Bernolli Equation:

The standard form of this eq. is:

$$y' + q(x)y = f(x)y^n \ldots (*) , \quad n \in \mathbb{R}$$

* Remark:

[1] if $[n=0]$, then the eq (*) becomes $y' + q(x)y = f(x)$
[linear in y]

[2] if $[n=1]$, the equation (*) becomes $y' + q(x)y = f(x)y$

$y' + [q(x) - f(x)]y = 0$ [linear in y]

[3] For $n \neq 0$ & $n \neq 1$ then we use the substitution $[v = y^{1-n}]$ to transform the eq (*) to linear in v, as:

(i) multiply the eq (*) by y^{-n}, then we have

$$y^{-n}y' + q(x)y^{1-n} = f(x)$$

(ii) since $v = y^{1-n} \rightarrow v' = (1-n)y^{-n}y'$

$$\frac{v'}{1-n} = y^{-n}y'$$

(iii) put (ii), & $v = y^{1-n}$ in eq. (i)

$$\frac{v'}{1-n} + q(x)v = f(x)$$

$$v' + (1-n)q(x)v = (1-n)f(x) \rightarrow \text{linear in } (v)$$

Remark:

$$y'^{(1-n)} + q(x)y^{(1-n)} = f(x)y^n \quad (\text{linear in } y)$$

مش خطية

$y' \to v$, $y \to v$

$$\Downarrow$$

$$v' + q(x)(1-n)v = (1-n)f(x) \quad (\text{linear in } v)$$

Ex: Solve $\quad xy' + y = \ln x \, y^2$

Soln: $\quad y' + \frac{1}{x} y = \frac{\ln x}{x} y^{2} \rightarrow n$ $\rightarrow$ Bernoulli in y

$$\Downarrow$$

$v = y^{1-n}$

$\hookrightarrow v = y^{1-2}$ $\qquad v' + \underbrace{\frac{-1}{x}}_{p(x)} v = \underbrace{\frac{-\ln x}{x}}_{f(x)}$ $\rightarrow$ linear in (v)

$v = y^{-1} = \frac{1}{y}$

① $\mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{-1}{x}\,dx} = e^{-\ln x} = \frac{1}{x}$

integrating factor

for linear eq. ② $v = \frac{1}{1/x} \int \frac{1}{x} \cdot \frac{-\ln x}{x}\,dx + \frac{c}{1/x}$

التكامل بالأجزاء $\qquad v = x \int \frac{-\ln x}{x^2}\,dx + C(x)$

$u = \ln x \qquad dv = \frac{-1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = \frac{1}{x}$ $\qquad \left[v = x \left[\frac{\ln x}{x} + \frac{1}{x} \right] + Cx \right]$

$\frac{1}{x} \ln x - \int \frac{1}{x^2}\,dx$

$\frac{1}{x} \ln x + \frac{1}{x}$ $\qquad \hookrightarrow v = \ln x + 1 + C(x)$.

$$\frac{1}{y} = \ln x + 1 + C(x)$$

$$y = \frac{1}{\ln x + 1 + Cx}$$

Ex: Solve $\frac{dy}{dx} = \frac{1}{\frac{x}{y} + yx^2}$

Soln: $\frac{dx}{dy} = \frac{x}{y} + yx^2$

$$v = x^{1-n} = x^{1-2} = x^{-1} = \frac{1}{x}$$

$$x' - \frac{1}{y}x = yx^{2} \xrightarrow{} n=2$$

$$\Downarrow$$

$$v' + \underbrace{\frac{1}{y}}_{p(y)} v = \underbrace{-y}_{g(y)} \rightarrow \text{linear in } v$$

