* Equation of form $\frac{dy}{dx} = G(ax + by + c)$

To solve the eq. $\frac{dy}{dx} = G(ax+by+c)$ we use the substitution $v = ax+by+c$ which reduce the above DE into separable.

Ex: solve : $\frac{dy}{dx} = 2 + \sqrt{y-2x+3}$

$$v = y-2x+3$$

$$v+2x-3 = y$$

$$\frac{dv}{dx} + 2 = \frac{dy}{dx}$$

$$\frac{dv}{dx} + \cancel{2} = \cancel{2} + \sqrt{v}$$

$$\int \frac{1}{\sqrt{v}}\, dv = \int dx$$

$$2\sqrt{v} = x + C$$

$$v = \left(\frac{1}{2}x + C\right)^2$$

$$y-2x+3 = \left(\frac{1}{2}x+C\right)^2$$

$$\left[\ \therefore\ y = 2x-3 + \left(\frac{1}{2}x+C\right)^2\ \right]$$

Ex: Solve $\dfrac{dy}{dx} = y-x-1 + (x-y+2)^{-1}$

$$\frac{dy}{dx} = y-x-1 + \frac{1}{-(y-x)+2}$$

let:

$v = y-x$

$v+x = y$

$\dfrac{dv}{dx}+1 = \dfrac{dy}{dx}$

$$\frac{dv}{dx} + 1 = v - 1 + \frac{1}{-v+2}$$

$$\frac{dv}{dx} = v-2 + \frac{1}{-v+2}$$

$$\frac{dv}{dx} = v-2 - \frac{1}{v-2}$$

$$\frac{dv}{dx} = \frac{(v-2)^2 - 1}{v-2}$$ توحيد مقامات

$$\frac{dv}{dx} = \frac{v^2-4v+3}{v-2}$$ فك التربيع

$$\times\frac{1}{2}\int \frac{(v-2)\times 2}{v^2-4v+3}\,dv = \int 1\,dx$$ separable

$$\frac{1}{2}\ln|v^2-4v+3| = x + C$$

$$e^{\ln|v^2-4v+3|} = e^{2x+C}$$

$$v^2-4v+3 = e^{2x}\cdot C$$

$\therefore$ Soln: $\left[\, (y-x)^2 - 4(y-x) + 3 = Ce^{2x} \,\right]$

Ex: Solve $\frac{dy}{dx} = \sin(x-y)$

$v = x - y$

$x - v = y$

$1 - \frac{dv}{dx} = \frac{dy}{dx}$

$1 - \frac{dv}{dx} = \sin(v)$

$-\frac{dv}{dx} = \sin(v) - 1$

$\frac{dv}{dx} = 1 - \sin(v)$

$\int \frac{1}{1-\sin(v)} dv = \int dx$

$= x + C$

← استخدام المرافق

$\int \frac{1}{1-\sin v} \cdot \frac{1+\sin v}{1+\sin v} dv$

$\int \frac{1+\sin v}{1-\sin^2 v} dv$

$\int \frac{1+\sin v}{\cos^2 v} dv$

$\int \frac{1}{\cos^2 v} + \frac{\sin v}{\cos v} \cdot \frac{1}{\cos v} dv$

$\int \sec^2 v + \int \tan v \cdot \sec v \, dv$

$\tan v + \sec v$

$\tan v + \sec v = x + C$

soln → $[\ \tan(x-y) + \sec(x-y) = x + C\ ]$

Ex: The substitution $v = x^2 \ln y$ transforms the DE

$x^2 y' + 4x^2 y \ln y = y \sin x$ into (linear in v)

$v = x^2 \ln y$

$v' = 2x \ln y + x^2 \frac{1}{y} y'$

$y v' = 2xy \ln y + x^2 y'$

$y v' - 2xy\ln y = x^2 y'$ → put this in DE

$y v' - 2xy\ln y + 4x^2 y \ln y = y \sin x$ نقسم على (y)

$v' - 2x \ln y + 4x^2 \ln y = \sin x$

$v' - 2x\left(\frac{v}{x^2}\right) + 4v = \sin x$

$v' - 2\left(\frac{v}{x}\right) + 4v = \sin x$ → $v' + \left(4 - \frac{2}{x}\right) v = \sin x$

