Case (3): If $\Delta = b^2 - 4ac < 0$ then we have complex roots say that $r = \alpha \pm \beta i$, where $i^2 = -1$ & we have the solution:

$$y_1 = e^{\alpha x}\cos(\beta x) \quad , \quad y_2 = e^{\alpha x}\sin(\beta x)$$

G.Soln: $y = c_1 y_1 + c_2 y_2$

$$= e^{\alpha x}\left(c_1\cos(\beta x) + c_2\sin(\beta x)\right)$$

Ex: Find the general soln. for

① $y'' - 4y' + 13y = 0$ $\qquad \Delta = b^2 - 4ac$

$r^2 - 4r + 13 = 0$ $\qquad = 16 - (4\times 1\times 13) = 16 - 52 = -36 < 0$

(complex)

$$r = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{-(-4) \pm \sqrt{-36}}{2} = 2 \pm 3i$$

$\therefore y_1 = e^{2x}.\cos(3x)$, $y_2 = e^{2x}\sin(3x)$

$\therefore$ G.Soln:-

$$y = e^{2x}\left(c_1\cos(3x) + c_2\sin(3x)\right)$$

② $y'' - 4y' = 0$

$r^2 - 4r = 0 \Rightarrow r(r-4) = 0$ $\qquad \therefore y_1 = e^{0x} = 1$

$r = 0, r = 4$ $\qquad y_2 = e^{4x} = e^{4x}$

G.Soln: $y = c_1 + e^{4x}c_2$

③ $y'' + 4y = 0$ $\qquad \alpha = 0$, $\beta = 2$

$r^2 + 4 = 0$ $\qquad y_1 = e^{0x}\cos(2x)$, $y_2 = e^{0x}\sin(2x)$

$r^2 = -4$ $\qquad y_1 = \cos(2x)$, $y_2 = \sin(2x)$

$r = \pm\sqrt{-4}$

$r = \pm 2i$ $\qquad \therefore y = c_1\cos(2x) + c_2(\sin 2x)$

Ex : Solve the (IVP) $y'' - 4y' - 5y = 0$ where $y(0) = -1$, $y'(0) = 4$

Soln :

$r^2 - 4r - 5 = 0$

$\Delta = b^2 - 4ac$

$= 16 - (-5 \times 4)$

$= (+)$

$(r+1)(r-5) = 0$

$r_1 = -1 \qquad r_2 = 5$

$y_1 = e^{-x} \qquad y_2 = e^{5x}$

G. Soln $\therefore y = C_1 e^{-x} + C_2 e^{5x}$

Since $y(0) = -1 \Rightarrow C_1 + C_2 = -1 \quad \dots (1)$

$y'(0) = 4 \Rightarrow y' = -C_1 + 5C_2 e^{5x}$

$4 = -C_1 + 5C_2 \quad \dots (2)$

(1) + (2)

$\cancel{C_1} + C_2 = -1$

$5C_2 - \cancel{C_1} = 4$

$\Rightarrow 6C_2 = 3 \quad , \quad C_1 = \frac{-3}{2}$

$C_2 = \frac{1}{2}$

$y = \frac{-3}{2} e^{-x} + \frac{1}{2} e^{5x}$

Ex : If $y_1 = xe^{3x}$ is a soln to DE $ay'' + (b-3)y' + Cy = 0$

Find (a, b, c).

$(r - r_1) \Rightarrow (r-3)(r-3) = 0$

$r^2 - 6r + 9 = 0$

$y'' - 6y' + 9y = 0$

$a = 1 \quad , \quad b - 3 = -6 \quad , \quad C = 9$

$b = -3$

- H.W : If $y_1 = e^{3x}$, $y_2 = e^{-x}$ are soln to DE $ay'' - by' + (C-2)y = 0$

Find (a, b, c).

$\alpha = -3$ $\quad \beta = 2$

Ex: If $e^{-3x}\sin(2x)$ is a soln to DE $y'' - by' - 2cy = 0$ Find b, c

Soln:

$r = \alpha \pm \beta i$

$r = -3 \pm 2i$

$(r+3)^2 = (\pm 2i)^2$ تربيع الطرفين

$(r+3)^2 = 4 * -1$

$r^2 + 6r + 13 = 0$

$y'' + 6y' + 13y = 0$

$-b = 6$

$[b = -6]$

$-2C = 13$

$\left(C = -\frac{13}{2}\right)$

* Euler Equation (second order & homog.)

