(H.W) ① $L^{-1}\left[\dfrac{s+8}{(s+3)^2+81}\right]$

② $L^{-1}\left[\dfrac{s}{s^2-4s+5}\right]$.

③ $L^{-1}\left[\dfrac{2}{s^2+3s-4}\right]$.

④ $L^{-1}\left[\dfrac{3}{(s^2+1)(s^2+5)}\right]$.

⑤ $L^{-1}\left[\ln\left(\dfrac{s^2}{s^2+16}\right)\right]$.

⑥ $L^{-1}\left[\dfrac{2}{s^3+3s^2+2s}\right]$

Unit - Step Function :

$$u(t-a) = \begin{cases} 0 & 0 \leq t < a \\ 1 & t \geq 0 \end{cases}$$

Remark :

$\boxed{1}$ $L[u(t-a)] = \dfrac{e^{-as}}{s}$

$\boxed{2}$ $L[u(t-a)\,f(t)] = e^{-as}\,L\{f(t+a)\}$

$\boxed{3}$ $L^{-1}[e^{-as}F(s)] = u(t-a)\cdot L^{-1}\{F(s)\}_{t \to t-a}$

Ex: Evaluate :

① $\mathcal{L}\{u(t-3)\} = \frac{e^{-3s}}{s}$

② $\mathcal{L}\{(u(t-4))^{20}\} = L\{u(t-4)\} = \frac{e^{-4s}}{s}$

ممكن هدول يكونوا
$(\mathcal{L}^{-1})$
يعني الremark بالعكس

③ $\mathcal{L}\{u(t-2)t^2\} = e^{-2s} L\{(t+2)^2\} = e^{-2s} L\{t^2+4t+4\}$

$$= e^{-2s}\left[\frac{2!}{s^3} + 4\cdot\left(\frac{1!}{s^2}\right) + \frac{4}{s}\right]$$

$$= e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$$

.(استخدام متطابقات)

④ $L\left[u\left(t-\frac{\pi}{6}\right)\cos t\right] = e^{-\frac{\pi}{6}s} L\left\{\cos\left(t+\frac{\pi}{6}\right)\right\}$

$$= e^{-\frac{\pi}{6}s} L\left[\cos t\cdot \overset{\frac{\sqrt{3}}{2}}{\cos\frac{\pi}{6}} \mp \sin t\ \underline{\underline{\overset{\frac{1}{2}}{\sin\frac{\pi}{6}}}}\right]$$

$$= e^{-\frac{\pi}{6}s}\left[\frac{\sqrt{3}}{2} L\{\cos t\} - \frac{1}{2} L\{\sin t\}\right]$$

$$= e^{-\frac{\pi}{6}s}\left[\frac{\sqrt{3}}{2}\cdot\frac{s}{s^2+1} - \frac{1}{2}\cdot\frac{1}{s^2+1}\right]$$

⑤ $\mathcal{L}\{u(t-2)e^{3t}\} = e^{-2s} L\{e^{3(t+2)}\} = e^{-2s} L\{e^{3t}\cdot \underline{\underline{\overset{\text{ثابت}}{e^{6}}}}\}$

يمكن الحل بطريقتان

$$= e^{-2s}\cdot e^{6} L\{e^{3t}\} = e^{6-2s}\cdot\frac{1}{s-3} = \frac{e^{6-2s}}{s-3}$$

⑥

.الطريقة الثانية لسؤال (5)

$$\mathcal{L}\{u(t-2)e^{3t}\} = \mathcal{L}\{u(t-2)\}\Big|_{s\to s-3}$$

$$= \frac{e^{-2s}}{s}\Bigg|_{s\to s-3} = \frac{e^{-2(s-3)}}{s-3} = \frac{e^{-2s+6}}{s-3}$$

⑥ $\mathcal{L}^{-1}\left[\frac{(s-1)}{\underset{-}{e}^{2s}(s-1)^2+16\underset{-}{e}^{2s}}\right] = L^{-1}\left[\frac{(s-1)e^{-2s}}{(s-1)^2+16}\right]$

$= u(t-2)\left[\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+6}\right\}\right]_{t\to t-2} = u(t-2)\left[e^{t} L^{-1}\left[\frac{s}{s^2+16}\right]\right]_{t\to t-2}$

$= u(t-2)\left[e^{t}\cos 4t\right]_{t\to t-2}$

$= u(t-2)\left[e^{t-2}\cos(4t-8)\right]$

Ex: let :

$$g(t) = \begin{cases} t & 0<t<2 \\ 3 & 2<t<5 \\ e^{-4t} & 5<t \end{cases}$$

$\xleftarrow{u}$ $\xleftarrow{u}$

ملاحظة: ① لازم الفترات تكون مرتبة ترتيب تصاعدي ⇐

② اول فترة لازم تكون t أقل من عدداً أكبر من صفر

① write the function g(t) in terms of unit step function.

