Ordinary Differential Equations :

chapter (1) : Introduction :

Def: An equation containing the derivative of one or more dependent Variable (s), with respect to one or more independent variable(s) is said to be differentiable equation (DE) (الاختصار).

Ex: → (differentiable eq. أمثلة)

1) $y'' - 2xy = 5$
2) $\frac{dy}{dx} - 3xy = e^x$
3) $\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial z \partial x} = 0$
4) $u_{xz} - u_x + 4u_z = e^{xz}$

• **Remark:** DE's are classified by: type, order, and linearity.

1 The Type:

DE

Ordinary DE (ODE)
(containing the ordinary derivative)
$\left[\frac{dy}{dx}, y'''\right]$

Ex:

1) $y'' - 2xy = 5$
2) $\frac{dy}{dx} - 3xy = e^x$

Partial DE (PDE)
(containing the Partial derivative)
$\left[\frac{\partial y}{\partial x}, y_x, y_{zx}\right]$

Ex:

1) $\frac{\partial y}{\partial x} - \frac{\partial^2 y}{\partial z \partial x} = 0$
2) $u_{xz} - u_x + 4u_z = e^{xz}$

* Partial derivative (• مثال توضيحي للمشتقة الجزئية)

Ex: $f(x,y) = x\sin(xy) + 3xy^2 + y + 2x + 5$

Find:

$$f_x = \frac{\partial f}{\partial x} = (1)(\sin(xy)) + x\cos(xy)\,y + 3y^2 + 0 + 2 + 0$$

$$f_y = \frac{\partial f}{\partial y} = x\cos(xy)\,x + 6yx + 1 + 0 + 0$$

2 The Order:

(The order of a DE is the order of the highest derivative in this equation)

Ex:

1) $x^2y'' - 2xy' + 3y = e^x$

(ODE) with second order

2) $\dot{y}\,y''' + 5y = 3$

(ODE) with 3^{rd} order

3) $\frac{d^2y}{dx^2}\left(y^{(11)} + 5y\right) + 5xy = 0 \rightarrow$ (ODE) with 13^{th} order

4) $\frac{\partial^2 y}{\partial x \partial z} - \frac{\partial y}{\partial x} = \frac{\partial y}{\partial z} \rightarrow$ (PDE) with 2^{nd} order

5) $U_{xy} + U_{xyz} + (U_x)^4 = 5$

(PDE) with 3^{rd} order

ملاحظة ⇐ الفرق بين (3) / 3

مثال $(U_x)^3$ و $(U_x)^{(3)}$

هاي يعني المشتقة مضروبة ثلاث مرات (قوة)

هاي يعني مشتقة الثالثة

[3] Linearity: (The (DE) (ODE) is linear if can be written as:

$$a_n(x)\, y^{(n)} + a_{n-1}(x)\, y^{(n-1)} + \cdots + a_1(x)\, y' + a_0(x)\, y = g(x) \;)$$

Ex:

① $x^2 y'' - 3xy' + 2y = \ln x$

(linear)

② $y'' - (2y+3)y' + 5e^x = 0$

(Non - linear)

③ $y^{(5)} - 2(y')^{2} + 5xy = \ln y$

(Non - linear)

ملاحظة: (y ومشتقاتها الاسس لهم واحد غير موجودة داخل (ln) أو جذر أو داخل (sin...) أو غير مضروبة ببعض أو غير موجودة بالمقام + معاملاتهم ثوابت أو اقترانات تعتمد على (x))

(شروط حتى يكون linerality)

Solution of (DE)

Explicit Solution

($y = f(x)$)

Function

Ex $y = x + e^x$

Implicit solution

(relation between x, y)

