Poisson Distribution The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t, is

avg number

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is the average number of outcomes per unit time, distance, area, or volume and $e = 2.71828\ldots$.

Table A.2 contains Poisson probability sums,

$$P(r; \lambda t) = \sum_{x=0}^{r} p(x; \lambda t),$$

for selected values of λt ranging from 0.1 to 18.0. We illustrate the use of this table with the following two examples.

Theorem 5.4: $\boxed{\text{Both the mean and the variance of the Poisson distribution } p(x; \lambda t)}$ $\boxed{\text{are } \lambda t.}$ $\rightarrow \lambda t \sim$ interval

particle

Example 5.17: During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

number of outcome

4 in 1ms

Solution: Using the Poisson distribution with $x = 6$ and $\lambda t = 4$ and referring to Table A.2, we have

$= P(X=6) = F(6) - F(5)$

$$p(6; 4) = \frac{e^{-4}4^6}{6!} = \sum_{x=0}^{6} p(x; 4) - \sum_{x=0}^{5} p(x; 4) = 0.8893 - 0.7851 = 0.1042.$$

25

expected value $= \mu \rightarrow \lambda t \rightarrow$ avg number $= np$

$r = x$

Example 5.19: In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

possion = Binomial

$p \approx 0$

$n >$

(a) What is the probability that in any given period of 400 days there will be an accident on one day? $\boxed{x=1}$

(b) What is the probability that there are at most three days with an accident?

Solution: Let X be a binomial random variable with $n = 400$ and $p = 0.005$. Thus, $np = 2$. Using the Poisson approximation,

$\lambda t = np$

from table $\rightarrow$ $(0.4060 - 0.1355)$

(a) $P(X = 1) = e^{-2}2^1 = 0.271$ and

(b) $P(X \leq 3) = \sum_{x=0}^{3} e^{-2}2^x/x! = 0.857.$ 0.135335+0.270671 +0.270671 +0.180447

table 3

$\boxed{p = \frac{1}{1000}}$

Example 5.20: In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

$\boxed{n}$

Solution: This is essentially a binomial experiment with $n = 8000$ and $p = 0.001$. Since p is very close to 0 and n is quite large, we shall approximate with the Poisson distribution using

$\mu = np = \mu = (8000)(0.001) = 8.$

Hence, if X represents the number of bubbles, we have

$F(6)$ table

$$P(X < 7) = \sum_{x=0}^{6} b(x; 8000, 0.001) \approx p(x; 8) = 0.3134.$$

أقل من 7

number of outcome عدد
Interval بمعدل

Example 1: A secretary makes 2 errors per page, on average.

What is the probability that on the next page he or she will make

$\mu = \lambda t = 2 * 1 = 2$

(a) 4 or more errors? $P(x \geq 4)$ عدد الأكثر

$= 1 - P(x < 4) = 1 - 0.857$

$f(3)$

(b) no errors?

table من ال $P(x=0)$

$\boxed{0.1353}$

$P(Y \geq 4) = 1 - P(Y < 4) = 1 - P(Y \leq 3)$

a) 0.143

$P(x < 7) = P(x = 6)$

$P(x \leq 7) = P(x = 7)$

$P(x > 7) = 1 - P(x \leq 7)$

ch 5

Binomial distribution

A.1

$$b(x; n, p) = \binom{n}{x} p^{x} q^{n-x} \quad (q = 1 - p)$$

جزء من كل

$\boxed{\mu = np}$ $\boxed{\underset{\text{Variance}}{\sigma^2} = npq}$

Hypergeometric distribution:

مثل جرات دخلناها توافيق

$(\mu \pm 2\sigma)$

$$h(x; N, n, k) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}$$

$\boxed{\mu = \frac{nk}{N}}$

$$\sigma^2 = \frac{N-n}{N-1} \cdot n \cdot \frac{k}{N}\left(1 - \frac{k}{N}\right)$$