This eq. has the form

$$ax^2y'' + bxy' + Cy = 0 \qquad , x > 0$$

To solve this (DE) we use the substutition $(x = e^t \leftrightarrow \ln x = t)$ then the above (DE) becomes:

$$\left(a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + Cy = 0\right)$$

Ex: Find the G. Soln to $x^2y'' + 4xy' + 2y = 0 \quad , x > 0$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + Cy = 0$$

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+1)(r+2) = 0$$

$r_1 = -1 \quad r_2 = -2$

$y_1(t) = e^{-t} = x^{-1}$

$y_2(t) = e^{-2t} = x^{-2}$

G.Soln

$y = C_1y_1 + C_2y_2$

$y = C_1x^{-1} + C_2x^{-2}$

Ex: Find the general solution

① $x^2y'' + 3xy' + y = 0 \quad , x>0$

Euler eq. $\Rightarrow$ $[\, x=e^t \iff \ln x = t \,]$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \rightarrow r^2+2r+1=0$$

$$(r+1)(r+1)=0$$

$$[r=-1] \quad [r=-1] \quad (\text{مكرر})$$

$y_1(t) = e^{-t} = x^{-1}$

$y_2(t) = te^{-t} = x^{-1}\ln x$

(G.Soln $\Rightarrow y = C_1x^{-1} + C_2x^{-1}\ln x$)

② $x^2y'' + xy' + 9y \quad , x>0$

Euler eq. $\Rightarrow$ $[x=e^{+t} \iff \ln x = t]$

$$a\frac{d^2y}{dt} + (b-a)\frac{dy}{dt} + cy = 0$$

$$\frac{d^2y}{dt} + 9y = 0$$

$$r^2+9=0 \Rightarrow r^2=-9 \quad \alpha \quad \beta$$

$$r = \pm\sqrt{-9} = \pm 3i$$

$y_1(t) = e^{0t}\cos(3t) \quad , \quad y_2 = e^{0t}\sin(3t)$

$\hookrightarrow y_1 = \cos(3\ln x) \quad , \quad y_2 = \sin(3\ln x)$

G.Soln $\Rightarrow y = C_1\cos(3\ln x) + C_2\sin(3\ln x)$.

③ $x^2y'' - xy' + 5y = 0$

Euler eq. $\Rightarrow$ $[x=e^{+t} \iff \ln x = t]$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0$$

$$\frac{d^2y}{dt^2} + -2\frac{dy}{dt} + 5y = 0 \rightarrow r^2-2r+5=0$$

$$\Delta = b^2-4ac = 4-(4\times1\times5) = -16<0$$

complex

$$r = \frac{2\pm\sqrt{-16}}{2} = \frac{\overset{\alpha}{1} \pm \overset{\beta}{2}i}{2}$$

$y_1(t) = e^t\cos(2t) \rightarrow x\cos(2\ln x)$

$y_2(t) = e^t\sin(2t) \rightarrow x\sin(2\ln x)$

G.S $\Rightarrow y = C_1x\cos(2\ln x) + C_2x\sin(2\ln x)$.

Ex: If $x^{-2}\cos(3\ln x)$ is a soln. to DE $ax^2y'' + bxy' + Cy = 0$

Find a, b, C

Euler eq. $\rightarrow [x = e^t \Leftrightarrow \ln x = t]$

$y_1 = x^{-2}\cos(3\ln x) \rightarrow y_1(t) = e^{-2t}\cos(3t)$

α, β

$r = \alpha \pm \beta i \rightarrow r = -2 \pm 3i$

$(r+2)^2 = (\pm 3i)^2$

$r^2 + 4r + 4 = -9 \Rightarrow r^2 + 4r + 13 = 0$

$\Delta = 16 - (4 \times 13 \times 1) = \ominus$

$r^2 + 4r + 13 = 0$

$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0 \rightarrow a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + Cy = 0$