① $\mu(y) = e^{\int p(y)\,dy} = e^{\int \frac{1}{y} dy} = y$

② $v = \frac{1}{y} \int -y \cdot y \, dy + \frac{c}{y}$

$$= \frac{1}{y} \int -y^2 \, dy + \frac{c}{y}$$

$$= \frac{1}{\cancel{y}} \left(\frac{-y^{\cancel{3}}}{3}\right) + \frac{c}{y}$$

$$= \frac{-y^2}{3} + \frac{c}{y} = \frac{-y^3 + c}{3y}$$

$[v = x^{-1}]$

$$\frac{1}{x} = \frac{-y^3 + c}{3y}$$

$$x = \frac{3y}{-y^3 + c}$$

Ex: Solve $x^2 y' - x^2 y = 2xy - x^3 y^3$

$$x^2 y' - (x^2+2x) y = -x^3 y^3$$

$$y' - \left(\frac{x^2+2x}{x^2}\right) y = \frac{-x^3}{x^2} y^3$$

$$y' - \underbrace{\left(1+\frac{2}{x}\right)}_{p(x)} y = \underbrace{-x}_{f(x)} y^{3} \quad \rightarrow \text{Bernolli in } y$$

$$v = y^{1-n} = y^{1-3} = y^{-2} = \frac{1}{y^2}$$

$$\Downarrow$$

$$v' + \underbrace{2\left(1+\frac{2}{x}\right)}_{p(x)} v = \underbrace{2x}_{g(x)} \quad \rightarrow \text{linear in } v$$

① $\mu(x) = e^{\int p(x)dx} = e^{\int\left(2+\frac{4}{x}\right)dx} = e^{2x+4\ln x} = e^{2x}\cdot x^4$

② $v = \frac{1}{e^{2x}\cdot x^4}\int x^4 e^{2x}\cdot 2x \, dx + \frac{C}{e^{2x}\cdot x^4}$

$$v = \frac{1}{e^{2x}x^4}\int \underbrace{2x^5 e^{2x}}_{\text{أجزاء}} + \frac{C}{e^{2x}\cdot x^4}$$

$'$	$\int$
$+\ 2x^5$	e^{2x}
$-\ 10x^4$	$2e^{2x}$
$+\ 40x^3$	$4e^{2x}$
$-\ 120x^2$	$8e^{2x}$
$+\ 240x$	$16e^{2x}$
$-\ 240$	$32e^{2x}$
0	$64e^{2x}$

$$v = \frac{1}{\cancel{e^{2x}}\cancel{x^4}}\left(4x^5 - 40x^4 + 320x^3 - 1920x^2 + 7680x - 15360\right)\cancel{e^{2x}} + \frac{C}{e^{2x}x^4}$$

$$v = 4x - 40 + 320x^{-1} - 1920x^{-2} + 7680x^{-3} - 1536x^{-4} + \frac{C}{e^{2x}x^4}$$

آخر خطوة نعوض بدل $v \leftarrow \frac{1}{y^2}$

وما ننسى نقلب المعادلة

(H.w): Solve $y' - y = x\sqrt{y}$

* Exact Equation:

The DE $M(x,y)\,dx + N(x,y)\,dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{M(x,y)}{N(x,y)}$

if it is exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad (M_y = N_x)$$

To solve the exact equation, we suppose that the Solution is

$$f(x,y) = C \quad \text{such that}$$

$$\frac{\partial f}{\partial x} = M(x,y), \quad \frac{\partial f}{\partial y} = N(x,y)$$

if $f(x,y) = C$, then $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

(Exact).

Ex: Solve

$$(\underbrace{\cos x \sin x - xy^2}_{M})\,dx + (\underbrace{y - yx^2}_{N})\,dy = 0$$

$M_y \overset{?}{=} N_x$ أول خطوة يجب أن نتأكد من أنها Exact

$M_y = 0 - 2xy$

$N_x = 0 - 2yx$

$\therefore M_y = N_x \quad \therefore$ Exact

Suppose the solution of this DE is $f(x,y) = C$ such that

[1] $\frac{\partial f}{\partial x} = M$

$$\int \frac{\partial f}{\partial x}dx = \int (\cos x \sin x - xy^2)\,dx$$

$$f(x,y) = \int \left(\frac{1}{2}\sin(2x) - xy^2\right)dx$$

$$= -\frac{1}{4}\cos(2x) - \frac{x^2y^2}{2} + g(y)$$

في تكملة السؤال.