Negative Binomial distribution:

وجود حد أدنى للنجاح

4 out of 7

$$b^{*}(x; k, p) = \binom{x-1}{k-1} p^{k} q^{x-k}$$

لحد النجاح

Geometric distribution:

$\boxed{g(x; p) = pq^{x-1}}$

First success

$\boxed{\mu = \frac{1}{p}}$ $\boxed{\sigma^2 = \frac{1-p}{p^2}}$

Poisson distribution:

$$P(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^{x}}{x!}$$

A. 2

$$\lambda t = \mu = np$$

$$\mu = \sigma^2 = \lambda t$$

عدد خلال وقت

لمان سليمان أحمد علي

1/75.

Chapter 6

Some Continuous Probability Distributions

1

1) **Uniform Distribution** The density function of the continuous uniform random variable X on the interval $[A, B]$ is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A \le x \le B, \\ 0, & \text{elsewhere.} \end{cases}$$

probability ال constant

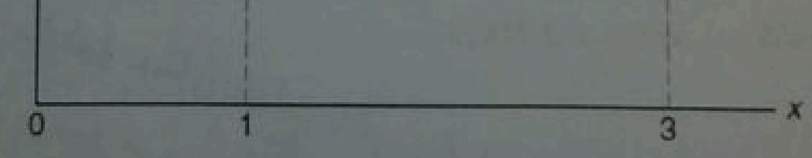


Figure 6.1: The density function for a random variable on the interval [1, 3].

$$f(1,3) = \begin{cases} \frac{1}{2}, & 1 \le x \le 3 \\ 0 \end{cases}$$

2

Example 6.1: Suppose that a large conference room at a certain company can be reserved for no more than 4 hours. Both long and short conferences occur quite often. In fact, it can be assumed that the length X of a conference has a uniform distribution on the interval $[0, 4]$.

(a) What is the probability density function?

(b) What is the probability that any given conference lasts at least 3 hours?

Solution: (a) The appropriate density function for the uniformly distributed random variable X in this situation is

$$\frac{1}{B-A} \leftarrow f(x) = \begin{cases} \frac{1}{4}, & 0 \le x \le 4, \\ 0, & \text{elsewhere.} \end{cases} \rightarrow \int_0^4 = 1$$

تكاملها من

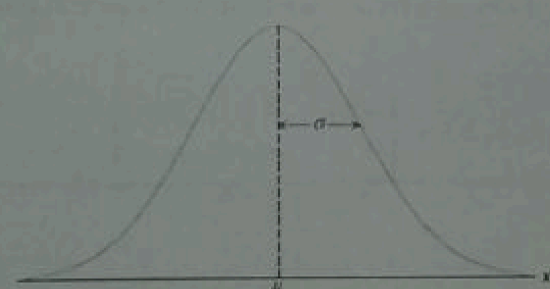
(b) $P[X \ge 3] = \int_3^4 \frac{1}{4}\, dx = \frac{1}{4}$.

Theorem 6.1: The mean and variance of the uniform distribution are

$$\boxed{\mu = \frac{A+B}{2}} \text{ and } \boxed{\sigma^2 = \frac{(B-A)^2}{12}}.$$

$\boxed{\mu = 2}$ $\boxed{\sigma^2 = \frac{4}{3}}$

Normal Distribution



Normal Distribution

The density of the normal random variable X, with mean μ and variance σ^2, is

$$\boxed{n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}, \quad -\infty < x < \infty,}$$

where $\pi = 3.14159\ldots$ and $e = 2.71828\ldots$.

بدنا نكامله عشان نطلع ال probability

لذلك نستخدم ال table

أسهل !!