$\therefore (a = 1)$, $b - 1 = 4 \rightarrow (b = 5)$, $(C = 13)$

$\therefore$ the Euler eq. $\rightarrow x^2y'' + 5xy' + 13y = 0$

Ex: If $x^4\ln x$ is a soln to DE $ax^2y'' + bxy' + Cy = 0$, $x > 0$

Find a, b, C $(x = e^t \leftrightarrow \ln x = t)$

$y_1(t) = x^4\ln x = te^{4t}$ (دليل على أنه مكرر)

$(r - r_1)(r - r_2) = 0$

$(r-4)(r-4) = 0$

$r^2 - 8r + 16 = 0$

$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} + 16y = 0$

$\left(a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + Cy\right)$

$\therefore$ the Euler eq is

$x^2y'' - 7xy' + 16y = 0$

$\therefore (a = 1)$, $b - 1 = -8$, $(C = 16)$

$(b = -7)$

⊛ Linear 2^{nd} order non-homog. with constant coefficients:

$$ay'' + by' + cy = g(x)$$

to solve this (DE)

① Find the homog. soln say y_1, y_2 & $y_h = c_1 y_1 + c_2 y_2$.

② Find the particular soln for non-homog. eq. say (y_p)

③ the g. soln is $(y = y_h + y_p)$

Remark: To find the particular soln we use (y_p)

① undermined coefficients.

② variation of parameters.

Methods (1): Undermined Coefficients:

To use this method the function $g(x)$ must be of the form: exponential, polynomial, or (sin, cos).

$g(x)$

poly. — $e^{\alpha x}$ — sin or cos.

$g(x)$		y_p
(polynomial)		
3	$\rightarrow$	$(A)\,x^m$
x	$\rightarrow$	$(Ax+B)\,x^m$
x^2+x	$\rightarrow$	$(Ax^2+Bx+C)\,x^m$

$m \Rightarrow$ عدد مرات ظهور (الصفر) في جذورها

$g(x)$ [exponential]		y_p
$e^{\alpha x}$	$\rightarrow$	$(Ae^{\alpha x})\,x^n$
$xe^{\alpha x}$	$\rightarrow$	$((Ax+B)\,e^{\alpha x})\,x^n$

$n \Rightarrow$ عدد مرات ظهور عدد (α) في جذور المعادلة التربيعية ،

$g(x)$ [$\cos(\beta x)$ or $\sin(\beta x)$]	y_p
$\cos(\beta x)$	$\rightarrow (A\cos(\beta x) + B\sin(\beta x))x^k$
$x\cos(\beta x)$	$\rightarrow ((Ax+B)\cos(\beta x) + (Cx+D)\sin(\beta x))x^k$
$e^{\alpha x}\cos(\beta x)$	$\rightarrow (Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x))x^k$
$xe^{\alpha x}\cos(\beta x)$	$\rightarrow ((Ax+B)e^{\alpha x}\cos(\beta x) + (Cx+D)e^{\alpha x}\sin(\beta x))x^k$

$K \Leftarrow$ عدد مرات ظهور العدد $(\alpha + \beta i)$ في جذور المعادلة المميزة

Ex: Find the general solution to $y'' - 2y' - 3y = 4x - 5 + 6xe^{3x}$

Soln: ① $y'' - 2y' - 3y = 0$

homog. $r^2 - 2r - 3 = 0$ $\qquad y_1 = e^{-x}$, $y_2 = e^{3x}$

$(r+1)(r-3) = 0$ $\qquad [\, y_h = C_1 e^{-x} + C_2 e^{3x} \,]$

$r_1 = -1$, $r_2 = 3$

② $g(x) = 4x - 5 + 6xe^{3x}$ $\qquad 4x - 5 \rightarrow x = \frac{5}{4}$

$y_p = (Ax+B)x^0 + ((Cx+D)e^{3x})x^1$

$[\, y_p = (Ax+B) + (Cx^2 + Dx)e^{3x} \,]$ The form of particular soln.