(linear in v)

Ex: Solve $x^2\,dy = (x^2 + 3xy + y^2)\,dx$

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}$$

$$\frac{dy}{dx} = 1 + 3\frac{y}{x} + \left(\frac{y}{x}\right)^2$$

let $v = \frac{y}{x}$

$xv = y$

$v + x\frac{dv}{dx} = \frac{dy}{dx}$

$$v + x\frac{dv}{dx} = 1 + 3v + v^2$$

$$x\frac{dv}{dx} = 1 + 2v + v^2$$

$$\int \frac{1}{v^2 + 2v + 1}\,dv = \int \frac{1}{x}\,dx \rightarrow \text{(separable)}$$

$$\int \frac{1}{(1+v)^2}\,dv = \ln|x| + C.$$

$$\frac{-1}{1+v} = \ln|x| + C$$

$$\left[\therefore \quad \frac{-1}{1 + \left(\frac{y}{x}\right)} = \ln|x| + C. \right]$$

Chapter (3): Linear second order (DE)

" The linear second order (DE) is $a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = g(x)$ " $(*)$

Remark: ① If $[g(x) = 0]$, then the eq $(*)$ becomes:

$$a_2(x)\,y'' + a_1(x)\,y' + a_0(x)\,y = 0$$

which is called the (homogenoues) but if $[g(x) \neq 0]$, then the eq $(*)$ is called the (non-homogenoues)

② If $a_2(x) = a$, $a_1(x) = b$, $a_0(x) = c$, then the eq $(*)$ becomes $[ay'' + by' + cy = g(x)]$ $\begin{cases} =0 & \text{homog.} \\ \neq 0 & \text{Non homog.} \end{cases}$

linear second order DE with constant coefficents.

③ The eq $(*)$ can be written as: $L[y]_{(x)} = g(x)$ where $L[y]_{(x)} = a_2(x)y'' + a_1(x)y' + a_0(x)\,y$ which is a linear operator.

[Theorem]: If $y_1, y_2, \dots, y_n$ are solution to DE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = 0$$

then

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n \text{ is another soln to DE}$$

Def: The Wronskain to the functions $y_1, y_2, \dots, y_n$ is given as:

$$W[y_1, y_2, \dots y_n](x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Remark: $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Ex: Find $\begin{vmatrix} 2 & 3 \\ -4 & 5 \end{vmatrix} = 2*5 - (3*-4) = 22$

Ex: Find

1. $W[\sin x, \cos x] = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -\sin^2 x - \cos^2 x$
$= -1$

2. $W[\cosh x, \sinh x] = \begin{vmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{vmatrix} = \cosh^2 x - \sinh^2 x$
$= 1$

3. $W[x, e^x] = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = xe^x - e^x = e^x(x-1)$

4. $W[\cosh x, x^3, e^x] = \begin{vmatrix} \cosh x & x^3 & e^x \\ \sinh x & 3x^2 & e^x \\ \cosh x & 6x & e^x \end{vmatrix} \begin{matrix} \cosh x & x^3 \\ \sinh x & 3x^2 \\ \cosh x & 6x \end{matrix}$

$= (\cosh x \cdot 3x^2 . e^x + e^x x^3 \cosh x + 6xe^x \sinh x) -$
$(x^3 e^x \sinh x + 6xe^x \cosh x + 3x^2 e^x \cosh x)$
$= x^3 e^x (\cosh x - \sinh x) + 6xe^x(\sinh x - \cosh x)$

5. $W[1, x, x^2, x^3, e^x] = \begin{vmatrix} 1 & x & x^2 & x^3 & e^x \\ 0 & 1 & 2x & 3x^2 & e^x \\ 0 & 0 & 2 & 6x & e^x \\ 0 & 0 & 0 & 6 & e^x \\ 0 & 0 & 0 & 0 & e^x \end{vmatrix}$

$= 1*1*2*6*e^x$
$= 12e^x$

Def: We say that the set $\{y_1(x), y_2(x) \cdots y_n(x)\}$ is Linearly independent if

$$c_1 y_1 + c_2 y_2 + \cdots c_n y_n = 0, \text{ then } c_1 = c_2 = \cdots = c_n = 0$$

or

$$W[y_1, \cdots y_n](x) \neq 0, \text{ for } x \in I$$

Ex: $\{e^x, 0, 3x\}$ is L. independent ?

