13.2) polar form of complex numbers, powers & roots:

$z = x + iy \Rightarrow re^{i\theta}$

① $r = |z| \Rightarrow$ reduis = moduluse = norm = magnitude $= \sqrt{x^2 + y^2}$

② $\theta = \tan^{-1}\left[\frac{y}{x}\right]$

$x = r\cos\theta \quad , \quad y = r\sin\theta$

Ex: let $z_1 = 1 + i$, Find the polar form? [cartesian ⇒ polar]

$r = \sqrt{x^2+y^2} = \sqrt{1+1} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4} \Rightarrow \left[z = \sqrt{2}\, e^{i\frac{\pi}{4}}\right]$

Ex: let $z = \underbrace{\sqrt{2}}_{r}\, e^{i\overbrace{\frac{\pi}{4}}^{\theta}}$ Find the $x+iy$ form ??

$x = \sqrt{2}\cos\frac{\pi}{4} = 1 \quad , \quad y = \sqrt{2}\sin\frac{\pi}{4} = 1 \Rightarrow [z = 1 + i]$

• Eular formula

$\Rightarrow e^{i\theta} = \cos\theta + i\sin\theta$

• Principle value of complex numbers:

$Arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$-\pi \leq Arg(z) \leq \pi$

$arg(z) = Arg(z) \pm 2\pi n$

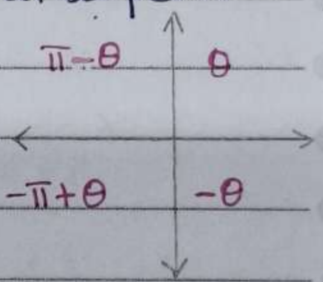
$n = \pm 1, \pm 2, \ldots$

* How to find the angle θ ?

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

• مهم جدًا التأكد من الربع الموجود فيه

Ex: Find θ for the following complex numbers:



① $z_1 = 1 + i \;(1,1) \Rightarrow \tan^{-1}(1) = \frac{\pi}{4}$ الأول

$-\frac{3\pi}{4} \Leftarrow$ ② $z_2 = -1 - i \;(-1,-1) \rightarrow \tan^{-1}\left(\frac{-1}{-1}\right) = -\pi + \frac{\pi}{4}$ الثالث

③ $z_3 = 1 - i \;(1,-1) \rightarrow \tan^{-1}\left(\frac{1}{-1}\right) = -\frac{\pi}{4}$ الرابع

④ $z_4 = -1 + i \;(-1,1) \rightarrow \frac{3\pi}{4}$ الثاني

Ex: let $z_1=1+i$ find $Arg(z)$ & $arg(z)$?

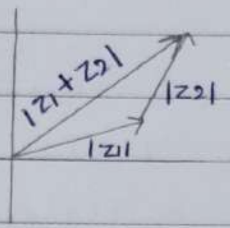
$$Arg(z) = \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$arg(z) = \frac{\pi}{4} \pm 2\pi n$$

$n = \pm 1, \pm 2 \ldots$

- **Triangle Inequality**

$$|z_1+z_2| \leq |z_1| + |z_2|$$



- **Rules:**

[1] $|z_1 z_2| = |z_1||z_2|$

[2] $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

- **Power of complex numbers:**

$$(1+i)^2 = 1+2i+i^2 = 2i$$

$$[z]^n = [x+iy]^n = [re^{i\theta}]^n$$

n = integer $= \pm 1, \pm 2, \ldots$

$$= r^n e^{i\theta n}$$

$$= r^n [\cos(\theta n) + i\sin(\theta n)]$$

Ex: Find $(1+i)^{10}$ in simple form?

$$= r^{10} e^{i\theta 10} \rightarrow r = \sqrt{2}, \theta = \frac{\pi}{4}$$

$$= (\sqrt{2})^{10} e^{i\frac{10\pi}{4}} = 32\, e^{i\frac{5\pi}{2}}$$

$= 2^{\frac{10}{2}}$

$= 2^5$

$$= 32\left[\cos\left(\frac{5\pi}{2}\right) + i\sin\left(\frac{5\pi}{2}\right)\right]$$

$$= 32i$$

- **Multiplication & division of complex numbers:**

let $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

① $z_1 \cdot z_2 = r_1 \cdot r_2\, e^{i(\theta_1+\theta_2)} \Rightarrow$ (جمع)

$$= r_1 r_2 [\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

② $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)} \Rightarrow$ (طرح)

$$= \frac{r_1}{r_2} [\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

Ex: let $z_1 = 2e^{i\frac{\pi}{4}}$, $z_2 = 3e^{i\frac{\pi}{2}}$

Find $z_1 \cdot z_2$ & $\frac{z_1}{z_2}$

[1] $z_1 \cdot z_2 = 6e^{i\left(\frac{3\pi}{4}\right)}$

$$= 6\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right]$$

[2] $\frac{z_1}{z_2} = \frac{2}{3} e^{i(-\pi/4)}$

$$= \frac{2}{3}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$$

$$= \frac{2}{3}\left[\cos\left(\frac{\pi}{4}\right) - i\sin\left(\frac{\pi}{4}\right)\right]$$