To find A, B, C & D

$y_p' = A + (2Cx+D)e^{3x} + (Cx^2+Dx) * 3e^{3x}$

$y_p'' = (2Cx+D) * 3e^{3x} + e^{3x}(2C) + (Cx^2+Dx) \cdot 9e^{3x} + 3e^{3x}(2Cx+D)$

(put y', y'', y in DE)

$e^{3x}(6Cx+3) + 2C + 9Cx^2 + 9Dx + 6Cx + 3D) + (-2A - 4Cx - 2D - 6Cx^2 - 6Dx)e^{3x}$
$-3((Ax+B) + (Cx^2+Dx)e^{3x}) = 4x - 5 + 6xe^{3x}$

الثابت $\quad -2A - 3B = -5 \rightarrow B = \frac{23}{9}$

معامل x $\quad -3A = 4 \rightarrow A = \frac{-4}{3}$

معامل e^{3x} $\quad 2C + 6D - 2D = 0 \rightarrow D = \frac{-3}{8}$

معامل xe^{3x} $\quad 12C - 4C = 6$

$\hookrightarrow C = \frac{6}{8} = \frac{3}{4}$

$$\therefore y_p = -\frac{4}{3}x + \frac{23}{9} + \left(\frac{3}{4}x^2 - \frac{3}{8}x\right)e^{3x}$$

G.Soln $\Rightarrow$ $y = y_h + y_p$

Ex: Find the general form of the particular soln to:

① $y'' + 4y = \cos(2x) + e^{2x} + 3$

① Homog.

$y'' + 4y = 0$ $\qquad y_1 = \cos(2x)$

$r^2 + 4 = 0 \rightarrow r = \pm\sqrt{-4}$ $\qquad y_2 = \sin(2x)$

$r = \pm 2i$ $\qquad y_h = \cos(2x)C_1 + C_2\sin(2x)$.

✱② $y_p = (A\cos(2x) + B\sin(2x))x^1 + (Ce^{2x})x^0 + (D)x^0$

$= xA\cos(2x) + Bx\sin(2x) + Ce^{2x} + D$

② $y'' + y = \sin x \cos(2x)$ • ملاحظة استخدام متطابقة

① $r^2 + 1 = 0$ $\qquad y_1 = \cos x$, $y_2 = \sin x$

$r^2 = -1$

$r = \pm\sqrt{-1} = \pm i$

② $g(x) = \sin x \cos(2x)$

$= \frac{1}{2}[\sin(-x) + \sin(3x)]$

$= -\frac{1}{2}\sin(x) + \frac{1}{2}\sin(3x)$

$$y_p = (A\cos x + B\sin x)x^1 + (C\cos(3x) + D\sin(3x))x^0$$

✱ $y_p = Ax\cos x + Bx\sin x + C\cos(3x) + D\sin(3x)$

③ $x^2 y'' + 3xy' - 8y = (\ln x)^3 - \ln x$, $x>0$

Euler eq.

① Homog. $x^2 y'' + 3xy' - 8y = 0$ $\rightarrow (x = e^t) \leftrightarrow (\ln x = t)$

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 8 = 0$$

$$r^2 + 2r - 8 = 0$$

$$(r+4)(r-2) = 0$$

$$(r_1 = -4) \quad (r_2 = 2)$$

$$y_1(t) = e^{-4t} \rightarrow x^{-4}$$

$$y_2(t) = e^{2t} \rightarrow x^2$$

② Non-Homog. $g(t) = t^3 - t$

$$y_p(t) = \left(A(t^3) + Bt^2 + Ct + D\right)t^0$$

$$\circledast \; y_{p_2}(x) = A(\ln x)^3 + B(\ln x)^2 + C\ln x + D.$$

④ $y'' + 4y = \cos^2 x$

① Homog. $r^2 + 4 = 0$

$$r = \pm\sqrt{-4} = \pm 2i$$

$y_1 = \cos(2x)$, $y_2 = \sin(2x)$

② Non-Homog.

$$g(x) = \cos^2(x) = \underbrace{\frac{1}{2}} + \underbrace{\frac{1}{2}\cos(2x)}.$$

$$\circledast \; y_p = A + \left(B\cos(2x) + D\sin(2x)\right)x$$

$$= A + Bx\cos(2x) + Dx\sin(2x)$$

⑤ $x^2y'' - xy' + y = 2x - 3\cos(\ln x)$.