1)

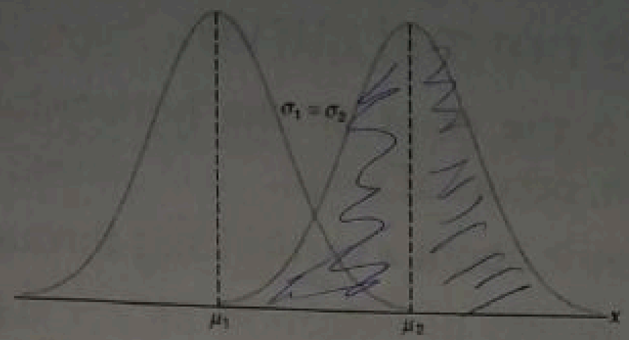


Figure 6.3: Normal curves with $\mu_1 < \mu_2$ and $\boxed{\sigma_1 = \sigma_2}$.

s.D

Two normal curves having the same standard deviation but different means. The two curves are identical in form but are centered at different positions along the horizontal axis

التوزيع حول ال center متساوي

2)

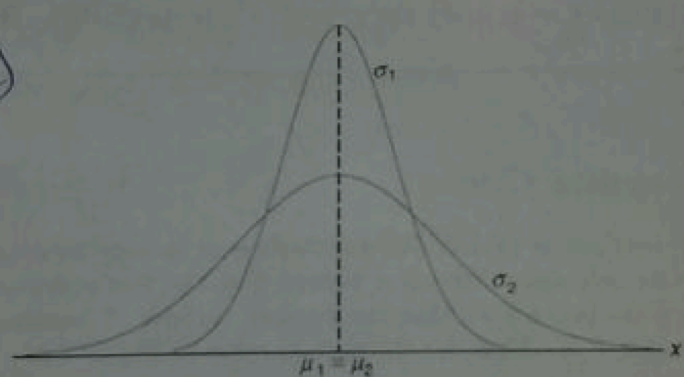


Figure 6.4: Normal curves with $\mu_1 = \mu_2$ and $\sigma_1 < \sigma_2$.

Two normal curves with the same mean but different standard deviations

3)

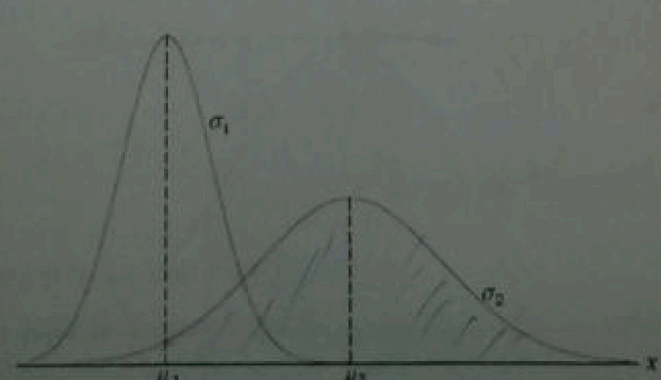


Figure 6.5: Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 < \sigma_2$.

1) symetirc around the center

2) أعلى قيمة $x = \mu$

3) يمين $\mu + \sigma$ يسار $\mu - \sigma$

Two normal curves having different means and different standard deviations

Properties of the normal curve:

1. The mode, which is the point on the horizontal axis where the curve is a maximum, occurs at $x = \mu$..
2. The curve is symmetric about a vertical axis through the mean μ.
3. The curve has its points of inflection at $\boxed{x = \mu \pm \sigma}$; it is concave downward if $\mu-\sigma<X<\mu+\sigma$ and is concave upward otherwise.
4. The normal curve approaches the horizontal axis asymptotically as we proceed in either direction away from the mean.
5. The total area under the curve and above the horizontal axis is equal to 1.

Theorem 6.2: The mean and variance of $n(x; \mu, \sigma)$ are μ and σ^2, respectively. Hence, the standard deviation is σ.