② Find $\mathcal{L}\{g(t)\}$.

Soln ⇒

① $g(t) = t + u(t-2)(3-t) + u(t-5)(e^{-4t}-3)$

② $L\{g(t)\} = \mathcal{L}\{t\} + \mathcal{L}\{u(t-2)(3-t)\} + L\{u(t-5)(e^{-4t}-3)\}$.

$= \frac{1}{s^2} + e^{-2s} L\{3-(t+2)\} + e^{-5s} L\{e^{-4(t+5)} - 3\}$

$= \frac{1}{s^2} + e^{-2s}\left[\frac{1}{s} - \frac{1}{s^2}\right] + e^{-5s}\left[\frac{e^{20}}{s+4} - \frac{3}{s}\right]$

- Find $L\{g(t)\}$ where

$$g(t) \begin{cases} 0 & 0<t<1 \\ t & ①<t<2 \\ 0 & 2<t<2\pi \\ \cos t & 2\pi<t \end{cases}$$

$\Rightarrow g(t) = 0 + u(t-1)(t-0) + u(t-2)(0-t) + u(t-2\pi)(\cos t-0)$

ونكمل الحل.

(H.W) ⇒ Evaluate

① $\mathcal{L}\{u(t-5)t^2 + u(t-5)\sin t\}$

② $\mathcal{L}^{-1}\left\{\frac{2}{e^{3s}(s-7)^{10}}\right\} = \mathcal{L}^{-1}\left\{\frac{2e^{-3s}}{(s-7)^{10}}\right\}$

$= 2u(t-3)\left[\mathcal{L}^{-1}\left[\frac{1}{(s-7)^{10}}\right]\right]_{t\to t-3}$

$= 2u(t-3)\left[e^{7t}\,\mathcal{L}^{-1}\left[\frac{1}{s^{10}}\right]\right]_{t\to t-3}$

$= 2u(t-3)\,e^{7t-21}\,\frac{(t-3)^9}{9!}$

Solving Initial Value problem:

Remark:

① $L\{y^{(n)}\} = s^n L\{y\} - s^{n-1}y(0) - s^{n-2}y'(0) - \dots - y^{(n-1)}(0)$.

(n=1)

② $L\{y'\} = s\,L\{y\} - y(0)$

③ $L\{y''\} = s^2\{y\} - s\,y(0) - y'(0)$

(n=2)

To solve an IVP by Laplace transforms

① Take the laplace transform of both sides of DE.

② Find $L\{y(t)\} = Y(s)$

③ Take $\mathcal{L}^{-1}$ to both sides in part (2).

Ex: Use laplace transform to solve the I.V.P.

$$\ddot{y} - 4\dot{y} + 4y = t^3 e^{2t} \quad , \; y(0)=0 \; , \; \dot{y}(0)=0$$

Soln: $L\{\ddot{y}\} - 4L\{\dot{y}\} + 4L\{y\} = L\{t^2 e^{2t}\}$

$$s^2 L\{y\} - \overset{=0}{\cancel{s\,y(0)}} - \overset{=0}{\cancel{\dot{y}(0)}} - 4\left(s\,L\{y\} - \overset{=0}{\cancel{y(0)}}\right) + 4\,L\{y\} = L\{t^3\}\Big|_{s \to s-2}$$

$$s^2 L\{y\} - 4s\,L\{y\} + 4L\{y\} = \frac{3!}{s^4}\Bigg|_{s \to s-2}$$

$$(s^2 - 4s + 4)\,L\{y\} = \frac{6}{(s-2)^4}$$

$$L\{y\} = \frac{6}{(s-2)^4(s^2-4s+4)} = \frac{6}{(s-2)^6}$$

$$Y(s) = L\{y(t)\} = \frac{6}{(s-2)^6}$$

Now ;

$$L^{-1}\{Y(s)\} = y(t) = L^{-1}\left\{\frac{6}{(s-2)^6}\right\}$$

shifting

$$y(t) = 6e^{2t}\,L^{-1}\left[\frac{1}{s^6}\right]$$

$$y(t) = 6e^{2t} \cdot \frac{t^5}{\boxed{5!} \rightarrow 120} = \frac{1}{20}\,t^5 e^{2t} .$$