Ex: $x^2 + y^2 = 25$

Ex: Show that $4x^2 - y^2 = 4$ is an implicit solution to

$$y \cdot \frac{dy}{dx} - 4x = 0$$

نشتق ضمني

$$4x^2 - y^2 = 4$$

$$\left(8x - 2y\frac{dy}{dx} = 0\right) \div (-2)$$

$$-4x + y\frac{dy}{dx} = 0$$

$$y\frac{dy}{dx} - 4x = 0$$

طريقة ثانية

$$y\frac{dy}{dx} - 4x = 0$$

$$\left[\, y\frac{dy}{dx} = 4x \,\right]$$

نعوضها

$$y\frac{dy}{dx} - 4x = 0$$

$$4x - 4x = 0 \;\checkmark$$

Ex: Show that $y = f(x) = x^2 - x^{-1}$ is an explicit solution for $\frac{d^2y}{dx^2} - \frac{2}{x^2} y = 0$

$y = f(x) = x^2 - x^{-1}$

$f'(x) = 2x + x^{-2}$

$f''(x) = 2 - 2x^{-3}$

$2 - 2x^{-3} - \frac{2}{x^2}(x^2 - x^{-1}) \stackrel{?}{=} 0$

$\cancel{2} - \cancel{2x^{-3}} - \cancel{2} + \cancel{2x^{-3}} \stackrel{?}{=} 0$

$\therefore = 0$ ✓

[LHS = RHS]

Ex: Determine for which value (s) of m the function $y = f(x) = e^{mx}$ is a solution to DE $y'' + 6y' + 5y = 0$

$y = e^{mx}$

$y' = me^{mx}$

$y'' = m^2 e^{mx}$

$y'' + 6y' + 5y = 0$

$m^2 e^{mx} + 6me^{mx} + 5e^{mx} = 0$

$\underbrace{e^{mx}}_{\neq 0}(m^2 + 6m + 5) = 0$

$m^2 + 6m + 5 = 0$

$(m+1)(m+5) = 0$

$[m = -1]$ $[m = -5]$

The intial Value problem (IVP)

" solve the following (IVP) : $\frac{d^n y}{dx^n} = f(x, y, y', \ldots, y^{(n-1)})$

with $y(x_0) = y_0$, $y'(x_0) = y_1$, ... $y^{(n-1)}(x_0) = y_{n-1}$ "

(صيغة السؤال)

Theorem: (نظرية)

Suppose $p(x)$ & $f(x)$ are cont. on interval $(a,b) = I$ that contains x_0, then there exists a unique solution to the following (IVP)

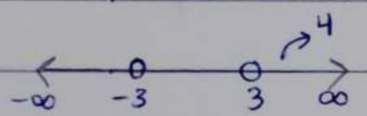
(1) $\frac{dy}{dx} + p(x)y = f(x)$ with $y(x_0) = y_0$ on interal I

Ex: find the largest interal such that the (IVP) has a unique solution $(x^2-9)y' + 2y = \ln|20-4x|$, $y(4) = -3$

Soln:
$$y' + \underbrace{\frac{2}{(x^2-9)}}_{p(x)} y = \underbrace{\frac{\ln|20-4x|}{(x^2-9)}}_{f(x)}$$

$P(x) = \frac{2}{(x^2-9)}$

$x^2-9=0 \rightarrow x = \pm 3$



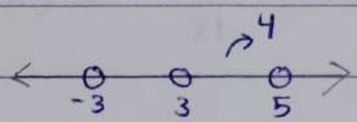
$(3,\infty)$

$g(x) = \frac{\ln|20-4x|}{(x^2-9)}$

$x^2-9=0 \rightarrow x = \pm 3$, $\ln|20-4x| \geq 0$

$20-4x = 0$

$x = 5$



$(3,5)$

$\therefore$ the largest interval is $(3,\infty) \cap (3,5) = (3,5)$

Ex: Find the largest interval such that the (IVP) has a unique solution:

1. $(x^2-4)y' + \frac{x-3}{\ln(x-2)} y = \frac{\sin x}{x-6}$, $y(\frac{5}{2}) = 3$

2. $2xy' + x\tan x\, y = \frac{2}{e^x}$, $y(-1) = 100$

Chapter (2) : First Order (ODE)

This DE has the form $\frac{dy}{dx} = F(x,y)$

1)
- Separable Equation

This equation can be written as $\frac{dy}{dx} = f(x) \cdot h(y)$

Ex: ① $\frac{dy}{dx} = xe^{x+y} \rightarrow \frac{dy}{dx} = \underbrace{xe^{x}}_{f(x)} \cdot \underbrace{e^{y}}_{h(x)}$

$\therefore$ Separable.