S.N باستخدام الـ table

نحوّل x إلى Z

μ أعلى نقطة في المنحنى

Areas under the Normal Curve

The curve of any continuous probability distribution or density function is constructed so that the area under the curve bounded by the two ordinates $x = x_1$ and $x = x_2$ equals the probability that the random variable X assumes a value between $x = x_1$ and $x = x_2$. Thus, for the normal curve in Figure 6.6,

$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x; \mu, \sigma)\, dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\, dx$$

is represented by the area of the shaded region.

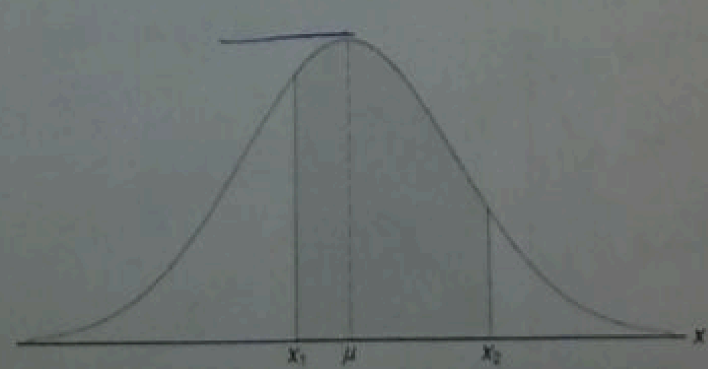


Figure 6.6: $P(x_1 < X < x_2)$ = area of the shaded region.

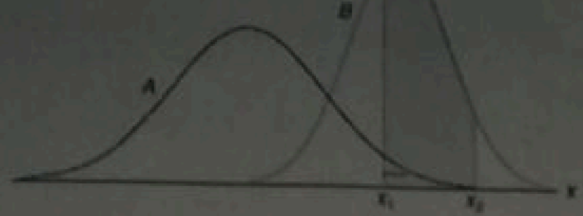


Figure 6.7: $P(x_1 < X < x_2)$ for different normal curves.

of a normal random variable Z with mean 0 and variance 1. This can be done by means of the transformation

$$\boxed{Z = \frac{X - \mu}{\sigma}.}$$

$\mu = 0$
$\sigma^2 = 1$

Definition 6.1: The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.

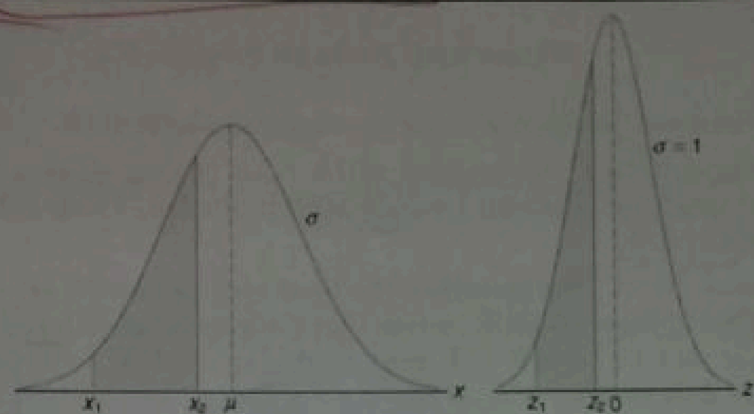


Figure 6.8: The original and transformed normal distributions.

الأول

$\rightarrow \mu = 0 \quad \sigma^2 = 1$

حل مثال

يجي عليه سؤال بالامتحان

Example 6.2: Given a standard normal distribution, find the area under the curve that lies

(a) to the right of $Z = 1.84$ and

(b) between $z = -1.97$ and $z = 0.86$.