No because $0 \in \{ \; \}$

Ex: Find b such that the set $\{x^2, 2b-1, e^x\}$ dependent set:

$$2b-1=0$$

$$b = \frac{1}{2}$$

Fundemental set of solution:

Def: A set $\{y_1, y_2\}$ is called fundemental soln for

$$L[y] = a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

(if)

① $y_1, y_2 \ldots$ are soln to DE

② $W[y_1, y_2] \neq 0$ (y_1, y_2 are L. independent)

المهم

Abel's Theorem:

If y_1, y_2 are soln for $y'' + p(x)y' + q(x)y = 0$, the

$$W[y_1, y_2] = C e^{-\int p(x)\,dx}$$

Ex: If y_1, y_2 are soln for $xy'' + 2y' + xy = 0$, Find $W[y_1, y_2](x)$.

Soln:

$$y'' + \underbrace{\frac{2}{x}}_{p(x)} y' + y = 0$$

دائماً p(x) هي ↑ معامل المشتقة التي تأتي بعد أعلى مشتقة مباشرة.

$$W[y_1, y_2] = C e^{-\int \frac{2}{x}\,dx}$$

$$= C e^{-2\ln|x|} = C x^{-2} = \frac{C}{x^2}$$

Ex: If $x^2 y'' - 2y' + (3+x) y = 0$ if $w(y_1, y_2)(2) = 3$, find $w[y_1, y_2](5)$.

soln: $y'' - \frac{2}{x^2} y' + \frac{(3+x)}{x^2} y = 0$

$w[y_1, y_2] = ce^{-\int -\frac{2}{x^2} dx} = ce^{\int 2x^{-2} dx} = ce^{-\frac{2}{x}}$

since $w(y_1, y_2)(2) = 3$ $\quad ce^{-1} = 3 \Rightarrow [c = 3e]$

$\therefore w(y_1, y_2)(5) = 3e e^{-\frac{2}{5}}$

$= 3e^{\frac{3}{5}}$ #

* Reduction of order :

If y_1 is the soln for DE :

$y'' + p(x) y' + q(x) y = 0$, then the second solution is:

$$y_2 = y_1 \cdot \int \frac{e^{-\int p(x) dx}}{(y_1)^2} dx$$

and the general soln is $y = c_1 y_1 + c_2 y_2$

Ex: If $y_1 = e^x$ is a soln for the DE $xy'' - (x+1) y' + y = 0$, $x > 0$ find the second soln & the general soln.

Soln: $y'' - \frac{(x+1)}{x} y' + \frac{1}{x} y = 0$ $\qquad \int 1 + \frac{1}{x} = x + \ln x$

$y_2 = e^x \cdot \int \frac{e^{-\int -\frac{(x+1)}{x} dx}}{(e^x)^2} dx = e^x \cdot \int \frac{e^x \cdot x}{e^{2x}} dx$

parts

$= e^x \cdot \int x e^{-x} dx$

$= e^x \cdot (-x e^{x} - e^{-x})$

$= -x - 1$

or $y_2 = x + 1$

	D	I
$\oplus$	x	e^{-x}
$\ominus$	1	$-e^{-x}$
	0	$+e^{-x}$

G.Soln

$y = c_1 e^x + c_2(-x - 1)$

Ex: If $y_1 = \cos(\ln x)$ is a soln for DE $x^2 y'' + x y' + y = 0$, find the general soln.

soln: $y_2 = y_1 \cdot \int \frac{e^{-\int p(x)\,dx}}{(y_1)^2}\,dx$

$\therefore y'' + \frac{1}{x} y' + \frac{1}{x^2} y = 0$

$y_2 = \cos(\ln x) \cdot \int \frac{e^{-\int \frac{1}{x} dx}}{(\cos(\ln x))^2}\,dx$

$y_2 = \cos(\ln x) \cdot \int \frac{x^{-1}}{(\cos(\ln x))^2}\,dx$

$\int \frac{x^{-1}}{\cos^2(\ln x)}$

$\int \frac{1}{x \cos^2(\ln x)}\,dx$

$\int \frac{\sec^2(\ln x)}{x}\,dx$ نعوض

$\int \sec^2(u)\,du$ $\quad u = \ln x$

$du = \frac{1}{x} dx$

$x\,du = dx$

$y_2 = \cos(\ln x) \cdot \int \sec^2(u)\,du$

$= \cos(\ln x) \cdot \tan(\ln x)$

$= \sin(\ln x)$

[G. Soln]

$y = C_1 \cos(\ln x) + C_2 \sin(\ln x)$.