Ex: Use laplace to solve the I.V.P

$$y'' - 6y' + 13y = 0 \quad , y(0) = 0 \quad , y'(0) = -3$$

Soln:

$$L\{y''\} - 6L\{y'\} + 13L\{y\} = 0$$

$$s^2 L\{y\} - s\cancel{y(0)} - \overset{=-3}{y'(0)} - 6(sL\{y\} - \cancel{y(0)}) + 13(L\{y\}) = 0$$

$$s^2 L\{y\} \cancel{+3} - 6s L\{y\} + 13L\{y\} = 0 \quad -3$$

$$(s^2 - 6s + 13)\, L\{y\} = -3 \Rightarrow L\{y\} = \frac{-3}{s^2 - 6s + 13}$$

Now;

$$Y^{-1}(s) = y(t) = L^{-1}\left\{\frac{-3}{s^2 - 6s + 13}\right\}$$

اكمال مربع $\nearrow$ $s^2 - 6s + \underset{\downarrow}{\underline{\underline{9}}} - 9 + 3$

$\rightarrow (s-3)^2 + 4 \qquad 9 = \left(-\frac{6}{2}\right)^2$

$$y(t) = -3e^{3t} L^{-1}\left\{\frac{1}{s^2 + 4}\right\}$$

$$= -3e^{3t} \frac{\sin(2t)}{2}$$

$$y(t) = \frac{-3}{2} e^{3t} \sin(2t)$$

Ex: Use laplace method to solve the I.V.P

$$\dot{y} - 2y = g(t) \quad , \quad y(0) = 0$$

where $g(t) = \begin{cases} 0 , & 0 \le t < 1 \\ 2 , & t > 1 \end{cases}$

Soln:

⊙ $g(t) = 0 + u(t-1)(2-0)$

$\therefore g(t) = 2\,u(t-1)$

$$L\{\dot{y}\} - 2\,L\{y\} = 2L\{u(t-1)\}$$

$$s\,L\{y\} - \overset{=0}{\cancel{y(0)}} - 2\,L\{y\} = 2L\{u(t-1)\}$$

$$(s-2)\,L\{y\} = \frac{2e^{-s}}{s}$$

$$Y(s) = L\{y\} = \frac{2e^{-s}}{s(s-2)}$$

$\boxed{\text{Now}}$

$$Y^{-1}(s) = y(t) = L^{-1}\left\{\frac{2e^{-s}}{s(s-2)}\right\}$$

كسور جزئية

$$= 2u(t-1)\left[L^{-1}\left\{\frac{1}{s(s-2)}\right\}\right]_{t \to t-1}$$

$$\frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$1 = A(s-2) + B(s)$

$\boxed{s=0}$

$1 = -2A \Rightarrow A = \frac{-1}{2}$

$\boxed{s=2}$

$1 = 2B \Rightarrow B = \frac{1}{2}$

$$= 2u(t-1)\left[L^{-1}\left\{\frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-2}\right\}\right]_{t \to t-1}$$

$$= 2 \cdot \frac{1}{2}\,u(t-1)\left[L^{-1}\left\{\frac{-1}{s} + \frac{1}{s-2}\right\}\right]_{t \to t-1}$$

$$\therefore y(t) = u(t-1)\left[-1 + e^{2t}\right]\Big|_{t \to t-1}$$

$$= u(t-1)\left[-1 + e^{2t-2}\right]$$

(فكرة جديدة)

Ex: Use laplace to solve the (IVP)

$$y'' + ty' - 2y = -2 \qquad y(0)=0 \text{ ، } y'(0)=0$$

$$L\{y''\} + \underline{L\{ty'\}} - 2L\{y\} = L\{-2\}$$

$$s^2 L\{y\} - \cancel{sy(0)} - \cancel{y'(0)} + (-1)^1 \frac{d}{ds}\left[L\{y'\}\right] - 2L\{y\} = \frac{-2}{s}$$

$$s^2 L\{y\} - \frac{d}{ds}\left[sL\{y\} - \cancel{y(0)}\right] - 2L\{y\} = \frac{-2}{s}$$

$$s^2 Y(s) - \frac{d}{ds}\left[sY(s)\right] - 2Y(s) = -\frac{2}{s}$$

$$s^2 Y(s) - \left[Y(s) + sY'(s)\right] - 2Y(s) = -\frac{2}{s}$$

$$\frac{-sY'(s)}{-s} + \frac{(s^2-3)Y(s)}{-s} = \frac{-2}{s} \cdot \frac{1}{-s}$$ ⇒ linear (1) order.