② $\frac{dy}{dx} = \frac{x^2-2x+1}{xy-y} \rightarrow \frac{dy}{dx} = \frac{x^2-2x+1}{(x-1)} \cdot \frac{1}{y}$

$$\frac{dy}{dx} = \frac{(x-1)^2}{x-1} \cdot \frac{1}{y}$$

$$\frac{dy}{dx} = (x-1) \cdot \frac{1}{y}$$ $\therefore$ separable.

③ $\frac{dy}{dx} = \sin(x+y)$ (Not separable)

Remark: A (DE) $\frac{dy}{dx} = f(x,y)$ is separable if :

$$F(x,y) \cdot \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial F}{\partial x} \cdot \frac{\partial F}{\partial y}$$

Remark: To solve the separable eq. $\frac{dy}{dx} = f(x) \cdot h(y)$

$\Rightarrow \frac{dy}{h(y)} = f(x)\,dx$

$\Rightarrow \int \frac{1}{h(y)}\,dy = \int f(x)\,dx + C$

Ex: Solve $(1+x)\,dy - y\,dx = 0$

$$(1+x)\,dy = y\,dx$$

$$\int \frac{1}{y}\,dy = \int \frac{1}{1+x}\,dx$$

$$[\ln|y| = \ln|1+x| + C] \Rightarrow \text{implicit soln.}$$

$$\overset{e}{\Rightarrow} \quad |y| = e^{\ln|1+x| + C}$$

$$|y| = |1+x| \cdot e^{c}$$

$$|y| = C\,|1+x|$$

$$y = -C\,|1+x|$$

$$y = C\,|1+x|$$

explicit soln

Ex: Find the general solution to DE $xe^{-y}\sin x\,dx - y\,dy = 0$

$$x\sin x\,dx = ye^{y}\,dy \quad \text{(separable)}$$

$$\int x\sin x\,dx = \int ye^{y}\,dy \rightarrow$$

تكامل بالأجزاء

/	∫
y	$\oplus\ e^{y}$
1	$\ominus\ e^{y}$
0	e^{y}

$$[-x\cos x + \sin x + C = (y-1)\,e^{y}]$$

general soln (implicit soln).

تكامل بالأجزاء

/	∫
x	$\oplus\ \sin x$
1	$\ominus\ -\cos x$
0	$-\sin x$

Ex: Solve $\dfrac{dy}{dx} = \dfrac{xy+3x-y-3}{xy-2x+4y-8}$

Soln: $\dfrac{dy}{dx} = \dfrac{x(y+3)-(y+3)}{x(y-2)+4(y-2)}$

$$\frac{dy}{dx} = \frac{(y+3)(x-1)}{(y-2)(x+4)}$$

$$\frac{dy}{dx} = \frac{(y+3)}{(y-2)} \cdot \frac{(x-1)}{(x+4)}.$$

(Separable)

كسور جزئية (قسمة)

كسور جزئية (المشتقة)

$$\int \frac{(y-2)}{(y+3)}\,dy = \int \frac{(x-1)}{(x+4)}\,dx$$

$$\int \left(1 - \frac{5}{y+3}\right) dy = \int \left(1 - \frac{5}{x+4}\right) dx$$

$$y - 5\ln|y+3| = x - 5\ln|x+4| + C$$

$$y - 5\ln\left|\frac{y+3}{x+4}\right| = x + C.$$

Ex: let $\frac{dy}{dx} = \frac{xy+my-x-2}{xy-3y+Kx-3}$,

find the value of m & K such that the DE is separable

Soln: $\frac{dy}{dx} = \frac{y(x+m) - (x+2)}{y(x-3) + K\left(x - \frac{3}{K}\right)}$

$\therefore [m=+2]$ & $-3 = -\frac{3}{K}$

$[K=1]$

Ex: Solve the following (IVP) $\frac{dy}{dt} = e^{y-t} \cdot \frac{1}{y} \cdot (1+t^2)$ such that $y(0)=-2$