- **Demovieve Formula:**

$$[e^{i\theta}]^n = [e^{i\theta}]^n$$

$$e^{i\theta n} = [e^{i\theta}]^n$$

$$\cos(\theta n) + i\sin(\theta n) = [\cos\theta + i\sin\theta]^n$$

- $\theta deg = \theta rad \cdot \frac{180}{\pi}$
- $\theta rad = \theta deg \cdot \frac{\pi}{180}$

Ex: Find $(-1+i)^{-3}$ in simpleform?

$$(-1+i)^{-3} = r^n e^{i\theta n} \qquad r = \sqrt{2},\ \theta = \frac{3\pi}{4}$$

$$= (\sqrt{2})^{-3} e^{i(-\frac{9\pi}{4})}$$

$$= 2^{-\frac{3}{2}} \left[\cos\left(-\frac{9\pi}{4}\right) + i\sin\left(-\frac{9\pi}{4}\right)\right]$$

- **Root of complex numbers:**

$$\sqrt[n]{Z} = \sqrt[n]{(x+iy)}$$

$$= \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi K}{n}\right) + i\sin\left(\frac{\theta + 2\pi K}{2}\right)\right]$$

- $r = \sqrt{x^2 + y^2}$ • n = integer value
- $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ • $K = 0, 1, 2 \cdots (n-1)$

Ex: Find $\sqrt[3]{1}$ in simple form.

(الجذر الأول صفر)
(الجذر الثاني 1)

$\Rightarrow r = \sqrt{1} = 1,\ \theta = \tan^{-1}\left(\frac{0}{1}\right) = 0,\ n = 3,\ K = 0, 1, 2$

- For $K=0$ the 1^{st} root:

$$= 1[\cos(0) + i\sin(0)] = 1$$

- For $K=1$ the 2^{nd} root

$$= 1\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right] = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

- For $K=2$ the 3^{ed} root

$$= 1\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right] = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Solve:

1. $\sqrt[3]{-i}$
2. $\sqrt[2]{1-i}$
3. $(1-i)^{-10}$

No.

- **Rules**

1. $|Re(z)| \leq |z|$
2. $|IM(z)| \leq |z|$
3. $|z_1+z_2|^2 + |z_1-z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

13.5) Exponential Form of complex numbers:

- $e^z = e^{x+iy}$

$= e^x \cdot e^{iy}$

$= e^x [\cos y + i\sin y]$

$= \underbrace{e^x \cos y}_{\text{Real}} + \underbrace{i e^x \sin y}_{\text{IM}}$

- $|e^z| = \sqrt{x^2+y^2}$

$= \sqrt{(e^x\cos y)^2 + (e^x \sin y)^2}$

$= \sqrt{e^{2x}\cos^2 y + e^{2x}\sin^2 y}$

$= e^x\sqrt{\cos^2 y + \sin^2 y} = e^x$

- $\theta = \tan^{-1}\left(\frac{e^x \sin y}{e^x \cos y}\right) =$

$\tan^{-1}(\tan y) = y$

- $\arg(e^z) = y \pm 2\pi n \quad , n = \pm 1, \pm 2 \dots$

- $[e^z]' = e^z$
- $e^{z_1} e^{z_2} = e^{z_1+z_2}$
- $\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$
- $e^{i\pi} \; , \; e^{2\pi i}$

$e^{i\pi} = \cos\pi + i\sin\pi = -1$

$e^{2i\pi} = \cos 2\pi + i\sin 2\pi = 1$

Ex: Write $e^{1.4-0.6i}$ in simple form & Find the magnitude & θ.

$e^{1.4} \cdot e^{-0.6i} = e^{1.4}(\cos(-0.6) + i\sin(-0.6))$

$= e^{1.4}(\cos(0.6) - i\sin(0.6))$ → ملاحظة وانتباه عند ادخال الزاوية على الآلة الحاسبة لايجاد الجواب النهائي التحويل من rad لـ deg. أو العكس.

→ $|e^z| = e^{1.4}$

→ $\theta = -0.6$

→ $\arg(z) = -0.6 \pm 2\pi n$

No.

