F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt} = 0 \Rightarrow \langle F_x, F_y, F_z \rangle \cdot \langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \rangle = 0$$

The point $P(x_0, y_0, z_0)$ is on the curve $C \;\Rightarrow\; \vec{r}(t_0) = \langle x_0, y_0, z_0 \rangle$

The equation (2) at the point $P(x_0, y_0, z_0) \Rightarrow\; \nabla F(x_0, y_0, z_0) \cdot \vec{r}'(t_0) = 0$

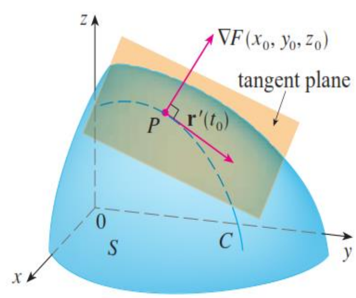
Which means that ∇F is perpendicular to the tangent vector $\vec{r}'(t)$ of C at the point P

Remark 14.6.20: Theorem 19 says the following:

∇F is normal to the surface $S: F(x, y, z) = k$ at any point on S.

So, we have the following Theorem:

Theorem 14.6.21: Let $S: F(x, y, z) = k$ be a surface, $P(x_0, y_0, z_0)$ be a point on S and let $\nabla F(x_0, y_0, z_0) = \langle a, b, c \rangle$. Then



(1) The equation of the tangent plane to the surface S at the point P is:

$$ax + by + cz = ax_0 + by_0 + cz_0$$

(2) Parametric equations of the normal line to the surface S at the point P are $x = x_0 + at, y = y_0 + bt, z = z_0 + ct$

Example 14.6.22: Find the equations of the tangent plane and the normal line at the point $(-2,1,3)$ to the ellipsoid $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$.

Solution:

- First we simplify the equation of the surface:
  $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3 \Rightarrow\; 9x^2 + 36y^2 + 4z^2 - 108 = 0$. So, let $F(x, y, z) = 9x^2 + 36y^2 + 4z^2 - 108$
  $\Rightarrow\; \nabla F = \langle 18x, 72y, 8z \rangle \Rightarrow\; \nabla F(-2,1,3) = \langle -36, 72, 24 \rangle\ (\div\ 12) \Rightarrow$ vector is $\langle -3, 6, 2 \rangle$
- The equation of the tangent plane at $(-2,1,3)$ is:
  $$-3x + 6y + 2z = -3(-2) + 6(1) + 2(3) = 18 \;\;\Rightarrow\;\; -3x + 6y + 2z = 18$$
- Param. equations of the normal line at $(-2,1,3)$ are $x = -2 - 3t,\; y = 1 + 6t,\; z = 3 + 2t$

ملاحظة: يجوز استخدام متجه موازٍ للمتجه $\langle -3,6,2 \rangle$ مثلاً بقسمته على 3 فيصبح المتجه $\langle -1, 2, \frac{2}{3} \rangle$ وبالتالي تصبح معادلات الخط العامودي (normal line) كما يلي: $x = -2 - t, y = 1 + 2t, z = 3 + \frac{2}{3}t$

Example 14.6.23: Find the equations of the tangent plane and the normal line at the point $(-2,1,5)$ to the surface $z = x^2 + y^2$.

Solution:

- Surface: $z = x^2 + y^2 \Rightarrow x^2 + y^2 - z = 0$
- Let $F(x,y,z) = x^2 + y^2 - z \Rightarrow \nabla F = \langle 2x, 2y, -1\rangle \Rightarrow \nabla F(-2,1,3) = \langle -4,2,-1\rangle$
  - The equation of the tangent plane at $(-2,1,5)$ is:
    $$-4x + 2y - z = -4(-2) + 2(1) - (5) = 5 \Rightarrow -4x + 2y - z = 5$$
  - The equations of the normal line at $(-2,1,5)$ are: $x = -2 - 4t,\ y = 1 + 2t,\ z = 5 - t$

Example 14.6.24: At what point the surface $y = x^2 + z^2$ is tangent to the plane parallel to the plane $x + y + 3z = 1$.

Solution: Let (x, y, z) be the pt. of tangency.

- Surface: $y = x^2 + z^2 \Rightarrow x^2 - y + z^2 = 0 \Rightarrow F(x,y,z) = x^2 - y + z^2 \Rightarrow \nabla F = \langle 2x, -1, 2z\rangle$
- Plane: $x + y + 3z = 1 \Rightarrow \vec{v} = \langle 1,1,3\rangle$
  - $\nabla F \,//\, \langle 1,1,3\rangle \Rightarrow \langle 2x,-1,2z\rangle \,//\, \langle 1,1,3\rangle \Rightarrow \frac{2x}{1} = \frac{-1}{1} = \frac{2z}{3}$ (Ratio Method)
    $\Rightarrow 2x = -1, 2z = -3 \Rightarrow x = -\frac{1}{4}$ and $z = -\frac{3}{4}$.
  - To find y: We substitute $x = -\frac{1}{4}, z = -\frac{3}{4}$ in $y = x^2 + z^2$:
    $$y = \left(-\frac{1}{4}\right)^2 + \left(-\frac{3}{4}\right)^2 = \frac{10}{16} \Rightarrow \text{The point is } \left(-\frac{1}{4}, \frac{10}{16}, -\frac{3}{4}\right)$$

Example 14.6.25: At what point the surface $x^2 - y^2 + z^2 - 2x = 1$ has a normal line parallel to the line $x = 4t,\ y = 1 - 2t,\ z = -2t$.

Solution: Let (x, y, z) be the pt. at which the normal line parallel to the given line

- Surface: $x^2 - y^2 + z^2 - 2x = 1 \Rightarrow x^2 - y^2 + z^2 - 2x - 1 = 0$
  $F(x,y,z) = x^2 - y^2 + z^2 - 2x - 1 \Rightarrow \nabla F = \langle 2x - 2, -2y, 2z\rangle$……..①
- Line $x = 4t,\ y = 1 - 2t,\ z = -2t$: A vector parallel to it is $\langle 4,-2,-2\rangle$
- The normal line parallel to the line $x = 4t,\ y = 1 - 2t,\ z = -2t \Rightarrow \nabla F \,//\, \langle 4,-2,-2\rangle$
  $\Rightarrow \langle 2x - 2, -2y, 2z\rangle \,//\, \langle 4,-2,-2\rangle$:
  - Ratio Method: $\frac{2x-2}{4} = \frac{-2y}{-2} = \frac{2z}{-2} \Rightarrow \frac{x-1}{2} = y = -z \Rightarrow \frac{x-1}{2} = y, -z = y$
    $\Rightarrow x = 2y + 1, z = -y$ substitute in these equation in $x^2 - y^2 + z^2 - 2x = 1$
    We have $(2y + 1)^2 - y^2 + (-y)^2 - 2(2y + 1) = 1$
    $\Rightarrow 4y^2 + 4y + 1 - y^2 + y^2 - 4y - 2 = 1 \Rightarrow 4y^2 = 2 \Rightarrow y^2 = \frac{1}{2} \Rightarrow y = \pm\frac{1}{\sqrt{2}}$
    - $y = -\frac{1}{\sqrt{2}} \Rightarrow x = 2y + 1 = 1 - \frac{2}{\sqrt{2}}, z = -\frac{1}{\sqrt{2}}$
    - $y = \frac{1}{\sqrt{2}} \Rightarrow x = 2y + 1 = 11\frac{2}{\sqrt{2}}, z = \frac{1}{\sqrt{2}}$
    - We have two pts.: $\left(1 - \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(1 + \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Example 14.6.26: At what points does the normal line through the point $(1,1,2)$ on the ellipsoid $4x^2 + y^2 + 4z^2 = 21$ intersects the sphere $x^2 + y^2 + z^2 = 6$

Solution:

- Surface: $4x^2 + y^2 + 4z^2 = 21 \Rightarrow 4x^2 + y^2 + 4z^2 - 21 = 0$
  $\Rightarrow F(x,y,z) = 4x^2 + y^2 + 4z^2 - 21 \Rightarrow \nabla F = \langle 8x, 2y, 8z\rangle$
  $\Rightarrow \nabla F(1,1,2) = \langle 8,2,16\rangle \Rightarrow \langle 8,2,16\rangle$ // normal line $(\div 2) \Rightarrow \langle 4,1,8\rangle$ // normal line
- Equations of the normal line are: $x = 1 + 4t, y = 1 + t, z = 2 + 8t$
- The normal line intersects the sphere $x^2 + y^2 + z^2 = 6$. So,
  - Substitute $(x = 1 + 4t, y = 1 + t, z = 2 + 8t)$ in the equation $x^2 + y^2 + z^2 = 6$:

    $(1 + 4t)^2 + (1 + t)^2 + (2 + 8t)^2 = 6$

    $\Rightarrow 1 + 8t + 16t^2 + 1 + 2t + t^2 + 4 + 32t + 64t^2 = 6 \Rightarrow 81t^2 + 42t = 0$

    $\Rightarrow t(80t + 42) = 0 \Rightarrow t = 0$ or $t = -\frac{42}{81}$:

    - $t = 0 \Rightarrow \begin{cases} x = 1 + 4t \Rightarrow x = 1 + 0 \Rightarrow x = 1 \\ y = 1 + t \Rightarrow y = 1 + 0 \Rightarrow y = 1 \\ z = 2 + 8t \Rightarrow z = 2 + 0 \Rightarrow z = 2 \end{cases}$
    - $t = -\frac{42}{81}: \Rightarrow \begin{cases} x = 1 + 4t \Rightarrow x = 1 - 4\left(\frac{42}{81}\right) \Rightarrow x = -\frac{87}{81} \\ y = 1 + t \Rightarrow y = 1 - \left(\frac{42}{81}\right) \Rightarrow y = \frac{39}{81} \\ z = 2 + 8t \Rightarrow z = 2 - 8\left(\frac{42}{81}\right) \Rightarrow z = -\frac{174}{81} \end{cases}$
- The points are: $(1,1,2)$ and $\left(-\frac{87}{81}, \frac{39}{81}, -\frac{174}{81}\right)$

Example 14.6.27: Where does the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1,1,2)$ intersects the paraboloid a second time.

Solution:

- Surface: $z = x^2 + y^2 \Rightarrow x^2 + y^2 - z = 0 \Rightarrow F(x,y,z) = x^2 + y^2 - z$
  $\Rightarrow \nabla F = \langle 2x, 2y, -1\rangle \Rightarrow \nabla F(1,1,2) = \langle 2,2,-1\rangle$
- The equations of the normal line are: $x = 1 + 2t, y = 1 + 2t, z = 2 - t$
- The normal line intersects the paraboloid $z = x^2 + y^2$. So:
  - Substitute $(x = 1 + 2t, y = 1 + 2t, z = 2 - t)$ in the equation $z = x^2 + y^2$:

    $2 - t = (1 + 2t)^2 + (1 + 2t)^2 \Rightarrow 2 - t = 2(1 + 4t + 4t^2) \Rightarrow 8t^2 + 9t = 0$

    $\Rightarrow t(8t + 9) = 0 \Rightarrow t = 0$ or $t = -\frac{9}{8}$

    - $t = 0: \Rightarrow \begin{cases} x = 1 + 2t \Rightarrow x = 1 + 0 \Rightarrow x = 1 \\ y = 1 + 2t \Rightarrow y = 1 + 0 \Rightarrow y = 1 \\ z = 2 - t \Rightarrow z = 2 - 0 \Rightarrow z = 2 \end{cases}$
    - $t = -\frac{9}{8}: \Rightarrow \begin{cases} x = 1 + 2t \Rightarrow x = 1 + 2\left(-\frac{9}{8}\right) \Rightarrow x = -\frac{10}{8} \\ y = 1 + 2t \Rightarrow y = 1 + 2\left(-\frac{9}{8}\right) \Rightarrow y = -\frac{10}{8} \\ z = 2 - t \Rightarrow z = 2 - \left(-\frac{9}{8}\right) \Rightarrow z = \frac{25}{8} \end{cases}$
- The points are: $(1,1,2)$ and $\left(-\frac{10}{8}, -\frac{10}{8}, \frac{25}{8}\right)$

$\Rightarrow$ the normal line to the paraboloid $z = x^2 + y^2$ at the point $(1,1,2)$ intersects the paraboloid a second time at $\left(-\frac{10}{8}, -\frac{10}{8}, \frac{25}{8}\right)$.

Example 14.6.28: Show that every plane that is tangent to the cone $z^2 = x^2 + y^2$ passes through the origin.

Solution: Let (a, b, c) be a point on the cone $z^2 = x^2 + y^2 \Rightarrow c^2 = a^2 + b^2$ …………..①

Now, we find the equation of the tangent plane to the cone:

- $z^2 = x^2 + y^2 \;\Rightarrow\; x^2 + y^2 - z^2 = 0$. Let $F(x, y, z) = x^2 + y^2 - z^2 \Rightarrow \nabla F = \langle 2x, 2y, -2z \rangle \Rightarrow \nabla F(a, b, c) = \langle 2a, 2b, -2c \rangle$
  $\nabla F(a, b, c) \perp$ tangent plane $\Rightarrow \langle 2a, 2b, -2c \rangle \perp$ tangent plane $\div 2$
  $\Rightarrow \langle a, b, -c \rangle \perp$ tangent plane and (a, b, c) is a point on the tangent plane
- The equation of the tangent plane is: $ax + by - cz = a^2 + b^2 - c^2 = 0$ (by equation ①)
  $\Rightarrow\; ax + by - cz = 0$ …………②
- substituting the origin in the equation ② we have:
  - $a(0) + b(0) - c(0) = 0 \Rightarrow$ The origin satisfies the equation ②
    So, the origin lies on the tangent plane which means that:
    the tangent plane passes through the origin.

Example 14.6.29: Show that every normal line to the sphere $x^2 + y^2 + z^2 = r^2$ passes through the center of the sphere.

Solution: Let (a, b, c) be a point on the sphere $x^2 + y^2 + z^2 = r^2$.

- First, we find the equations of the normal line to the sphere $x^2 + y^2 + z^2 = r^2$ at (a, b, c):

  $x^2 + y^2 + z^2 = r^2 \Rightarrow x^2 + y^2 + z^2 - r^2 = 0$. Let $F(x, y, z) = x^2 + y^2 + z^2 - r^2$

  $\Rightarrow \nabla F = \langle 2x, 2y, 2z \rangle \Rightarrow \nabla F(a, b, c) = \langle 2a, 2b, 2c \rangle$

- $\nabla F(a, b, c)$ // normal line: $\Rightarrow \langle 2a, 2b, 2c \rangle$ // normal line $\div 2$

  $\Rightarrow \langle a, b, c \rangle$ // normal line and (a, b, c) is a point on the normal line

- Equations of the normal line: $x = a + at, y = b + bt, z = c + ct$
  - To show that the normal line passes through the center of $x^2 + y^2 + z^2 = r^2$:

    Observe that the center of the sphere is $(0,0,0)$:

$$\text{So, taking } t = -1 \;\Rightarrow\; \begin{Bmatrix} x = a + at & \Rightarrow & x = a - a = 0 \\ y = b + bt & \Rightarrow & y = b - b = 0 \\ z = c + ct & \Rightarrow & z = c - c = 0 \end{Bmatrix}$$

  - the normal line passes through the origin which is the center of the sphere.

Section 14.7: Maximum and Minimum Values

Definition 14.7.1: A function $f(x, y)$ is said to have:

(1) a local maximum value at a point $(a, b) \in \text{Dom}(f)$ if $f(a, b) \geq f(x, y)$ for all $(x, y) \in D$, where D is a disk in Domain f centered at (a, b). The number $f(a, b)$ is called a local maximum value of f.

(2) a local minimum value at a point $(a, b) \in \text{Dom}(f)$ if $f(a, b) \leq f(x, y)$ for all $(x, y) \in D$, where D is a disk in Domain f centered at (a, b). The number $f(a, b)$ is called a local minimum value of f.

(3) an absolute maximum value at a point $(a, b) \in \text{Dom}(f)$ if $f(a, b) \geq f(x, y)$ for all $(x, y) \in \text{Dom}(f)$. The number $f(a, b)$ is called the absolute maximum value of f.

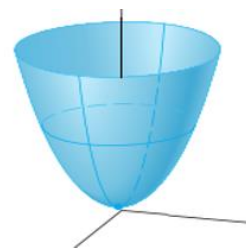
(4) an absolute minimum value at a point $(a, b) \in \text{Dom}(f)$ if $f(a, b) \leq f(x, y)$ for all $(x, y) \in \text{Dom}(f)$. The number $f(a, b)$ is called the absolute minimum value of f.

(5) a local extremum at a point (a, b) if f has a local maximum or minimum value at (a, b).

(6) an absolute extremum at a point (a, b) if f has an absolute maximum or minimum value at (a, b).

Example 14.7.2: Find the absolute and local extrema of the function $f(x, y) = 2x^2 + y^2$

Solution: First, we give a graph of the function f:



From the graph we see that:

- f has a local minimum value at $(0,0)$. This local minimum value is $f(0,0) = 0$
- f has an absolute minimum value at $(0,0)$.
- The absolute minimum value is $f(0,0) = 0$

Part 1: Local Maximum and Minimum Values

Definition 14.7.3: A function $f(x, y)$ is said to have a critical point at $(a, b) \in Dom(f)$ if:

- $f_x(a, b) = 0$ and $f_y(a, b) = 0$

or

- $f_x(a, b)$ does not exist

or

- $f_y(a, b)$ does not exist

Example 14.7.4: Find the values of a and b that makes the function f has a critical point at $(1, -1)$, where $f(x, y) = x^2y + 3axy^2 - bxy$.

Solution:

- $f_x = 2xy + 3ay^2 - by$ and $f_y = x^2 + 6axy - bx$
- $(1, -1)$ is a critical point $f_x(1, -1) = 0$ and $f_y(1, -1) = 0$
  - $f_x(1, -1) = 0 \Rightarrow -2 + 3a + b = 0 \Rightarrow 3a + b = 2 \ldots\ldots\ldots$ ①
  - $f_y(1, -1) = 0 \Rightarrow 1 - 6a - b = 0 \Rightarrow 6a + b = 1 \ldots\ldots\ldots$ ②

② - ① $\Rightarrow 3a = -1 \Rightarrow a = -\frac{1}{3}$. ① $\Rightarrow 3a + b = 2 : 1 + b = 2 \Rightarrow b = 1$

Theorem 14.7.5: If a function $f(x, y)$ has a local maximum or minimum value at (a, b) and $f_x(a, b), f_y(a, b)$ both exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$

The Second Derivative Test 14.7.6: Suppose that the second partial derivatives of the function $f(x,y)$ are continuous on a disk centered at a point (a,b) and suppose that $f_x(a,b)=0$ and $f_y(a,b)=0$.

Let $D(a,b)=f_{xx}(a,b)f_{yy}(a,b)-\left[f_{xy}(a,b)\right]^2$

(1) If $D(a,b)>0$ and $f_{xx}(a,b)>0$, then f has a local minimum value at (a,b). This local minimum value equals to $f(a,b)$.

(2) If $D(a,b)>0$ and $f_{xx}(a,b)<0$, then f has a local maximum value at (a,b). This local maximum value equals to $f(a,b)$.

(3) If $D(a,b)<0$, then f has neither a local maximum value nor a local minimum value at (a,b). In this case we say that f has a saddle point at (a,b).

Example 14.7.7: Find and classify the critical points of the function f as local maximum, local minimum, or saddle point, where $f(x,y)=2x^3+6xy^2-3y^3-150x$. Moreover find the local maximum and minimum values of f.

Solution:

- **Step 1:** We must find the critical points of f: $f_x=6x^2+6y^2-150$ and $f_y=12xy-9y^2$
  - $f_x=0\Rightarrow 6x^2+6y^2-150=0\Rightarrow(6x^2+6y^2=150)\div 6\Rightarrow x^2+y^2=25\ldots\ldots ①$
  - $f_y=0\Rightarrow(12xy-9y^2=0)\div 3\Rightarrow 4xy-3y^2=0\Rightarrow y(4x-3y)=0$. So,

$$y=0\ldots\ldots ② \text{ or } y=\frac{4}{3}x\ldots\ldots ③$$

Case1: Eqs. ①&②: $x^2+y^2=25$ and $y=0$

① $\Rightarrow x^2=25\Rightarrow x=\pm5\Rightarrow$ two critical points $(\pm5,0)$

Case2: Eqs. ①&②: $x^2+y^2=25$ and $y=\frac{4}{3}x$:

① $\Rightarrow x^2+\frac{16}{9}x^2=25\Rightarrow\frac{25}{9}x^2=25\Rightarrow x^2=9\Rightarrow x=\pm3$

- If $x=-3$: $y=\frac{4}{3}x\Rightarrow y=-4\Rightarrow$ one point $(-3,-4)$
- If $x=3$: $y=\frac{4}{3}x\Rightarrow y=4\Rightarrow$ one point $(3,4)$

f has four critical points: $(-5,0),(5,0),(-3,-4),(3,4)$

- **Step 2:** We classify the critical points:
  - $f_{xx}=12x$, $f_{yy}=12x-18y$, and $f_{xy}=12y$ $D=f_{xx}f_{yy}-\left[f_{xy}\right]^2$

$$D=12x(12x-18y)-(12y)^2$$

- **At $(-5,0)$**: $D(-5,0)=12(-5)\big(12(-5)\big)>0$ and $f_{xx}(-5,0)=12(-5)<0$:
  - $\Rightarrow f$ has a local maximum value at the point $(-5,0)$
  - A local maximum value of f is $f(-5,0)=2(-5)^3-150(-5)=500$
- **At $(5,0)$**: $D(-5,0)=12(5)\big(12(5)\big)>0$ and $f_{xx}(5,0)=12(5)>0$:
  - $\Rightarrow f$ has a local minimum value at the point $(5,0)$
  - A local minimum value of f is $f(5,0)=2(5)^3-150(5)=-500$
- **At $(-3,-4)$**: $D(-3,-4)=12(-3)\big(12(-3)-18(-4)\big)-\big(12(-4)\big)^2<0$
  - $\Rightarrow f$ has a saddle point at $(-3,-4)$
- **At $(3,4)$**: $D(3,4)=12(3)\big(12(3)-18(4)\big)-\big(12(4)\big)^2<0$
  - $\Rightarrow f$ has a saddle point at $(3,4)$

Example 14.7.8: Find the local maximum and local minimum values of the function

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

Solution:

❖ **Step 1:** We must find the critical points of f: $f_x = 4x^3 - 4y$ and $f_y = 4y^3 - 4x$

➢ $f_x = 0 \Rightarrow 4x^3 - 4y = 0 \Rightarrow y = x^3$ ……….. ①

➢ $f_y = 0 \Rightarrow 4y^3 - 4x = 0 \Rightarrow x = y^3$ ………..②

Substitute equation ① in ②: $x = (x^3)^3 \Rightarrow x = x^9 \Rightarrow x = 0, -1, 1$

- If $x = 0$: ① $\Rightarrow y = 0^3 = 0 \Rightarrow$ one critical points $(0,0)$
- If $x = -1$: ① $\Rightarrow y = (-1)^3 = -1 \Rightarrow$ one critical points $(-1,-1)$
- If $x = 1$: ① $\Rightarrow y = (1)^3 = 1 \Rightarrow$ one critical points $(1,1)$

❖ f has three critical points: $(0,0), (-1,-1), (1,1)$

❖ **Step 2:** We classify the critical points: $f_{xx} = 12x^2$, $f_{yy} = 12y^2$ and $f_{xy} = -4$

$$D = f_{xx}f_{yy} - [f_{xy}]^2 \Rightarrow D = 12x^2(12y^2) - (-4)^2$$

➢ **At $(0,0)$**: $D(0,0) = -16 < 0$: $\Rightarrow f$ has a saddle point at $(0,0)$
- f has neither a local maximum nor a local minimum at $(0,0)$

➢ **At $(-1,-1)$**: $D(-1,-1) = 12(12) - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0$:
- f has a local minimum value at the point $(-1,-1)$
- $f(-1,-1) = -1$ is a local minimum value of f

➢ **At $(1,1)$**: $D(1,1) = 12(12) - 16 > 0$ and $f_{xx}(1,1) = 12 > 0$:
- f has a local minimum value at the point $(1,1)$
- $f(1,1) = -1$ is a local minimum value of f

Example 14.7.9: Find and classify the critical points of the function f as local maximum, local minimum, or saddle point, where $f(x,y) = x^2 + y^2 - 2x - 6y + 12$.

Solution: $f_x = 2x - 2 = 0 \Rightarrow x = 1$ and $f_y = 2y - 6 = 0 \Rightarrow y = 3$

❖ f has only one critical point: $(1,3)$

➢ $f_{xx} = 2$, $f_{yy} = 2$, and $f_{xy} = 0 \Rightarrow D = f_{xx}f_{yy} - [f_{xy}]^2 \Rightarrow D = 4$

➢ **At $(1,3)$:** $D(1,3) = 4 > 0$ and $f_{xx}(1,3) = 2 > 0$:

$\Rightarrow f$ has a local minimum value at the point $(1,3)$

Part 2: Absolute Maximum and Minimum Values of Functions with Only One Critical point

Part 2.1: Absolute Extrema for functions with Domain $\mathbb{R}^2$ and with only one critical pt

Theorem 14.7.10: Let $f(x,y)$ be with domain $\mathbb{R}^2$ and has only one critical point at (a,b).