$\frac{dy}{dt} = \frac{e^y}{e^t} \frac{1}{y} (1+t^2)$

separable

إجزاء ← $\int y e^{-y} dy = \int e^{-t} (1+t^2) dt$ → إجزاء

① ②

$(-y-1)e^{-y} = (-1-t^2-2t-2)e^{-t} + C$

①

/	∫
y (+)	e^{-y}
1 (−)	$-e^{-y}$
0	e^{-y}

$\therefore (-y-1)e^{-y}$

②

/	∫
$1+t^2$ (+)	e^{-t}
$2t$ (−)	$-e^{-t}$
2 (+)	$+e^{-t}$
0	$-e^{-t}$

$\therefore (-1-t^2-2t-2)e^{-t}$

since $y(0)=-2$ (t, y)

$(-(-2)-1)e^{+2} = (-1-0-0-2)e^0 + C$

$e^2 = -3+C$

$C = e^2+3$

$\therefore$ the soln of IVP is:

$(y+1)e^{-y} = (3+t^2+2t)e^{-t} - (e^2+3)$

2) • **Linear Equation:** The standard form of this equation is:

$$y' + P(x)y = g(x) \quad \dots \circledast$$

To solve this equation:

① Find the (integrating factor) as:

$$\mu(x) = e^{\int p(x)\,dx}$$

② Multiply both sides of eq ($*$) by $\mu(x)$, then we have:

$$\mu(x)\left[y' + p(x)y = g(x)\right]$$

$$\left[\mu(x)y' + \mu(x)p(x)y\right] = \mu(x)g(x)$$

$$\hookrightarrow \frac{d}{dx}\left[\mu(x)y\right] = \mu(x)g(x).$$

③ Integrate both sides (with respect x)

$$\int \frac{d}{dx}\left[\mu(x)y\right] = \int \mu(x)g(x)\,dx$$

$$\hookrightarrow \mu(x)y = \int \mu(x)g(x)\,dx + C$$

$$\left[y = \frac{1}{\mu(x)}\int \mu(x)g(x)\,dx + \frac{C}{\mu(x)}\right] \;**$$

Remark: ① $\mu'(x) = p(x)\,e^{\int p(x)\,dx}$

$$= p(x)\,\mu(x)$$

$$\mu(x) = e^{\int p(x)\,dx}$$

② $p(x) = \dfrac{\mu'(x)}{\mu(x)}$

③ $\dfrac{d}{dx}\left[\mu(x)y\right] = \mu'(x)y + \mu(x)y'$

$$= P(x)\mu(x)y + \mu(x)y'$$

Ex: Solve $(x^2+1)\frac{dy}{dx} + xy - x = 0$

soln:

$$(x^2+1)\,y' + xy - x = 0 \rightarrow (x^2+1)\,y' + xy = x$$

$$y' + \frac{x}{x^2+1}\,y = \frac{x}{x^2+1}$$

$$\mu(x) = e^{\int p(x)\,dx} \qquad p(x) = \frac{x}{x^2+1}, \quad g(x) = \frac{x}{x^2+1}$$

$$= e^{\int \frac{x}{x^2+1}\,dx}$$

$$= e^{\frac{1}{2}\ln|x^2+1|}$$

$$= (x^2+1)^{1/2}$$

- the solution is:

$$y(x) = \frac{1}{\mu(x)} \int \mu(x)\,g(x)\,dx + \frac{c}{\mu(x)}$$

$$= \frac{1}{(x^2+1)^{1/2}} \int \cancel{(x^2+1)^{1/2}} * \frac{x}{\cancel{x^2+1}} + \frac{c}{(x^2+1)^{1/2}}$$

$$= \frac{1}{(x^2+1)^{1/2}} \int x\,\underbrace{(x^2+1)^{-1/2}}_{\text{استخدام تكامل بالتعويض.}}\,dx + \frac{c}{(x^2+1)^{1/2}}$$