① Euler eq (Homog)

$x^2y'' - xy' + y = 0$ $[x = e^t \longleftrightarrow \ln x = t]$

$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + 1 = 0$ $b - a = -1 - 1 = -2$

$y_1 = e^t = x$, $y_2 = e^t * (t) = \ln x (x)$

$r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$ $y_h = C_1 x + C_2 x \ln x$

$r_1 = 1$ $r_2 = 1$ (مكرر)

② Non-Homog.

$g(t) = 2e^t - 3\cos(t)$

$y_p(t) = (Ae^t)t^2 + (B\cos(t) + C\sin(t))x^0$

* $y_p(x) = Ax(\ln x)^2 + B\cos(\ln x) + C\sin(\ln x)$.

⑥ $y^{(4)} - 2y^{(3)} + y^{(2)} = 1 + e^x$

① Homog.

$y^{(4)} - 2y^{(3)} + y^{(2)} = 0$ $\therefore y_1 = 1$, $y_2 = x$

$r^4 - 2r^3 + r^2 = 0$ $y_3 = e^x$, $y_4 = xe^x$

$r^2(r^2 - 2r + 1) = 0$

$r^2 = 0$ $r^2 - 2r + 1 = 0$

$[r_1 = 0]$, $[r_2 = 0]$ $(r-1)(r-1) = 0$

$[r_3 = 1]$ $[r_4 = 1]$

② Non-Homog.

$G(x) = 1 + e^x$

* $y_p = (A)x^2 + (Be^x)x^2$

$= Ax^2 + Bx^2e^x$

⑦ $y''' + 4y'' + 3y' = x^2\cos x - 3xe^{-3x}$

① Homog.

$y^{(3)} + 4y^{(2)} + 3y' = 0$ $\quad\therefore\ y_1 = 1,\ y_2 = e^{-x},\ y_3 = e^{-3x}$

$r^3 + 4r^2 + 3r = 0$

$r(r^2 + 4r + 3) = 0$

$(r=0)\ (r+1)(r+3)$

$\quad 1 \qquad (r_2 = -1)\ (r_3 = -3)$

② Non - Homog

$g(x) = x^2\cos x - 3xe^{-3x}$

$y_p = \left((Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x\right)x^0 +$

$\left((Gx + W)e^{-3x}\right)x^1$

$* \ y_p = [(Ax^2 + Bx + C)\cos x + (Dx^2 + Ex + F)\sin x] + (Gx^2 + Wx)e^{-3x}$

⑧ $y^{(3)} - 2y^{(2)} + y^{(1)} = x + 2e^x - \cos(3x)$

① Homog.

$y^{(3)} - 2y^{(2)} + y^{(1)} = 0$ $\qquad y_1 = 1,\ y_2 = e^x,\ y_3 = xe^x$

$r^3 - 2r^2 + r = 0$

$r(r^2 - 2r + 1) = 0$

$r_1 = 0 \quad (r-1)(r-1)$

$\qquad r_2 = 1 \quad r_3 = 1$

② Non - Homog.

$G(x) = x + 2e^x - \cos(3x)$

$y_p = (Ax + B)x^1 + (Ce^x)x^2 + (D\cos(3x) + G\sin(3x))x^0$

$* \ y_p = Ax^2 + Bx + Cx^2e^x + D\cos(3x) + G\sin(3x)$

Ex: Find the Cauchy-Euler equation of second order having general soln :

$$y(x) = c_1 x + c_2 x \ln x + x(\ln x)^2$$

soln. $\rightarrow$ Euler eq. $\Rightarrow (x = e^t \leftrightarrow \ln x = t)$

$$\text{G.Soln} \Rightarrow \quad y(t) = \underbrace{(c_1 e^t + c_2 t e^t)}_{y_h} + \underbrace{(e^t \; t^2)}_{y_p}$$

G.Soln $\rightarrow$ $y = y_h + y_p$

Homog : $y_h = C_1 e^t + C_2 t e^t$

$r_1 = 1$, $r_2 = 1$ $\rightarrow$ $(r - r_1)(r - r_2) = 0$

$(r-1)(r-1) = 0$

$b-1 = -2$ $\quad r^2 - 2r + 1 = 0$

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = 0 \Leftarrow \frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$$