(1) If $f(a,b)$ is a local maximum value of the function f, then $f(a,b)$ is an absolute maximum value of f.

(2) If $f(a,b)$ is a local minimum value of the function f, then $f(a,b)$ is an absolute minimum value of f.

Example 14.7.11: Find the absolute maximum and minimum values of the function $f(x,y) = x^2 + y^2 - 2x - 6y + 12$.

Solution: From Example 9 we see that this function has only one critical point at $(1,3)$ and at this point f has a local minimum value. So by Theorem 10, f has an absolute minimum value at $(1,3)$

$\Rightarrow$ The absolute minimum value of f is $f(1,3) = 2$.

Observe that the function f has no absolute maximum value because it has only one critical point.

Example 14.7.12:

(1) Find the point on the plane $x + 2y + z = 4$ which is the closest to the point $(1,0,-2)$.

(2) Find the shortest distance from the point $(1,0,-2)$ to the plane $x + 2y + z = 4$.

Solution:

(1) Let (x,y,z) be a pt on the plane $x + 2y + z = 4$ which is the closest to the pt $(1,0,-2)$. The distance from the point $(1,0,-2)$ to the point (x,y,z) is

$$d = \sqrt{(x-1)^2 + (y-0)^2 + (z-(-2))^2}$$

$$\Rightarrow d^2 = (x-1)^2 + y^2 + (z+2)^2 \ldots\ldots\ldots\ldots (1)$$

Since $z = 4 - x - 2y$ substituting this in the equation (1) we have:

$$\Rightarrow d^2 = (x-1)^2 + y^2 + (4 - x - 2y + 2)^2$$

$$= (x-1)^2 + y^2 + (6 - x - 2y)^2$$

Let $f(x,y) = d^2 \Rightarrow f(x,y) = (x-1)^2 + y^2 + (6 - x - 2y)^2$

Want to find the point at which the absolute minimum value of f occurs:

- $f_x = 2(x-1) + 2(6 - x - 2y)(-1) \Rightarrow f_x = 4x + 4y - 14 = 0 \Rightarrow 2x + 2y = 7 \ldots\ldots\ldots (1)$
- $f_y = 2y + 2(6 - x - 2y)(-2) \Rightarrow f_y = 4x + 10y - 24 = 0 \Rightarrow 2x + 5y = 12 \ldots\ldots\ldots (2)$

  $(2) - (1) \Rightarrow 3y = 5 \Rightarrow y = \frac{5}{3}$. $(1) \Rightarrow 2x + 2\left(\frac{5}{3}\right) = 7 \Rightarrow x = \frac{11}{6}$

  The function f has only one critical point at $\left(\frac{11}{6}, \frac{5}{3}\right)$

- $f_{xx} = 4, f_{yy} = 10$, and $f_{xy} = 4 \Rightarrow D = (f_{xx})(f_{yy}) - [f_{xy}]^2$

  $D = 40 - 16 = 24 > 0$ and $f_{xx} = 4 > 0 \Rightarrow f$ has a local minimum value at $\left(\frac{11}{6}, \frac{5}{3}\right)$
- Since f has only one critical point at $\left(\frac{11}{6}, \frac{5}{3}\right)$ and $f\left(\frac{11}{6}, \frac{5}{3}\right)$ is a local minimum value of f, then f has an absolute minimum value at the pt $\left(\frac{11}{6}, \frac{5}{3}\right)$

  $\Rightarrow$ The closest pt is when $x = \frac{11}{6}$ and $y = \frac{5}{3}$. To find the value of z:

  Substitute $x = \frac{11}{6}$ and $y = \frac{5}{3}$ in the eq of the plane: $x + 2y + z = 4$

  $\Rightarrow z = 4 - \frac{11}{6} - 2\left(\frac{5}{3}\right) = -\frac{7}{6}$

  $\therefore$ The closest pt is $\left(\frac{11}{6}, \frac{5}{3}, -\frac{7}{6}\right)$

(2) The absolute minimum value of f is $f\left(\frac{11}{6}, \frac{5}{3}\right)$

$\Rightarrow$ The shortest distance is $d = \sqrt{f\left(\frac{11}{6}, \frac{5}{3}\right)} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{6}\right)^2} = \frac{5\sqrt{6}}{6}$

Another Solution for Part (2): Since $x + 2y + z = 4$ is an equation of a plane, we can use the law of distance from a point and a plane:

$$x + 2y + z = 4 \Rightarrow \ x + 2y + z - 4 = 0, \text{ point } (1,0,-2)$$

$$\text{The shortest distance is } d = \frac{|1 + 2(0) + (-2) - 4|}{\sqrt{1^2 + 2^2 + 1^2}} = \frac{|-5|}{\sqrt{6}} = \frac{5}{\sqrt{6}} = \frac{5\sqrt{6}}{6}$$

Part 2.2: Absolute Extrema for functions with known Ranges

Remark 14.7.13: Let $f(x,y)$ be a function with range$(f) = S$, where S is a set in $\mathbb{R}$, then

- The absolute maximum value of f = maximum value in S
- The absolute minimum value of f = minimum value in S

Example 14.7.14: Find the absolute maximum and minimum values of the function

$$f(x,y) = 5 - \sqrt{9 - x^2 - y^2}$$

Solution: First we find the range of f: Let $z = 5 - \sqrt{9 - x^2 - y^2}$

- $\sqrt{9 - x^2 - y^2} \geq 0 \ \Rightarrow -\sqrt{9 - x^2 - y^2} \leq 0 \Rightarrow 5 - \sqrt{9 - x^2 - y^2} \leq 5 \ \Rightarrow \ z \leq 5$……..①
- $z = 5 - \sqrt{9 - (x^2 + y^2)}$: $(x^2 + y^2) \geq 0 \Rightarrow 9 - (x^2 + y^2) \leq 9 \Rightarrow \sqrt{9 - (x^2 + y^2)} \leq \sqrt{9}$

  $\Rightarrow \ \sqrt{9 - (x^2 + y^2)} \leq 3 \Rightarrow \ 5 - \sqrt{9 - (x^2 + y^2)} \geq 5 - 3 \Rightarrow z \geq 2$…………②

①&② $\Rightarrow$ range$(f) = [2,5]$

- The absolute maximum value of f = maximum value in $[2,5] = 5$
- The absolute minimum value of f = minimum value in $[2,5] = 2$

The following is Example 11 with another solution:

Example 14.7.15: Find the absolute maximum and minimum values of the function

$$f(x,y) = x^2 + y^2 - 2x - 6y + 12.$$

Solution: First we find range(f): Let $z = x^2 + y^2 - 2x - 6y + 12$

$$\Rightarrow z = (x-1)^2 + (y-3)^2 + 2$$

$$(x-1)^2 + (y-3)^2 \geq 0 \Rightarrow \ (x-1)^2 + (y-3)^2 + 2 \geq 2 \ \Rightarrow \ z \geq 2 \Rightarrow \text{range}(f) = [2,\infty)$$

- The absolute minimum value of f is 2
- f has no absolute maximum value.

Part 2.3: Absolute Extrema for functions over closed bounded sets

(a) Closed sets

Definition 14.7.16:

(1) A closed set in $\mathbb{R}^2$ is a set that contains all its boundary points, where a boundary point of a set D is a point (a,b) such that every disk with center (a,b) contains points in D and also points not in D.

(2) A bounded set in $\mathbb{R}^2$ is a set that is contained within some disk.

(b) Sets that are not closed

Extreme Value Theorem for Functions of two variables 14.7.17:

If $f(x,y)$ is continuous on a closed, bounded set D in $\mathbb{R}^2$, then attains an absolute maximum value $f(x_1,y_1)$ and an absolute minimum value $f(x_2,y_2)$ at some points (x_1,y_1) and (x_2,y_2) in D.

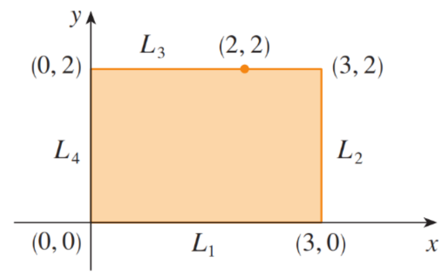
Remark 14.7.18: To find the absolute maximum and minimum values of a continuous function $f(x,y)$ on a closed, bounded set D:

Step 1. Find the values of at the critical points of $f(x,y)$ in D.

Step 2. Find the extreme values of $f(x,y)$ on the boundary of D.

Step 3. The largest of the values from steps 1 and 2 is the absolute maximum value and the smallest of these values is the absolute minimum value.

Example 14.7.19: Find the absolute maximum and minimum values of the function $f(x,y) = x^2 - 2xy + 2y$ on the rectangle $D = \{(x,y): 0 \le x \le 3, 0 \le y \le 2\}$.



Solution:

Step 1: We find the critical points of f in D:

$$f_x = 2x - 2y \text{ and } f_y = -2x + 2$$

$$\left.\begin{aligned} f_x = 0 &\Rightarrow 2x - 2y = 0 \Rightarrow y = x \\ f_y = 0 &\Rightarrow -2x + 2 = 0 \Rightarrow x = 1 \end{aligned}\right\} \Rightarrow y = 1 \Rightarrow (1,1) \text{ is a critical point of } f$$

Check: Is $(1,1) \in D$? Yes

Step 2: We find the extreme values of f on the boundary of D:

Observe that the boundary of D consists of 4 line segments: L_1, L_2, L_3, L_4:

- On L_1 $(y=0)$: $g_1(x) = f(x,0) = x^2 - 2x(0) + 2(0) = x^2, 0 \le x \le 3$.
  - $g_1'(x) = 2x, 0 < x < 3 \Rightarrow g_1'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \notin \underbrace{(0,3)}_{\text{interval}}$
  - We only have the end points when $x = 0$ and $x = 3 \Rightarrow$ the points are $(0,0), (3,0)$
- On L_2 $(x=3)$: $h_1(y) = f(3,y) = (3)^2 - 2(3)y + 2y = 9 - 4y, 0 \le y \le 2$.
  - $h_1'(y) = -4, 0 < y < 2 \Rightarrow h_1'(y) \ne 0, \forall y \in \underbrace{(0,2)}_{\text{interval}}$
  - We only have the end points when $y = 0$ and $y = 2 \Rightarrow$ the points are $(3,0), (3,2)$.
- On L_3 $(y=2)$: $g_2(x) = f(x,2) = x^2 - 2x(2) + 2(2) = x^2 - 4x + 4, 0 \le x \le 3$.
  - $g_2'(x) = 2x - 4, 0 < x < 3 \Rightarrow g_2'(x) = 0 \Rightarrow 2x - 4 = 0 \Rightarrow x = 2 \in \underbrace{(0,3)}_{\text{interval}}$
  - We three points: when $x = 2$ and at the end points when $x = 0$ and $x = 3$ $\Rightarrow$ the points are $(2,2), (0,2), (3,2)$
- On L_4 $(x=0)$: $h_2(y) = f(0,y) = (0)^2 - 2(0)y + 2y = 2y, 0 \le y \le 2$.
  - $h_2'(y) \ne 0, \forall y \in \underbrace{(0,2)}_{\text{interval}}$
  - We only have the end points when $y = 0$ and $y = 2 \Rightarrow$ the points are $(0,0), (0,2)$.

Step 3: $f(x,y) = x^2 - 2xy + 2y$

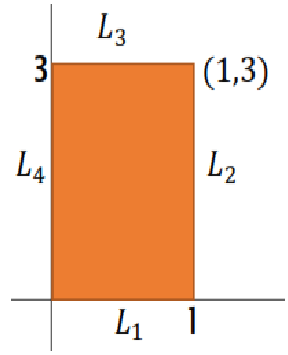
Points	(1,1)	(0,0)	(3,0)	(3,2)	(2,2)	(0,2)
$f(x,y)$	1	0	9	1	0	4

- The absolute maximum value of f is 9
- The absolute minimum value of f is 0

Example 14.7.20: Find the absolute maximum and minimum values of the function

$f(x,y) = x^2 - 2xy - y^2 + 8y - 1$ on the rectangle $D = \{(x,y): 0 \le x \le 1, 0 \le y \le 3\}$.

Solution:



Step 1: We find the critical points of f in D:

$f_x = 2x - 2y$ and $f_y = -2x + 2y + 8$

$$\left.\begin{aligned} f_x = 0 &\Rightarrow 2x - 2y = 0 &&\Rightarrow x - y = 0 \\ f_y = 0 &\Rightarrow -2x - 2y + 8 = 0 &&\Rightarrow x + y = 4 \end{aligned}\right\}$$

$\Rightarrow x = 2, y = 2 \Rightarrow (2,2)$ is a critical point of f

Check: Is $(2,2) \in D$? No. $\Rightarrow$ We do not have any critical point for f in step 1

Step 2: We find the extreme values of f on the boundary of D:

Observe that the boundary of D consists of 4 line segments: L_1, L_2, L_3, L_4:

- On L_1 $(y = 0)$: $g_1(x) = f(x,0) = x^2, 0 \le x \le 1$
  - $g_1'(x) = 2x, 0 < x < 1 \Rightarrow g_1'(x) = 0 \Rightarrow 2x = 0 \Rightarrow x = 0 \notin \underbrace{(0,1)}_{\text{interval}}$
  - We only have the end points when $x = 0$ and $x = 1 \Rightarrow$ the points are $(0,0), (1,0)$
- On L_2 $(x = 1)$: $h_1(y) = f(1,y) = -y^2 + 6y + 1, 0 \le y \le 3.$
  - $h_1'(y) = -2y + 6, 0 < y < 3$: $h_1'(y) = 0 \Rightarrow -2y + 6 = 0 \Rightarrow y = 3 \notin \underbrace{(0,3)}_{\text{interval}}$
  - We only have the end points when $y = 0$ and $y = 3$: $\Rightarrow$ the points are $(1,0), (1,3)$.
- On L_3 $(y = 3)$: $g_2(x) = f(x,3) = x^2 - 6x + 15, 0 \le x \le 1$:
  - $g_2'(x) = 2x - 6, 0 < x < 1$ $g_2'(x) = 0 \Rightarrow 2x - 6 = 0 \Rightarrow x = 3 \notin \underbrace{(0,1)}_{\text{interval}}$:
  - We only have the end points when $x = 0$ and $x = 1 \Rightarrow$ the points are $(0,3), (1,3)$
- On L_4 $(x = 0)$: $h_2(y) = f(0,y) = -y^2 + 8y, 0 \le y \le 3.$
  - $h_2'(y) = -2y + 8, 0 < y < 2$: $h_2'(y) = 0 \Rightarrow -2y + 8 = 0 \Rightarrow y = 4 \notin \underbrace{(0,3)}_{\text{interval}}$
  - We only have the end points when $y = 0$ and $y = 3 \Rightarrow$ the points are $(0,0), (0,3)$

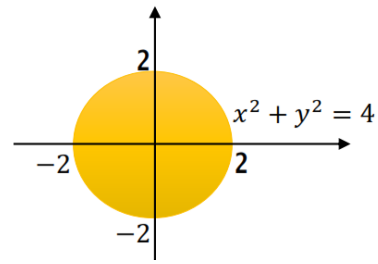
Step 3: $f(x,y) = x^2 - 2xy - y^2 + 8y - 1$

Points	(0,0)	(1,0)	(1,3)	(0,3)
$f(x,y)$	-1	0	9	14

- The absolute maximum value of f is 14
- The absolute minimum value of f is -1

Example 14.7.21: Find the absolute maximum and minimum values of the function

$f(x,y) = xy^2$ on the disk $D = \{(x,y): x^2 + y^2 \leq 4\}$.



Solution:

Step 1: We find the critical points of f in D:

$f_x = y^2$ and $f_y = 2xy$

- $f_x = 0 \Rightarrow y^2 = 0 \Rightarrow y = 0 \ldots\ldots\ldots$ ①
- $f_y = 0 \Rightarrow 2xy = 0 \Rightarrow \begin{cases} x = 0 \ldots. ② \\ y = 0 \ldots. ③ \end{cases}$

Case 1: ①&② $\Rightarrow y = 0 \,\&\, x = 0 \Rightarrow$ we have one critical point $(0,0)$ in D

Case 2: ①&③ $\Rightarrow y = 0 \,\&\, y = 0 \Rightarrow y = 0, \forall x \in \underbrace{[-2,2]}_{\text{interval}}$

$\Rightarrow$ we have infinitely many critical points $(x,0), x \in \underbrace{[-2,2]}_{\text{interval}}$ in D

Step 2: We find the extreme values of f on the boundary of D:

Observe that the boundary of D is the circle: $x^2 + y^2 = 4$

❖ **On $x^2 + y^2 = 4$:** $y^2 = 4 - x^2$. So, let $g(x) = f(x,y)|_{y^2=4-x^2} = x(4 - x^2) = 4x - x^3$

$\Rightarrow g(x) = 4x - x^3, -2 \leq x \leq 2$

- $g'(x) = 4 - 3x^2, -2 < x < 2 \Rightarrow g'(x) = 0 \Rightarrow 4 - 3x^2 = 0$

  $\Rightarrow x^2 = \frac{4}{3} \Rightarrow x = \pm\frac{2}{\sqrt{3}} \in \underbrace{(-2,2)}_{\text{interval}}$

- We have 4 points when $x = \pm\frac{2}{\sqrt{3}}$ and at the end points when $x = \pm 2$:
  - If $x = \frac{2}{\sqrt{3}}$: $y^2 = 4 - \left(\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3} \Rightarrow y = \pm\frac{\sqrt{8}}{\sqrt{3}}$

    $\Rightarrow$ We have two points: $\left(\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right), \left(\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$
  - If $x = -\frac{2}{\sqrt{3}}$: $y^2 = 4 - \left(-\frac{2}{\sqrt{3}}\right)^2 = 4 - \frac{4}{3} = \frac{8}{3} \Rightarrow y = \pm\frac{\sqrt{8}}{\sqrt{3}}$

    $\Rightarrow$ We have two points: $\left(-\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right), \left(-\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$
  - $x = -2 \Rightarrow y^2 = 4 - (-2)^2 = 0 \Rightarrow$ point $(-2,0)$
  - $x = 2 \Rightarrow y^2 = 4 - (2)^2 = 0 \Rightarrow$ point $(2,0)$

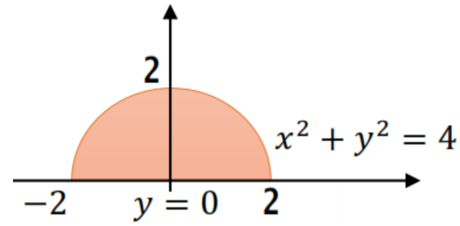
Step 3: $f(x,y) = xy^2$

Point	$(x,0), x \in [-2,2]$	$\left(\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right)$	$\left(\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$	$\left(-\frac{2}{\sqrt{3}}, -\frac{\sqrt{8}}{\sqrt{3}}\right)$	$\left(-\frac{2}{\sqrt{3}}, \frac{\sqrt{8}}{\sqrt{3}}\right)$
$f(x,y)$	0	$\frac{16}{3\sqrt{3}}$	$\frac{16}{3\sqrt{3}}$	$-\frac{16}{3\sqrt{3}}$	$-\frac{16}{3\sqrt{3}}$

- The absolute maximum value of f is $\frac{16}{3\sqrt{3}}$
- The absolute minimum value of f is $-\frac{16}{3\sqrt{3}}$

Example 14.7.22: Find the absolute maximum and minimum values of the function $f(x,y) = x^4 + 2x^2y^2 + y^4 + 8y - 1$ on the half disk $D = \{(x,y): x^2 + y^2 \le 4, y \ge 0\}$.

Solution:



Step 1: We find the critical points of f in D:

- $f_x = 4x^3 + 4xy^2$ and $f_y = 4x^2y + 4y^3 + 8$

  $f_x = 0 \Rightarrow 4x^3 + 4xy^2 = 0 \Rightarrow 4x(x^2 + y^2) = 0$

  $\Rightarrow \begin{cases} x = 0 \ldots\ldots\ldots \text{①} \\ x^2 + y^2 = 0 \Rightarrow x = 0 \text{ and } y = 0 \ldots\ldots\ldots \text{②} \end{cases}$

  $f_y = 0 \Rightarrow 4x^2y + 4y^3 + 8 = 0 \ldots\ldots\ldots \text{③}$

Case 1: ① & ③: $x = 0$ & $4x^2y + 4y^3 + 8 = 0 \Rightarrow 4(0)^2y + 4y^3 + 8 = 0$

$\Rightarrow 4y^3 + 8 = 0 \Rightarrow y^3 = -2 \Rightarrow y = -\sqrt[3]{2}$

$\Rightarrow (0, -\sqrt[3]{2})$ but $(0, -\sqrt[3]{2}) \notin D$. So in this case we do not have critical point of f.

Case 2: ② & ③: $x = 0, y = 0$ & $4x^2y + 4y^3 + 8 = 0$

$\Rightarrow 4(0)^2(0) + 4(0)^3 + 8 = 0 \Rightarrow 8 = 0$ which is impossible.

So, again in this case we do not have any critical point of f

Step 2: We find the extreme values of f on the boundary of D:

The boundary of D consists of two parts: $C: x^2 + y^2 = 4$ and $L: y = 0$

- On $C: x^2 + y^2 = 4$: Observe that $f(x,y) = (x^2 + y^2)^2 + 8y - 1$

  Let $h(y) = f(x,y)|_{x^2 = 4 - y^2} = (4)^2 + 8y - 1 = 8y + 15$

  $\Rightarrow h(y) = 8y + 15, 0 \le y \le 2 \Rightarrow h'(y) = 8, 0 < y < 2$

  $\Rightarrow h'(y) \ne 0, \forall y \in \underbrace{(0,2)}_{\text{interval}}$

  We have only the end points which are when $y = 0$ and $y = 2$:

  - $y = 0: x^2 = 4 - y^2 \Rightarrow x^2 = 4 - (0)^2 = 4 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

    $\Rightarrow (-2,0), (2,0).$
  - $y = 2: x^2 = 4 - y^2 \Rightarrow x^2 = 4 - (2)^2 = 0 \Rightarrow x^2 = 0$

    $\Rightarrow x = 0 \Rightarrow (0,2).$
- On $L: y = 0$: $g(x) = f(x,0) = x^4 + 2x^2(0)^2 + (0)^4 + 8(0) - 1$

  $g(x) = x^4, -2 \le x \le 2 \Rightarrow g'(x) = 4x^3, -2 < x < 2$

  - $g'(x) = 0 \Rightarrow 4x^3 = 0 \Rightarrow x = 0$
  - We have a point (when $x = 0$) and the end points (when $x = -1, x = 2$) $\Rightarrow$ The points $(-2,0), (2,0), (0,0)$

Step 3: $f(x,y) = x^4 + 2x^2y^2 + y^4 + 8y - 1$

point	$(-2,0)$	$(2,0)$	$(0,2)$	$(0,0)$
$f(x,y)$	15	15	31	-1

- The absolute maximum of f is 31
- The absolute minimum of f is -1

Example 14.7.23: Find the absolute maximum and minimum values of the function $f(x,y) = 2x + y - xy$ in the closed triangular region with vertices $(0,0), (0,2), (4,0)$.

Solution:

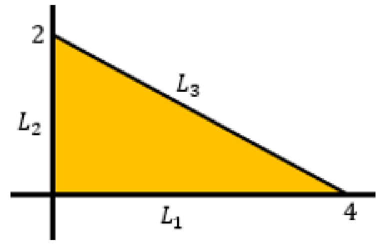
Step 1: We find the critical points of f in D:

$f_x = 2 - y$ and $f_y = 2 - x$

$f_x = 0 \Rightarrow\ 2 - y = 0\ \Rightarrow y = 2$

$f_y = 0 \Rightarrow 1 - x = 0\ \Rightarrow x = 1$

The point $(1,2) \notin D \Rightarrow\ f$ has no critical points inside D.



We do not have any critical point of f in Step 1

Step 2: We find the extreme values of f on the boundary of D:

The boundary of D consists of two parts: L_1, L_2, and L_3.

- On L_1: $y = 0$. Let $g(x) = f(x,y)|_{y=0} = 2x, 0 \le x \le 4$.
  - $g'(x) = 2, 0 < x < 4 \Rightarrow g'(x) \neq 0, \forall x \in \underbrace{(0,4)}_{\text{interval}}$
  - We only have the end points when $x = 0$ and $x = 4$

    $\Rightarrow$ The points are $(0,0), (4,0)$.
- On L_2: $x = 0$: $h(y) = f(x,y)|_{x=0} = y, 0 \le y \le 2$.
  - $h'(y) = 1, 0 < y < 2\ \Rightarrow h'(y) \neq 0, \forall y \in \underbrace{(0,2)}_{\text{interval}}$
  - We only have the end points when $y = 0$ and $y = 2$

    $\Rightarrow$ The points are $(0,0), (0,2)$.