$$\mu(s) = e^{\int \frac{-s^2+3}{s} ds} = e^{-\frac{s^2}{s}+3\ln s} = s^3 e^{-\frac{s^2}{2}}$$

$$Y(s) = \frac{1}{s^3 e^{-\frac{s^2}{s}}}\left[\int \frac{2}{\cancel{s^2}} \cdot \overset{s}{\cancel{s^3}} e^{-\frac{s^2}{2}} ds + C\right]$$

$$= \frac{e^{\frac{s^2}{2}}}{s^3}\left(-2e^{-\frac{1}{2}s^2}\right) + \frac{Ce^{\frac{s^2}{2}}}{s^3}$$

$$Y(s) = \frac{-2}{s^3} + \frac{Ce^{s^2/2}}{s^3}$$ take $C=0$ because

$$\lim_{s\to\infty} L\{y\}_{(s)} = 0$$

$$Y(s) = \frac{-2}{s^3}$$

Now;

$$y(t) = L^{-1}\left(\frac{-2}{s^3}\right) = -t^2$$

H.W : Use laplace transform to solve the IVP

① $y'' + 2y' + 2y = e^{-t}$, $y(0) = 0$, $y'(0) = 1$

② $y' + y = f(t)$, $y(0) = 0$, $f(t) = \begin{cases} 0 & t<2 \\ 5 & t>2 \end{cases}$

③ $y'' + ty' - 2y = -2$ $y(0) = 0$, $y'(0) = 0$ ⇒ (محلولة)

Chapter (6): Series Solution of DE's:

Def: ① A Taylor series of a function $f(x)$ about $x = x_0$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}$$

② A maclurin Series of $f(x)$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(0)\, x^n}{n!}$$

[Theorem]: If $P_1(x), \dots, P_n(x), g(x)$ are continuous at x_0 & $y(x)$ is a function that has derivatives of all orders at x_0 & $y(x)$ is soln for the (IVP):

$$y^{(n)} + P_1(x)\, y^{(n-1)} + \dots + P_n(x)\, y = g(x) \quad , \text{where}$$

$$y(x_0) = y_0 \quad , y'(x_0) = y_1 \quad , \dots , \quad y^{(n-1)}(x_0) = y_{n-1}$$

Then y has a power series expansion in the form:

$$y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)(x-x_0)^n}{n!}$$

$$= y(x_0) + y'(x_0)(x-x_0) + \frac{y''(x_0)(x-x_0)^2}{2!} + \dots$$

Ex: Find at least the {Five} non zero terms in a power Series expansion of the soln of intial value problem (IVP)

$$y'' + y' - xy = 3 \qquad y(0) = 1 \;,\; y'(0) = -2$$

Soln: $x_0 = 0$ ← لانه لا تساوي صفر اذاً محسوب

- $y'' + y' - xy = 3$ → put in this DE $x = 0$

$y''(0) + -2 - 0 = 3$

$y''(0) = 5$ ⇒

- نشتقه $y'' + y' - xy = 3$

$$y''' + y'' - y - xy' = 0$$

$$y'''(0) + \overset{\nearrow 5}{y''(0)} - \overset{\nearrow 1}{y(0)} - \cancel{0 * y'(0)} = 0$$

$$y'''(0) = -4$$

- نشتقه $y''' + y'' - y - xy' = 0$

$$y^{(4)} + \overset{\nearrow -4}{y^{(3)}} - \overset{\nearrow -2}{y'} - (x\overset{\nearrow 5}{y''} + \overset{\nearrow -2}{y'}) = 0$$ * نعوض

$$y^{(4)}(0) = 0$$

- نشتقه $y^{(4)} + y^{(3)} - y' - xy'' - y' = 0$

$$y^{(5)} + \underset{\swarrow 0}{y^{(4)}} - \underset{\swarrow 5}{y''} - \underset{\swarrow 0}{x}\underset{\swarrow -4}{y'''} - \underset{\swarrow 5}{y''} - \underset{\swarrow 5}{y''} = 0$$ نعوض *