$$u = x^2+1$$
$$du = 2x\,dx$$
$$dx = \frac{du}{2x}$$

$$= \frac{1}{(x^2+1)^{\frac{1}{2}}} \int \cancel{x}\,u^{-1/2}\,\frac{du}{2\cancel{x}} + \frac{c}{(x^2+1)^{1/2}}$$

$$= \frac{1}{\cancel{(x^2+1)^{\frac{1}{2}}}} \left[\cancel{u^{1/2}}\right] + \frac{c}{(x^2+1)^{1/2}}$$

$$\therefore\; y = 1 + \frac{c}{(x^2+1)^{\frac{1}{2}}}$$

Ex: solve $xy' - (x-1)y = x^2$

soln: $y' - \frac{(x-1)}{x} y = x$ (linear in y)

$\mu(x) = e^{\int p(x)\,dx}$

$= e^{\int -1+\frac{1}{x}\,dx}$

$= e^{-x+\ln|x|}$

$= e^{-x} \cdot x$

$p(x) = -\frac{(x-1)}{x}$, $g(x) = x$

اختصار

$= -1+\frac{1}{x}$

(*) The solution is:

$y = \frac{1}{\mu(x)} \int \mu(x)\, g(x) + \frac{c}{\mu(x)}$

$= \frac{1}{xe^{-x}} \int \underbrace{xe^{-x} \cdot x}_{\text{اجزاء}} + \frac{c}{xe^{-x}}$

$= \frac{1}{x\cancel{e^{-x}}} \left(\cancel{e^{-x}}(-x^2-2x-2)\right) + \frac{c}{xe^{-x}}$

$= \frac{1}{x} [-x^2-2x-2] + \frac{c}{xe^{-x}}$

$\therefore\ y = -x-2-\frac{2}{x} + \frac{c}{xe^{-x}}$

x^2	e^{-x}
$2x$ (+)	$-e^{-x}$
2 (−)	e^{-x}
0 (+)	$-e^{-x}$

$-x^2e^{-x} - 2xe^{-x} - 2e^{-x}$

$e^{-x}(-x^2-2x-2)$

Ex: Solve the (IVP) $\frac{dy}{dx} = \frac{1}{e^y - x}$ where $y(1)=0$.

Soln:

$\frac{dx}{dy} = e^y - x$

$x' + x = e^y$ → Linear in(x)

$p(y) = 1$, $g(y) = e^y$

$\mu(y) = e^{\int p(y)\,dy}$

$= e^{\int 1\,dy}$

$= e^y$

∴ solution :

$x\ \cancel{y} = \frac{1}{e^y} \int e^y \cdot e^y\, dy + \frac{c}{e^y}$

$x = \frac{e^y}{2} + \frac{c}{e^y}$ ⟸

then since $y(1) = 0$

$$x = \frac{e^y}{2} + \frac{C}{e^y} \Rightarrow 1 = \frac{e^0}{2} + \frac{C}{e^0}$$

$$\left[C = \frac{1}{2} \right]$$

$\therefore$ the solution of IVP is

$$x = \frac{e^y}{2} + \frac{1}{2} e^{-y}$$

$$= \frac{e^y + e^{-y}}{2} = \cosh(y)$$

$$\therefore \ x = \frac{e^y + e^{-y}}{2} = \cosh(y)$$

Ex: Solve $(x+2)^2 y' = 5 - 4y - 2xy$

Soln:

$$(x+2)^2 \cdot y' + \underbrace{4y + 2xy}_{} = 5$$

$$(x+2)^2 y' + 2(x+2)y = 5$$

$$y' + \frac{2}{(x+2)} y = \frac{5}{(x+2)^2} \qquad p(x) = \frac{2}{x+2}, \ g(x) = \frac{5}{(x+2)^2}$$

- $\mu(x) = e^{\int p(x)\,dx}$

$$= e^{\int \frac{2}{x+2} dx}$$

$$= (x+2)^2$$

$$\star \quad y = \frac{1}{(x+2)^2} \int \cancel{(x+2)^2} \cdot \frac{5}{\cancel{(x+2)^2}} + \frac{C}{(x+2)^2}$$

$$y = \frac{5x + C}{(x+2)^2}$$