$P(Z > 1.84) = 1 - P(Z < 1.84)$

ما نفرق المساواة

لأنه continuous

$1 - 0.9671 =$

$P(-1.97 \leq Z \leq 0.86)$

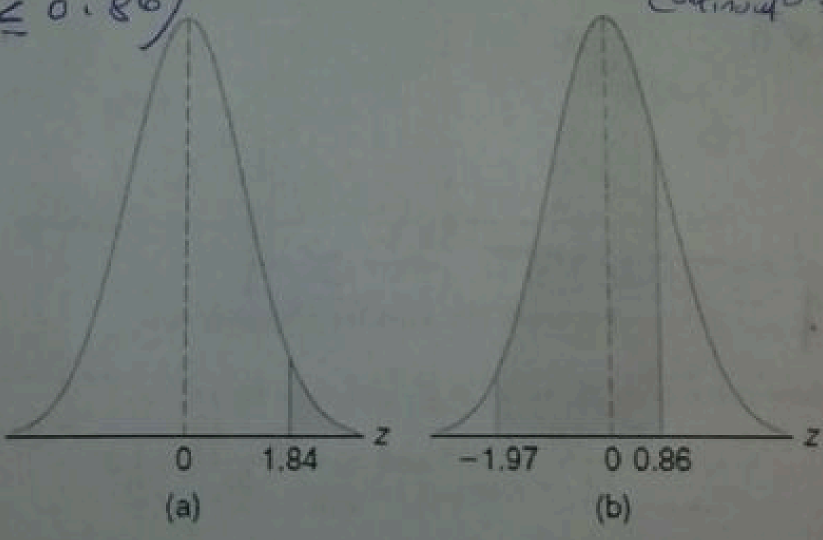


Figure 6.9: Areas for Example 6.2.

***Solution*:** See Figure 6.9 for the specific areas.

(a) The area in Figure 6.9(a) to the right of $z = 1.84$ is equal to 1 minus the area in Table A.3 to the left of $z = 1.84$, namely, $1 - 0.9671 = 0.0329$.

(b) The area in Figure 6.9(b) between $z = -1.97$ and $z = 0.86$ is equal to the area to the left of $z = 0.86$ minus the area to the left of $z = -1.97$. From Table A.3 we find the desired area to be $0.8051 - 0.0244 = 0.7807$. ∎

Example 6.3: Given a standard normal distribution, find the value of k such that

(a) $P(Z > k) = 0.3015$ and

(b) $P(k < Z < -0.18) = 0.4197$.

k ?

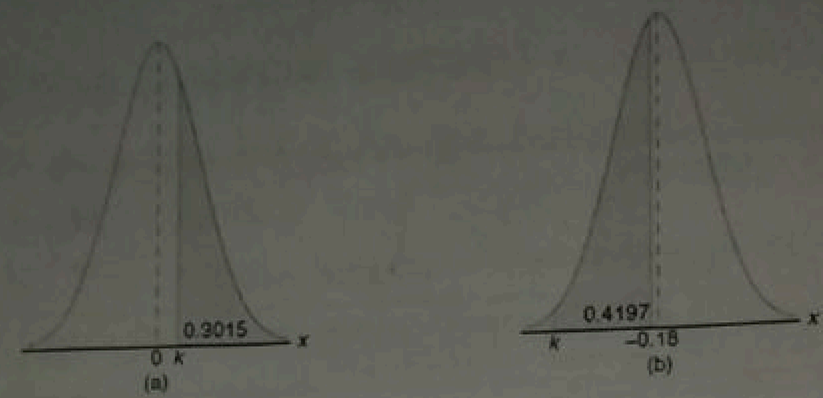


Figure 6.10: Areas for Example 6.3.

Solution: Distributions and the desired areas are shown in Figure 6.10.

(a) In Figure 6.10(a), we see that the k value leaving an area of 0.3015 to the right must then leave an area of 0.6985 to the left. From Table A.3 it follows that $k = 0.52$.

(b) From Table A.3 we note that the total area to the left of -0.18 is equal to 0.4286. In Figure 6.10(b), we see that the area between k and -0.18 is 0.4197, so the area to the left of k must be $0.4286 - 0.4197 = 0.0089$. Hence, from Table A.3, we have $k = -2.37$.

