Ex: Find eigen value + eigen vector for

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

① $\det [A - \lambda I] = 0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{bmatrix} = 0$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda = 5, -3, -3$$

② for $\lambda = 5$ the x is:

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tilde{A} = [A \,\vdots\, b]$

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 2 & -4 & -6 & 0 \\ -1 & -2 & -5 & 0 \end{array}\right] \Rightarrow \text{pivot}$$

$$\begin{array}{r} \\ R_2 + \frac{2}{7}R_1 \leftarrow \\ R_3 - \frac{1}{7}R_1 \leftarrow \end{array} \left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 0 & -24/7 & -48/7 & 0 \\ 0 & -16/7 & -32/7 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} -7 & 2 & -3 & 0 \\ 0 & \frac{-24}{7} & -48/7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \Rightarrow R_3 - 2/3 R_2 \end{array}$$

$$7x_1 + 2x_2 - 3x_3 = 0$$

$$-24/7\, x_2 - \frac{48}{7} x_3 = 0$$

$\boxed{} \Rightarrow$ (معادلة ناقصة)

$$x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

③ for $\lambda = -3$ the x is:

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\tilde{A} = [A \,\vdots\, b]$

$$\text{pivot} \leftarrow \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \rightarrow R_2 - 2R_1 \\ \rightarrow R_3 + R_1 \end{array}$$

$$x_1 + 2x_2 - 3x_3 = 0$$

$\boxed{}$
$\boxed{}$ معادلتين ناقصات

$$x = \begin{bmatrix} 1 \\ \\ \end{bmatrix}$$

No.

Ex: Find eigen value + eigen vector for

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$\det [A - \lambda I] = 0$

$$\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = 0$$

$\lambda^2 + 1 = 0$

$\lambda = \pm \sqrt{-1} = \pm i$

- for $\lambda = i$ the x is :

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x = \begin{bmatrix} 1 \\ i \end{bmatrix}$

- for $\lambda = -i$ the x is:

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

(8.3) Symmetric, skew-symmetric & orthogonal Matrix

Def: a real square Matrix $A = [a_{jk}]$ is called

① symmetric if $A^T = A$

② skew-symmetric if $A^T = -A$

③ orthogonal if $A^T = A^{-1}$

Ex: ① symmetric

$$A = \begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}$$

② skew-symmetric

$$A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$$

③ Orthogonal

$$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$$

No.

* for any square Matrix (A)

① symmetric Matrix $(R) = \frac{1}{2}[A+A^T]$

② Skew-symmetric Matrix $(S) = \frac{1}{2}[A-A^T]$

Ex: let $A = \begin{bmatrix} 9 & 5 & 2 \\ 2 & 3 & -8 \\ 5 & 4 & 3 \end{bmatrix}$, find (R, S)

$A^T = \begin{bmatrix} 9 & 2 & 5 \\ 5 & 3 & 4 \\ 2 & -8 & 3 \end{bmatrix} \rightsquigarrow R = \begin{bmatrix} 9 & 3.5 & 3.5 \\ 3.5 & 3 & -2 \\ 3.5 & -2 & 3 \end{bmatrix}$

$S = \begin{bmatrix} 0 & 1.5 & -1.5 \\ -1.5 & 0 & -6 \\ 1.5 & 6 & 0 \end{bmatrix}$

* Theorem:

① The eigen value of symmetric Matrix are Real

② The eigen value of skew-symmetric Matrix are pure imaginary or zero

$\hookrightarrow (z = iy)$

Ex: Find eigen value of $A = \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}$

det $[A-\lambda I]x = 0$

$\begin{bmatrix} -\lambda & 9 & -12 \\ -9 & -\lambda & 20 \\ 12 & -20 & -\lambda \end{bmatrix}$

$-\lambda^3 - 625\lambda = 0$

$-\lambda[\lambda^2 + 625] = 0$

$\lambda = 0, \pm 25i$

No.

* Orthogonal Matrix :

EX: $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$A^T = A^{-1}$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} C^T$

$= \frac{1}{1} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

Theorem :

① The determinent of an orthogonal Matrix is (1) OR (-1)

② The eigen value of orthogonal Matrix A are Real or [complex conjegate] in pairs & have [absolute value] of (1)

$\hookrightarrow z = x+yi \Rightarrow \bar{z} = x-yi$ $\qquad \hookrightarrow |z| = \sqrt{x^2+y^2} = 1$

EX: find det. + eigen value of

$A = \begin{bmatrix} 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \\ 1/3 & 2/3 & -2/3 \end{bmatrix}$ $\rightarrow$ ① $\det[A] = -1$

② $\det[A - \lambda I] = 0$

$\begin{vmatrix} 2/3-\lambda & 1/3 & 2/3 \\ -2/3 & +2/3-\lambda & 1/3 \\ 1/3 & 2/3 & -2/3-\lambda \end{vmatrix} = 0$

$-\lambda^3 + \frac{2}{3}\lambda^2 + \frac{2}{3}\lambda - 1 = 0$ $\Rightarrow$ الحل باستخدام القسمة التركيبة.