$$f(x,y) = -\frac{1}{4}\cos(2x) - \frac{x^2y^2}{2} + g(y)$$

$$\frac{\partial f}{\partial y} = N \rightarrow \frac{\partial f}{\partial y} = y - yx^2$$

$$0 - x^2y + g'(y) = y - yx^2 \longrightarrow \cancel{-x^2y} + g'(y) = y - \cancel{yx^2}$$

$$\int g'(y) = \int y$$

[في هاي الخطوة عملنا على اشتقاق الـ $f(x,y)$ → بالنسبة لـ y]

$$g(y) = \frac{y^2}{2}$$

∴ the solution is

$$f(x,y) = C$$

$$-\frac{1}{4}\cos(2x) - \frac{x^2y^2}{2} + \frac{1}{2}y^2 = C$$

*Remark: if (DE) $M\,dx + N\,dy = 0$ is exact then the solution is:

$$\int M\,dx + \int \left(\text{الحدود التي لا تحتوي على } (x) \text{ في } (N)\right) dy = C$$

(or)

$$\int \left(\text{الحدود التي لا تحتوي على } y \text{ في } (M)\right) dx + \int N\,dy = C$$

EX: Solve $\underset{(M)}{(\cos x \sin x - xy^2)}\,dx + \underset{(N)}{(y - yx^2)}\,dy = 0$

(نفس السؤال يلي قبل ولكن طريقة مختصرة)

$$\int M\,dx + \int \left(\text{حدود لا تحتوي على } x \text{ في } N\right) dy = C$$

$$\int (\cos x \sin x - xy^2)\,dx + \int y\,dy = C$$

$$-\frac{1}{4}\cos(2x) - \frac{x^2y^2}{2} + \frac{y^2}{2} = C$$

Ex: Solve $\left(ye^{xy} + y^2 - \frac{y}{x^2} + x\right)dx + \left(xe^{xy} + 2xy + \frac{1}{x} + 3\right)dy$

$\underbrace{}_{M}$ $\underbrace{}_{N}$

$M_y = 1 \cdot e^{xy} + y \cdot e^{xy} \cdot x + 2y - \frac{1}{x^2} + 0$

$N_x = 1 \cdot e^{xy} + x\, e^{xy} \cdot y + 2y - \frac{1}{x^2} + 0$

$\therefore M_y = N_x \quad \therefore$ Exact

$\int M\, dx + \int$ الحدود الـ y لوحدها x في N $dy = C$

$\int \left(ye^{xy} + y^2 - \frac{y}{x^2} + x\right) dx + \int 3\, dy = C$

$\left[\frac{\cancel{y}e^{xy}}{\cancel{y}} + 0 + \frac{y}{x} + \frac{x^2}{2} + 3y = C\right]$

$\therefore e^{xy} + \frac{y}{x} + \frac{x^2}{2} + 3y = C$

Ex: Find the value of a & b which make

$\underbrace{(xy^3 + by^2)}_{M}\, dx + \underbrace{(ax^2y^2 + xy)}_{N}\, dy = 0$ exact

$\therefore$ exact $M_y = N_x$

$M_y = 3xy^2 + 2by \quad , \quad N_x = 2axy^2 + y$

$3xy^2 = 2axy^2 \quad , \quad 2by = y$

$3 = 2a \quad \quad 2b = 1$

$a = \frac{3}{2} \quad \quad b = \frac{1}{2}$

Remark: If $f(x,y) = 0$ or c is a solution to exact equation $M\,dx + N\,dy = 0$ then $M = \frac{\partial f}{\partial x}$, $N = \frac{\partial f}{\partial y}$

Ex: If $\cos(xy) - 2x + \ln y = e^{x} - 3y^{2} + 5$ is a solution for the exact eq. $M\,dx + N\,dy = 0$, then find the functions $M(x,y)$ & $N(x,y)$.