We need to find the equations of the lines L_1, L_2, and L_3:

- $L_1: y = 0$
- $L_2: x = 0$
- L_3: passes through $(4,0)$ and $(0,2)$
  - slope $= \frac{2}{-4} = -\frac{1}{2}$
  - Equation is:

  $y - 0 = -\frac{1}{2}(x - 4)$

  $\Rightarrow$ $L_3: x = 4 - 2y$

- On L_3: $x = 4 - 2y$:

  $h(y) = f(x,y)|_{x=4-2y} = 2(4-2y) + y - (4-2y)y, 0 \le y \le 2$

  $\Rightarrow h(y) = 2y^2 - 7y + 8, 0 \le y \le 2$
  - $h'(y) = 4y - 7 = 0\ \Rightarrow\ y = \frac{7}{4}$
  - We have three points (when $y = \frac{7}{4}$) and at the end points (when $y = 0, y = 2$):
    - $= \frac{7}{4}\ \Rightarrow\ x = 4 - 2\left(\frac{7}{4}\right) = \frac{1}{2} \Rightarrow$ The point $\left(\frac{1}{2}, \frac{7}{4}\right)$
    - $y = 0\ \Rightarrow\ x = 4 - 2(0) = 4\ \Rightarrow$ The point $(4,0)$
    - $y = 2\ \Rightarrow\ x = 4 - 2(2) = 0\ \Rightarrow$ The point $(0,2)$

Step 3: $f(x,y) = 2x + y - xy$

point	$(0,0)$	$(4,0)$	$(0,2)$	$\left(\frac{1}{2}, \frac{7}{4}\right)$
$f(x,y)$	0	8	2	$\frac{15}{8}$

- The absolute maximum of f is 8
- The absolute minimum of f is 0

Chapter 15: Multiple Integrals

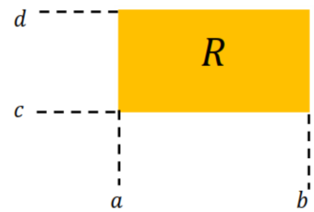
Section 15.1: Double Integrals over Rectangles

Section 15.2: Iterated Integrals

Fubini's Theorem 15.2.1:

Let $R = \{(x, y): a \le x \le b, c \le y \le d\}$ be a rectangle and let $f(x, y)$ be a continuous function on R. Then the double integral $\iint_R f(x,y)dA$ can be expressed as an iterated integral as:
$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dy\,dx = \int_c^d \int_a^b f(x,y)dx\,dy$

$$dA = dydx \quad \text{or} \quad dA = dxdy$$



Remark 15.2.2:

(1) $\int_a^b \int_c^d f(x,y)dy\,dx = \int_a^b \left(\int_c^d f(x,y)dy\right) dx$:

- Means: Compute $g(x) = \int_c^d f(x,y)dy$ by taking x as a constant, then compute $\int_a^b g(x)dx$.

(2) $\int_c^d \int_a^b f(x,y)dx\,dy = \int_c^d \left(\int_a^b f(x,y)dx\right) dy$:

- Means: Compute $h(y) = \int_a^b f(x,y)dx$ by taking y as a constant, then compute $\int_c^d h(y)dy$.

(3) The rectangle $R = \{(x, y): a \le x \le b, c \le y \le d\}$ can be expressed as:

$$R = \{(x, y): a \le x \le b, c \le y \le d\} = [a, b] \times [c, d].$$

Properties of Double Integrals 15.2.3:

Let $f(x, y)$ and $g(x, y)$ be continuous functions on a rectangle R o. Then

(1) $\iint_R (f + g)dA = \iint_R fdA + \iint_R gdA$

(2) $\iint_R (f - g)dA = \iint_R fdA - \iint_R gdA$

(3) $\iint_R cfdA = c\iint_R fdA$, where c is a constant.

(4) If $f(x, y) \ge g(x, y)$ for all $(x, y) \in R$, then $\iint_R fdA \ge \iint_R gdA$

(5) If the variables x and y in $f(x, y)$ are separated, that is $f(x, y) = g(x)h(y)$ and f is continuous on the rectangle $R = [a, b] \times [c, d]$, then

$$\iint_R fdA = \int_a^b \int_c^d g(x)h(y)\,dy\,dx = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)$$

Example 15.2.4: Evaluate the double integral $\iint_R (x - 3y^2)dA$, where

$$R = \{(x, y): 0 \le x \le 2, -1 \le y \le 3\}.$$

Solution: $dA = dxdy$ or $dA = dydx$, we take $dA = dxdy$:

$$\iint_R (x - 3y^2)dA = \int_{-1}^3 \left(\int_0^2 (x - 3y^2)\,dx\right) dy = \int_{-1}^3 \left(\frac{x^2}{2} - 3xy^2\right]_0^2 \Big) dy = \int_{-1}^3 (2 - 6y^2)\,dy$$

$$= 2y - \frac{6y^3}{3}\Big]_{-1}^3 = -48$$

Example 15.2.5: Compute the following double integral: $\iint_R x\sin(xy)dA$, where $R = [1,2] \times [0, \frac{\pi}{2}]$.

Solution: $dA = dydx$ or $dA = dxdy$:

- $dA = dydx \Rightarrow$ we integrate $\int ysin(xy)\,dy$ by substitution
- $dA = dxdy \Rightarrow$ if we use we integrate $\int ysin(xy)\,dx$ by parts

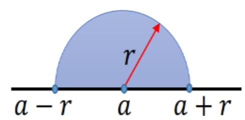
So, we take $dA = dydx$:

$$\iint_R x\sin(xy)dA = \int_1^2 \left(\int_0^{\frac{\pi}{2}} x\sin(xy)\,dy\right)dx = \int_1^2 \left(x\left(-\frac{\cos(xy)}{x}\right)\Big]_0^{\frac{\pi}{2}}\right)dx = -\int_1^2 \left(\cos\left(\frac{\pi}{2}x\right) - 1\right)dx$$

$$= -\left(\frac{\sin\left(\frac{\pi}{2}x\right)}{\frac{\pi}{2}} - x\right)\Big]_1^2 = 1 + \frac{2}{\pi}$$

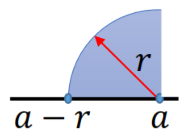
Remark 15.2.6:

Recall that in 2D: $y = \sqrt{r^2 - (x-a)^2}$ is the equation of the semicircle:



So,

- $\int_{a-r}^{a+r} \sqrt{r^2 - (x-a)^2}\,dx = \frac{1}{2}\pi r^2$
- $\int_{a}^{a+r} \sqrt{r^2 - (x-a)^2}\,dx = \frac{1}{4}\pi r^2$



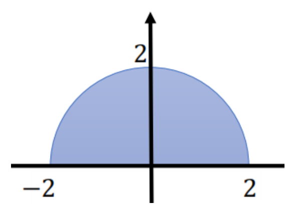
Example 15.2.7: Evaluate the double integral $\iint_R \sqrt{4-x^2}dA$, where $R = [-2,2] \times [0,3]$.

Solution: $dA = dydx$ or $dA = dxdy \Rightarrow$ we choose $dA = dydx$ (why?)

$y = \sqrt{4-x^2}$ is an equation of a semicircle:

$$\iint_R \sqrt{4-x^2}dA = \int_{-2}^{2} \left(\int_0^3 \sqrt{4-x^2}dy\right)dx = 3\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

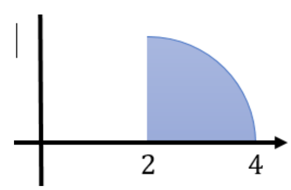
$$= 3\left(\frac{1}{2}\right)\pi(2)^2 = 6\pi$$



Example 15.2.8: Evaluate the double integral $\iint_R \sqrt{4x-x^2}dA$, where $R = [2,4] \times [0,3]$.

Solution: $dA = dydx$ or $dA = dxdy \Rightarrow$ we choose $dA = dydx$ (why?)

- $y = \sqrt{4x-x^2} \Rightarrow y^2 = 4x - x^2 \Rightarrow x^2 - 4x + y^2 = 0$
  $\Rightarrow x^2 - 4x + 4 + y^2 = 4 \Rightarrow (x-2)^2 + y^2 = 4$
  $\Rightarrow y = \sqrt{4x-x^2}$ is upper part of the circle
  But $2 \le x \le 4 \Rightarrow y = \sqrt{4x-x^2}$ is the quarter of the circle



- $\iint_R \sqrt{4x-x^2}dA = \int_2^4 \left(\int_0^3 \sqrt{4x-x^2}dy\right)dx$

$$= 3\int_2^4 \sqrt{4x-x^2}dx = 3\int_2^4 \sqrt{4-(x-2)^2}\,dx = 3\left(\frac{1}{4}\right)\pi(2)^2 = 3\pi$$

Example 15.2.9: Evaluate the iterated integral $\int_0^{\pi}\int_0^{\pi/12} \sin\left(\frac{x}{3}\right)\cos(2y)\,dydx$.

Solution: Observe that the variables in $\sin\left(\frac{x}{3}\right)\cos(2y)$ are separated, so

$$\int_0^{\pi}\int_0^{\pi/12} \sin\left(\frac{x}{3}\right)\cos(2y)\,dydx = \left(\int_0^{\pi}\sin\left(\frac{x}{3}\right)dx\right)\left(\int_0^{\pi/12}\cos(2y)\,dy\right)$$
$$= -3\cos\left(\frac{x}{3}\right)\Big]_0^{\pi}\ \frac{\sin(2y)}{2}\Big]_0^{\frac{\pi}{12}}$$
$$= -\frac{3}{2}\left(\cos\left(\frac{\pi}{3}\right)-\cos(0)\right)\left(\sin\left(\frac{\pi}{6}\right)-\sin(0)\right)$$
$$= \frac{3}{8}$$

Example 15.2.10: Evaluate the iterated integral $\int_0^2\int_0^1 f(x,y)\,dydx$, where

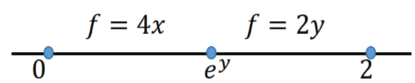
$$f(x,y) = \begin{cases} 2y & ,x \ge e^y \\ 4x & ,x < e^y \end{cases}$$

Solution: Since the function f is defined by 2 formulas when $x \ge e^y$ and $x < e^y$ so we first integrate with respect to x and then y:

$$\Rightarrow \int_0^2\int_0^1 f(x,y)\,dydx = \int_0^1\left(\int_0^2 f(x,y)\,dx\right)dy$$
$$= \int_0^1\left(\int_0^{e^y} f\,dx + \int_{e^y}^2 f\,dx\right)dy$$
$$= \int_0^1\left(\int_0^{e^y} 4x\,dx + \int_{e^y}^2 2y\,dx\right)dy = \int_0^1\left(2x^2\Big]_0^{e^y} + (2-e^y)2y\right)dy$$
$$= \int_0^1 (2e^{2y}+4y-2ye^y)\,dy = e^{2y}+2y^2-(2ye^y-2e^y)\Big]_0^1 = e+1$$

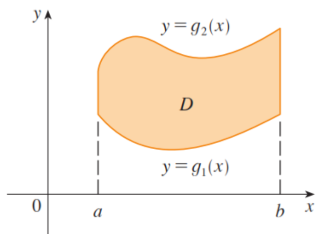
Differentiate	Integrate
$2y$	e^y
2	e^y
0	e^y

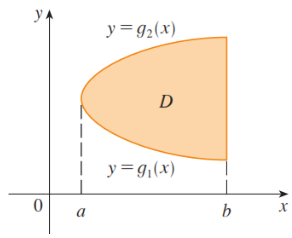
$$\Rightarrow \int 2ye^y\,dy = 2ye^y - 2e^y$$

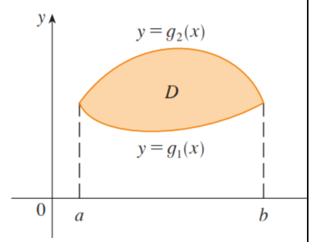


Section 15.3: Double Integrals over General Regions

15.3.1: Type 1 Regions: Let $D=\{(x,y): a\le x\le b, g_1(x)\le y\le g_2(x)\}$





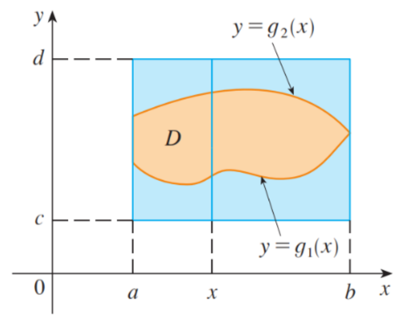


لاحظ أن المنحنيات التي تحد المنطقة تكتب بجعل y موضوع القانون: $y = f(x)$

and let $f(x,y)$ be a continuous function on D.

Then the double integral: $dA = dydx$

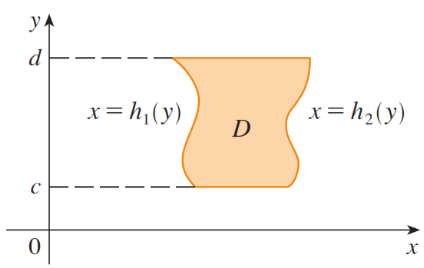
$$\iint_D f(x,y)dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$$

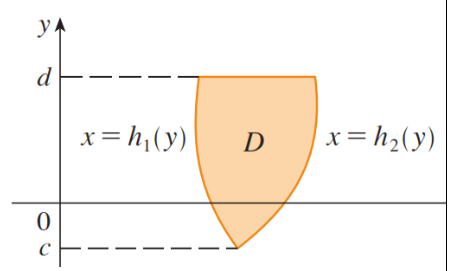


15.3.2: Type 2 Regions: Let $D=\{(x,y): h_1(y)\le x\le h_2(y), c\le y\le d\}$

لاحظ أن المنحنيات التي تحد المنطقة تكتب بجعل x موضوع القانون:

$x = h(y)$

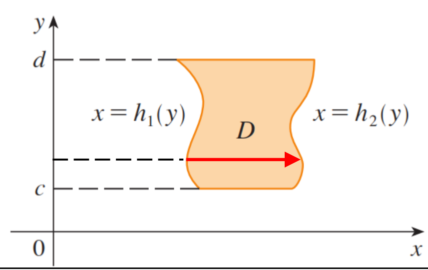




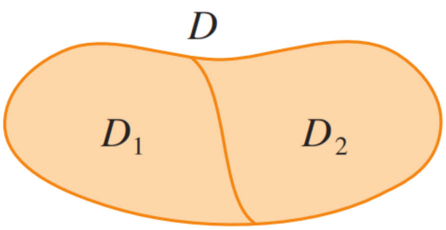
and let $f(x,y)$ be a continuous function on D.

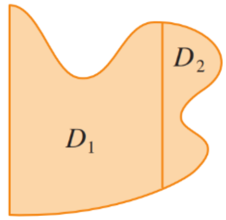
Then the double integral: $dA = dxdy$

$$\iint_D f(x,y)dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)dx\,dy$$



Remark 15.3.3: If the region D consists of two (or more) regions of type 1 (or Type 2) as in the figures:





Then $\iint_D f(x,y)dA = \iint_{D_1} f(x,y)dA + \iint_{D_2} f(x,y)dA$

Example 15.3.4: Sketch the region and change the order of integration:

(1) $\displaystyle\int_1^2 \int_0^{\ln(x)} f(x,y)\,dy\,dx$ (2) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{2} f(x,y)\,dx\,dy$

Solution:

(1) $dA = dydx$ Type 1 Region

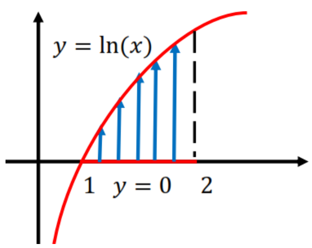
Curves: $y = 0 \rightarrow y = \ln(x)$

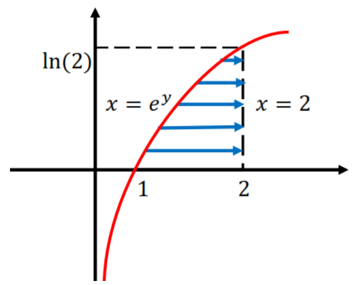
$x = 1 \rightarrow x = 2$

$dA = dxdy$ Type 2 Region

Curves: $x = e^y \rightarrow x = 2$

$y = 0 \rightarrow y = \ln(2)$





$\int_1^2 \int_0^{\ln(x)} f(x,y)\,dy\,dx = \int_0^{\ln(2)} \int_{e^y}^{2} f(x,y)\,dx\,dy$

(1) $dA = dxdy$ Type 2 Region

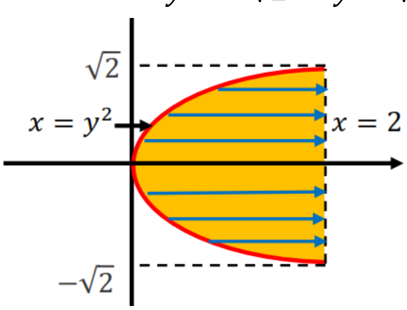
Curves: $x = y^2 \rightarrow x = 2$

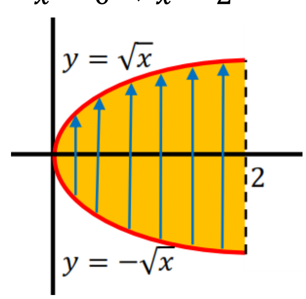
$y = -\sqrt{2} \rightarrow y = \sqrt{2}$

$dA = dydx$ Type 1 Region

Curves: $y = -\sqrt{x} \rightarrow y = \sqrt{x}$

$x = 0 \rightarrow x = 2$





$\Rightarrow \int_{-\sqrt{2}}^{\sqrt{2}} \int_{y^2}^{2} f(x,y)\,dx\,dy = \int_0^2 \int_{-\sqrt{x}}^{\sqrt{x}} f(x,y)\,dy\,dx$

Example 15.3.5: Sketch the region and change the order of integration:

$$\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} f(x,y)\,dx\,dy$$

Solution:

$dA = dxdy$ Type 2 Region

Curves: $x = \frac{y^2}{2} - 3 \ \rightarrow \ x = y + 1$

$y = -2 \rightarrow y = 4$

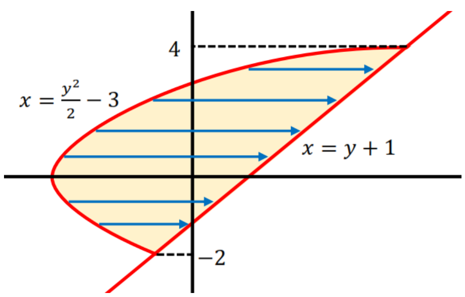
To find the pints of intersection of

$x = \frac{y^2}{2} - 3$ and $x = y + 1$:

$\frac{y^2}{2} - 3 = y + 1 \Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y-4)(y+2) = 0$

$\Rightarrow y = -2$ or $y = 4$



$dA = dydx$ Type 1 Region

$x = \frac{y^2}{2} - 3 \ \Rightarrow y^2 = 2x + 6$

$\Rightarrow y = \pm\sqrt{2x+6}$ ……①

$x = y + 1 \ \Rightarrow \ y = x - 1$

Intersection: $y = -2 \ \Rightarrow x = -1$

$y = 4 \ \Rightarrow \ x = 5$

We have 2 Regions:

Region D_1:

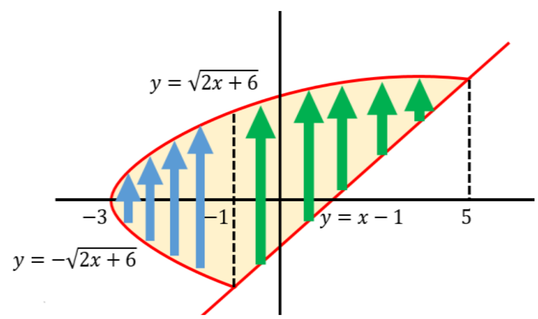
$y = -\sqrt{2x+6} \ \rightarrow y = \sqrt{2x+6}$

$x = -3 \ \rightarrow x = -1$

Region D_2:

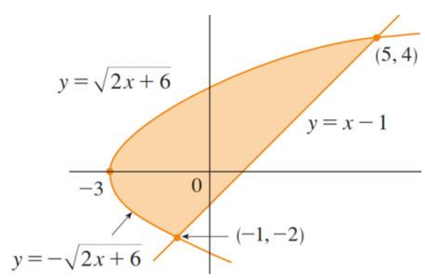
$y = x - 1 \ \rightarrow y = \sqrt{2x+6}$

$x = -1 \ \rightarrow x = 5$



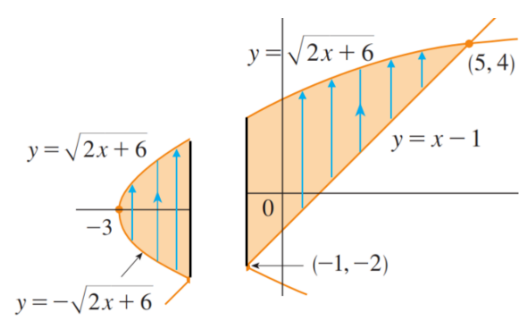
$$\Rightarrow \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} f(x,y)\,dx\,dy = \int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,dy\,dx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)\,dy\,dx$$

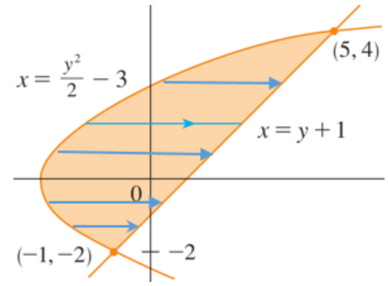
Example 15.3.6: Evaluate $\iint_D xy\,dA$, where D is the shaded region in the figure.



Solution:

- Curves are written as in Type 1 regions:
  $y = -\sqrt{2x+6}, y = \sqrt{2x+6}$ and $y = x - 1$
  So we have two regions as in the figure:





- Look to the region as Type 2 regions:
  We have one region and the equations of the curves are:
  $x = \frac{y^2}{2} - 3$ and $x = y + 1$

- Using Type 2 region is easier than using Type 1 regions: So, take $dA = dxdy$
  $\Rightarrow \iint_D xydA = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dxdy = \int_{-2}^{4} \frac{x^2}{2}y\Big]_{\frac{y^2}{2}-3}^{y+1} dy = \frac{1}{2}\int_{-2}^{4}(y+1)^2y - \left(\frac{y^2}{2} - 3\right)^2 y\,dy = 36$

Example 15.3.7: Evaluate $\iint_D xydA$, where D is the region bounded by $y = x - 1$ and $y^2 = 2x + 6$

Solution: Curves: $y = x - 1$ and $y^2 = 2x + 6 \Rightarrow x = y + 1$ and $x = \frac{y^2-6}{2} = \frac{y^2}{2} - 3$

$\Rightarrow dA = dxdy$ (Type 2 Regions)

Intersection of curves: $y + 1 = \frac{y^2}{2} - 3 \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y - 4)(y + 2) = 0$

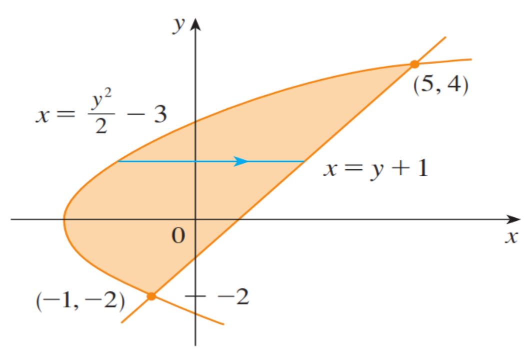
$\Rightarrow y = -2$ or $y = 4 \Rightarrow -2 \le y \le 4$

ملاحظة : $\int_{-2}^{4}\int_{\text{منحنى الحد الأدنى}}^{\text{منحنى الحد الأعلى}} xydxdy$

لتحديد المنحنى في الحد الأدنى والمنحنى في الحد الأعلى في حدود التكامل الأول وبدون رسم: بما أن $-2 \le y \le 4$ نأخذ نقطة اختبار في الفترة $[-2,4]$ ولتكن مثلاً $y = 0$ ثم نعوضها في المنحنيين فالذي قيمتة أصغر يكون المنحنى في الحد الأدنى والذي قيمته أكبر يكون المنحنى في الحد الأعلى

- $x = y + 1 \Rightarrow x = 0 + 1 = 1$
- $x = \frac{y^2}{2} - 3 \Rightarrow x = \frac{(0)^2}{2} - 3 = -3$
  - $x = \frac{y^2}{2} - 3$ (lower curve in integral)
  - $x = y + 1$ (upper curve in integral)

$$\Rightarrow \iint_D xydA = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dxdy = \int_{-2}^{4} \frac{x^2}{2}y\Big]_{\frac{y^2}{2}-3}^{y+1} dy = 36$$



Example 15.3.8: Compute $\iint_D (x+2y)dA$, where D is the region enclosed by

$$y = 2x^2 \text{ and } y = 1 + x^2.$$

Solution:

- Curves: $y = 2x^2$ and $y = 1 + x^2 \Rightarrow dA = dydx$ (Type 1 Regions)
- Intersection of curves: $2x^2 = 1 + x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1 \Rightarrow -1 \le x \le 1$

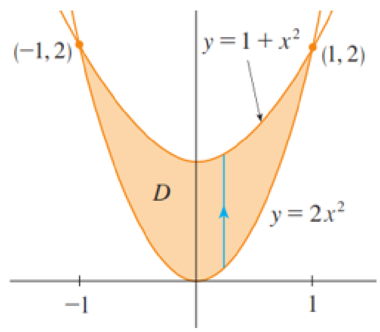
ملاحظة : $\int_{-1}^{1} \int_{\text{منحنى الحد الأدنى}}^{\text{منحنى الحد الأعلى}} (x+2y)dydx$