$\cancel{\frac{d^2y}{dt^2}}$ $\quad a = 1$, $b - a = -2$, $c = 1$

$b = -1$

Non-Homog:

$y_p(t) = e^t t^2$ is a particular soln. for $\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = g(t)$

$y'_p(t) = 2te^t + t^2 e^t$

$y''_p(t) = 2e^t + 2te^t + 2te^t + t^2 e^t$

$= 2e^t + 4te^t + t^2 e^t$

put y_p, y'_p, y''_p in DE

$$2e^t + 4te^t + t^2e^t - 2(2te^t + t^2e^t) + e^t t^2 = g(t)$$

$$2e^t + \cancel{4te^t} + \cancel{t^2e^t} - \cancel{4te^t} - \cancel{2t^2e^t} + \cancel{e^t t^2} = g(t)$$

$$2e^t = g(t)$$

$(g(x) = 2x)$

$\therefore$ Euler eq : $ax^2y'' + bxy' + cy = g(x)$

$$[x^2y'' - xy' + y = 2x]$$

Method (2): Variation of Parameters:

$$ay'' + by' + cy = f(x)$$

1. Find y_1, y_2 For Homog. DE : $ay'' + by' + cy = 0$

$$y_h = c_1 y_1 + c_2 y_2$$

2. Find $w[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

3. Find $v_1 = - \int \frac{g(x) \cdot y_2}{w} \, dx$

$$g(x) = \frac{f(x)}{a}$$

معامل y''

$$v_2 = \int \frac{g(x) \cdot y_1}{w} \, dx$$

4. The particular soln is :

$$y_p = y_1 \cdot v_1 + y_2 \cdot v_2$$

5. The general soln $y = y_h + y_p$

Ex: Find the general soln for:

$$y'' - 2y' + y = \frac{e^x}{x^2+1}$$

→ لا يمكن حلها على الطريقة الأولى

اجباري الحل على الطريقة الثانية

Soln:

① Homog. $r^2 - 2r + 1 = 0$

$(r-1)(r-1) = 0$ $\therefore y_1 = e^x, \ y_2 = xe^x$

$r_1 = 1 \quad r_2 = 1$ $y_h = c_1 e^x + c_2 xe^x$

② $w[y_1, y_2] = w[e^x, xe^x] = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - (xe^x \cdot e^x)$

$= \cancel{xe^{2x}} + e^{2x} - \cancel{xe^{2x}}$

$= e^{2x}$

③ $$v_1 = \ominus \int \frac{g(x)\, y_2}{w}\, dx \qquad g(x) = \frac{f(x)}{a} = \frac{\frac{e^x}{x^2+1}}{1}$$

$$= -\int \frac{\frac{e^x}{x^2+1} \cdot xe^x}{e^{2x}}\, dx \qquad * \; g(x) = \frac{e^x}{x^2+1}$$

$$= -\int \frac{x\cancel{e^{2x}}}{x^2+1} \cdot \frac{1}{\cancel{e^{2x}}}\, dx$$

$$= \underset{\times\frac{1}{2}}{-}\int \frac{x^{\times 2}}{x^2+1}\, dx = -\frac{1}{2}\ln|x^2+1| + C$$

$$v_2 = \int \frac{g(x)\, y_1}{w}\, dx$$

$$= \int \frac{\frac{e^x}{x^2+1} \cdot e^x}{e^{2x}}\, dx$$

$$= \int \frac{\cancel{e^{2x}}}{x^2+1} \cdot \frac{1}{\cancel{e^{2x}}}\, dx$$

$$= \int \frac{1}{x^2+1}\, dx = \tan^{-1}(x) + C$$

④ $$y_p = y_1 v_1 + y_2 v_2 = e^x \cdot -\frac{1}{2}\ln|x^2+1| + xe^x \tan^{-1}(x)$$

⑤ $$y = y_h + y_p$$

$$= c_1 e^x + c_2 x e^x - \frac{1}{2} e^x \ln|x^2+1 + x e^x \tan^{-1}(x)$$

Ex: Find the general soln for DE :

$$y'' + 4y = \tan(2x)$$

استخدام الطريقة الثانية

① Homog.