$\lambda = -1, \quad \frac{5+i\sqrt{11}}{6}, \quad \frac{5-i\sqrt{11}}{6}$

$|z| = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{\sqrt{11}}{6}\right)^2} = 1$

19/11

No.

* Similar Matrix, similarity transformation let $(n \times n)$ Matrix $\hat{A}$ is called similar to $(n \times n)$ Matrix A iff

$$\hat{A} = P^{-1} A P \Rightarrow P = \text{non singular Matrix}$$

ملاحظة: الضرب عملية غير تبديلية و

اذاً لا يجوز ضرب $P^{-1}P$ يعني

AI لان الترتيب مهم

Theorem:

① If $\hat{A}$ is similar to A, then $\hat{A}$ has the same eigen value as A.

② If x is an eigen value of A, then $y = P^{-1}x$ is the eigen vector of $\hat{A}$ corresponding to same eigen value

$$\begin{tikzcd}
A \arrow[r, "\hat{A} = P^{-1}AP"] \arrow[d] & \hat{A} \arrow[d] \\
\lambda \arrow[r, "="] \arrow[d] & \lambda \arrow[d] \\
x \arrow[r, "y = P^{-1}x"] & y
\end{tikzcd}$$

EX: let $A = \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix}$ & $P = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$, find eigen value & eigen vector for A and $\hat{A}$.

① $\det[A - \lambda I] = 0$

$$\begin{bmatrix} 6-\lambda & -3 \\ 4 & -1-\lambda \end{bmatrix} = 0$$

$$(6-\lambda)(-1-\lambda) + 12 = 0$$

$$\lambda = 2, 3$$

* For $\lambda = 2$ x is:

$$\begin{bmatrix} 4 & -3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

* for $\lambda = 3$ x is:

$$\begin{bmatrix} 3 & -3 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\hat{A} = P^{-1} A P$$

$$= \frac{1}{1} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 & -3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$\Rightarrow$ تكملة

No.

$$\hat{A} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\det[\hat{A} - \lambda I] = 0$$

$$\begin{bmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{bmatrix} = 0$$

$$(3-\lambda)(2-\lambda) - 0 = 0$$

$$\lambda = 2, 3$$

$$y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

* for $\lambda = 2$ the (y) is

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

* for $\lambda = 3$ the (y) is

$$\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- $y_1 = P^{-1} x_1$

$$= \frac{1}{1} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- $y_2 = P^{-1} x_2$

$$= \frac{1}{1} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Theorem: diagonalization of Matrix if an $(n \times n)$ Matrix A has abasis of eigen vectors then

$$D = X^{-1} A X$$

↳ diagonal Matrix ⇒ $\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

with eigen vector of A as element on main diagonal

$$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

No.

x = eigen vector of A as column vector.

$D^2 = D \times D = x^{-1} A \underbrace{x \cdot x^{-1}} A x$

$= x^{-1} A I A x$

$= x^{-1} A^2 x$ $\qquad [\, D^m = x^{-1} A^m x \,]$

Ex: Find D for A?

$$A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

$D = x^{-1} A x$

① $\det [A - \lambda I] = 0$

$$\begin{bmatrix} 7.3-\lambda & 0.2 & -3.7 \\ -11.5 & 1-\lambda & 5.5 \\ 17.7 & 1.8 & -9.3-\lambda \end{bmatrix}$$

$$7.3-\lambda \begin{bmatrix} 1-\lambda & 5.5 \\ 1.8 & -9.3-\lambda \end{bmatrix} - 0.2 \begin{bmatrix} -11.5 & 5.5 \\ 17.7 & -9.3-\lambda \end{bmatrix} + -3.7 \begin{bmatrix} -11.5 & 1-\lambda \\ 17.7 & 1.8 \end{bmatrix}$$

$\hookrightarrow -\lambda^3 - \lambda^2 + 12\lambda = 0$

$-\lambda [\lambda^2 + \lambda - 12] = 0$

$\lambda = 0, 3, -4$

- for $\lambda = 0$ the x is:

$$x = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$$

- for $\lambda = 3$ the x is:

$$x = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$$

- for $\lambda = -4$ the x is:

$$x = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\therefore x = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} \Longrightarrow x^{-1} = \frac{1}{|X|} C^T = \begin{bmatrix} -0.7 & 0.2 & 0.2 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}$$

No.

* $D = X^{-1} A X$

$$= \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix} \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* for $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Trace of $A = a_{11} + a_{22} + a_{33}$