لتحديد المنحنى في الحد الأدنى والمنحنى في الحد الأعلى في حدود التكامل الأول وبدون رسم: بما أن $-1 \le x \le 1$ نأخذ نقطة اختبار في الفترة $[-1,1]$ ولتكن مثلاً $x = 0$ ثم نعوضها في المنحنيين فالذي قيمتة أصغر يكون المنحنى في الحد الأدنى والذي قيمته أكبر يكون المنحنى في الحد الأعلى

- $y = 2x^2 \Rightarrow y = 2(0)^2 = 0$
- $y = 1 + x^2 \Rightarrow y = 1 + (0)^2 = 1$
  - $y = 2x^2$ (lower curve in integral)
  - $y = 1 + x^2$ (upper curve in integral)

$$\Rightarrow \iint_D (x+2y)dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dy\,dx$$

$$= \int_{-1}^{1}(xy + y^2)\Big]_{2x^2}^{1+x^2} dx = \frac{32}{15}$$



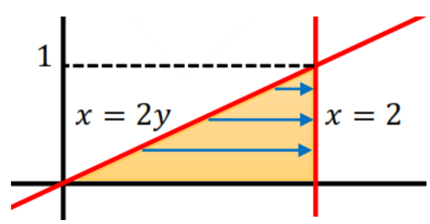
Example 15.3.9: Compute the following iterated integrals:

(1) $\int_0^1 \int_{2y}^{2} e^{x^2}\, dx\, dy$ (2) $\int_0^{\frac{1}{2}} \int_{2x}^{1} \sin(y^2)\, dy\, dx$ (3) $\int_0^4 \int_{\sqrt{y}}^{2} e^{x^3}\, dx\, dy$

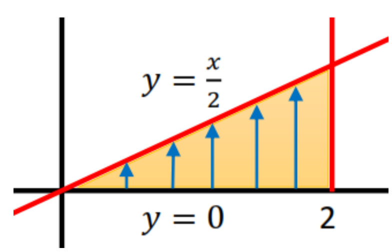
Solution:

(1) Observe that we cannot compute $\int_{2y}^{2} e^{x^2}\, dx$

- $dA = dxdy \Rightarrow$ We have Type 2 Region:
  - Curves: $x = 2y \rightarrow x = 2$
  - $y = 0 \rightarrow y = 1$



- Go to type 1 Regions: $\Rightarrow dA = dydx$
  - Curves: $y = 0 \rightarrow y = \frac{x}{2}$
  - $x = 0 \rightarrow x = 2$

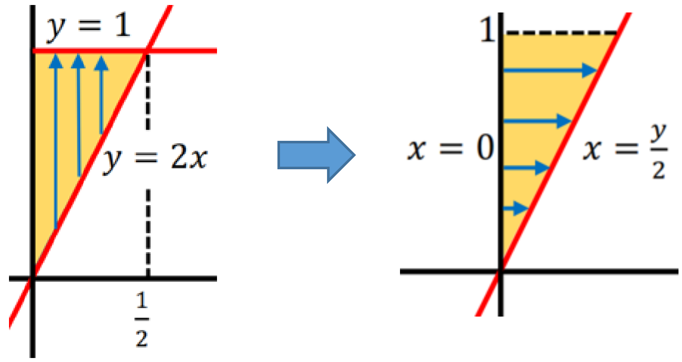


- $\int_0^1 \int_{2y}^{2} e^{x^2}\, dx\, dy = \int_0^2 \int_0^{\frac{x}{2}} e^{x^2}\, dy\, dx = \int_0^2 \frac{x}{2} e^{x^2}\, dx$

$$= \frac{e^{x^2}}{4}\Bigg]_0^2 = \frac{e^4 - 1}{4}$$

(2) Observe that we cannot compute $\int_{2x}^{1} \sin(y^2)\, dy$

- $dA = dydx \Rightarrow$ We have Type 1 Region
  - Curves: $y = 2x \to y = 1$
  - $x = 0 \to x = \frac{1}{2}$
- Go to type 2 Regions: $\Rightarrow dA = dxdy$
  - Curves: $x = 0 \to x = \frac{y}{2}$
  - $y = 0 \to y = 1$



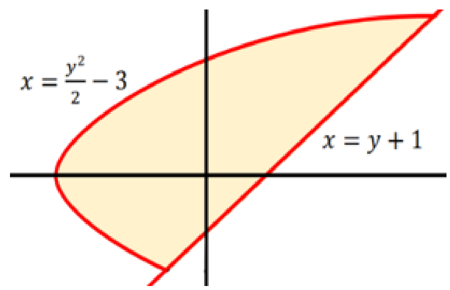
$$\int_0^{\frac{1}{2}} \int_{2x}^{1} \sin(y^2)\, dy\, dx = \int_0^1 \int_0^{\frac{y}{2}} \sin(y^2)\, dx\, dy$$

$$= \int_0^1 \frac{y}{2} \sin(y^2)\, dy = -\frac{\cos(y^2)}{4}\Big]_0^1 = -\frac{\cos(1) - 1}{4} = \frac{1 - \cos(1)}{4}$$

(3) Exercise

Rule 15.3.10: Let D be a region in the xy-plane. Then the area of D is $\iint_D 1 dA$

Example 15.3.11: Find the area of the shaded region in the figure:



Solution:

- First we find the points of intersections of curves

$\frac{y^2}{2} - 3 = y + 1 \Rightarrow y^2 - 2y - 8 = 0 \Rightarrow (y - 4)(y + 2) = 0$

$\Rightarrow y = -2$ or $y = 4$

- We take Type 2 regions: $dA = dxdy$

$$\text{Area} = \iint_D 1 dA = \int_{-2}^{4} \int_{\frac{y^2}{2}-3}^{y+1} 1\, dx\, dy = \int_{-2}^{4} \left(y + 1 - \frac{y^2}{2} + 3\right) dy = \underbrace{\cdots\cdots\cdots}_{\text{أكمل الحل}}$$

Example 15.3.12: Find the area of the region bounded by the curves $y = e^{2x}, y = 2$, and $x = 4$.

Solution:

- Observe that the curves are: $y = e^{2x}, y = 2$
  - $x = 4 \quad \to \quad x =$????? (we find it from intersection of curves):
  - $e^{2x} = 2 \Rightarrow 2x = \ln 2 \Rightarrow \ x = \frac{\ln 2}{2}$ (since $\frac{\ln 2}{2} \cong 0.34$) we have: $x = \frac{\ln 2}{2} \to \ x = 4$
- To identify the upper and lower curves in integral (take a value of x between $\frac{\ln 2}{2}$ and 4):
  - Take $x = 3$:

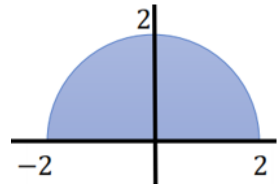
$$\left.\begin{array}{l} y = e^{2x} \ \Rightarrow \ y = e^6 \cong (2.7)^6 \\ \ \ y = 2 \ \Rightarrow \ y = 2 \end{array}\right\} \Rightarrow \begin{array}{l} y = 2 \text{ (lower curve in integral)} \\ y = e^{2x} \text{ (upper curve in integral)} \end{array}$$

$$\text{Area} = \iint_D 1 dA = \int_{\frac{\ln 2}{2}}^{4} \int_2^{e^{2x}} 1\, dy\, dx = \int_{\frac{\ln 2}{2}}^{4} (e^{2x} - 2) dx = \frac{e^{2x}}{2} - 2x \Big]_{\frac{\ln 2}{2}}^{4} = \underbrace{\cdots\cdots\cdots}_{\text{أكمل الحل}}$$

Example 15.3.12: Find each of the following iterated integrals:

(1) $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} -3\,dy\,dx$ (2) $\iint_D 2\,dA$, where $D=\{(x,y): x^2+y^2\le 3, x\ge 0, y\ge 0\}$

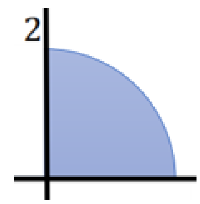
Solution:



(1) Curves: $y=0 \rightarrow y=\sqrt{4-x^2}$

- $x=-2 \rightarrow x=2$

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} -3\,dy\,dx = -3\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 1\,dy\,dx = -3Area = -3\left(\frac{\pi 2^2}{2}\right) = -6\pi$$

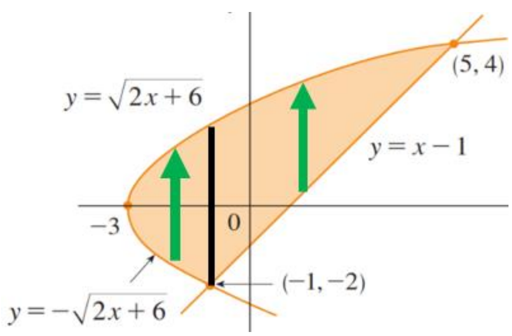


(2) $$\iint_D 2\,dA = 2\iint_D 1\,dA = 2(\text{Area of } D) = 2\left(\frac{\pi 2^2}{4}\right) = 2\pi$$

Example 15.3.13: Combine the sum of the two double integrals as a single double integral:

$$\int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,dydx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)\,dydx$$

Solution:

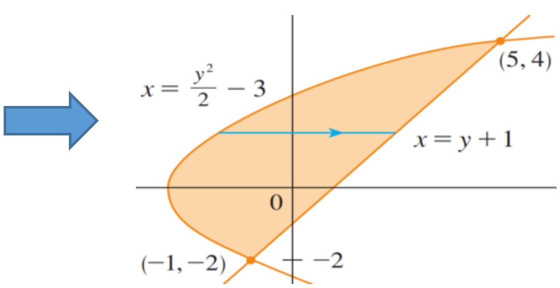


$\int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,dydx$:

- $y=-\sqrt{2x+6} \rightarrow y=\sqrt{2x+6}$,
- $-3\le x\le -1$

$\int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)\,dydx$:

- $y=x-1 \rightarrow y=\sqrt{2x+6}$
- $-1\le x\le 5$



- Curves:
  - $y=\pm\sqrt{2x+6} \Rightarrow y^2=2x+6 \Rightarrow x=\frac{y^2}{2}-3$ (lower curve in integral)
  - $y=x-1 \Rightarrow x=y+1$ (upper curve in integral)
  - $-2\le y\le 4$

$$\int_{-3}^{-1}\int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} f(x,y)\,dydx + \int_{-1}^{5}\int_{x-1}^{\sqrt{2x+6}} f(x,y)\,dydx = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} f(x,y)\,dxdy$$

Exercise 15.3.14: Combine the sum of the two double integrals as a single double integral

$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} f(x,y)\,dydx + \int_{0}^{1}\int_{0}^{1-x} f(x,y)\,dydx$$

15.4. Double Integrals in Polar Coordinates

Let (x,y) be a point in Cartesian (or rectangular coordinates). Then this point can be written in polar coordinates as (r,θ), where $x = r\cos(\theta)$, $y = r\sin(\theta)$

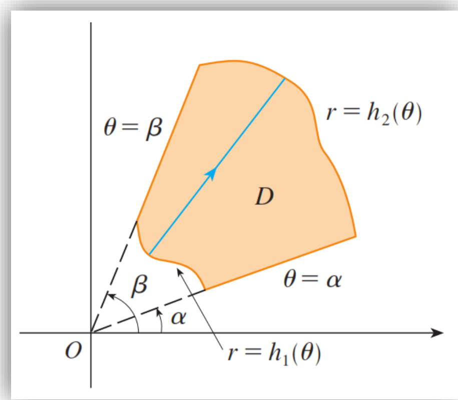
$r^2 = x^2 + y^2 \Leftrightarrow r = \sqrt{x^2+y^2}$ and $\tan(\theta) = \frac{y}{x} \Leftrightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

θ	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	0 or 2π	$\frac{\pi}{2}$	π or $-\pi$	$\frac{3\pi}{2}$ or $-\frac{\pi}{2}$
$\sin(\theta)$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	0	1	0	-1
$\cos(\theta)$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	1	0	-1	0
$\tan(\theta)$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	1	0	………	0	………

15.4.1 Polar Regions (Type 3 Regions):

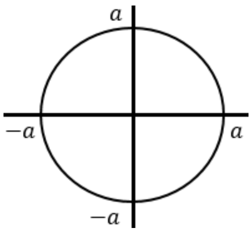
Let $D = \{(r,\theta): h_1(\theta) \le r \le h_2(\theta), \alpha \le \theta \le \beta\}$, where $0 < \beta - \alpha \le 2\pi$. Then

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f\big(r\cos(\theta), r\sin(\theta)\big) r\,dr\,d\theta$$

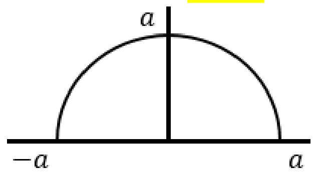


Remark 15.4.2: Let $a > 0$:

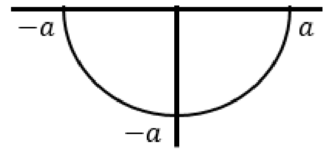
<table>
<tr>
<td rowspan="2">$x^2 + y^2 = a^2 \;\Leftrightarrow\; r = a$



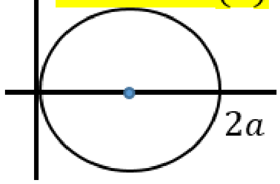
$0 \le \theta \le 2\pi$</td>
<td colspan="2">$x^2 + y^2 = a^2 \Rightarrow y^2 = a^2 - x^2 \;\Rightarrow\; y = \sqrt{a^2 - x^2}$ or $y = -\sqrt{a^2 - x^2}$</td>
</tr>
<tr>
<td>$y = \sqrt{a^2 - x^2} \Leftrightarrow r = a$



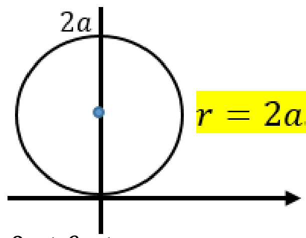
$0 \le \theta \le \pi$</td>
<td>$y = -\sqrt{a^2 - x^2} \Leftrightarrow r = a$



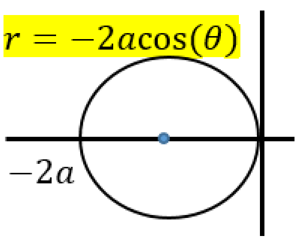
$\pi \le \theta \le 2\pi \Leftrightarrow -\pi \le \theta \le 0$</td>
</tr>
<tr>
<td>$r = 2a\cos(\theta)$



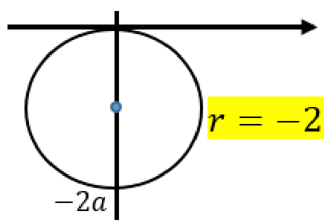
$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$</td>
<td>$r = 2a\sin(\theta)$



$0 \le \theta \le \pi$</td>
<td rowspan="2"></td>
</tr>
<tr>
<td>$r = -2a\cos(\theta)$



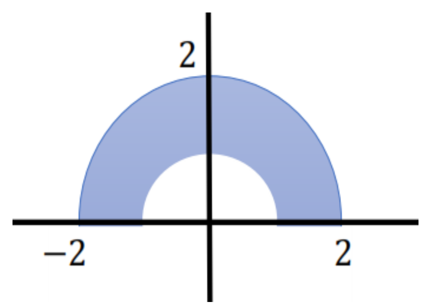
$\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$</td>
<td>$r = -2a\sin(\theta)$



$\pi \le \theta \le 2\pi$</td>
</tr>
</table>

Example 15.4.3: Evaluate $\iint_D (3x + 4y^2)dA$ where D is the region in the upper half-plane bounded by $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$

Solution: **Region:**



- $x^2 + y^2 = 1 \Rightarrow r^2 = 1 \Rightarrow \; r = 1$
- $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow \; r = 2$
- $0 \le \theta \le \pi$

$$\iint_D (3x + 4y^2)dA = \int_0^{\pi}\int_1^2 \left(3r\cos(\theta) + 4\big(r\sin(\theta)\big)^2\right) r\,dr\,d\theta$$

$$= \int_0^{\pi}\left(\int_1^2 (3r^2\cos(\theta) + 4r^3\sin^2(\theta))\,dr\right)d\theta = \int_0^{\pi} r^3\cos(\theta) + r^4\sin^2(\theta)\Big]_1^2\,d\theta$$

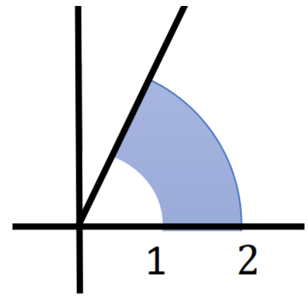
$$= \int_0^{\pi} (7\cos(\theta) + 15\sin^2(\theta))\,d\theta = \int_0^{\pi}\left(7\cos(\theta) + 15\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right)\right)d\theta$$

$$= 7\sin(\theta) + 15\left(\frac{\theta}{2} - \frac{1}{4}\sin(2\theta)\right)\Big]_0^{\pi} = \cdots\cdots$$

Example 15.4.4: Evaluate $\iint_D \arctan\left(\frac{y}{x}\right) dA$ where

$$D = \{(x,y): 1 \le x^2 + y^2 \le 4, 0 \le y \le \sqrt{3}x\}$$

Solution: Region:



- $x^2 + y^2 = 1 \Rightarrow r^2 = 1 \Rightarrow r = 1$
- $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$
- $y \le \sqrt{3}x \Rightarrow \frac{y}{x} = \sqrt{3} \Rightarrow \tan(\theta) = \sqrt{3} \Rightarrow \theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$
- $0 \le \theta \le \frac{\pi}{3}$

$$\iint_D \arctan\left(\frac{y}{x}\right) dA = \iint_D \tan^{-1}\left(\frac{y}{x}\right) dA = \int_0^{\frac{\pi}{3}} \int_1^2 \theta \, r dr d\theta$$

$$= \left(\int_1^2 r dr\right)\left(\int_0^{\frac{\pi}{3}} \theta \, d\theta\right) = \left(\frac{r^2}{2}\Big]_1^2\right)\left(\frac{\theta^2}{2}\Big]_0^{\frac{\pi}{3}}\right) = \frac{3\pi^2}{36}$$

Example 15.4.5: Evaluate the following iterated integrals:

(1) $\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{0} e^{x^2+y^2} dy\, dx$

(2) $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{0} e^{x^2+y^2} dy\, dx$

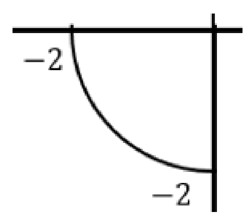
(3) $\int_{0}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy\, dx$

(4) $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy\, dx$

Solution:

(1) **Region:** $y = -\sqrt{4-x^2} \to y = 0$ and $-2 \le x \le 0$

Polar Region: $r = 0 \to r = 2$ and $-\pi \le \theta \le -\frac{\pi}{2}$

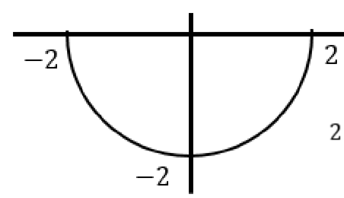


$$\int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{0} e^{x^2+y^2} dy\, dx = \int_{-\pi}^{-\frac{\pi}{2}} \int_0^2 e^{r^2}\, r dr d\theta$$

$$= \left(\int_{-\pi}^{-\frac{\pi}{2}} 1\, d\theta\right)\left(\int_0^2 r e^{r^2} dr\right) = \left(\frac{\pi}{2}\right)\left(\frac{e^{r^2}}{2}\Big]_0^2\right) = \frac{\pi(e^4-1)}{4}$$

(2) **Region:** $y = -\sqrt{4-x^2} \to y = 0$ and $-2 \le x \le 2$

Polar Region: $r = 0 \to r = 2$ and $-\pi \le \theta \le 0$

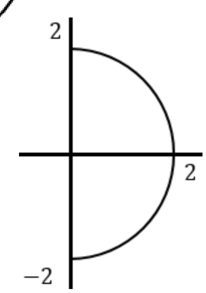


$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{0} e^{x^2+y^2} dy\, dx = \int_{-\pi}^{0} \int_0^2 e^{r^2}\, r dr d\theta$$

$$= \left(\int_{-\pi}^{0} 1\, d\theta\right)\left(\int_0^2 r e^{r^2} dr\right) = \cdots$$

(3) **Region:** $y = -\sqrt{4-x^2} \to y = \sqrt{4-x^2}$ and $0 \le x \le 2$

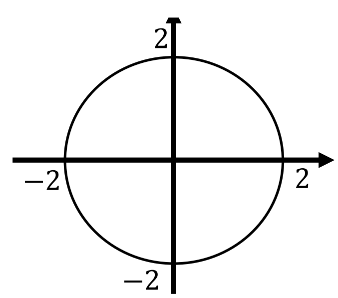
Polar Region: $r = 0 \to r = 2$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



$$\int_{0}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^2 e^{r^2}\, r dr d\theta = \ldots.$$

(4) **Region:** $y = -\sqrt{4-x^2} \to y = \sqrt{4-x^2}$ and $-2 \le x \le 2$

Polar Region: $r = 0 \to r = 2$ and $0 \le \theta \le 2\pi$



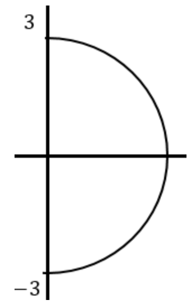
$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} e^{x^2+y^2} dy\, dx = \int_0^{2\pi} \int_0^2 e^{r^2}\, r dr d\theta = \cdots\cdots$$

Example 15.4.6: Evaluate the following iterated integrals:

(1) $\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\sin(x^2+y^2)dx\,dy$ (2) $\int_{0}^{\frac{3}{\sqrt{2}}}\int_{y}^{\sqrt{9-y^2}}\sin(x^2+y^2)dx\,dy$

(3) $\int_{-\frac{3}{\sqrt{2}}}^{0}\int_{-y}^{\sqrt{9-y^2}}\sin(x^2+y^2)dx\,dy$

Solution:



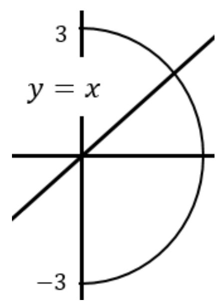
(1) **Region:** $x=0\to x=\sqrt{9-y^2}$ and $-3\le y\le 3$

Polar Region: $r=0\to r=3$ and $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}}\sin(x^2+y^2)dx\,dy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{3}\sin(r^2)\,r dr d\theta$$

$$=\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}1\,d\theta\right)\left(\int_{0}^{3}r\sin(r^2)\,dr\right)=(\pi)\left(-\frac{\cos(r^2)}{2}\Big]_{0}^{3}\right)=\frac{\pi(1-\cos(9))}{2}$$

(2) **Region:** $x=y\to x=\sqrt{9-y^2}$ and $0\le y\le\frac{3}{\sqrt{2}}$



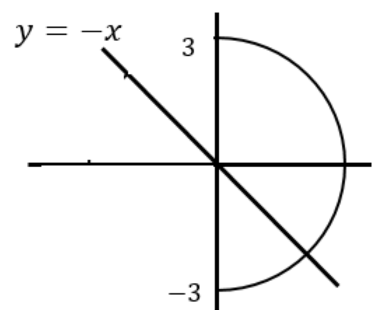
- Intersection of Curves: $y=\sqrt{9-y^2}\Rightarrow y^2=9-y^2$
  $y^2=\frac{9}{2}\Rightarrow\ y=\pm\frac{3}{\sqrt{2}}$

Polar Region: $r=0\to r=3$ and $0\le\theta\le\frac{\pi}{4}$

$$\int_{0}^{\frac{3}{\sqrt{2}}}\int_{y}^{\sqrt{9-y^2}}\sin(x^2+y^2)dx\,dy=\int_{0}^{\frac{\pi}{4}}\int_{0}^{3}\sin(r^2)\,r dr d\theta$$

$$=\left(\int_{0}^{\frac{\pi}{4}}1\,d\theta\right)\left(\int_{0}^{3}r\sin(r^2)\,dr\right)=\ldots\ldots.$$

(3) **Region:** $x=-y\to x=\sqrt{9-y^2}$ and $0\le y\le\frac{3}{\sqrt{2}}$



- Intersection of Curves: $-y=\sqrt{9-y^2}\Rightarrow y^2=9-y^2$
  $y^2=\frac{9}{2}\Rightarrow\ y=\pm\frac{3}{\sqrt{2}}$

Polar Region: $r=0\to r=3$ and $\frac{\pi}{4}\le\theta\le 0$

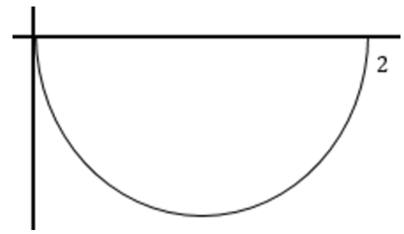
$$\int_{-\frac{3}{\sqrt{2}}}^{0}\int_{-y}^{\sqrt{9-y^2}}\sin(x^2+y^2)\,dx\,dy=\int_{-\frac{\pi}{4}}^{0}\int_{0}^{3}r\sin(r^2)dr\,d\theta=\ldots\ldots$$