** Linear, homog. second Order with constant coefficients:

$ay'' + by' + cy = 0$, $a/b/c$ are constant and $a \neq 0$

To solve this DE, we suppose that the soln has the form $y = e^{rx}$, then $y' = re^{rx}$, $y'' = r^2 e^{rx}$ put this in DE

$a(r^2 e^{rx}) + b(re^{rx}) + c(e^{rx}) = 0$

$\underbrace{e^{rx}}_{\neq 0}(ar^2 + br + c) = 0$

$\therefore$ [$ar^2 + br + c = 0$] is called the auxiliary eq for DE

then there will be three forms of the G. Soln.

* Case (1) : If $\Delta = b^2 - 4ac > 0$, then we have two distinct real roots, say $\underline{\underline{r_1}}, \underline{\underline{r_2}}$ (حلان حقيقيان مختلفان)

$\Rightarrow$ the soln of DE are $[y_1 = e^{r_1 x}]$ $[y_2 = e^{r_2 x}]$

& the G. Soln is $y = C_1 y_1 + C_2 y_2$

Ex : ① Solve $\quad y'' + 5y' + 6y = 0$

soln. :

$$r^2 + 5r + 6 = 0 \rightarrow \text{auxiliary eq.}$$

$$(r+3)(r+2) = 0$$

$$[r_1 = -2], [r_2 = -3]$$

$(y_1 = e^{-2x})$, $(y = e^{-3x})$ $\Rightarrow$ [G. Soln $y = C_1 e^{-2x} + C_2 e^{-3x}$]

② Solve $\quad y'' - 4y = 0$

Soln : $r^2 - \cancel{5}4 = 0$ $\quad (y_1 = e^{2x})$ $(y_2 = e^{-2x})$

$$r = \pm 2$$

$[r_1 = 2]$ $[r_2 = -2]$ $\quad$ [G. Soln $\Rightarrow y = C_1 e^{2x} + C_2 e^{-2x}$]

③ Solve $\quad y'' - 5y' = 0$

Soln $\quad r^2 - 5r = 0$ $\quad (y_1 = e^{0} = 1)$ $(y_2 = e^{5x})$

$$r(r-5) = 0$$

[G. Soln $\Rightarrow y = C_1(1) + C_2(e^{5x})$]

$[r_1 = 0]$ $[r_2 = 5]$

* Case (2) : If $\Delta = b^2 - 4ac = 0$, then we have one repate real root, say r_1, r_1 [جذر واحد مكرر]

$$y_1 = e^{r_1 x} \quad , \quad y_2 = x e^{r_1 x} \rightarrow \text{G.Soln is}$$

$$y = C_1 y_1 + C_2 y_2$$

Ex: ① Solve $y'' + 12y' + 36y = 0$

$$r^2 + 12r + 36 = 0$$

$$(r+6)(r+6) = 0$$

$(r_1 = -6)$ $(r_2 = -6)$ (مكرر) $\rightarrow (y_1 = e^{-6x})$, $(y_2 = x e^{-6x})$

$$\left[\text{G.Soln} \Rightarrow C_1 e^{-6x} + C_2 \underline{\underline{x}} e^{-6x} \right]$$

ع وجود (x) نستخدم حتى تكون هما مستقلين (independent

② Solve $4y'' + 20y' + 25y = 0$

$$4r^2 + 20r + 25 = 0$$

$$(2r+5)(2r+5) = 0$$

$\left(r_1 = -\frac{5}{2}\right)$ $\left(r_2 = -\frac{5}{2}\right)$

$\left(y_1 = e^{-\frac{5}{2}x}\right)$, $\left(y_2 = x e^{-\frac{5}{2}x}\right)$

G.soln $\Rightarrow$

$$\left[y = C_1 e^{-5/2 x} + C_2 e^{-\frac{5}{2}x} (x) \right]$$