Soln:

$$\cos(xy) - 2x + \ln y - e^{x} + 3y^{2} - 5 = 0$$

$$M = \frac{\partial f}{\partial x}$$

$$= -\sin(xy) \cdot y - 2 + 0 - e^{x} + 0 + 0$$

$$N = \frac{\partial f}{\partial y} = -\sin(xy) \cdot x - 0 + \frac{1}{y} - 0 + 6y - 0$$

Remark: If $M\,dx + N\,dy = 0$ is exact eq. then

$\boxed{1}$ $M(x,y) = \int N_x \, dy + g(x)$

$\boxed{2}$ $N(x,y) = \int M_y \, dx + h(y)$

exact

$M_y = N_x$

$\int \frac{\partial M}{\partial y} dy = \int N_x \, dy$

$M = \int N(x)\, dy + g(x)$

Ex: Find the general function $N(x,y)$ so that $\left(ye^{xy} + y^{2} - \frac{y}{x^{2}}\right) dx + N(x,y)\, dy = 0$ exact

$$N(x,y) = \int M_y \, dx + g(y)$$

$$M_y = 1 \cdot e^{xy} + y \cdot e^{xy} \cdot x + 2y - \frac{1}{x^{2}}$$

$$\int \left(e^{xy} + \underbrace{yxe^{xy}}_{\text{اجزاء}} + 2y - \frac{1}{x^{2}}\right) dx + g(y)$$

$$\frac{e^{xy}}{y} + \cancel{y}x\frac{e^{xy}}{\cancel{y}} - \cancel{y}\frac{e^{xy}}{y^{2}} + 0 + \frac{1}{x} + g(y)$$

/	$\int$
$\oplus$ yx	e^{xy}
$\ominus$ y	$\frac{e^{xy}}{y}$
0	$\frac{e^{xy}}{y^{2}}$

Ex: Find the general function $M(x,y)$ so that

$$\underbrace{\left(4\,M(x,y) + 2xy\right)}_{M} dx + \underbrace{\left(\sec^2 y + \frac{x}{y}\right)}_{N} dy = 0 \quad \text{is exact}$$

$$M = \int N_x \, dy + g(x) \qquad N_x = 0 + \frac{1}{y}$$

$$4\,M(x,y) + 2xy = \int \frac{1}{y}\, dy + g(x)$$

$$4\,M(x,y) + 2xy = \ln|y| + g(x)$$

$$M(x,y) = \frac{1}{4}\left(\ln|y| + g(x) - 2xy\right)$$

* Special Integrating Factor :

If the eq $M(x,y)\,dx + N(x,y)\,dy = 0 \cdots (*)$ isn't exact ($M_y \neq N_x$), then

① If the function $\frac{M_y - N_x}{N}$ depends only on x, then the special integrating factor is

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

Morever, the eq $(*)$ becomes exact if it is multiply by $\mu(x)$.

② If $\frac{N_x - M_y}{M}$ depends on (y) only, then the special integrating factor is :

$$\mu(y) = e^{\int \frac{N_x - M_y}{M_y} dy}$$

The eq $(*)$ becomes exact if it is multiply by $\mu(y)$.

Ex: solve $y\,dx + 2x\,dy = ye^{y}\,dy$

Soln:

$$\underbrace{y}_{M}\,dx + (\underbrace{2x - ye^{y}}_{N})\,dy = 0$$

$M_y = 1$, $N_x = 2$ $\therefore$ Not exact $M_y \neq N_x$

$$\Rightarrow \frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y} \rightarrow$$ يعتمد فقط على (y)

$$\mu(y) = e^{\int \frac{1}{y} dy} = e^{\ln y} = y$$

↳ special integrating factor

⇒ multiply the DE by $\mu(y) = y$

$$y\,dx + 2x\,dy = ye^{y}\,dy$$

$$\left(y\,dx + (2x - ye^{y})\,dy = 0 \right) \times y$$

$$y^2\,dx + (2xy - y^2 e^{y})\,dy = 0$$

↳ $M_y = 2y$

$N_x = 2y - 0$ $\therefore$ Exact.