Example 15.4.7: Evaluate the following iterated integrals:

(1) $\displaystyle\int_0^2 \int_{-\sqrt{2x-x^2}}^{0} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$ (2) $\displaystyle\int_0^1 \int_x^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx$

Solution:

(1) **Region:** $y = -\sqrt{2x-x^2} \to y = 0$ and $0 \le x \le 2$

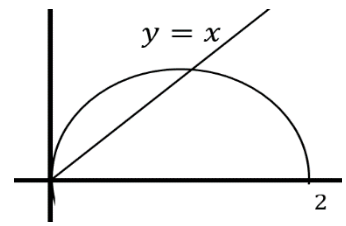


- $y = -\sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 - 2x + y^2 = 0$

  $x^2 - 2x + 1 + y^2 = 1 \Rightarrow (x-1)^2 + y^2 = 1$

Polar Region: $r = 0 \to r = 2\cos\theta$ and $-\frac{\pi}{2} \le \theta \le 0$

$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{0} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx = \int_{-\frac{\pi}{2}}^{0} \int_0^{2\cos\theta} \frac{1}{r} r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{0} \int_0^{2\cos\theta} 1\,dr\,d\theta = \ldots\ldots$$



(2) **Region:** $y = x \to y = \sqrt{2x-x^2}$ and $0 \le x \le 1$

- Sketch the region:

  $y = \sqrt{2x-x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 - 2x + y^2 = 0$

  $x^2 - 2x + 1 + y^2 = 1 \Rightarrow (x-1)^2 + y^2 = 1$

- Intersection of curves:
  - $x = \sqrt{2x-x^2} \Rightarrow x^2 = 2x - x^2 \Rightarrow 2x^2 - 2x = 0$

    $\Rightarrow x(x-1) = 0 \Rightarrow x = 0,1$

Polar Region: $r = 0 \to r = 2\cos\theta$ and $\frac{\pi}{4} \le \theta \le \frac{\pi}{2}$

$$\int_0^1 \int_x^{\sqrt{2x-x^2}} \frac{1}{\sqrt{x^2+y^2}}\,dy\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{1}{r} r\,dr\,d\theta = \ \ldots\ldots.$$

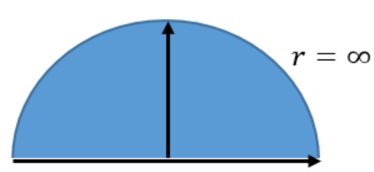
Example 15.4.8: Evaluate the following iterated improper integrals:

(1) $\int_{-\infty}^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\,dydx$ (2) $\int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\,dydx$

(3) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dydx$ (4) $\int_0^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dydx$

Solution:

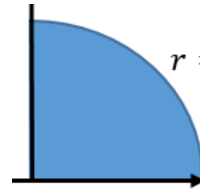
(1) **Region:** $y = 0 \to y = \infty$ and $-\infty < x < \infty \Rightarrow$ The region is the upper half plane:
Polar Region: $r = 0 \to r = \infty$ and $0 \le \theta \le \pi$



$$\int_{-\infty}^{\infty} \int_0^{\infty} e^{-(x^2+y^2)}\,dydx = \int_0^{\pi} \int_0^{\infty} e^{-r^2}\,r dr d\theta$$

$$= \left(\int_0^{\pi} 1\,d\theta\right)\left(\int_0^{\infty} re^{-r^2}\,dr\right) = \pi\, \frac{-e^{-r^2}}{2}\Bigg]_0^{\infty} = -\frac{\pi}{2}(e^{-\infty} - e^0) = \frac{\pi}{2}$$

(2) **Region:** $y=0\to y=\infty$ and $0\le x<\infty \Rightarrow$ The region is the first quadrant:
Polar Region: $r=0\to r=\infty$ and $0\le\theta\le\pi$



$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dydx=\int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\,rdrd\theta$$

$$=\left(\int_0^{\frac{\pi}{2}}1\,d\theta\right)\left(\int_0^\infty re^{-r^2}\,dr\right)=\frac{\pi}{2}\left.\frac{-e^{-r^2}}{2}\right]_0^\infty=-\frac{\pi}{2}(e^{-\infty}-e^0)=\frac{\pi}{4}$$

(3) Exercise

(4) Exercise

Example 15.4.9: Evaluate the following iterated improper integrals:

(1) $\int_0^\infty e^{-x^2}dx$ (2) $\int_{-\infty}^\infty e^{-x^2}dx$

Solution:

(1) Let $I=\int_0^\infty e^{-x^2}dx \Rightarrow I=\int_0^\infty e^{-y^2}dy$

$$\Rightarrow I^2=I\times I=\left(\int_0^\infty e^{-x^2}dx\right)\left(\int_0^\infty e^{-y^2}dy\right)=\int_0^\infty\int_0^\infty e^{-x^2}e^{-y^2}dy\,dx=\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dy\,dx$$

By Example 15.4.8 we find that $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}dy\,dx=\frac{\pi}{4}$. So, $I^2=\frac{\pi}{4}\Rightarrow I=\sqrt{\frac{\pi}{4}}=\frac{\sqrt{\pi}}{2}$

So, $\int_0^\infty e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$

(2) Exercise

Section 15.7: Triple Integrals

15.7.1 Fubini's Theorem for Triple integrals: If $f(x,y,z)$ is continuous on the rectangular box $B=\{(x,y,z): a\le x\le b, c\le y\le d, r\le z\le s\}=[a,b]\times[c,d]\times[r,s]$, then

$$\underbrace{\iiint_B f\,dV}_{\text{called triple integral}}=\underbrace{\int_a^b\int_c^d\int_r^s f\,dzdydx}_{\text{called iterated integrals}}=\underbrace{\int_r^s\int_c^d\int_a^b f\,dx\,dy\,dz}_{\text{called iterated integrals}}=\underbrace{\int_a^b\int_r^s\int_c^d f\,dy\,dz\,dx}_{\text{called iterated integrals}}=\cdots\cdots$$

Observe that $dV=dzdydx=dzdxdy=dxdydz=dxdzdy=dydxdz=dydzdx$

Example 15.7.2: Evaluate $\iiint_B xyz^2dV$, where $B=\{(x,y,z):\sqrt{2}\le x\le 2, 0\le y\le 4, -1\le z\le 1\}$

Solution: Take $dV=dxdydz$

$$\iiint_B xyz^2dV=\int_{-1}^1\int_0^4\int_{\sqrt{2}}^2 xyz^2\,dx\,dy\,dz=\int_{-1}^1\int_0^4 \frac{x^2}{2}\Big]_{\sqrt{2}}^2 yz^2\,dy\,dz=\int_{-1}^1\int_0^4 yz^2\,dy\,dz=\cdots=16$$

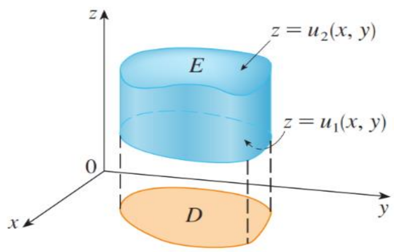
Observe that this example can be solved faster as: $\iiint_B xyz^2dV=\left(\int_{\sqrt{2}}^2 x\,dx\right)\left(\int_0^4 y\,dy\right)\left(\int_{-1}^1 z^2\,dz\right)=\cdots$

15.7.3 Triple Integrals for Non-Rectangular Box Regions:

(1) Let E be the solid in $3D$ such that $u_1(x,y)\le z\le u_2(x,y)$ and the region D is the projection of S on the xy-plane.

Then: $\iiint_E f dV=\iint_D\left(\int_{u_1(x,y)}^{u_2(x,y)} f\,dz\right)dA$

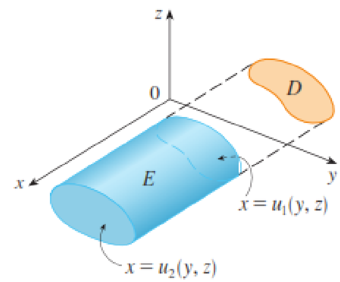
- Take dA as: $dA=dydx$ or $dA=dxdy$ or $dA=rdrd\theta$



(2) Let E be the solid in $3D$ such that $u_1(y,z)\le x\le u_2(y,z)$ and the region D is the projection of S on the yz-plane.

Then: $\iiint_E f dV=\iint_D\left(\int_{u_1(y,z)}^{u_2(y,z)} f\,dx\right)dA$

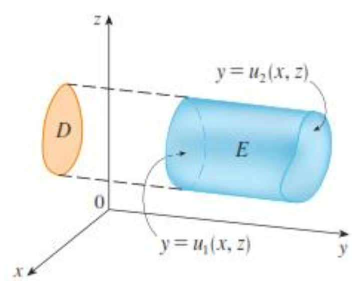
- Take dA as: $dA=dydz$ or $dA=dzdy$ or $dA=rdrd\theta$



(3) Let E be the solid in $3D$ such that $u_1(x,z)\le y\le u_2(x,z)$ and the region D is the projection of S on the xy-plane.

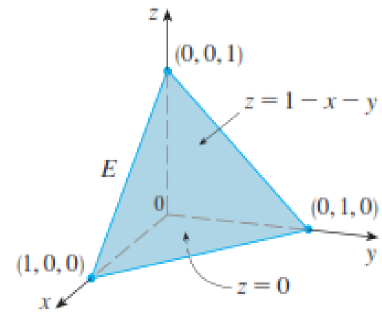
Then $\iiint_E f dV=\iint_D\left(\int_{u_1(x,z)}^{u_2(x,z)} f\,dy\right)dA$

- Take dA as: $dA=dxdz$ or $dA=dzdx$ or $dA=rdrd\theta$



Example 15.7.4: Evaluate $\iiint_E zdV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution:



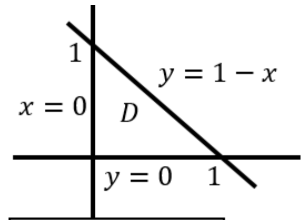
- First we must sketch the graph of the solid.

  Its graph is given in the next figure:
- The surfaces are:

  $z = 0$ (lower surface) and $z = 1 - x - y$ (upper surface)
- $dV = dzdA$

  $\Rightarrow \iiint_E zdV = \iint_D \left(\int_0^{1-x-y} z\,dz\right) dA$
- Now we must sketch the graph of the projection of the solid on the xy-plane:
  - The projection region D is bounded by:

    $$x = 0, y = 0, y = 1 - x$$
  - $dA = dydx$: $y = 0 \to y = 1 - x$ and $0 \le x \le 1$



$dA = dydx$
$dV = dzdydx$

$$\Rightarrow \iiint_E zdV = \iint_D \left(\int_0^{1-x-y} z\,dz\right) dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} \frac{z^2}{2}\Big]_0^{1-x-y} dy\,dx$$

$$= \frac{1}{2}\int_0^1 \int_0^{1-x} (1-x-y)^2\,dy\,dx = \frac{1}{2}\int_0^1 \frac{(1-x-y)^3}{-3}\Big]_0^{1-x} dx$$

$$= \frac{1}{6}\int_0^1 (1-x)^3\,dx = \frac{1}{6}\frac{(1-x)^4}{-4}\Big]_0^1 = \frac{1}{24}$$

ملاحظة: لدراسة التكاملات الثلاثية لا بد من رسم الجسم لتحديد السطوح وتحديد منطقة الاسقاط للجسم. ولكن قد يتساءل البعض كيف يقوم برنامج حاسوبي (**software**) بحساب التكاملات؟ الجواب: هناك خوارزميات رياضية تستخدم في عمل البرامج الرياضية التي تحساب التكاملات دون الحاجة للقيام برسم الأجسام لتحديد السطوح ومناطق الاسقاط للأجسام. ومن هذه الخوارزميات ما يلي:

① نحدد معادلات السطوح (Surfaces) ويفترض أن يكون عددها 2 وغير معقدة الشكل

② المعادلات المتبقية توضع ضمن معادلات منطقة الإسقاط (Projection Region) ثم نرسم هذه المعادلات في بعدين (حسب المتغيرات في المعادلات): اذا كانت المنطقة المرسومة مغلقة فننتقل للخطوة ③، أما اذا لم تكن المنطقة مغلقة فنقم بعمل تقاطع للسطوح (التي في الخطوة 1) ونضيف معادلات التقاطع الى معادلات منطقة الإسقاط(في بعض الحالات نستثني بعض المعادلة الناتجة من تقاطع السطوح لأنها لا تمثل حد للمنطقة المغلقة)

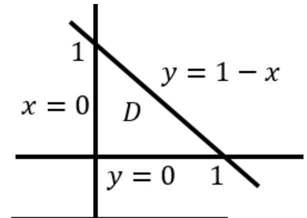
Now we resolve Example 15.7.4 using another algorithm: Recall that Example 15.7.4 says that:

Evaluate $\iiint_E zdV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

- **Surfaces:** $z = 0$ and $z = 1 - x - y \Rightarrow dV = dzdA$
- **Projection Region** D: bounded by $x = 0$, $y = 0$ (not closed)
  - We must add the curve of intersection of the surfaces to region D

$$1 - x - y = 0 \Leftrightarrow y = 1 - x$$

  - **Projection Region** D is bounded by: $x = 0, y = 0, y = 1 - x$
    - $dA = dydx$: $y = 0 \to y = 1 - x$ and $0 \le x \le 1$



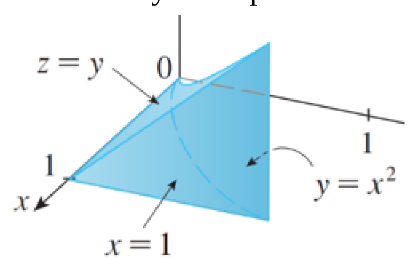
$dA = dydx$
$dV = dzdydx$

لتحديد السطح في الحد الأدنى والسطح في الحد الأعلى في حدود التكامل الأول وبدون رسم: نأخذ نقطة اختبار في المنطقة D ولتكن مثلاً (0,0) ثم نعوضها في معادلتي السطحين فالذي قيمة z أصغر يكون السطح في الحد الأدنى والذي قيمة z له أكبر يكون السطح في الحد الأعلى

$z = 0$ at $(x, y) = (0,0) \Rightarrow z = 0$
$z = 1 - x - y$ at $(x, y) = (0,0) \Rightarrow z = 1$
$\Rightarrow$
- ❖ $z = 0$ (lower surface)
- ❖ $z = 1 - x - y$ (upper surface)

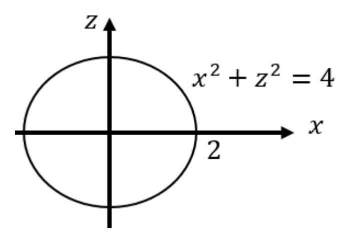
$$\Rightarrow \iiint_E f dV = \iint_D \left(\int_0^{1-x-y} z\,dz\right) dA = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx = \cdots = \frac{1}{24}$$

Example 15.7.5: Find $\iiint_E \sqrt{x^2 + z^2}dV$, where E is the region bounded by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$.



Solution:

- **Surfaces:** $y = x^2 + z^2$ and $y = 4 \Rightarrow dV = dydA$
- **Projection Region** D:
  bounded by …….. (there are no curves). So, we must add the curve of intersection of surfaces:
  - $x^2 + z^2 = 4$
  - **The Projection Region** D is enclosed by: $x^2 + z^2 = 4$
  - The graph of the projection region D is given:
    - The Projection Region D is:
      $r = 0 \to r = 2$ and $0 \le \theta \le 2\pi \Rightarrow dA = rdrd\theta$



لتحديد السطح في الحد الأدنى والسطح في الحد الأعلى في حدود التكامل الأول: نأخذ نقطة اختبار في المنطقة D ولتكن مثلاً (0,0) ثم نعوضها في معادلتي السطحين فالذي قيمة y له أصغر يكون السطح في الحد الأدنى والذي قيمة y له أكبر يكون السطح في الحد الأعلى

$y = x^2 + z^2$ at $(x, z) = (0,0) \Rightarrow y = 0$
$y = 4$ at $(x, z) = (0,0) \Rightarrow y = 4$
$\Rightarrow$
- ❖ $y = x^2 + z^2$ (lower surface)
- ❖ $y = 4$ (upper surface)

$$\iiint_E f dV = \iint_D \left(\int_{x^2+z^2}^4 \sqrt{x^2 + z^2}\,dy\right) dA = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 r\,dy\,rdrd\theta = \int_0^{2\pi} \int_0^2 r^2(4 - r^2)\,dr\,d\theta$$

$$= \cdots = \frac{128\pi}{15}$$

Volumes 15.7.6: Let E be the solid in $3D$ such that $u_1(x,y) \leq z \leq u_2(x,y)$ and the region D is the projection of S on the xy-plane.

(1) The volume of the solid E can expressed as a triple integral as:

$$V = \iiint_E 1\, dV = \iint_D \left(\int_{u_1(x,y)}^{u_2(x,y)} 1\, dz \right) dA$$

(2) The volume of the solid E can expressed as a duble integral as:

$$V = \iint_D (u_2(x,y) - u_1(x,y))\, dA$$

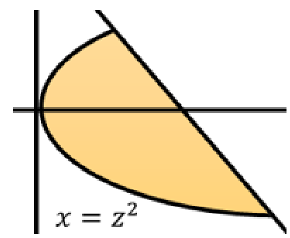
Example 15.7.7:

(1) Use triple integral to find the volume of the solid enclosed by the parabolic cylinder $y = z^2$ and the planes $y = x, z = 2 - x$.

(2) Use double integral to find the volume of the solid enclosed by the parabolic cylinder $y = z^2$ and the planes $y = x, z = 0, z = 2 - x$.

Solution:

(1)



- **Surfaces:** $y = z^2, y = x \Rightarrow dV = dydA$
- **Projection Region** D: $z = 2 - x$ (not a closed region)
  - The curve of intersection of surfaces: $z^2 = x$
  - The **Projection Region** D is bounded by: $z = 2 - x, z^2 = x$
    - Curves: $x = 2 - z, x = z^2$ and $?? \leq z \leq ??$. So we must find the intersections of curves:

      $$z^2 = 2 - z \Rightarrow z^2 + z - 2 = 0 \Rightarrow (z+2)(z-1) = 0 \Rightarrow z = -2,1$$
    - $dA = dxdz$ with $x = 2 - z, x = z^2$ and $-2 \leq z \leq 1$
- **Lower and upper surfaces**:

$$\begin{Bmatrix} y = z^2 \text{ at } (x,z) = (2,0) \Rightarrow y = 0 \\ y = x \text{ at } (x,z) = (2,0) \Rightarrow y = 2 \end{Bmatrix} \Rightarrow \begin{Bmatrix} y = z^2 \text{ (lower surface)} \\ y = x \text{ (upper surface)} \end{Bmatrix}$$

$$V = \iiint_E 1\, dV = \iint_D \left(\int_{z^2}^{x} 1\, dy \right) dA = \int_{-2}^{1} \int_{z^2}^{2-z} \int_{z^2}^{x} 1\, dy\, dx\, dz = \int_{-2}^{1} \int_{z^2}^{2-z} (x - z^2) dx\, dz = \cdots$$

(2) $V = \iint_D (x - z^2)\, dA = \int_{-2}^{1} \int_{z^2}^{2-z} (x - z^2) dx\, dz = \cdots$

Example 15.7.8:

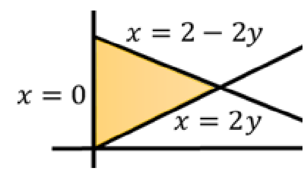
(1) Use triple integral to find the volume of the tetrahedron T enclosed by the planes $x + 2y + z = 2, x = 2y, x = 0, z = 0$.

(2) Use triple integral to find the volume of the tetrahedron T enclosed by the planes $x + 2y + z = 2, x = 2y, x = 0, z = 0$.

Solution:

(1)

- **Surfaces:** $z = 2 - x - 2y$, $z = 0 \Rightarrow dV = dzdA$
- **Projection Region D:** $x = 2y$, $x = 0$ (not a close region). So, we must add the intersection of surfaces:
  - Intersection of surfaces: $2 - x - 2y = 0 \Rightarrow x = 2 - 2y$
  - **The Projection Region** is enclosed by: $x = 2y, x = 0, x = 2 - 2y$
  - The graph of the region D is given in the next figure:



  - $dA = dydx \Rightarrow y = \frac{x}{2} \to y = \frac{2-x}{2}$ and $0 \le x \le ???$

    So, we must find the intersection of curves:
    - Intersection of curves: $\frac{x}{2} = \frac{2-x}{2} \Rightarrow x = 2 - x \Rightarrow 2x = 2 \Rightarrow x = 1$
    - $dA = dydx \Rightarrow y = \frac{x}{2} \to y = \frac{2-x}{2}$ and $0 \le x \le 1$
- **Lower and upper surfaces**:

$$\begin{Bmatrix} z = 2 - x - 2y \text{ at } (x, y) = (0,0) \Rightarrow \ z = 2 \\ z = 0 \text{ at } (x, y) = (0,0) \Rightarrow \ z = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} z = 0 \text{ (lower surface)} \\ z = 2 - x - 2y \text{ (upper surface)} \end{Bmatrix}$$

$$\text{Volume} = \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} \int_0^{2-x-2y} 1 \, dz \, dy \, dx = \cdots$$

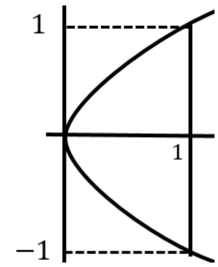
(2) By Part (1) we have Volume $= \int_0^1 \int_{\frac{x}{2}}^{\frac{2-x}{2}} (2 - x - 2y) \, dy \, dx = \cdots$

Example 15.7.9:

(1) Use triple integral to find the volume of the solid enclosed by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0, x = 1$.

(2) Use double integral to find the volume of the solid enclosed by the parabolic cylinder $x = y^2$ and the planes $x = z, z = 0, x = 1$.

Solution:



- **Surfaces:** $z = x,\ z = 0 \Rightarrow dV = dzdA$
- **Projection Region D**: $x = y^2,\ x = 1$ (closed region see the figure)
  - Intersection of curves: $y^2 = 1 \Rightarrow y = -1$ or $y = 1$
  - $dA = dxdy \Rightarrow x = y^2 \to x = 1$ and $-1 \le y \le 1$
- **Lower and upper surfaces**:

$$\begin{Bmatrix} z = x \text{ at } (x,z) = (1,0) \Rightarrow z = 1 \\ z = 0 \text{ at } (x,z) = (1,0) \Rightarrow z = 0 \end{Bmatrix} \Rightarrow \begin{Bmatrix} z = 0 \text{ (lower surface)} \\ z = x \text{ (upper surface)} \end{Bmatrix}$$

(1) Volume $= \iiint_E 1dV = \int_{-1}^{1}\int_{y^2}^{1}\int_0^x 1\,dz\,dx\,dy = \int_{-1}^{1}\int_{y^2}^{1}(x-0)\,dx\,dy = \int_{-1}^{1} \frac{x^2}{2}\Big]_{y^2}^{1} dy$

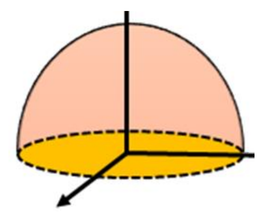
$$= \frac{1}{2}\int_{-1}^{1}(1-y^4)dy = \frac{4}{5}$$

(2) Volume $= \iint_D (x-0)dA = \int_{-1}^{1}\int_{y^2}^{1}(x-0)\,dx\,dy = \cdots = \frac{4}{5}$

Example 15.7.10: Compute $\iiint_E -12dV$, where $E = \{(x,y,z): x^2 + y^2 + z^2 \le 9, z \ge 0\}$

Solution: $\iiint_E -12dV = -12\iiint_E 1dV = -12 \times$ Volume of E (observe that E is a hemisphere)

$$= -12 \times \frac{\text{Volume of the sphere}}{2} = -12 \times \frac{\frac{4}{3}\pi(3)^3}{2} = -216\pi$$



Example 15.7.11: Let $I = \int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx$

(1) Express the iterated integral I as a triple integral and sketch the solid.
(2) Rewrite the iterated integral I in a different order, integrating first with respect to x, then z, and then y.
(3) Rewrite the iterated integral I in a different order, integrating first with respect to y, then x, and then z.

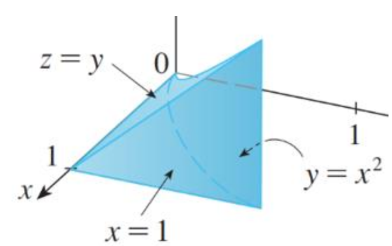
Solution:

(1) First we give the equations of the boundary surfaces of the solid E. We can do this by one of two ways:

(a) by sketching the solid E

or

(b) by deleting the equations that results from surface or curve intersections.