$r^2 + 4 = 0$ $\qquad y_1 = \cos(2x)$, $y_2 = \sin(2x)$

$r^2 = -4 \Rightarrow r = \pm 2i$ $\qquad y_h = C_1 \cos(2x) + C_2 \sin(2x)$

② $w[y_1, y_2] = w[\cos(2x), \sin(2x)] = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos 2x \end{vmatrix} = 2\cos^2(2x) + 2\sin^2(2x)$

$= 2 \times 1 = 2$

③ $v_1 = -\int \frac{(\tan(2x)) \cdot \sin(2x)}{2} dx$ $\qquad \frac{\sin(2x)}{\cancel{\cos(2x)}}$

$* \; g(x) = \frac{f(x)}{a} = \frac{\tan(2x)}{1} = \tan(2x)$

$= -\frac{1}{2} \int \frac{\sin^2(2x)}{\cos(2x)} dx$

$= -\frac{1}{2} \int \frac{1 - \cos^2(2x)}{\cos(2x)} dx$ (توزيع البسط على المقام)

$= \frac{-1}{2} \int (\sec(2x) - \cos(2x)) dx$

$= \frac{-1}{2} \left[\frac{1}{2} \ln|\sec(2x) + \tan(2x)| - \frac{1}{2} \sin(2x) \right]$

$= -\frac{1}{4} \ln|\sec(2x) + \tan(2x)| + \frac{1}{4} \sin(2x)$.

$V_2 = \int \frac{g(x) \cdot y_1}{w} dx = \int \frac{\tan(2x) \cdot \cos(2x)}{2} dx$

$= \frac{1}{2} \int \sin(2x) dx = \frac{1}{2} \left(\frac{-\cos(2x)}{2} \right) = \frac{-1}{4} \cos(2x)$

④ $y_p = y_1 v_1 + y_2 v_2 = \Big(\cos(2x) \cdot \frac{-1}{4} \ln|\sec(2x) + \tan(2x)| - \frac{1}{2}\sin(2x) +$
$\sin(2x) \cdot -\frac{1}{4} \cos(2x) \Big)$

⑤ $y = y_h + y_p = C_1 \cos(2x) + C_2 \sin(2x) +$

$y_1 \; , y_2$

Ex: let $L[y](x) = y'' + P(x)y' + q(x)y$ & $\{x^{-6}, x^{-1}\}$ be a fundamental soln set to the DE $L[y](x) = 0$

If $y_p = x^{-6}u_1 + x^{-1}u_2$ is a particular soln for DE $L[y](x) = \frac{e^x}{x^2}$, find u_2

$f(x)$

Soln: ① $y_1 = x^{-6}$, $y_2 = x^{-1}$

② $w\{y_1, y_2\} = \begin{vmatrix} x^{-6} & x^{-1} \\ -6x^{-7} & -x^{-2} \end{vmatrix} = -x^{-8} - -6x^{-8}$
$= -x^{-8} + 6x^{-8} = 5x^{-8}$

③ $g(x) = \frac{f(x)}{a} = \frac{e^x}{x^2}$

$$u_2 = \int \frac{g(x) \cdot y_1}{w} dx = \int \frac{\frac{e^x}{x^2} \cdot x^{-6}}{5x^{-8}} dx$$

$$= \int \frac{e^x x^{-8}}{5x^{-8}} dx = \frac{1}{5} \int e^x dx = \frac{1}{5} e^x$$

Theorm: If y_1 is a soln for DE: $L[y](x) = ay'' + by' + cy = f_1(x)$ & y_2 is a soln for DE: $L[y]_{(x)} = ay'' + by' + cy = f_2(x)$ then $y = c_1 y_1 + c_2 y_2$ is a soln for (DE): $L[y]_{(x)} = ay'' + by' + cy = C_1 f_1(x) + C_2 f_2(x)$

$f_1(x)$

Ex: If $y_1 = \frac{3}{2}x - \frac{9}{4}$ is a soln for (DE) $y'' + 3y' + 2y = 3x$ & $y_2 = \frac{e^{3x}}{2}$ is a soln for (DE) $y'' + 3y' + 2y = 10e^{3x} \rightarrow f_2(x)$ Find the soln for the (DE) $y'' + 3y' + 2y = \underbrace{-9x + 20e^{3x}}_{f(x)}$

