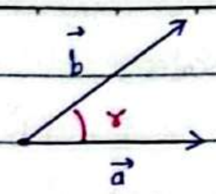
1/12

No.

* 9.2 : Dot product



$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\gamma$$

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|\vec{b}| = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

- $\vec{a}\cdot\vec{b} = 0$ if $\vec{a} = 0$ OR $\vec{b} = 0$
- for $\vec{a} = [a_1, a_2, a_3]$
  $\vec{b} = [b_1, b_2, b_3]$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Theorem: $\vec{a}, \vec{b}$ are orthogonal if $\gamma = 90, \pi/2$ so $\vec{a}\cdot\vec{b} = 0$

Ex: let $\vec{a} = [1,2,0]$, $\vec{b} = [3,-2,1]$ find γ between $\vec{a}, \vec{b}$?

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\gamma$$

$$(1)(3) + (2)(-2) + (0)(1) = \sqrt{5}\sqrt{14}\cos\gamma$$

$$\gamma = 96.865^\circ$$

Rules: ① $[q_1\vec{a} + q_2\vec{b}]\cdot\vec{c} = q_1\vec{a}\vec{c} + q_2\vec{b}\vec{c}$

② $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$

③ $\vec{a}\cdot\vec{a} \geq 0$

④ $\vec{a}\cdot\vec{a} = 0$, if $\vec{a} = 0$

⑤ $[\vec{a} + \vec{b}]\cdot\vec{c} = \vec{a}\cdot\vec{c} + \vec{b}\cdot\vec{c}$

⑥ $|\vec{a}\cdot\vec{b}| \leq |\vec{a}||\vec{b}|$
↳ (cauchy - Schwarz inequality)

⑦ $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
↳ Trainglure inequality

⑧ $|\vec{a} + \vec{b}|^2 + |\vec{a} - \vec{b}|^2 = 2\left[|\vec{a}|^2 + |\vec{b}|^2\right]$
↳ parallegram equality

Notes:

① $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

② $i\cdot j = j\cdot i = k\cdot j = j\cdot k = k\cdot i = i\cdot k = 0$

smile for life

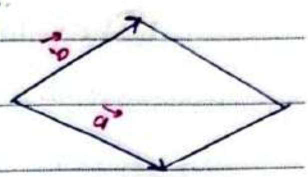
(9.3) Cross product

$$\vec{v} = \vec{a} \times \vec{b}$$

① if $\vec{a} = 0$ OR $\vec{b} = 0$ then $\vec{v} = \vec{a} \times \vec{b} = 0$

② if $\vec{a} \neq 0$, $\vec{b} \neq 0$ then $|\vec{v}| = |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin \gamma$

③ $|\vec{b}|$: Area of parallegram contralled by $\vec{a}, \vec{b}$



④ if (γ) between $\vec{a}, \vec{b}$ is zero OR 180 same OR opposite direction

$\sin(0) = \sin(180) = 0$, $\vec{v} = \vec{a} \times \vec{b} = 0$

if $\vec{a} \neq 0$, $\vec{b} \neq 0$ ⑤ $\vec{v} = \vec{a} \times \vec{b}$ then $\vec{v}$ is perpendecular to plane formed by $\vec{a}, \vec{b}$

⑥ $\vec{a} = [a_1, a_2, a_3]$, $\vec{b} = [b_1, b_2, b_3]$

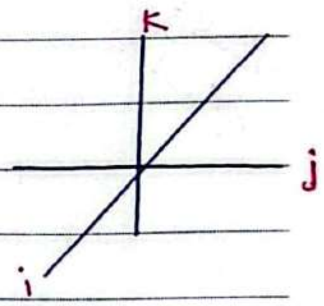
$$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= i \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - j \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + k \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= i\,[a_2 b_3 - a_3 b_2] - j\,[a_1 b_3 - a_3 b_1] + k\,[a_1 b_2 - a_2 b_1]$$

Ex: if $\vec{a} = [1,1,0]$, $\vec{b} = [3,0,0]$ find $\vec{v} = \vec{a} \times \vec{b}$?

$$\vec{v} = \vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = -3k = [0,0,-3]$$



Notes: ① $i \times j = k$, $j \times k = i$, $k \times i = j$

② $j \times i = -k$, $k \times j = -i$, $i \times k = -j$

Rules:

① $(l\vec{a}) \times \vec{b} = l(\vec{a} \times \vec{b}) = \vec{a} \times (l\vec{b})$

② $\vec{a} \times [\vec{b} + \vec{c}] = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

③ $[\vec{a} + \vec{b}] \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

الترتيب مهم

④ $\vec{a} \times \vec{b} = -[\vec{b} \times \vec{a}]$

⑤ $[\vec{a} \times \vec{b}] \times \vec{c} \neq \vec{a} \times [\vec{b} \times \vec{c}]$. الأولوية للي بين أقواس

⊛ **Scalar Triple product:**

Def: for vector $\vec{a}, \vec{b}, \vec{c}$

$$(\vec{a}\ \vec{b}\ \vec{c}) = \vec{a} \cdot [\vec{b} \times \vec{c}] = [\vec{a} \times \vec{b}] \times \vec{c} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$|(\vec{a}\ \vec{b}\ \vec{c})|$ = volume of Box

* **Theorem:** $\vec{a}, \vec{b}, \vec{c}$ are linearly independent if $(\vec{a}\ \vec{b}\ \vec{c}) \neq 0$
if $(\vec{a}\ \vec{b}\ \vec{c}) = 0$ linearly dependent.