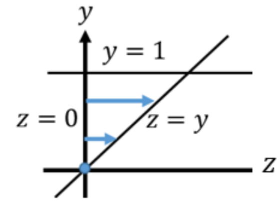
We will use the second way to determine the surface E:

- Equations: $\underbrace{z = 0, z = y}_{\text{Surfaces}}$, $\underbrace{y = 0, y = x^2}_{\text{Curves}}$ && $\underbrace{x = 0,\ x = 1}$
  (Surfaces: لا يلغي أي منهم; Curves: ممكن الغاء بعضهم; $x = 0,\ x = 1$: ممكن الغاء بعضهم أو كلهم)
  - Intersection of surfaces: $z = 0, z = y \Rightarrow y = 0$ (no need for this equation)
  - Intersection of curves: $y = 0, y = x^2 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$ (no need for these equations)
- The triple integral is: $I = \iiint_E f dV$, where E is the solid bounded by:

$$z = 0, z = y, y = x^2, \text{and } x = 1$$

(2) Integrating first with respect to x, then z, and then $y \Rightarrow \overbrace{x=???}^{\text{surfaces}}, \overbrace{z=???}^{\text{curves}}, \overbrace{???\le y \le ???}^{\text{for the Projection Region}}$

- ❖ From Part (1) we find that the solid E is bounded by: $z = 0, z = y, y = x^2$, and $x = 1$
  - ➢ **Surfaces:** $x = \sqrt{y}$ and $x = 1 \Rightarrow dV = dx dz dy$
  - ➢ **Projection Region D:**
    - ▪ Curves: $z = 0, z = y$ (not a closed region)
    - ▪ Intersection of surfaces: $\sqrt{y} = 1 \Rightarrow y = 1$
    - ▪ The Projection region D is bounded by: $z = 0, z = y, y = 1$
    - ▪ The graph of D given in the next figure:



$z = 0 \to z = y$ and $0 \le y \le 1$

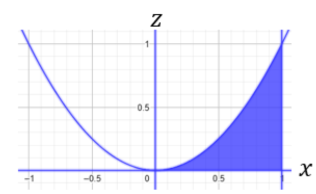
  - ➢ **Lower and Upper Surfaces:**

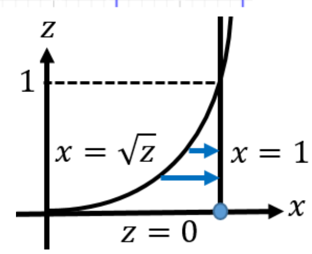
$$\begin{Bmatrix} x = \sqrt{y} \text{ at } (0,0) \Rightarrow & x = 0 \\ x = 1 \text{ at } (0,0) \Rightarrow & x = 1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} x = \sqrt{y} \text{ (lower surface)} \\ x = 1 \text{ (upper surface)} \end{Bmatrix}$$

$$I = \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\, dx\, dz\, dy$$

(3) Integrating first with respect to y, then x, and then $z \Rightarrow \overbrace{y=???}^{\text{surfaces}}, \overbrace{x=???}^{\text{curves}}, \overbrace{???\le z \le ???}^{\text{for the Projection Region}}$

- ❖ From Part (1) we find that the solid E is bounded by: $z = 0, z = y, y = x^2$, and $x = 1$
  - ➢ **Surfaces:** $y = z$ and $y = x^2 \Rightarrow dV = dy dx dz$
  - ➢ **Projection Region D:**
    - ▪ Curves: $z = 0, x = 1$ (not a closed region)
    - ▪ Intersection of surfaces: $z = x^2$ (add this eq. to the region D)
    - ▪ The Projection region D is bounded by: $z = 0, x = 1, z = x^2$
    - ▪ The graph of D given in the next figure:
    - ▪ The Projection region is with $dA = dx dz$ which means that:
      - • Curves: $x = \sqrt{z} \to x = 1$ and $0 \le z \le 1$





  - ➢ **Lower and Upper Surfaces:**

$$\begin{Bmatrix} y = z \text{ at } (x,z) = (1,0) \Rightarrow & y = 0 \\ y = x^2 \text{ at } (x,z) = (1,0) \Rightarrow & y = 1 \end{Bmatrix} \Rightarrow \begin{Bmatrix} y = z \text{ (lower surface)} \\ y = x^2 \text{ (upper surface)} \end{Bmatrix}$$

$$I = \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\, dy\, dx\, dz$$

Example 15.7.12: Rewrite the iterated integral $\int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} \int_0^{2-y} f(x,y,z)\, dz\, dy\, dx$ in a different order, integrating first with respect to x, then y, and then z.

Solution: $\int_0^4 \int_{\frac{x}{2}}^{\sqrt{x}} \int_0^{2-y} f\, dz\, dy\, dx = \int_0^2 \int_0^{2-z} \int_{y^2}^{2y} f\, dx\, dy\, dz.$

Example 15.7.13: Express the iterated integral $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f\, dy\, dz\, dx$ in a different order, integrating first with respect to z, then y, and then x.

Solution: $\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f\, dy\, dz\, dx = \int_0^1 \int_0^{1-x^2} \int_0^{1-x^2} f\, dz\, dy\, dx.$

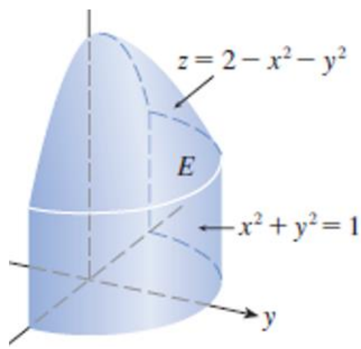
Exercise 15.7.14:

(1) Let $\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} f\, dy\, dz\, dx.$
- (a) Express the iterated integral I as a triple integral
- (b) Rewrite the iterated integral I in a different order, integrating first with respect to z, then x, and then y.

(2) Express the iterated integral $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} f\, dz\, dy\, dx$ in different order:
- (a) First integrate with respect to x, then y, then z
- (b) First integrate with respect to y, then x, then z

(3) Find the volume of the solid E below the surface $z = 2x + y$ and above the region $R = [0,1] \times [0,2]$.

(4) Set up in cylindrical coordinates, the volume of the solid E (whose graph is given on the right)



(5) Set up the integral in polar coordinates of the volume V of the solid that lies inside the cylinder $x^2 + y^2 = 2y$, under the paraboloid $z = x^2 + y^2$, and above xy-plane.

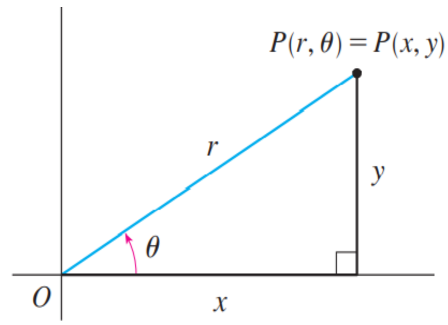
15.8. Triple Integrals in Cylindrical Coordinates

Definition 15.8.1: Let $P(x, y, z)$ be a point in rectangular (Cartesian) coordinates.

The cylindrical coordinates of P are: $P(r, \theta, z)$, where

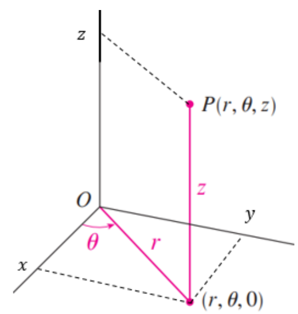
rectangular $\Rightarrow$ Cylindrical: $(x, y, z) \Rightarrow (r, \theta, z)$

- $r = \sqrt{x^2 + y^2} \iff r^2 = x^2 + y^2$
- $\theta = \tan^{-1}\left(\frac{y}{x}\right) \iff \tan\theta = \frac{y}{x}, 0 \le \theta < 2\pi$
- $z = z$



Cylindrical $\Rightarrow$ rectangular: $(r, \theta, z) \Rightarrow (x, y, z)$

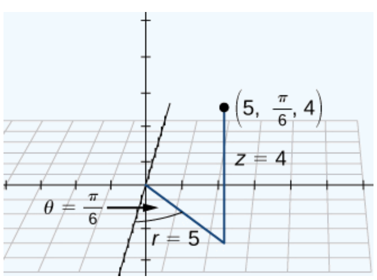
$$x = r\cos\theta, y = r\sin\theta, \ z = z$$



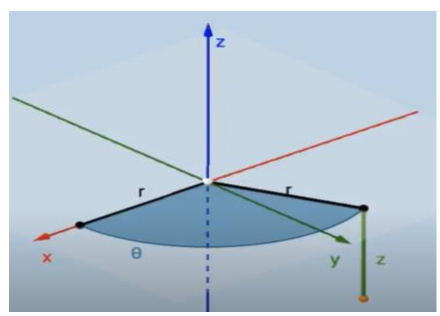
How to Plot a point $P(r, \theta, z)$ in Cylindrical Coordinates?

Example 15.8.2: Plot the points with cylindrical coordinates $\left(5, \frac{\pi}{6}, 4\right), \left(2, \frac{2\pi}{3}, -2\right)$ and find their rectangular coordinates.

Solution:



$\left(5, \frac{\pi}{6}, 4\right)$



$\left(2, \frac{2\pi}{3}, -2\right)$

The Rectangular coordinates are:

- $\left(5, \frac{\pi}{6}, 4\right) \Rightarrow r = 5, \theta = \frac{\pi}{6}, z = 4$:
  - $x = r\cos\theta = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2}$
  - $y = r\sin\theta = 5\sin\frac{\pi}{6} = \frac{5}{2}$
  - $z = 4$

The rectangular coordinates of the point $\left(5, \frac{\pi}{6}, 4\right)$ are $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 4\right)$

- $\left(2, \frac{2\pi}{3}, -2\right) \Rightarrow r = 2, \theta = \frac{2\pi}{3}, z = -2$
  - $x = r\cos\theta = 2\cos\frac{2\pi}{3} = 2\left(-\cos\frac{\pi}{3}\right) = -2 \times \frac{1}{2} = -1$
  - $y = r\sin\theta = 2\sin\frac{2\pi}{3} = 2\left(\sin\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$
  - $z = -2$

The rectangular coordinates of the point $\left(2, \frac{2\pi}{3}, -2\right)$ are $\left(-1, \sqrt{3}, -2\right)$

Example 15.8.3: Find the cylindrical coordinates of the points whose Cartesian (or rectangular) coordinates are: $A(3,-3,-2), B(-3,3,7), C(-2,-3,1), D(2,0,-1),$
$E(-2,0,1), F(0,2,-1), G(0,-2,3).$

Solution:

- $A(3,-3,-2): \Rightarrow x = 3, y = -3, z = -2$
  (observe that the point $(x,y) = (3,-3)$ in the 4th quadrant $\Rightarrow \frac{3\pi}{2} \le \theta \le 2\pi$
  $\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$
  $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{-3}{3} = -1$ and recall that $\frac{3\pi}{2} \le \theta \le 2\pi \Rightarrow \theta = 2\pi - \frac{\pi}{4} \Rightarrow \theta = \frac{7\pi}{4}$
  The cylindrical coordinates of the point $A(3,-3,-2)$ are: $\left(3\sqrt{2}, \frac{7\pi}{4}, -2\right)$
- $B(-3,3,7): \Rightarrow x = -3, y = 3, z = 7$
  (observe that the point $(x,y) = (-3,3)$ in the 2nd quadrant $\Rightarrow \frac{\pi}{2} \le \theta \le \pi$
  $\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$
  $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{3}{-3} = -1$ and recall that $\frac{\pi}{2} \le \theta \le \pi \Rightarrow \theta = \pi - \frac{\pi}{4} \Rightarrow \theta = \frac{3\pi}{4}$
  The cylindrical coordinates of the point $B(-3,-3,7)$ are: $\left(3\sqrt{2}, \frac{3\pi}{4}, 7\right)$
- $C(-2,-3,1): \Rightarrow x = -2, y = -3, z = 1$
  (observe that the point $(x,y) = (-2,-3)$ in the 3rd quadrant $\Rightarrow \pi \le \theta \le \frac{3\pi}{2}$
  $\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{4+9} = \sqrt{13}$
  $\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{-3}{-2} = \frac{3}{2}$ and recall that $\pi \le \theta \le \frac{3\pi}{2} \Rightarrow \theta = \pi + \tan^{-1}\left(\frac{3}{2}\right)$
  The cylindrical coordinates of the point $C(-2,-3,1)$ are: $\left(\sqrt{13}, \pi + \tan^{-1}\left(\frac{3}{2}\right), 1\right)$
- $D(2,0,-1): \Rightarrow x = 2, y = 0, z = -1$
  (observe that the point $(x,y) = (2,0)$ on the positive x-axis $\Rightarrow \theta = 0$
  $\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{4+0} = 2$
  The cylindrical coordinates of the point $D(2,0,-1)$ are: $(2,0,-1)$
- $E(-2,0,1): \Rightarrow x = -2, y = 0, z = 1$
  (observe that the point $(x,y) = (-2,0)$ on the negative x-axis $\Rightarrow \theta = \pi$
  $\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{4+0} = 2$
  The cylindrical coordinates of the point $E(-2,0,1)$ are: $(2,\pi,1)$
- $F(0,2,-1): \Rightarrow x = 0, y = 2, z = -1$
  (observe that the point $(x,y) = (0,2)$ on the positive y-axis $\Rightarrow \theta = \frac{\pi}{2}$
  $\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{0+4} = 2$
  The cylindrical coordinates of the point $F(0,2,-1)$ are: $\left(2, \frac{\pi}{2}, -1\right)$
- $G(0,-2,3): \Rightarrow x = 0, y = -2, z = 3$
  (observe that the point $(x,y) = (0,-2)$ on the negative y-axis $\Rightarrow \theta = \frac{3\pi}{2}$
  $\Rightarrow r = \sqrt{x^2+y^2} = \sqrt{0+4} = 2$
  The cylindrical coordinates of the point $G(0,-2,3)$ are: $\left(2, \frac{3\pi}{2}, 3\right)$

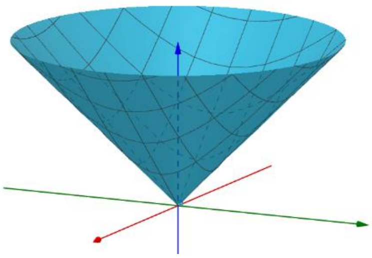
Example 15.8.4: Describe and sketch the curve whose equation in cylindrical coordinates is give by:

(1) $z = r$ (2) $z = -r$

(3) $z = \sqrt{9 - r^2}$

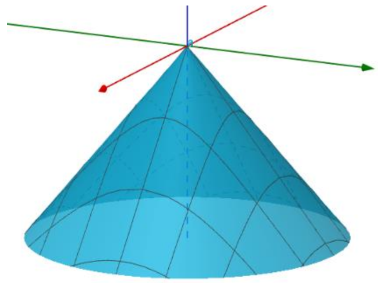
Solution:

$z = r \Rightarrow z = \sqrt{x^2 + y^2}$



The surface is a cone

$z = -r \Rightarrow z = -\sqrt{x^2 + y^2}$



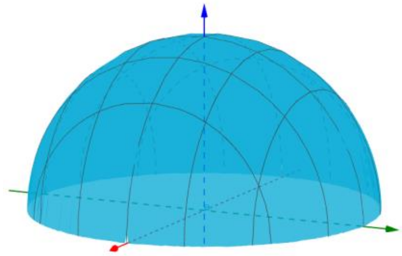
The surface is a cone

$z = \sqrt{9 - r^2} \Rightarrow z = \sqrt{9 - (x^2 + y^2)}$ (this means $z \geq 0$)

$\Rightarrow z^2 = 9 - (x^2 + y^2)$

$\Rightarrow x^2 + y^2 + z^2 = 9, z \geq 0$

The surface is the hemisphere centered at $(0,0,0)$ of radius 3



Example 15.8.5: Write the equation $z = x^2 - y^2$ in cylindrical coordinates

Solution: $z = x^2 - y^2 \Rightarrow z = (r\cos\theta)^2 - (r\sin\theta)^2 \Rightarrow z = r^2(\cos^2\theta - \sin^2\theta)$

$\Rightarrow z = r^2 \cos(2\theta)$

Example 15.8.6: Identify (give the name) of the surface whose equation is given in cylindrical coordinates and find its equation in Cartesian coordinates:

(1) $r = 5$ (2) $z = 4 - r^2$ (3) $\theta = \frac{\pi}{4}$ (4) $\theta = \frac{2\pi}{3}$

(5) $\theta = 0$ (6) $\theta = \pi$ (7) $\theta = \frac{\pi}{2}$ (8) $\theta = \frac{3\pi}{2}$

Solution:

(1) $r = 5 \Rightarrow r^2 = 25 \Rightarrow x^2 + y^2 = 25$ (the surface is a cylider)

(2) $z = 4 - r^2 \Rightarrow z = 4 - (x^2 + y^2) \Rightarrow 4 - z = x^2 + y^2$ (the surface is a paraboloid)

(3) $\theta = \frac{\pi}{4}$ (in the 1st quadrant $\Rightarrow x \geq 0, y \geq 0) \Rightarrow \tan(\theta) = \tan\left(\frac{\pi}{4}\right) \Rightarrow \frac{y}{x} = 1$

$\Rightarrow y = x$ with $x \geq 0) \Rightarrow y = x, x \geq 0$ ia a half plane.

(4) $\theta = \frac{2\pi}{3}$ (in the 2nd quadrant $\Rightarrow x \leq 0, y \geq 0) \Rightarrow \tan(\theta) = \tan\left(\frac{2\pi}{3}\right) \Rightarrow \frac{y}{x} = -\sqrt{3}$

$\Rightarrow y = -\sqrt{3}x$ with $x \leq 0 \Rightarrow y = -\sqrt{3}x, x \leq 0$ ia a half plane.

(5) $\theta = 0 \Rightarrow y = 0$ and $x \geq 0 \Rightarrow y = 0, x \geq 0$ is the half xz-plane

(6) $\theta = \pi \Rightarrow y = 0$ and $x \leq 0 \Rightarrow y = 0. x \leq 0$ is the half xz-plane

(7) $\theta = \frac{\pi}{2} \Rightarrow x = 0$ and $y \geq 0 \Rightarrow x = 0. y \geq 0$ is the half yz-plane

(8) $\theta = \frac{3\pi}{2} \Rightarrow x = 0$ and $y \leq 0 \Rightarrow x = 0. y \leq 0$ is the half yz-plane

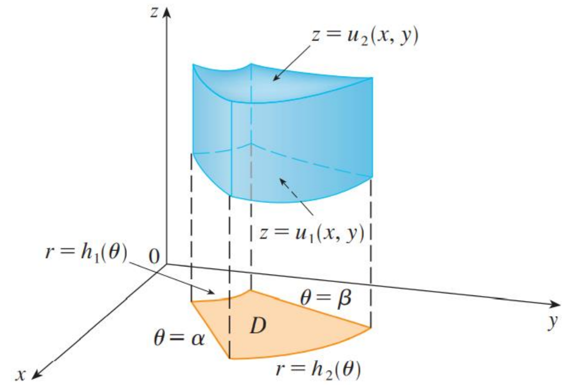
Evaluating Triple Integrals with Cylindrical Coordinates

Suppose that E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates (see Figure 6). In particular, suppose that f is continuous and

$$E = \{(x, y, z) \mid (x, y) \in D,\ u_1(x, y) \leqslant z \leqslant u_2(x, y)\}$$

where D is given in polar coordinates by

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$



It is known that

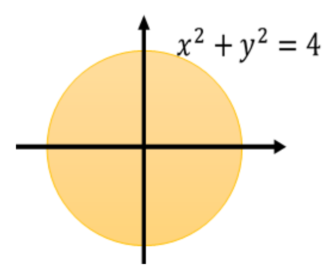
$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz \right] dA$$

⇒ The triple integral can written in cylindrical coordinates as:

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta$$

Example 15.8.7: Set up (do not evaluate) in cylindrical coordinates the volume of the solid bounded by the plane $z = 1$ and the paraboloid $z = 5 - x^2 - y^2$.

Solution:



- **Surfaces:** $z = 1, z = 5 - x^2 - y^2 \Rightarrow dV = dzdA$
- **Projection Region D:**
  - Intersection of surfaces: $5 - x^2 - y^2 = 1$
  - The Projection Region is the region inside $x^2 + y^2 = 4$
  - $dA = rdrd\theta \Rightarrow r = 0 \to r = 2$ and $0 \le \theta \le 2\pi$
- **Lower and Upper Surfaces:**

  $z = 1$ (lower surface) and $z = 5 - x^2 - y^2$ (upper surface) (why?)

Volume $= \iint_D \int_1^{5-x^2-y^2} 1\, dzdA = \iint_D \left(4 - (x^2 + y^2)\right) dA = \int_0^{2\pi} \int_0^2 \int_1^{5-r^2} 1dzrdrd\theta$

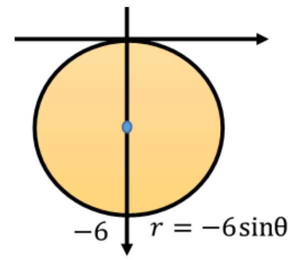
$= \int_0^{2\pi} \int_0^2 \int_1^{5-r^2} rdzdrd\theta$

Example 15.8.8: Set up (do not evaluate) as a triple integral the volume of the solid lies under the cone $z=\sqrt{x^2+y^2}$ and above the xy-plane and inside the cylinder $x^2+y^2=-6y$.

Solution: The solid is bounded by $z=\sqrt{x^2+y^2}, z=0, x^2+y^2=-6x$.

- **Surfaces:** $z=\sqrt{x^2+y^2}, z=0$
- **Projection Region D:** $\overbrace{\text{inside}}^{\text{من نص السؤال}}$ $x^2+y^2=-6y$ (closed region)
  - Sketch the region D: $x^2+y^2=-6y \Rightarrow x^2+\underbrace{y^2+6y}_{\text{إكمال مربع}}=0$

    $\Rightarrow x^2+\underbrace{y^2+6y+9}_{\text{إكمال مربع}}=\underbrace{\quad 9 \quad}_{\text{بسبب إكمال المربع}} \quad \Rightarrow \quad x^2+(y+3)^2=9$



- **Lower and Upper Surfaces:**
  $z=0$ (lower surface) and $z=\sqrt{x^2+y^2}$ (upper surface) (why?)

$$\text{Volume}=\iiint_E 1dV=\iint_D \int_0^{\sqrt{x^2+y^2}} 1\,dz\,dA=\int_{-\pi}^{0}\int_0^{-6\sin\theta}\int_0^{r} r\,dz\,dr\,d\theta$$

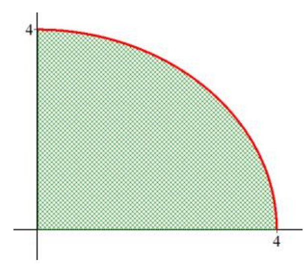
Example 15.8.9: Express the triple integral $I=\iiint_E (x+y+z)dV$ in cylindrical coordinates, where E is the solid in the first octant that lies under the paraboloid $z=48-3x^2-3y^2$

Solution: Observe that the solid is in the first octant which means that: $x\ge 0, y\ge 0, z\ge 0$

$\Rightarrow$ the solid is bounded by: $z=48-3x^2-3y^2, x=0, y=0, z=0$ in the first octant

$dV=dzdA$

- **Surfaces:** $z=48-3x^2-3y^2, z=0 \Rightarrow dV=dzdA$
- **Projection Region D:** $x=0, y=0$ in the 1st quadrant (not closed)
  - Intersection of surfaces: $48-3x^2-3y^2=0$
    $\Rightarrow x^2+y^2=16$ (add this equation to the region)
  - The Projection Region is bounded by:
    $x=0, y=0, x^2+y^2=16$ in the 1st quadrant
  - $dA=rdrd\theta \Rightarrow 0\le r\le 2$ and $0\le\theta\le\frac{\pi}{2}$



- **Lower and Upper Surfaces:**
  $z=0$ (lower surface) and $z=48-3x^2-3y^2$ (upper surface) (why?)

$$I=\iiint_E (x+y+z)dV=\iint_D\int_0^{48-3x^2-3y^2}(x+y+z)dz\,dA$$

$$=\int_0^{\frac{\pi}{2}}\int_0^4\int_0^{48-3r^2}(r\cos\theta+r\sin\theta+z)rdzdrd\theta$$

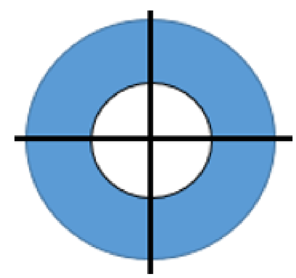
$$=\int_0^{\frac{\pi}{2}}\int_0^4\int_0^{48-3r^2}(r^2\cos\theta+r^2\sin\theta+rz)dzdrd\theta$$

Example 15.8.10: Set up as triple integral the volume of the solid inside the sphere $x^2+y^2+z^2=4$ and outside the cylinder $x^2+y^2=1$.