0.2789

نبحث عن K ← من ال A.3 table

probability

Example 6.4: Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.

X → normal

Z → standard

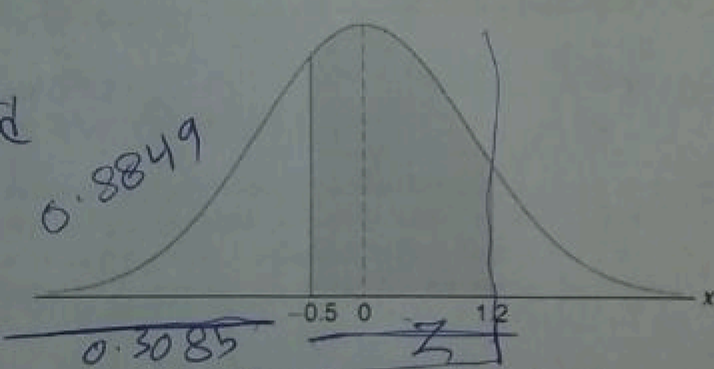


Figure 6.11: Area for Example 6.4.

Solution: The z values corresponding to $x_1 = 45$ and $x_2 = 62$ are

$$z_1 = \frac{45 - 50}{10} = -0.5 \text{ and } z_2 = \frac{62 - 50}{10} = 1.2.$$

Therefore,

$$P(45 < X < 62) = P(-0.5 < Z < 1.2).$$

$P(-0.5 < Z < 1.2)$ is shown by the area of the shaded region in Figure 6.11. This area may be found by subtracting the area to the left of the ordinate $z = -0.5$ from the entire area to the left of $z = 1.2$. Using Table A.3, we have

$$P(45 < X < 62) = P(-0.5 < Z < 1.2) = P(Z < 1.2) - P(Z < -0.5)$$
$$= 0.8849 - 0.3085 = 0.5764.$$

Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.

Solution: The normal probability distribution with the desired area shaded is shown in Figure 6.12. To find $P(X > 362)$, we need to evaluate the area under the normal curve to the right of $x = 362$. This can be done by transforming $x = 362$ to the corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from 1. We find that

(نحول X لـ Z)

$$z = \frac{362 - 300}{50} = 1.24.$$

Hence,

$$P(X > 362) = P(Z > 1.24) = 1 - P(Z < 1.24) = 1 - 0.8925 = 0.1075.$$

لـ الأقل من دائماً

discrete ← X
continous ← Z

#

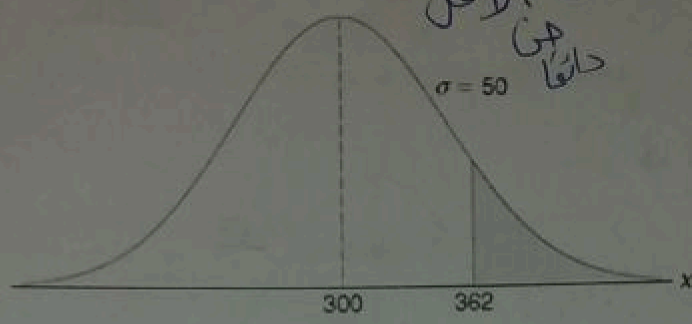
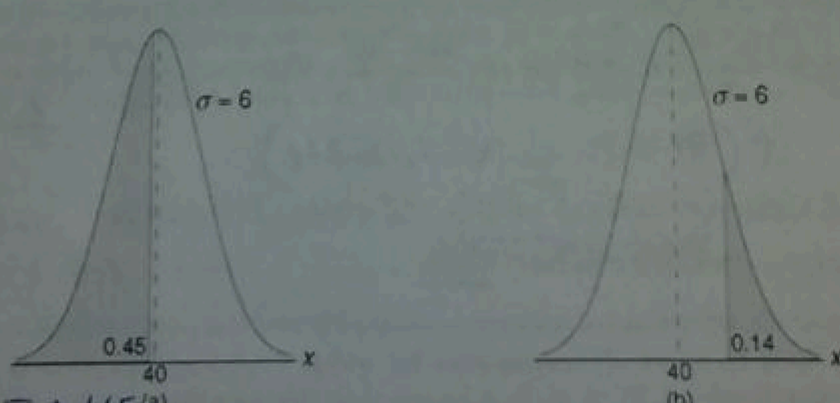


Figure 6.12: Area for Example 6.5.