اذا حتى نعرف اذا حلنا صح

لازم تكون (Exact)

↳ $M_y = N_x$

The solution is

$$\int M\,dx + \int \text{الحدود التي لا تحتوي على } x \text{ في } N = C$$

$$\int y^2\,dx + \underbrace{\int -y^2 e^{y}\,dy}_{\text{(parts)}} = C$$

/	∫
⊕ $-y^2$	e^{y}
⊖ $-2y$	e^{y}
⊕ -2	e^{y}
0	e^{y}

$$\Rightarrow y^2 x + (-y^2 + 2y - 2)e^{y} = C$$

Ex: Solve $\underbrace{y}_{M}\,dx + \underbrace{(y^2-x)}_{N}\,dy = 0$

Soln:

$M_y = 1$ & $N_x = 0-1$ $\Rightarrow$ $\therefore$ Not exact $M_y \neq N_x$

$\rightarrow$ $\dfrac{N_x - M_y}{M} = \dfrac{-1-1}{y} = -\dfrac{2}{y}$ → يعتمد فقط على (y)

$\mu(y) = e^{\int -\frac{2}{y}\,dy} = e^{-2\ln y} = y^{-2} = \dfrac{1}{y^2}$

multiply the (DE) by $\mu(y) = \dfrac{1}{y^2}$, then we have:

$$\left(y\,dx + (y^2-x)\,dy = 0\right) * \frac{1}{y^2}$$

$$\frac{1}{y}\,dx + \left(1 - \frac{x}{y^2}\right)dy = 0$$

$M_y = \dfrac{-1}{y^2}$, $N_x = 0 - \dfrac{1}{y^2}$ $\therefore$ Exact $M_y = N_x$

$$\int \frac{1}{y}\,dx + \int 1\,dy = C$$

$$\frac{x}{y} + y = C$$

Ex: If $\mu(y) = y^k$ is an integrating factor to the non-exact eq.:

$$xy\,dx + (2x^2 + 3y^2 - 20)\,dy = 0$$

Find the value of (k).

$$\left(xy\,dx + (2x^2+3y^2-20)\,dy = 0\right) y^k$$

$$\underbrace{xy^{k+1}}_{M}\,dx + (\underbrace{2x^2y^k + 3y^{k+2} - 20y^k}_{N})\,dy = 0$$

$\therefore$ $M_y = N_x$ $\Rightarrow$ $(k+1)y^k x = 4xy^k$

$$k+1 = 4$$

$$[k = 3]$$

Remark: If $\mu(x,y) = x^p y^q$ is an integrating Factor to the non-exact (DE) $M\,dx + N\,dy = 0$, then Find p & q, we use condition

$$M_y - N_x = p\,\frac{N}{x} - q\,\frac{M}{y}$$

هاي المعادلة بس

ل $\mu(x,y) = x^p y^q$ طريقة مختصرة بدل ضرب $\mu(x,y)$ في (DE)

ثم نغيّر من non-exact ← exact

Ex: Find the integrating Factor $\mu(x,y) = x^p y^q$ to the Non-exact DE

$$(12 + 5xy)\,dx + (6xy^{-1} + 3x^2)\,dy = 0$$

$M_y = 0 + 5x$

$N_x = 6y^{-1} + 6x$

$$5x - 6y^{-1} - 6x = p\left(\frac{6xy^{-1} + 3x^2}{x}\right) - q\left(\frac{12 + 5xy}{y}\right)$$

$$-6y^{-1} - x = (6py^{-1} + p3x) - q12y^{-1} - 5qx$$

$-\cancel{x} = 3p\cancel{x} - 5q\cancel{x} \quad \rightarrow \quad (-1 = 3p - 5q) \times -2$

$-6\cancel{y^{-1}} = 6p\cancel{y^{-1}} - q12\cancel{y^{-1}} \quad \rightarrow \quad -6 = 6p - 12q$

$\Downarrow$

$2 = \cancel{-6p} + 10q$

$-6 = \cancel{6p} - 12q$

$-4 = -2q$

$\Leftarrow [q = 2]$

$-6 = 6(p) - 12(2)$

$-6 = 6p - 24 + 6$

$+6$

$-6p = -18$

$[p = 3]$

$\therefore \mu(x,y) = x^3 y^2$

$\mu(x,y)$

Ex: Find the integrating Factor for the non-exact eq

$$(\underbrace{3x^2+y}_{M})\,dx + (\underbrace{x^2y-x}_{N})\,dy = 0$$

Soln:

$My = 0 + 1$, $Nx = 2xy - 1$ $\quad\therefore$ Non exact ($My \neq Nx$)

$$\Rightarrow \frac{My - Nx}{N} = \frac{1 - 2xy + 1}{x(xy-1)} = \frac{2-2xy}{x(xy-1)} = \frac{-2\cancel{(xy-1)}}{x\cancel{(xy-1)}} = \frac{-2}{x}$$

يعتمد فقط على x

$$\mu(x) = e^{\int \frac{-2}{x}} = e^{-2\ln x} = x^{-2} = \boxed{\frac{1}{x^2}}$$

Ex: If the integrating factor $\mu(x) = x$ to the Non-exact DE $(x^2+y)\,dx + f(x)\,dy = 0$, Find the faction $f(x)$ such that $f(1) = 2$

Soln:

$$\left[(x^2+y)\,dx + f(x)\,dy = 0\right] * x$$

$$(x^3+yx)\,dx + x f(x)\,dy = 0$$

$My = 0 + x$

$Nx = x f'(x) + f(x)$

$$My = Nx$$

$$x = x f'(x) + f(x)$$

$$\int x = \int (x f(x))'$$

$$C + \frac{x^2}{2} = x f(x)$$

$$\therefore f(x) = \frac{x}{2} + \frac{C}{x}$$

$$\therefore f(1) = 2 \longrightarrow 2 = \frac{1}{2} + \frac{C}{1}$$

$$\left[C = \frac{3}{2}\right]$$

$$\therefore f(x) = \frac{x}{2} + \frac{3}{2x}$$

* Homogeneous Equation :

Def : the function $H(x,y)$ is said to be Homogeneous Function of order(n), if $H(tx, ty) = t^n H(x,y)$.

Ex: show that the function $G(x,y) = x^2 + 2xy + y^2$ is Homog. of order 2

$$G(tx, ty) = (tx)^2 + 2(tx)(ty) + (ty)^2$$
$$= t^2x^2 + 2t^2xy + t^2y^2$$
$$= t^2(x^2 + 2xy + y^2) = t^{2} G(x,y)$$

$n = 2$

order of the function.

Ex: Is the function $G(x,y) = x^2 + xy^2$ Homog ?

No, $G(tx, ty) = t^2x^2 + tx\,t^2y^2$
$$= t^2(x^2 + txy^2) \neq t^n G(x,y)$$

Remark: The (DE) $\frac{dy}{dx} = f(x,y)$ is Homog. eq if the function $f(x,y)$ can be written as
$$f(x,y) = h\left(\frac{y}{x}\right).$$

Def: The DE $M(x,y)dx + N(x,y)dy = 0$ is said to be homog. if $M(x,y)$ & $N(x,y)$ are homog. of the same order

Ex: $\underbrace{(x^2+y^2)}_{M}dx + (\underbrace{yx + 2y^2}_{N})dy = 0$, Is DE homog. ?

$M(tx, ty) = t^2x^2 + t^2y^2 = t^2(x^2+y^2)$

$N(tx, ty) = ty \cdot tx + 2t^2y^2 = t^2yx + 2t^2y^2 = t^2(yx + 2y^2)$

$\therefore$ $M(x,y)$ homog. & $N(x,y)$ homog. of order (2).

Ex: $\underbrace{(x^2+xy)}_{M}dx + (\underbrace{2xy+y^2+5}_{N})dy = 0$, Is DE homog. ??

$M(tx,ty) = t^2x^2 + t^2xy = t^2(x^2+xy)$ Homog.

$N(tx,ty) = 2t^2xy + t^2y^2 + 5$ Not Homog.

Ex: $\dfrac{dy}{dx} = \dfrac{4x^3 - xy^2}{2y^3 + x^2y}$, Is DE homog. ?? homog.