Soln:

$f(x) = c_1 f_1(x) + C_2 f_2(x)$

$-9x + 20e^{3x} = C_1(3x) + C_2(10e^{3x})$

$-9 = 3C_1 \qquad 20 = 10C_2$

$(C_1 = -3) \qquad (C_2 = 2)$

$\therefore$ the soln is

$y_1 = c_1 y_1 + C_2 y_2$

$= -3\left(\frac{3}{2}x - \frac{9}{4}\right) + 2\frac{e^{3x}}{2}$

EX = Find the general soln for DE

(انتباه) $y'' - 2y' - 5 = 0$ (Non-Homog)

$\hookrightarrow y'' - 2y' = 5$

① $r^2 - 2r = 0$ $\qquad y_1 = e^0 = 1 \quad , y_2 = e^{2x}$

$r(r-2) = 0$

$r_1 = 0 \qquad r_2 = 2$

② $g(x) = 5 \rightarrow y_p = (A)x^1 = AX$

③ $y_p = AX$

$y'_p = A \quad , y''_p = 0$

$0 - 2A = 5 \Rightarrow A = \frac{-5}{2}$

$\therefore y_p(x) = \frac{-5}{2} X$

Ex: If $w[2, x, f(x)] = 8x$, find the function $f(x)$ where $f(0) = 2$, $f'(0) = 3$

$\therefore f(x) = \frac{2}{3}x^3 + 3X + 2$

$w[2, x, f(x)] = 8x$

$$= \begin{vmatrix} 2 & x & f(x) \\ 0 & 1 & f'(x) \\ 0 & 0 & f''(x) \end{vmatrix} = 8X$$

$= 2 \times 1 \times f''(x) = 8X$

$f''(x) = 4X \rightarrow \int f''(x) = \int 4x \, dx$

$f'(0) = 3 \rightarrow 3 = C_1$ $\qquad C_1$ نعوض $\leftarrow \quad \int f'(x) = \int (2x^2 + C_1)dx$ $\quad 3$

$f(0) = 2 \rightarrow 2 = 0 + 0 + C_2$ $\qquad C_2$ نحسب $\leftarrow \quad f(x) = \frac{2x^3}{3} + 3X + C_2$

Ex: If $W[x, y(x)] = x^2 e^x$, find the function $y(x)$ where $y(1) = e^2$

Soln: $w[x, y(x)] = \begin{vmatrix} x & y(x) \\ 1 & y'(x) \end{vmatrix} = x^2 e^x$

$xy' - y = x^2 e^x$ (linear in y)

$\hookrightarrow y' - \frac{1}{x} y = x e^x$

$\mu(x) = e^{\int \frac{-1}{x} dx}$
$= e^{-\ln x} = \frac{1}{x}$

$y = x \int \frac{1}{\cancel{x}} \cdot \cancel{x} e^x \, dx + \underline{Cx}$

$= x \int e^x \, dx + Cx \quad = x e^x + Cx$

Since $y(1) = e^2$

$e^2 = e + C \Rightarrow C = e^2 - e$

Remark: The (DE) $ay'' + by' + cy = 0$ can be written as $L[y]_{(x)} = 0$, where $L[y]_{(x)} = ay'' + by' + cy$ is a linear operator

Ex: If $L[y](x) = 3y'' - 4y' + 5y$ Find $L[y](x)$ where $y = x^3 - 4x^2 + 1$

$\hookrightarrow y' = 3x^2 - 8x$
$y'' = 6x - 8$

$L[y]_{(x)} = 3(6x-8) - 4(3x^2 - 8x) + 5(x^3 - 4x^2 + 1)$
$= 18x - 24 - 12x^2 + 32x + 5x^3 - 20x^2 + 5$
$= 5x^3 - 32x^2 + 50x - 19$

Ex: If $L(y)_{(x)} = y'' - 2y' + 3y$ then Find the G.Soln for $L[y](x) = 0$

Soln: $y'' - 2y' + 3y = 0$
$r^2 - 2r + 3 = 0$ . . . كمّل