Solution:

- **Surfaces:** $z=-\sqrt{4-(x^2+y^2)}$, $z=\sqrt{4-(x^2+y^2)} \Rightarrow dV=dzdA$
- **Projection Region D:** $\overbrace{\text{outside}}^{\text{من نص السؤال}}\ x^2+y^2=1$ (not closed region)
  - Intersection of surfaces: $-\sqrt{4-(x^2+y^2)}=\sqrt{4-(x^2+y^2)}$

    $\Rightarrow 2\sqrt{4-(x^2+y^2)}=0 \Rightarrow x^2+y^2=4$
  - The Projection Region D is between $x^2+y^2=1$ and $x^2+y^2=4$
  - $dA=rdrd\theta \Rightarrow r=1 \rightarrow r=2$ and $0\le\theta\le2\pi$



- **Lower and Upper Surfaces:**

  $z=-\sqrt{4-(x^2+y^2)}$ (lower surface)

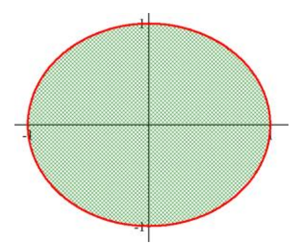
  and $z=\sqrt{4-(x^2+y^2)}$ (upper surface) (why?)

$$\text{Volume}=\iiint_E 1dV=\iint_D \int_{-\sqrt{4-(x^2+y^2)}}^{\sqrt{4-(x^2+y^2)}} 1\,dz\,dA=\int_0^{2\pi}\int_1^2\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta$$

Example 15.8.11: Use triple integrals to find the volume of the solid that lies within the cylinder $x^2+y^2=1$, below the plane $z=1$ and above the paraboloid $z=1-x^2-y^2$.

Solution:

$dV=dzdA$



- **Surfaces:** $z=1$, $z=1-x^2-y^2$
- **Projection Region D:** within $x^2+y^2=1$
  - $dA=rdrd\theta \Rightarrow r=0 \rightarrow r=1$ and $0\le\theta\le2\pi$
- **Lower and Upper Surfaces:**

  $z=1-x^2-y^2$ (lower surface) and $z=1$ (upper surface) (why?)

$$V=\iiint_E 1dV=\iint_D \int_{1-x^2-y^2}^{1} 1dz\,dA=\int_0^{2\pi}\int_0^1\int_{1-r^2}^{1} 1dzrdrd\theta$$

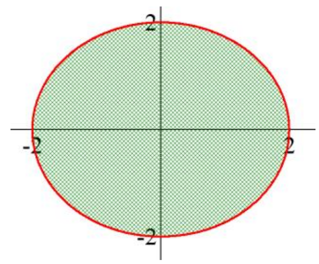
$$=\int_0^{2\pi}\int_0^1\big(1-(1-r^2)\big)r\,drd\theta$$

$$=\int_0^{2\pi}\int_0^1 r^3\,drd\theta=\frac{\pi}{2}$$

Example 15.8.12: Express each of the following integrals in cylidrical coordinates and then evaluate it:

(1) $$I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$$

(2) $$II = \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{0} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$$



Solution:

(1)

- **Projection Region:** $y = -\sqrt{4-x^2} \to y = \sqrt{4-x^2}$

  and $-2 \le x \le 2$

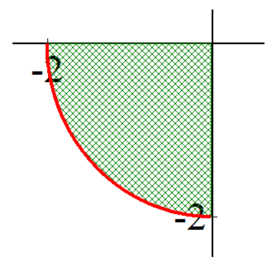
  - $dA = r dr d\theta \Rightarrow r = 0 \ \to r = 2$ and $0 \le \theta \le 2\pi$

$$\therefore\ I = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx = \int_0^{2\pi} \int_0^2 \int_r^2 r^2 dz\, r dr d\theta = \int_0^{2\pi} \int_0^2 \int_r^2 r^3 dz\, dr d\theta$$

- To evaluate the iterated triple integrals:

$$I = \int_0^{2\pi} \int_0^2 \int_r^2 r^3 dz\, dr d\theta = \int_0^{2\pi} \int_0^2 r^3(2-r) dr d\theta = \int_0^{2\pi} \int_0^2 (2r^3 - r^4) dr d\theta = \cdots$$

(2)



- **Projection Region:** $y = -\sqrt{4-x^2}, y = 0$ and $-2 \le x \le 0$
  - $dA = r dr d\theta \Rightarrow r = 0 \ \to r = 2$ and $\pi \le \theta \le \frac{3\pi}{2}$

$$II = \int_{-2}^{0} \int_{-\sqrt{4-x^2}}^{0} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dydx$$

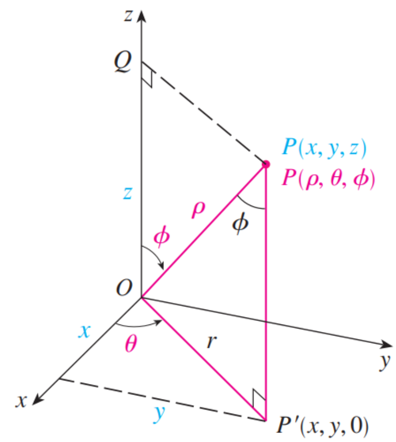
$$= \int_{\pi}^{2\pi} \int_0^2 \int_r^2 r^2 dz\, r dr d\theta = \int_{\pi}^{\frac{3\pi}{2}} \int_0^2 \int_r^2 r^3 dz\, dr d\theta$$

- To evaluate the iterated triple integrals:

$$II = \int_{\pi}^{\frac{3\pi}{2}} \int_0^2 \int_r^2 r^3 dz\, dr d\theta = \int_{\pi}^{\frac{3\pi}{2}} \int_0^2 r^3(2-r) dr d\theta = \int_{\pi}^{\frac{3\pi}{2}} \int_0^2 (2r^3 - r^4) dr d\theta = \cdots$$

15.9. Triple Integrals in Spherical Coordinates

Definition 15.9.1: Let $P(x, y, z)$ be a point in rectangular (Cartesian) coordinates. The spherical coordinates of P are: $P(\rho, \theta, \varphi)$:



- rectangular ⇒ spherical: $(x, y, z) \Rightarrow (\rho, \theta, \varphi)$, where
  $$\rho = \sqrt{x^2 + y^2 + z^2} \;\Leftrightarrow\; r^2 = x^2 + y^2$$
  $$\theta = \tan^{-1}\left(\frac{y}{x}\right) \;\Leftrightarrow\; \tan\theta = \frac{y}{x}, 0 \le \theta < 2\pi$$
  $$\cos\varphi = \frac{z}{\rho}, 0 \le \varphi \le \pi$$
- spherical ⇒ rectangular: $(\rho, \theta, \varphi) \Rightarrow (x, y, z)$
  $$x = \rho\sin\varphi\cos\theta$$
  $$y = \rho\sin\varphi\sin\theta$$
  $$z = \rho\cos\varphi$$
- spherical ⇒ Cylindrical: $(\rho, \theta, \varphi) \Rightarrow (r, \theta, z)$
  $$r = \rho\sin\varphi$$
  $$\theta = \theta$$
  $$z = \rho\cos\varphi$$
- Cylindrical ⇒ spherical : $(r, \theta, z) \Rightarrow (\rho, \theta, \varphi)$
  $$\rho = \sqrt{r^2 + z^2} \;\Leftrightarrow\; r^2 = x^2 + y^2$$
  $$\theta = \theta$$
  $$\cos\varphi = \frac{z}{\rho}$$

How to Plot a point $P(r, \theta, z)$ in Spherical Coordinates?

Open the following website: https://dynref.engr.illinois.edu/rvs.html

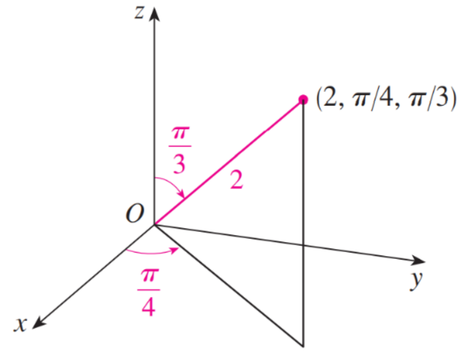
Example 15.9.2: The point $A\left(2,\frac{\pi}{4},\frac{\pi}{3}\right)$ is given in spherical coordinates.

(1) Plot the point showing its spherical coordinates.

(2) Find the rectangular and cylindrical coordinates of the point A.

Solution: $A\left(2,\frac{\pi}{4},\frac{\pi}{3}\right) \Rightarrow \rho = 2, \theta = \frac{\pi}{4}, \varphi = \frac{\pi}{3}$

(1)



(2)

- Rectangular coordinates:

$$x = \rho \sin\varphi \cos\theta \Rightarrow x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$y = \rho \sin\varphi \sin\theta \Rightarrow y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = 2 \times \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}} = \frac{\sqrt{3}}{\sqrt{2}}$$

$$z = \rho\cos\varphi \Rightarrow z = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$$

$\Rightarrow$ The Rectangular coordinates are $\left(\frac{\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}}, 1\right)$

- The cylidrical coordiates:

$$r = \rho\sin\varphi \Rightarrow r = 2\sin\frac{\pi}{3} = 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\theta = \theta \Rightarrow \theta = \frac{\pi}{4}$$

$$z = \rho\cos\varphi \Rightarrow z = 2\cos\frac{\pi}{3} = 2 \times \frac{1}{2} = 1$$

$\Rightarrow$ The cylidrical coordiates are: $\left(\sqrt{3}, \frac{\pi}{4}, 1\right)$

Example 15.9.3: The point $A\left(\sqrt{3}, \frac{2\pi}{5}, -1\right)$ is given in cylindrical coordinates. Find its spherical coordinates.

Solution:

$A\left(\sqrt{3}, \frac{2\pi}{5}, -1\right) \Rightarrow r = \sqrt{3}, \theta = \frac{2\pi}{5}, z = -1$

$$\rho = \sqrt{r^2 + z^2} \Rightarrow \rho = \sqrt{3+1} = 2$$

$$\theta = \theta \Rightarrow \theta = \frac{2\pi}{5}$$

$\cos\varphi = \frac{z}{\rho} \Rightarrow \cos\varphi = \frac{-1}{2}$. But $0 \le \varphi \le \pi \Rightarrow \varphi$ in the 2nd quadrant $\Rightarrow \varphi = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

The spherical coordinates are $\left(2, \frac{2\pi}{5}, \frac{2\pi}{3}\right)$

Example 15.9.4: The point $A(0,2,-2\sqrt{3})$ is given in rectangular coordinates. Find its spherical coordinates.

Solution:

$A(0,2,-2\sqrt{3}) \Rightarrow x = 0, y = 2, z = -2\sqrt{3}$

$\rho = \sqrt{x^2 + y^2 + z^2} \Rightarrow \rho = \sqrt{0 + 4 + 12} = \sqrt{16} = 4$

$\tan\theta = \frac{y}{x} \Rightarrow \tan\theta = \frac{2}{0}$ undefined, so to find θ plot the point $(x, y) = (0,2)$ in the xy-plane:

$\Rightarrow \theta = \frac{\pi}{2}$

$\cos\varphi = \frac{z}{\rho}$ with $0 \le \varphi \le \pi \Rightarrow \cos\varphi = \frac{-2\sqrt{3}}{4} \Rightarrow \cos\varphi = -\frac{\sqrt{3}}{2} \Rightarrow \varphi$ in the 2nd quadrant so:

$\varphi = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

The spherical coordinates are: $\left(4, \frac{\pi}{2}, \frac{5\pi}{6}\right)$

Example 15.9.5: Identify and sketch the surface whose eq is given in spherical coordinates by:

(1) $\rho = 3$ (2) $\rho = 4\cos\varphi$ (3) $\rho = \csc\varphi\cot\varphi$

(4) $\rho = 6\csc\varphi$ (5) $\rho = 6\sin\varphi\sin\theta$ (6) $\rho = 6\sec\varphi$

Solution:

We want to convert the equations from spherical coordinates to rectangular coordinates:

(1) $\rho = 3 \Rightarrow \rho^2 = 9 \Rightarrow x^2 + y^2 + z^2 = 9 \Rightarrow$ The surface $\rho = 3$ is the sphere centered at the origin of radius 3.

(2) $\rho = 4\cos\varphi$ (multiplying by ρ) $\Rightarrow \rho^2 = 4\rho\cos\varphi \Rightarrow x^2 + y^2 + z^2 = 4z$

$\Rightarrow x^2 + y^2 + z^2 - 4z = 0 \Rightarrow x^2 + y^2 + z^2 - 4z + 4 = 4 \Rightarrow x^2 + y^2 + (z-2)^2 = 4$

$\Rightarrow$ The surface $\rho = 4\cos\varphi$ is the sphere centered at $(0,0,2)$ of radius 2

(3) $\rho = \csc\varphi\cot\varphi \Rightarrow \rho = \frac{1}{\sin\varphi} \times \frac{\cos\varphi}{\sin\varphi} \Rightarrow \rho\sin^2\varphi = \cos\varphi$ (multiply by ρ)

$\Rightarrow \rho^2\sin^2\varphi = \rho\cos\varphi \Rightarrow (\rho\sin\varphi)^2 = \rho\cos\varphi \Rightarrow r^2 = z \Rightarrow z = x^2 + y^2$

$\Rightarrow$ The surface $\rho = \csc\varphi\cot\varphi$ is a paraboloid

(4) $\rho = 6\csc\varphi \Rightarrow \rho = 6 \times \frac{1}{\sin\varphi} \Rightarrow \rho\sin\varphi = 6 \Rightarrow r = 6 \Rightarrow r^2 = 36 \Rightarrow x^2 + y^2 = 36$

The surface $\rho = 6\csc\varphi$ is the cylinder $x^2 + y^2 = 36$

(5) $\rho = 6\sin\varphi\sin\theta$ (multiply by ρ) $\Rightarrow \rho^2 = 6\rho\sin\varphi\sin\theta \Rightarrow x^2 + y^2 + z^2 = 6y$

$\Rightarrow x^2 + y^2 - 6y + z^2 = 0 \Rightarrow x^2 + y^2 - 6y + 9 + z^2 = 9 \Rightarrow x^2 + (y-3)^2 + z^2 = 9$

$\Rightarrow$ The surface $\rho = 6\sin\varphi\sin\theta$ is the sphere centered at $(0,3,0)$ of radius 3

(6) $\rho = 6\sec\varphi \Rightarrow \rho = 6 \times \frac{1}{\cos\varphi} \Rightarrow \Rightarrow \rho\cos\varphi = 6 \Rightarrow z = 6$

The surface $\rho = 6\csc\varphi$ is the plane $z = 6$

Example 15.9.6: Identify and sketch the curve or the surface whose equation is given in spherical coordinates and find its equation in rectangular coordinates:

(1) $\varphi = 0$ (2) $\varphi = \pi$ (3) $\varphi = \frac{\pi}{2}$ (4) $\varphi = \frac{\pi}{4}$ (5) $\varphi = \frac{3\pi}{4}$

(6) $\varphi = \frac{\pi}{6}$ (7) $\varphi = \frac{5\pi}{6}$ (8) $\varphi = \frac{\pi}{3}$ (9) $\varphi = \frac{2\pi}{3}$

Solution:

(1) Geometrically, $\varphi = 0$ is the positive z-axis $\Rightarrow \varphi = 0$ is $z \geq 0, x = 0, y = 0$

(2) Geometrically, $\varphi = \pi$ is the negative z-axis $\Rightarrow \varphi = 0$ is $z \leq 0, x = 0, y = 0$

(3) Geometrically, $\varphi = \frac{\pi}{2}$ is the xy-plane $\Rightarrow \varphi = \frac{\pi}{2}$ is $z = 0$

(4) $\varphi = \frac{\pi}{4}$. Since $z = \rho\cos\varphi \Rightarrow z = \rho\cos\frac{\pi}{4} \Rightarrow z = \sqrt{x^2 + y^2 + z^2} \times \frac{1}{\sqrt{2}}$ (which means $z \geq 0$)

$\Rightarrow \sqrt{2}z = \sqrt{x^2 + y^2 + z^2} \Rightarrow 2z^2 = x^2 + y^2 + z^2 \Rightarrow z^2 = x^2 + y^2$

$\Rightarrow z = \pm\sqrt{x^2 + y^2}$. But $z \geq 0 \Rightarrow z = \sqrt{x^2 + y^2}$

$\therefore \varphi = \frac{\pi}{4}$ is the cone $z = \sqrt{x^2 + y^2}$

(5) $\varphi = \frac{3\pi}{4}$. Since $z = \rho\cos\varphi \Rightarrow z = \rho\cos\frac{3\pi}{4}$

$\Rightarrow z = \sqrt{x^2 + y^2 + z^2} \times \frac{-1}{\sqrt{2}}$ (which means $z \leq 0$)

$\Rightarrow -\sqrt{2}z = \sqrt{x^2 + y^2 + z^2} \Rightarrow 2z^2 = x^2 + y^2 + z^2 \Rightarrow z^2 = x^2 + y^2$

$\Rightarrow z = \pm\sqrt{x^2 + y^2}$. But $z \leq 0 \Rightarrow z = -\sqrt{x^2 + y^2}$

$\therefore \varphi = \frac{3\pi}{4}$ is the cone $z = -\sqrt{x^2 + y^2}$

(6) $\varphi = \frac{\pi}{6}$. Since $z = \rho\cos\varphi \Rightarrow z = \rho\cos\frac{\pi}{6}$

$\Rightarrow z = \sqrt{x^2 + y^2 + z^2} \times \frac{\sqrt{3}}{2}$ (which means $z \geq 0$)

$\Rightarrow 2z = \sqrt{3}\sqrt{x^2 + y^2 + z^2} \Rightarrow 4z^2 = 3(x^2 + y^2 + z^2) \Rightarrow z^2 = 3(x^2 + y^2)$

$\Rightarrow z = \pm\sqrt{3(x^2 + y^2)}$. But $z \geq 0 \Rightarrow z = \sqrt{3}\sqrt{x^2 + y^2}$

$\therefore \varphi = \frac{\pi}{6}$ is the cone $z = \sqrt{3}\sqrt{x^2 + y^2}$

(7) $\varphi = \frac{5\pi}{6}$. Since $z = \rho\cos\varphi \Rightarrow z = \rho\cos\frac{5\pi}{6}$

$\Rightarrow z = \sqrt{x^2 + y^2 + z^2} \times \frac{-\sqrt{3}}{2}$ (which means $z \leq 0$)

$\Rightarrow 2z = -\sqrt{3}\sqrt{x^2 + y^2 + z^2} \Rightarrow 4z^2 = 3(x^2 + y^2 + z^2) \Rightarrow z^2 = 3(x^2 + y^2)$

$\Rightarrow z = \pm\sqrt{3(x^2 + y^2)}$. But $z \leq 0 \Rightarrow z = -\sqrt{3}\sqrt{x^2 + y^2}$

$\therefore \varphi = \frac{\pi}{3}$ is the cone $z = -\sqrt{3}\sqrt{x^2 + y^2}$

(8) $\varphi = \frac{\pi}{3}$. Since $z = \rho\cos\varphi \Rightarrow z = \rho\cos\frac{\pi}{3}$

$\Rightarrow z = \sqrt{x^2 + y^2 + z^2} \times \frac{1}{2}$ (which means $z \geq 0$)

$\Rightarrow 2z = \sqrt{x^2 + y^2 + z^2} \Rightarrow 4z^2 = x^2 + y^2 + z^2 \Rightarrow z^2 = \frac{x^2+y^2}{3}$

$\Rightarrow z = \pm\sqrt{\frac{x^2+y^2}{3}}$. But $z \geq 0 \Rightarrow z = \frac{\sqrt{x^2+y^2}}{\sqrt{3}}$

$\therefore \varphi = \frac{\pi}{3}$ is the cone $z = \frac{\sqrt{x^2+y^2}}{\sqrt{3}}$

(9) $\varphi = \frac{2\pi}{3}$. Since $z = \rho\cos\varphi \Rightarrow z = \rho\cos\frac{2\pi}{3}$

$$\Rightarrow z = \sqrt{x^2 + y^2 + z^2} \times \frac{-1}{2} \text{ (which means } z \le 0)$$

$$\Rightarrow\ -2z = \sqrt{x^2 + y^2 + z^2} \Rightarrow 4z^2 = x^2 + y^2 + z^2 \Rightarrow z^2 = \frac{x^2+y^2}{3}$$

$$\Rightarrow z = \pm\sqrt{\frac{x^2+y^2}{3}}. \text{ But } z \le 0 \Rightarrow z = -\frac{\sqrt{x^2+y^2}}{\sqrt{3}}$$

$$\therefore \varphi = \frac{2\pi}{3} \text{ is the cone } z = -\frac{\sqrt{x^2+y^2}}{\sqrt{3}}$$

Remark 15.9.7:

(1) For $c > 0$: $\rho = c \iff x^2 + y^2 + z^2 = c^2$ (The sphere centered at the origin of radius c)

(2) $\varphi = 0 \iff x = 0, y = 0, z \ge 0$ (The positive z-axis including the origin)

(3) $\varphi = \pi \iff x = 0, y = 0, z \le 0$ (The negative z-axis including the origin)

(4) $\varphi = \frac{\pi}{2} \iff z = 0$ (The xy-plane)

(5) $0 < c < \frac{\pi}{2} \Rightarrow \varphi = c$ is a half-cone above the xy-plane

(6) $\frac{\pi}{2} < c < \pi \Rightarrow \varphi = c$ is a half-cone below the xy-plane

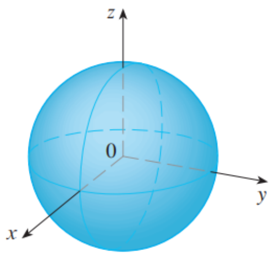


FIGURE 2 $\rho = c$, a sphere

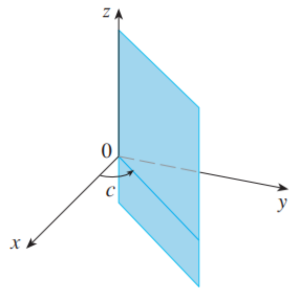


FIGURE 3 $\theta = c$, a half-plane

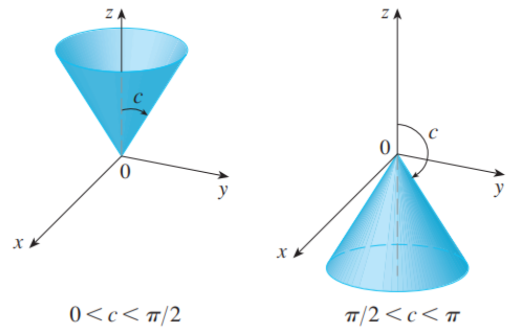


FIGURE 4 $\phi = c$, a half-cone

Example 15.9.8: Express each of the following equations in spherical coordinates:

(1) $z = -3$ (2) $x^2 + y^2 + z^2 = -6x$ (3) $z = -\sqrt{\frac{x^2+y^2}{3}}$

(4) $z^2 = x^2 + y^2$ (5) $z = \sqrt{x^2 + 2y^2}$

Solution:

(1) $z = -3 \Rightarrow \rho\cos\varphi = -3 \Rightarrow \rho = -\frac{3}{\cos\varphi} \Rightarrow \rho = -3\sec\varphi$

(2) $x^2 + y^2 + z^2 = -6x \Rightarrow x^2 + y^2 + z^2 + 6x = 0 \Rightarrow \rho^2 + 6\rho\sin\varphi\cos\theta = 0$ dividing by ρ
$\Rightarrow \rho + 6\sin\varphi\cos\theta = 0 \Rightarrow \rho = -6\sin\varphi\cos\theta$

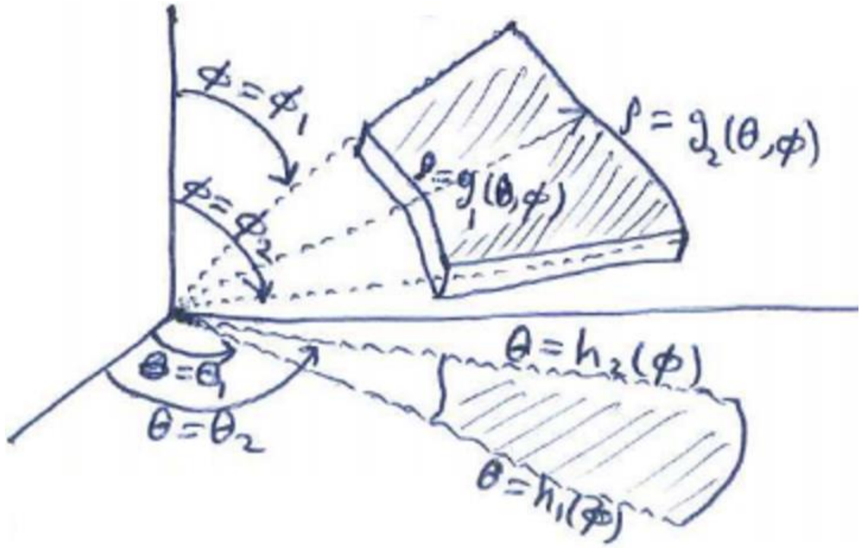
(3) $z = -\sqrt{\frac{x^2+y^2}{3}} \Rightarrow \varphi = \frac{2\pi}{3}$

(4) $z^2 = x^2 + y^2 \Rightarrow z = \sqrt{x^2 + y^2}$ or $z = -\sqrt{x^2 + y^2} \Rightarrow \varphi = \frac{\pi}{4}$ or $\varphi = \frac{3\pi}{4}$

(5) $z = \sqrt{x^2 + 2y^2} \Rightarrow z = \sqrt{x^2 + y^2 + y^2} \Rightarrow z = \sqrt{r^2 + y^2}$
$\Rightarrow \rho\cos\varphi = \sqrt{\rho^2\sin^2\varphi + \rho^2\sin^2\varphi\sin^2\theta} \Rightarrow \rho\cos\varphi = \rho\sin\varphi\sqrt{1 + \sin^2\theta}$
$\Rightarrow \cos\varphi = \sin\varphi\sqrt{1 + \sin^2\theta} \Rightarrow \cot\varphi = \sqrt{1 + \sin^2\theta}$

Rule 15.9.9: Let E be the solid given in spherical coordinates :

$$E = \{(\rho, \theta, \varphi): g_1(\theta,\varphi) \le \rho \le g_2(\theta,\varphi), h_1(\varphi) \le \theta \le h_2(\varphi), \varphi_1 \le \varphi \le \varphi_2\}$$



Then the triple integral $\iiint_E f\, dV$ can be expressed in spherical coordinates as:

$$\iiint_E f(x,y,z)\, dV = \int_{\varphi_1}^{\varphi_2} \int_{h_1(\varphi)}^{h_2(\varphi)} \int_{g_1(\theta,\varphi)}^{g_2(\theta,\varphi)} f\left(\overbrace{\rho \sin\varphi \cos\theta}^{x}, \overbrace{\rho \sin\varphi \sin\theta}^{y}, \overbrace{\rho \cos\varphi}^{z}\right) \overbrace{\rho^2 \sin\varphi\, d\rho d\theta d\varphi}^{dV}$$

Observe that $dV = \rho^2 \sin\varphi\, d\rho d\theta d\varphi$

Example 15.9.10:

(1) Using spherical coordinates, find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and inside the sphere $x^2+y^2+z^2 = z$.