Ex: let $\vec{a} = [2,0,3]$, $\vec{b} = [0,4,1]$, $\vec{c} = [5,6,0]$ find the volume of Box.

$$|(\vec{a}\ \vec{b}\ \vec{c})| = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 4 & 1 \\ 5 & 6 & 0 \end{bmatrix} = |-72| = 72 \text{ unit.}$$

• Volume of tetrahedrom $= \frac{1}{6}$ Volume of Box

في السؤال السابق $\Rightarrow \frac{1}{6}(72) = 12$ unit.

(9.4) vector + Scalar function

① vector function $\vec{v}$ that dep. on point p.

$\vec{v}(P) = [v_1(P), v_2(P), v_3(P)]$

ex: ① tangent vector ② normal vector

② Scalar function (f) that dep. on point p.

$f(p) = F(x,y,z)$

ex: ① Temp. ② pressure.

$\Rightarrow \vec{U}(x,y,z) = xyz\,\hat{i} + x^2y\,j + z^2xy\,k$

$f(x,y,z) = x^2y + z^2xy + z^2$

• **Dervative of vector function** :- the vector function is differentable iff.

① convergences $\lim_{n\to\infty} |\vec{a}_n - \vec{a}| = 0$

② continuity $\lim_{t\to t_0} \vec{v}(t) = \vec{v}(t_0)$

$\vec{v} = [v_1(t), v_2(t), v_3(t)]$

$\vec{v}' = [v_1'(t), v_2'(t), v_3'(t)]$

$\vec{v}'' = [v_1''(t), v_2''(t), v_3''(t)]$

ex: let $\vec{v}(t) = [t, t^2, 0]$ find $\vec{v}'(t)$ & $\vec{v}''(t)$?

$\vec{v}'(t) = [1, 2t, 0]$

$\vec{v}''(t) = [0, 2, 0]$

No.

Rules :

① $(c\vec{v})' = c\vec{v}'$ ② $(\vec{u}+\vec{v})' = \vec{u}' + \vec{v}'$

③ $(\vec{u}\cdot\vec{v})' = \vec{u}'\vec{v} + \vec{u}\cdot\vec{v}'$ ④ $(\vec{u}\times\vec{v})' = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$

⑤ $(\vec{u}\ \vec{v}\ \vec{w})' = (\vec{u}'\ \vec{v}\ \vec{w}) + (\vec{u}\ \vec{v}'\ \vec{w}) + (\vec{u}\ \vec{v}\ \vec{w}')$

* partial dervitave :-

$\vec{v}(t) = [\ v_1(t)\ ,\ v_2(t)\ ,\ v_3(t)\ ]$

$$\frac{\partial \vec{v}(t)}{\partial t_m} = \left[\frac{\partial v_1(t)}{\partial t_m} , \frac{\partial v_2(t)}{\partial t_m} , \frac{\partial v_3(t)}{\partial t_m} \right]$$

$$\frac{\partial^2 \vec{v}(t)}{\partial t_1 \partial t_m} = \left[\frac{\partial^2 v_1(t)}{\partial t_1 \partial t_m} , \frac{\partial^2 v_2(t)}{\partial t_1 \partial t_m} , \frac{\partial^2 v_3(t)}{\partial t_1 \partial t_m} \right]$$

Ex: let $\vec{r}(t_1, t_2) = a\cos t_1\, i + a\sin t_1\, j + t_2 K$

find $\frac{\partial \vec{r}}{\partial t_1}$, $\frac{\partial \vec{r}}{\partial t_2}$, $\frac{\partial^2 \vec{r}}{\partial t_1^2}$??

① $\frac{\partial \vec{r}}{\partial t_1} = -a\sin t_1\, i + a\cos t_1\, j + 0K$

② $\frac{\partial \vec{r}}{\partial t_2} = 0i + 0j + 1K$

③ $\frac{\partial^2 \vec{r}}{\partial t_1^2} = -a\cos t_1\, i - a\sin t_1\, j + 0K$

(9.7) gradent of scalar field directional derivative :-

$$\text{grad } f = \nabla f = \left[\frac{\partial f}{\partial x} , \frac{\partial f}{\partial y} , \frac{\partial f}{\partial z} \right]$$

$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} K$$

∇ = nablaf

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} K$$

No.

Ex: let $f(x,y,z) = 2y^3+4xz+3x$ find $\nabla f \Big|_{(1,1,1)}$?

Soln:

$$\nabla f = \left[0+4z+3 \,,\, 6y^2+0+0 \,,\, 0+4x+0\right]$$

$$= \left[4z+3 \,,\, 6y^2 \,,\, 4x\right] = \left[7 \,,\, 6 \,,\, 4\right]$$

⊛ directional dervitave :

- $D_{\vec{a}} f$ = direction dervitive of $f(x,y,z)$ in the direction of $\vec{a}$.

$$D_{\vec{a}} f = \frac{\vec{a}}{|\vec{a}|} \cdot \nabla f$$

* $\vec{a} = [a_1, a_2, a_3]$

* $|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

* $\bullet$ Dot product.

* $\nabla f = \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right]$

Ex: find the directional drvative of $f(x,y,z) = 2x^2+3y^2+z^2$ at point $P(2,1,3)$ in the direction of $\vec{a} = [1,0,-2]$.

Soln:

$$D_{\vec{a}} f = \frac{[1,0,-2]}{\sqrt{5}} \cdot [4x \,,\, 6y \,,\, 2z] = \frac{[1,0,-2]}{\sqrt{5}} \cdot [8,6,6]$$

$$= \frac{-4}{\sqrt{5}}$$

⊛ Laplace - equation :-

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

∇^2 = nable square $= \Delta$

Ex: let $f(x,y,z) = 4[x^2+y^2]-z^2$ find $\nabla^2 f$??

$\nabla^2 f = \Delta f$

$$\nabla^2 f = 8+8+(-2) = 14$$