$$y^{(5)} - 5 - 5 - 5 = 0 \Rightarrow y^{(5)}(0) = 15$$

$$\therefore\ y = \sum_{n=0}^{\infty} \frac{y^{(n)}(0)\, x^{n}}{n!}$$

$$= y(0) + y'(0)x + \frac{y''(0)\,x^2}{2!} + \frac{y'''(0)\,x^3}{3!} + \frac{y''''(0)\,x^4}{4!} + \frac{y^{(5)}(0)\,x^5}{5!}$$

$$= 1 + -2x + \frac{5}{2}x^2 - \frac{4}{6}x^3 + 0 + \frac{15}{120}x^5$$

Ex: Find the first 4 - Non zero terms in a power series expansion of the soln to (IVP)

$$y'' - xy = 5 + 6x + 3x^3 \quad , y(-1) = 1 \quad , \underline{\underline{y'(-1) = 0}}$$

مهمل

- $y''(-1) - (-1)y(-1) = 5 + 6(-1) + 3(-1)^3$

$$\left[y''(-1) = -5 \right]$$

- نشتق $\Rightarrow$ $y'' - xy = 5 + 6x + 3x^3$

$$y''' - xy' - y = 6 + 9x^2$$

$$y''(-1) - (-1)(y'(-1)) - y(-1) = 6 + 9(-1)^2$$

$$\left[y'''(-1) = 16 \right]$$

- نشتق $\Rightarrow$ $y''' - xy' - y = 6 + 9x^2$

$$y^{(4)} - y' - xy'' - y' = 18x$$

$$y^{(4)} - 2y' - xy'' = 18x$$

نعوض

$$\left[y^{(4)}(-1) = -13 \right]$$

$$\therefore y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)(x - x_0)^n}{n!} =$$

$$\sum_{n=0}^{\infty} \frac{y^{n}(-1)(x+1)^n}{n!}$$

$$= y(-1) + \frac{y'(-1)(x+1)}{1!} + \frac{y''(-1)(x+1)^2}{2!} + \frac{y'''(-x)(x+1)^3}{3!} +$$

$$\frac{y^{(4)}(-1)(x+1)^4}{4!}$$

$$= 1 + 0 - \frac{5}{2}(x+1)^2 + \frac{16}{6}(x+1)^3 - \frac{13}{24}(x+1)^4 .$$

Ex: Solve the DE (using series)

$$y'' - y = 0 \quad \text{about } x = 1 \nearrow x_0 = 1$$

- $x_0 = 1$, $y(1) = a$, $y'(1) = b$

- $y''(1) - y(1) = 0$

  $y''(1) - a = 0 \rightarrow [y''(1) = a]$

- نشتق $y'' - y = 0$

  $y''' - y' = 0$

  $\hookrightarrow y'''(1) - b = 0 \rightarrow [y'''(1) = b]$

- نشتق $y^{(4)} - y'' = 0$

  $y^{(4)} - y''(1) = 0 \rightarrow [y^{(4)}(1) = a]$

- نشتق $y^{(5)} - y^{(3)} = 0$

  $[y^{(5)} = b]$

$$\Rightarrow y = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)(\overset{x_0=1}{x - x_0})^4}{n!}$$

$$= y(1) + \frac{y'(1)(x-1)^1}{1!} + \frac{y''(1)(x-1)^2}{2!} + \frac{y^{(3)}(1)(x-1)^3}{3!} +$$

$$\frac{y^{(4)}(1)(x-1)^4}{4!} + \frac{y^{(5)}(x-1)^5}{5!}$$

$$= a + b(x-1) + \frac{a}{2}(x-1)^2 + \frac{b}{6}(x-1)^3 + \frac{a}{24}(x-1)^4 + \frac{b}{120}(x-1)^5$$

$$= a\left(1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{24}\right) + b\left((x-1) + \frac{(x-1)^3}{6} + \frac{(x-1)^5}{120}\right)$$

$$a \sum_{n=0}^{\infty} \underset{\text{زوجي}}{\frac{(x-1)^{2n}}{(2n)!}} + b \sum_{n=0}^{\infty} \underset{\text{فردي}}{\frac{(x-1)^{2n-1}}{(2n-1)!}}$$

H·W : Find the first 5-Non zero terms in a power series expansion of the soln to DE:

① $y'' - xy = 5 + 6x - 2x^3$, $y(0) = 1$, $y'(0) = 0$

② $y'' - xy = 5 + 6x - 2x^3$ about $x = -1$.