13.6) Trigonometric & hyperbolic Functions:

- $e^{iz} = \cos z + i \sin z \quad \dots \text{①}$
- $e^{-iz} = \cos(-z) + i\sin(-z)$

$= \cos z - i \sin z \quad \dots \text{②}$

① + ② $\Rightarrow$

$2\cos(z) + 0 = e^{iz} + e^{-iz}$

- $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2} \quad \dots \circledast$

① − ② $\Rightarrow$

$e^{iz} - e^{-iz} = 0 + 2i\sin z$

- $\sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} \quad \dots \circledast$
- $\tan(z) = \dfrac{\sin z}{\cos z} = \dfrac{e^{iz} - e^{iz}}{i(e^{iz} + e^{-iz})} \quad \dots \circledast$
- $\cot(z) = \dfrac{1}{\tan(z)} = \dfrac{i(e^{iz} + e^{-iz})}{e^{iz} - e^{iz}} \quad \dots \circledast$
- $\sec(z) = \dfrac{1}{\cos(z)} = \dfrac{2}{e^{iz} + e^{-iz}} \quad \dots \circledast$
- $\csc(z) = \dfrac{1}{\sin(z)} = \dfrac{2i}{e^{iz} - e^{-iz}} \quad \dots \circledast$

→ $[\cos(z)]' = -\sin(z)$

→ $[\sin(z)]' = \cos(z)$

→ $[\tan(z)]' = \sec^2(z)$

- $\cos(z) = \cos(x + iy)$

$= \cos x \cdot \cosh y - i \sin x \sinh y$

Ex: Find $\cos(\pi + 3i)$ in simple Form.

$= \cos\pi \cosh 3 - (\sin\pi \sinh 3)i$

$= -\cosh 3$

- $\sin(z) = \sin(x+iy)$
  $= \sin x \cosh y + (\cos x \sinh y)i$

Ex: Find $\sin(\pi+3i)$ in simple form.

$= \sin\pi \cosh 3 + (\cos\pi \sinh 3) \cdot i$
$= -(\sinh 3)i$

- Ex: $\tan(\pi+3i) = \dfrac{\sin(\pi+3i)}{\cos(\pi+3i)}$
  $= \dfrac{-(\sinh 3)i}{-\cosh 3}$
  $= i\tanh 3$

- $\cos^2(z) + \sin^2(z) = 1$
- $\cosh^2(z) = 1 + \sinh^2(z)$
- $|\cos z|^2 = \cos^2 x + \sinh^2 y$
- $|\sin z|^2 = \sin^2 x + \sinh^2 y$
- $\sin(z_1)\cos(z_2) = \frac{1}{2}\left[\sin(z_1+z_2) + \sin(z_1+z_2)\right]$
- $\cos(z_1 \pm z_2) = \cos z_1 \cos(z_2) \mp \sin(z_1)\sin(z_2)$
- $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1$
- Hyperbolic Functions:

$\cosh(z) = \dfrac{e^z + e^{-z}}{2}$, $\sinh(z) = \dfrac{e^z - e^{-z}}{2}$, $\tanh(z) = \dfrac{e^z - e^{-z}}{e^z + e^{-z}}$

$\operatorname{sech}(z) = \dfrac{2}{e^z + e^{-z}}$, $\operatorname{csch}(z) = \dfrac{2}{e^z - e^{-z}}$, $\coth(z) = \dfrac{e^z + e^{-z}}{e^z - e^{-z}}$

* $[\cosh(z)]' = \sinh(z)$.
* $[\sinh(z)]' = \cosh(z)$.

* Complex Trigonomitric & hyperbolic Function are related:

① $\cosh(iz) = \cos(z)$.
② $\sinh(iz) = i\sin(z)$,
③ $\cos(iz) = \cosh(z)$
④ $\sin(iz) = i\sinh(z)$.

- $\cosh(z) = \cosh(x+iy)$
  $= \cosh x \cos y + i\sinh x \sin y$

Ex: $\cosh(3+\pi i)$
$= \cosh 3 \cos\pi + i\sinh 3 \sin\pi$
$= -\cosh 3$

- $\sinh(z) = \sinh(x+iy)$
  $= \sinh x \cos y + i\cosh x \sin y$

Ex: Find $\sinh(3+\pi i)$
$= \sinh 3 \cos\pi + i\cosh 3 \sin\pi$
$= -\sinh 3$

Ex: $\tanh(3+\pi i)$
$= \dfrac{-\sinh 3}{-\cosh 3} = \tanh 3$.

- $\cosh(z_1+z_2) =$

$\cosh(z_1)\cosh(z_2) + \sinh(z_1)\sinh(z_2)$

- $\sinh(z_1+z_2)$

$\sinh(z_1)\cosh(z_2) + \cosh(z_1)\sinh(z_2)$

Ex: Find $\text{sech}(3+4i)$ in simple Form

$$= \frac{1}{\cosh(3+4i)} = \frac{1}{\cosh 3\cos 4 + i\sinh 3\sin 4}$$

الضرب بالمرافقة

$$= \frac{1}{x+iy} \cdot \frac{x-iy}{x-iy} \cdots\cdots \text{نكمل الحل}$$

Solve:

① $e^{5+2\pi i}$

② Find r & θ For $e^{2\pi+5i}$

③ $\cot(2\pi+3\pi i)$?

④ $\text{sech}(2\pi+\pi i)$?