(2) Using spherical coordinates, find the volume of the solid bounded below by the cone $z = \sqrt{x^2+y^2}$ and above by the sphere $x^2+y^2+z^2 = z$

Solution:

(1) $dV = \rho^2 \sin\varphi\, d\rho d\theta d\varphi$

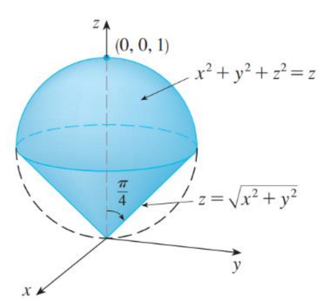
Step 1: We have to sketch the solid:

- $z = \sqrt{x^2+y^2}$ is a cone above the xy-plane
- $x^2+y^2+z^2 = z \Rightarrow x^2+y^2+z^2 - z = 0$

  $\Rightarrow x^2+y^2+\underbrace{z^2 - z + \frac{1}{4}}_{\text{إكمال مربع}} = \frac{1}{4}$

  $\Rightarrow x^2+y^2+\left(z-\frac{1}{2}\right)^2 = \frac{1}{4}$

  It is a sphere centered at $\left(0,0,\frac{1}{2}\right)$ of radius $\frac{1}{2}$

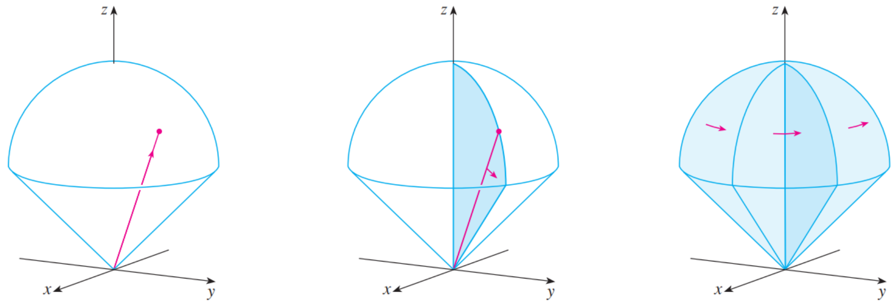


Step 2: We have to write the surfaces in spherical coordinates:

- $z = \sqrt{x^2+y^2} \Leftrightarrow \varphi = \frac{\pi}{4}$

- $x^2 + y^2 + z^2 = z \Rightarrow \rho^2 = \rho \cos\varphi \Rightarrow \rho = \cos\varphi$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, θ, and φ:



ρ varies from 0 to cos ϕ while ϕ and θ are constant. ϕ varies from 0 to $\pi/4$ while θ is constant. θ varies from 0 to 2π.

$0 \le \rho \le \cos\varphi$ $\quad$ $0 \le \varphi \le \frac{\pi}{4}$ $\quad$ $0 \le \theta \le 2\pi$

$$\Rightarrow V = \iiint_B 1\, dV = \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\cos\varphi} 1 \rho^2 \sin\varphi \, d\rho d\theta d\varphi = \frac{1}{3} \int_0^{\frac{\pi}{4}} \int_0^{2\pi} \cos^3\varphi \sin\varphi \, d\theta d\varphi$$

$$= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} \cos^3\varphi \sin\varphi \, d\varphi$$

$$= \frac{2\pi}{3} \int_0^{\frac{\pi}{4}} u^3 \sin\varphi \frac{du}{-\sin\varphi} = -\frac{2\pi}{3} \int_1^{\frac{1}{\sqrt{2}}} u^3 du = \frac{\pi}{8}$$

$u = \cos\varphi \Rightarrow d\varphi = \frac{du}{-\sin\varphi}$

- $\varphi = 0 \Rightarrow u = 1$
- $\varphi = \frac{\pi}{4} \Rightarrow u = \frac{1}{\sqrt{2}}$

(2) Part (2) is the same as part (1). So, Volume $= \frac{\pi}{8}$.

Exercise 15.9.11:

(1) Using spherical coordinates, find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 2z$.

(2) Using spherical coordinates, find the volume of the solid bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 2z$.

في الأمثلة القادمة لم نقم برسم الجسم وانما سنقوم برسمه بشكل يدوي اثناء حل المثال

Example 15.9.12: Evaluate $I = \iiint_B e^{(x^2+y^2+z^2)^{3/2}} dV$, where

(1) B is the unit ball $B = \{(x, y, z): x^2 + y^2 + z^2 \leq 1\}$
(2) B is the solid between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$
(3) B is the wedge between the two spheres $x^2 + y^2 + z^2 = 1$, $x^2 + y^2 + z^2 = 4$, and above the cone $z = \frac{\sqrt{x^2+y^2}}{\sqrt{3}}$

Solution:

(1) The solid is a ball $\Rightarrow dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

$x^2 + y^2 + z^2 \leq 1$ It is a sphere centered at $(0,0,0)$ of radius 1

Step 2: We have to write the surfaces in spherical coordinates:

$$x^2 + y^2 + z^2 = 1 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 1$$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, θ, and φ:

$$0 \leq \rho \leq 1 \qquad 0 \leq \varphi \leq \pi \qquad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow I = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{(\rho^2)^{3/2}} \rho^2 \sin\varphi \, d\rho d\theta d\varphi = \int_0^{\pi} \int_0^{2\pi} \int_0^{1} e^{\rho^3} \rho^2 \sin\varphi \, d\rho d\theta d\varphi$$

$$= \left(\int_0^{\pi} \sin\varphi \, d\varphi\right)\left(\int_0^{2\pi} 1 d\theta\right)\left(\int_0^{1} \rho^2 e^{\rho^3} d\rho\right)$$

$$= \left(\int_0^{\pi} \sin\varphi \, d\varphi\right)\left(\int_0^{2\pi} 1 d\theta\right)\left(\frac{1}{3}\int_0^{1} 3\rho^2 e^{\rho^3} d\rho\right)$$

$$= (-\cos\varphi]\!]_0^{\pi})(2\pi - 0)\left(\frac{1}{3} e^{\rho^3}]\!]_0^1\right) = 2(2\pi)\left(\frac{e-1}{3}\right) = \frac{4\pi}{3}(e-1)$$

(2) The solid is a part of a sphere $\Rightarrow dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 1$
  $\Rightarrow$ It is a sphere centered at $(0,0,0)$ of radius 1
- $x^2 + y^2 + z^2 = 4$
  $\Rightarrow$ It is a sphere centered at $(0,0,0)$ of radius 2

Step 2: We have to write the surfaces in spherical coordinates:

- $x^2 + y^2 + z^2 = 1 \Rightarrow \rho^2 = 1 \Rightarrow \rho = 1$
- $x^2 + y^2 + z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, θ, and φ:

$$1 \leq \rho \leq 2 \qquad 0 \leq \varphi \leq \pi \qquad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow I = \int_0^{\pi} \int_0^{2\pi} \int_1^{2} e^{(\rho^2)^{3/2}} \rho^2 \sin\varphi \, d\rho d\theta d\varphi = \int_0^{\pi} \int_0^{2\pi} \int_1^{2} e^{\rho^3} \rho^2 \sin\varphi \, d\rho d\theta d\varphi = \cdots$$

(3) The solid is a part of a sphere $\Rightarrow dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 1$
  $\Rightarrow$ It is a sphere centered at $(0,0,0)$ of radius 1
- $x^2 + y^2 + z^2 = 4$
  $\Rightarrow$ It is a sphere centered at $(0,0,0)$ of radius 2
- $z = \frac{\sqrt{x^2+y^2}}{\sqrt{3}}$ is a cone (below the xy-plane)

Step 2: We have to write the surfaces in spherical coordinates:

- $x^2 + y^2 + z^2 = 1 \Rightarrow \rho^2 = 1 \Rightarrow \rho = 1$
- $x^2 + y^2 + z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 2$
- $z = \frac{\sqrt{x^2+y^2}}{\sqrt{3}} \Leftrightarrow \varphi = \frac{\pi}{3}$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, θ, and φ:

$$1 \le \rho \le 2 \qquad 0 \le \varphi \le \frac{\pi}{3} \qquad 0 \le \theta \le 2\pi$$

$$\Rightarrow I = \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_1^2 e^{(\rho^2)^{3/2}} \rho^2 \sin\varphi \, d\rho d\theta d\varphi = \int_0^{\frac{\pi}{3}} \int_0^{2\pi} \int_1^2 e^{\rho^3} \rho^2 \sin\varphi \, d\rho d\theta d\varphi = \cdots$$

Example 15.9.13: Using spherical coordinates, find the volume of the solid bounded below by $z = -\sqrt{x^2 + y^2}$ and above by $x^2 + y^2 + z^2 = 4$

Solution: $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $z = -\sqrt{x^2 + y^2}$ is a cone (below the xy-plane)
- $x^2 + y^2 + z^2 = 4$
  $\Rightarrow$ It is a sphere centered at $(0,0,0)$ of radius 2

Step 2: We have to write the surfaces in spherical coordinates:

- $z = -\sqrt{x^2 + y^2} \Leftrightarrow \varphi = \frac{3\pi}{4}$
- $x^2 + y^2 + z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, θ, and φ:

$$0 \le \rho \le 2 \qquad 0 \le \varphi \le \frac{3\pi}{4} \qquad 0 \le \theta \le 2\pi$$

$$\Rightarrow V = \iiint_B 1 \, dV = \int_0^{\frac{3\pi}{4}} \int_0^{2\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho d\theta d\varphi$$

$$= \left(\int_0^2 \rho^2 d\rho\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{3\pi}{4}} \sin\varphi \, d\varphi\right) = \frac{16\pi}{3}\left(\frac{1}{\sqrt{2}} - 1\right)$$

Example 15.9.14: Using spherical coordinates, express the volume of the solid that lies inside the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane and below the cone $z = \sqrt{3x^2 + 3y^2}$.

Solution: $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 4$ is a sphere centered at the sorigin of radius 2
- $z = \sqrt{3x^2 + 3y^2}$ is a cone (above the xy-plane)

Step 2: We have to write the surfaces in spherical coordinates:

- $x^2 + y^2 + z^2 = 4 \Rightarrow \rho^2 = 4 \Rightarrow \rho = 2$
- $z = \sqrt{3x^2 + 3y^2} \Leftrightarrow \varphi = \frac{\pi}{6}$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \le \rho \le 2 \qquad \frac{\pi}{6} \le \varphi \le \frac{\pi}{2} \qquad 0 \le \theta \le 2\pi$$

$$\Rightarrow V = \iiint_B 1 \, dV = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

Example 15.9.15: Set up (do not evaluate) ***in spherical coordinates*** the volume of the solid in the first octant enclosed by $z = 2$ and $z = \sqrt{x^2 + y^2}$

Solution: $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $z = 2$ is a plane parallel to the xy-plane that passes through the point $(0,0,2)$
- $z = \sqrt{x^2 + y^2}$ is a cone (above the xy-plane)

Step 2: We have to write the surfaces in spherical coordinates:

- $z = 2 \Rightarrow \rho\cos\varphi = 2 \Rightarrow \rho = \frac{2}{\cos\varphi} \Rightarrow \rho = 2\sec\varphi$
- $z = \sqrt{x^2 + y^2} \Leftrightarrow \varphi = \frac{\pi}{4}$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \le \rho \le 2\sec\varphi \qquad 0 \le \varphi \le \frac{\pi}{4} \qquad 0 \le \theta \le \frac{\pi}{2}$$

$$\Rightarrow V = \iiint_B 1 \, dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{2\sec\varphi} \rho^2 \sin\varphi \, d\rho d\varphi d\theta$$

ملاحظة:

لو لم يحدد المثال السابق استخدام النظام الإحداثي الكروي (spherical coordinates) لايجاد الحجم، فما هو الأسلوب الأفضل والأسهل لإيجاد الحجم؟ الإجابة هي: النظام الأنسب والأسهل هو النظام الإحداثي الأسطواني (cylindrical coordinates).
وعليه فإن الحجم حسب النظام الأسطواني هو:

$$V = \iiint_B 1\, dV = \iint_D \int_0^{\sqrt{x^2+y^2}} 1 dzdA = \int_0^{\frac{\pi}{2}} \int_0^2 \int_0^r rdzdrd\theta$$

Example 15.9.16: Find the volume of the solid between $z = -\sqrt{3x^2+3y^2}, z = \sqrt{\frac{x^2+y^2}{3}}$ and inside the sphere $x^2+y^2+z^2 = 4$.

Solution: $dV = \rho^2 \sin\varphi\, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $z = -\sqrt{3x^2+3y^2}$ is a cone (below the xy-plane)
- $z = \sqrt{\frac{x^2+y^2}{3}}$ is a cone (above the xy-plane)
- $x^2+y^2+z^2 = 4$ is a sphere centered at the (0,0,0) of radius 2

Step 2: We have to write the surfaces in spherical coordinates:

- $z = -\sqrt{3x^2+3y^2} \Rightarrow \varphi = \frac{5\pi}{6}$
- $z = \sqrt{\frac{x^2+y^2}{3}} \Rightarrow \varphi = \frac{\pi}{3}$
- $x^2+y^2+z^2 = 4 \Leftrightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \le \rho \le 2 \qquad \frac{\pi}{3} \le \varphi \le \frac{5\pi}{6} \qquad 0 \le \theta \le 2\pi$$

$$\Rightarrow V = \iiint_B 1\, dV = \int_0^{2\pi} \int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} \int_0^2 \rho^2 \sin\varphi\, d\rho d\varphi d\theta = \cdots$$

Example 15.9.17: Set up the volume in spherical coordinates of each of the following solids:

(1) $E_1 = \{(x,y,z): x^2 + y^2 + z^2 \leq 4, z \leq 0\}$
(2) $E_2 = \{(x,y,z): x^2 + y^2 + z^2 \leq 4, x \geq 0\}$
(3) $E_3 = \{(x,y,z): x^2 + y^2 + z^2 \leq 4, x \leq 0\}$
(4) $E_4 = \{(x,y,z): x^2 + y^2 + z^2 \leq 4, y \geq 0\}$
(5) $E_5 = \{(x,y,z): x^2 + y^2 + z^2 \leq 4, y \leq 0\}$

Solution:

(1) $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 4, z \leq 0$ is the upper hemisphere centered at the $(0,0,0)$ of radius 2

Step 2: We have to write the surfaces in spherical coordinates:

- $x^2 + y^2 + z^2 = 4 \Leftrightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \leq \rho \leq 2 \qquad \frac{\pi}{2} \leq \varphi \leq \pi \qquad 0 \leq \theta \leq 2\pi$$

$$\Rightarrow V = \iiint_{E_1} 1 \, dV = \int_0^{2\pi} \int_{\frac{\pi}{2}}^{\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \cdots$$

(2) $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 4, x \geq 0$ is the front hemisphere centered at the $(0,0,0)$ of radius 2

Step 2: We have to write the surfaces in spherical coordinates:

- $x^2 + y^2 + z^2 = 4 \Leftrightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \leq \rho \leq 2 \qquad 0 \leq \varphi \leq \pi \qquad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

$$\Rightarrow V = \iiint_{E_1} 1 \, dV = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\pi} \int_0^2 \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \cdots$$

(3) $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 4, x \leq 0$ is the behind hemisphere centered at the $(0,0,0)$ of radius 2

Step 2: We have to write the surfaces in spherical coordinates:

- $x^2 + y^2 + z^2 = 4 \Leftrightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \le \rho \le 2 \qquad 0 \le \varphi \le \pi \qquad \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

$$\Rightarrow V = \iiint_{E_1} 1\, dV = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \int_0^{\pi} \int_0^{2} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \cdots$$

(4) $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 4, y \ge 0$ is the right hemisphere centered at the $(0,0,0)$ of radius 2

Step 2: We have to write the equation in spherical coordinates:

- $x^2 + y^2 + z^2 = 4 \Leftrightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \le \rho \le 2 \qquad 0 \le \varphi \le \pi \qquad 0 \le \theta \le \pi$$

$$\Rightarrow V = \iiint_{E_1} 1\, dV = \int_0^{\pi} \int_0^{\pi} \int_0^{2} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \cdots$$

(5) $dV = \rho^2 \sin\varphi \, d\rho d\theta d\varphi$

Step 1: We have to sketch the solid:

- $x^2 + y^2 + z^2 = 4, y \le 0$ is the left hemisphere centered at the $(0,0,0)$ of radius 2

Step 2: We have to write the equation in spherical coordinates:

- $x^2 + y^2 + z^2 = 4 \Leftrightarrow \rho = 2$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \le \rho \le 2 \qquad 0 \le \varphi \le \pi \qquad -\pi \le \theta \le 0$$

$$\Rightarrow V = \iiint_{E_1} 1\, dV = \int_{-\pi}^{0} \int_0^{\pi} \int_0^{2} \rho^2 \sin\varphi \, d\rho d\varphi d\theta = \cdots$$

Example 15.9.18: Converte the iterated triple integral to spherical coordinates:

$$\int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)dzdxdy$$

Solution:

- First we have to determine the silod to plot its graph:

$$z=\sqrt{x^2+y^2}, z=\sqrt{18-x^2-y^2}, x=0, x=\sqrt{9-y^2}, y=0, y=3$$

- **Surfaces:** $z=\sqrt{x^2+y^2}$ and $z=\sqrt{18-x^2-y^2}$
- **Projection Region:** $0 \le x \le \sqrt{9-y^2}$ and $0 \le y \le 3$
- The solid is bounded by: $z=\sqrt{x^2+y^2}$, $z=\sqrt{18-x^2-y^2}$
    - Since $x \ge 0, y \ge 0, z \ge 0$ we have that the solid is in the first octant
    - The solid is in the first octant bounded by $z=\sqrt{x^2+y^2}$, $z=\sqrt{18-x^2-y^2}$

Step 1: We have to sketch the solid:

- $z=\sqrt{x^2+y^2}$ cone above the xy-plane
- $z=\sqrt{18-x^2-y^2}$ is the upper hemisphere
- centered at the $(0,0,0)$ of radius $\sqrt{18}$

Step 2: We have to write the equation in spherical coordinates:

- $z=\sqrt{x^2+y^2} \Rightarrow \varphi=\frac{\pi}{4}$
- $z=\sqrt{18-x^2-y^2} \Rightarrow z^2=18-x^2-y^2$
  $\Rightarrow x^2+y^2+z^2=18 \Leftrightarrow \rho^2=18 \Rightarrow \rho=\sqrt{18}$

Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$0 \le \rho \le \sqrt{18} \qquad 0 \le \varphi \le \frac{\pi}{4} \qquad 0 \le \theta \le \frac{\pi}{2}$$

$$\Rightarrow \int_0^3 \int_0^{\sqrt{9-y^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{18-x^2-y^2}} (x^2+y^2+z^2)dzdxdy = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{18}} \rho^2 \underbrace{\rho^2 \sin\varphi \, d\rho d\varphi d\theta}_{dV=dzdxdy})$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \int_0^{\sqrt{18}} \rho^4 \sin\varphi \, d\rho d\varphi d\theta$$

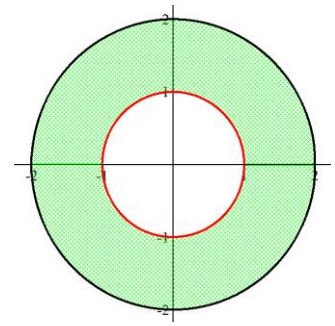
Example 15.9.19: Use rectangular, cylidrical, and spherical coordinates to set up triple integrals for the volume of the region inside the sphere $x^2 + y^2 + z^2 = 4$ and outside the cylinder $x^2 + y^2 = 1$.

Solution:

- **The Volume in rectangular coordinates:**
  - **Surfaces:** $z = -\sqrt{4 - x^2 - y^2}$ and $z = \sqrt{4 - x^2 - y^2}$
  - **Projection Region:** outside $x^2 + y^2 = 1$ (not closed region)
    - Intersection of surfaces: $\sqrt{4 - x^2 - y^2} = -\sqrt{4 - x^2 - y^2}$

      $\Rightarrow 2\sqrt{4 - x^2 - y^2} = 0$

      $\Rightarrow x^2 + y^2 = 4$ (add this eq. to the region)
    - The Projection Region: Between $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$
    - $dA = r dr d\theta \Rightarrow r = 1 \;\rightarrow r = 2$ and $0 \le \theta \le 2\pi$



  - The volume in rectangular coordinates is:

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx - \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\, dy\, dx$$

- **The volume using cylindrical coordinates:**
  - **Surfaces:** $z = -\sqrt{4 - x^2 - y^2}$ and $z = \sqrt{4 - x^2 - y^2}$
  - **Projection Region:**
    - Between $x^2 + y^2 = 4$ and $x^2 + y^2 = 1$
    - $dA = r dr d\theta \Rightarrow r = 1 \;\rightarrow r = 2$ and $0 \le \theta \le 2\pi$
  - The volume in cylidrical coordinates is: $V = \int_0^{2\pi} \int_1^2 \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r dz\, dr\, d\theta$

- **The volume using spherical coordinates:**

  Step 1: We have to sketch the solid:
  - $x^2 + y^2 + z^2 = 4$ sphere centered at (0,0,0) of radius 2
  - $x^2 + y^2 = 1$ cylinder

  Step 2: We have to write the equation in spherical coordinates:
  - $x^2 + y^2 + z^2 = 4 \Rightarrow \rho = 2$
  - $x^2 + y^2 = 1 \Rightarrow \; r^2 = 1 \Rightarrow r = 1 \Leftrightarrow \rho \sin\varphi = 1$

    $\Rightarrow \rho = \frac{1}{\sin\varphi} \Rightarrow \rho = \csc\varphi$

  - The values of φ depends on the points of intersection of the surfaces:

    $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1: \Rightarrow 1 + z^2 = 4 \Rightarrow z^2 = 3 \Rightarrow z = \pm\sqrt{3}$

    $\Rightarrow \rho\cos\varphi = \pm\sqrt{3}$ (but $x^2 + y^2 + z^2 = 4 \Rightarrow \rho = 2$)

    $\Rightarrow \rho\cos\varphi = \pm\sqrt{3}$ with $\rho = 2 \Rightarrow \cos\varphi = \pm\frac{\sqrt{3}}{2} \Rightarrow \varphi = \frac{\pi}{6}$ or $\varphi = \frac{5\pi}{6}$

  Step 3: Based on the graph in Steps 1 and 2: we must determine the range of ρ, φ, and θ:

$$\csc\varphi \le \rho \le 2 \qquad \frac{\pi}{6} \le \varphi \le \frac{5\pi}{6} \qquad 0 \le \theta \le 2\pi$$

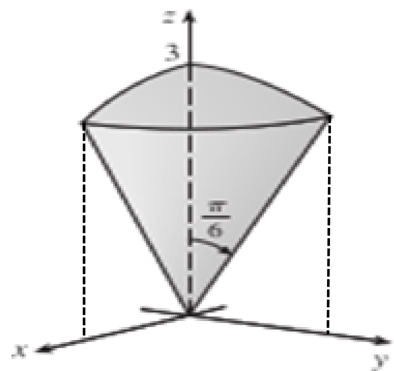
$$V = \int_0^{2\pi} \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \int_{\csc\varphi}^{2} \rho^2 \sin\varphi \, d\rho\, d\varphi\, d\theta$$

ملاحظة:

لو خيرت في طريقة ايجاد الحجم للمثال السابق فأي الانظمة الإحداثية تختار؟ من الواضح أن استخدام النظام الإسطواني هو الأنسب والأسهل. لذلك يجدر السؤال هنا كيف نستدل من نص السؤال على النظام الأنسب لحل المسألة؟ هل عرفت الجواب؟ إذا لم تعرف الإجابة فسأل لتتعلم.

Exercise 15.9.20:

(1) Let $\iiint_E \, dv = \int_0^A \int_0^B \int_0^C \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$,
where E is the solid in the first octant whose graph is on the right. Find A, B, C.



(2) Set up in spherical coordinates the volume of the solid inside the sphere $x^2 + y^2 + z^2 = 4x$ and bounded above by the cone $z = \sqrt{3x^2 + 3y^2}$.