To solve the homog. eq $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ we use the substitution $\left[v = \dfrac{y}{x}\right]$ to transform the homog. into separably

Ex: Solve $x^2 dy - (x^2+xy+y^2)dx = 0$

Soln: homog. homog.

$$x^2 dy = (x^2+xy+y^2)dx$$

$$\frac{dy}{dx} = \frac{x^2+xy+y^2}{x^2}$$

take $v = \dfrac{y}{x}$

$xv = y$ نشتق طرفي المعادلة

بالنسبة لـ(x) $v + x\dfrac{dv}{dx} = \dfrac{dy}{dx}$

$$\frac{dy}{dx} = 1 + \frac{y}{x} + \left(\frac{y}{x}\right)^2$$

$$\cancel{v} + x\frac{dv}{dx} = 1 + \cancel{v} + v^2$$

$$x\frac{dv}{dx} = 1 + v^2$$

$$\int \frac{1}{1+v^2}dv = \int \frac{1}{x}dx \quad \text{(seprable)}$$

$$\tan^{-1}(v) = \ln|x| + C$$

$$\left(\tan^{-1}\left(\frac{y}{x}\right) = \ln|x| + C\right)\tan x$$

$$\frac{y}{x} = \tan(\ln|x| + C)$$

Ex: solve $\frac{dy}{dx} = \frac{2y + \sqrt{x^2 - y^2}}{2x}$

take $v = \frac{y}{x}$

$xv = y$

$v + x\frac{dv}{dx} = \frac{dy}{dx}$

$v + x\frac{dv}{dx} = \frac{2}{2}\left(\frac{y}{x}\right) + \frac{1}{2}\frac{\sqrt{x^2 - y^2}}{x}$

$v + x\frac{dv}{dx} = \frac{y}{x} + \frac{1}{2}\sqrt{1 - \left(\frac{y}{x}\right)^2}$

$\cancel{v} + x\frac{dv}{dx} = \cancel{v} + \frac{1}{2}\sqrt{1 - v^2}$

$\frac{dv}{dx} = \frac{1}{2x}\sqrt{1 - v^2}$

$\int \frac{1}{\sqrt{1 - v^2}}\, dv = \int \frac{1}{2x}\, dx$ (separable).

$\sin^{-1}(v) = \frac{1}{2}\ln|x| + C$

$\left(\sin^{-1}\left(\frac{y}{x}\right) = \frac{1}{2}\ln|x| + C.\right)$ sin

$\frac{y}{x} = \sin\left(\frac{1}{2}\ln|x| + C\right).$

Ex: Solve $x\, dy = y(\ln y - \ln x + 1)\, dx$

Soln: $x\, dy = \left(y\ln\left(\frac{y}{x}\right) + y\right) dx$ Homog.

$\frac{dy}{dx} = \frac{\left(y\ln\left(\frac{y}{x}\right) + y\right)}{x}$

take $v = \frac{y}{x}$

$xv = y$

$v + x\frac{dv}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{y}{x}\ln\left(\frac{y}{x}\right) + \frac{y}{x}$

$\cancel{v} + x\frac{dv}{dx} = v\ln v + \cancel{v}$

$\int \frac{1}{v\ln v}\, dv = \int \frac{1}{x}\, dx$ (separable).

$\ln|\ln v| = \ln|x| + C$

$e^{\ln\left|\ln\frac{y}{x}\right|} = e^{\ln|x| + C}$

$\left|\ln\frac{y}{x}\right| = |x| * C.$

Ex: Find the value of k & m so that the DE is homog.

$$(3\sqrt{x\,y^3} - 3m + 4)\,dx + x^2 y^K\,dy = 0$$

$$(3\, x^{\frac{1}{2}} \cdot y^{\frac{3}{2}} - 3m + 4)\,dx + x^2 y^K\,dy = 0$$

$$\frac{1}{2} + \frac{3}{2} = 2 + K$$

$$2 = 2 + K$$

$$K = 0$$

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& $3m + 4 = 0 \rightarrow m = \frac{4}{3}$

(H.W) ① solve $(2\sqrt{xy} - y)\,dx - x\,dy = 0$ Homog.

② solve $(x^2 e^{-\frac{y}{x}} + y^2)\,dx = xy\,dy$ Homog.