Example 6.6: Given a normal distribution with $\mu = 40$ and $\sigma = 6$, find the value of x that has

(a) 45% of the area to the left and

(b) 14% of the area to the right.



a) $P(Z < \quad) = 0.45$

لـ بندور عليها في الاحتمالية وليس في قيم Z

$Z = -0.13$

تعرف القيمة الأخرى

Figure 6.13: Areas for Example 6.6.

Solution: (a) An area of 0.45 to the left of the desired x value is shaded in Figure 6.13(a). We require a z value that leaves an area of 0.45 to the left. From Table A.3 we find $P(Z < -0.13) = 0.45$, so the desired z value is -0.13. Hence,

$$x = (6)(-0.13) + 40 = 39.22.$$

(b) In Figure 6.13(b), we shade an area equal to 0.14 to the right of the desired x value. This time we require a z value that leaves 0.14 of the area to the right and hence an area of 0.86 to the left. Again, from Table A.3, we find $P(Z < 1.08) = 0.86$, so the desired z value is 1.08 and

$$x = (6)(1.08) + 40 = 46.48.$$

1) بنجيب 0.14 ونجيب Z عكسها بالإشارة

2) $1 - 0.14 = 0.86$

لـ بنقرأها من الجدول

σ

Example 6.7: A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.

Solution: First construct a diagram such as Figure 6.14, showing the given distribution of battery lives and the desired area. To find $P(X < 2.3)$, we need to evaluate the area under the normal curve to the left of 2.3. This is accomplished by finding the area to the left of the corresponding z value. Hence, we find that

$\mu = 3$
$\sigma = 0.5$
$x = 2.3$

$$z = \frac{2.3 - 3}{0.5} = -1.4,$$

and then, using Table A.3, we have

$$P(X < 2.3) = P(Z < -1.4) = 0.0808.$$

+ إذا أعطى بالسؤال standard
بحوّل لـ x لـ Z

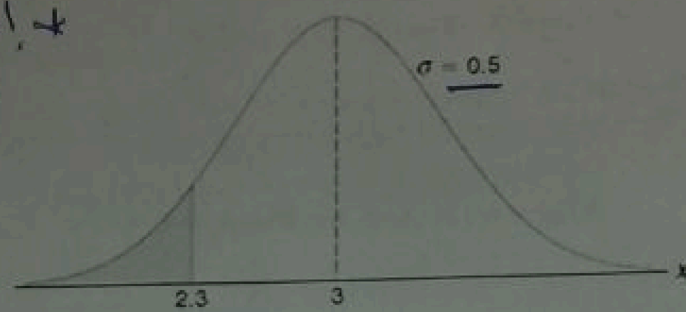


Figure 6.14: Area for Example 6.7.

Example 6.8: An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.

Solution: The distribution of light bulb life is illustrated in Figure 6.15. The z values corresponding to $x_1 = 778$ and $x_2 = 834$ are

$\mu = 800$
$\sigma = 40$

$$z_1 = \frac{778 - 800}{40} = -0.55 \text{ and } z_2 = \frac{834 - 800}{40} = 0.85.$$

$Z = \frac{x-}{\sigma}$

Hence,

$P(778 < x < 834)$

$$P(778 < X < 834) = P(-0.55 < Z < 0.85) = P(Z < 0.85) - P(Z < -0.55)$$
$$= 0.8023 - 0.2912 = 0.5111.$$

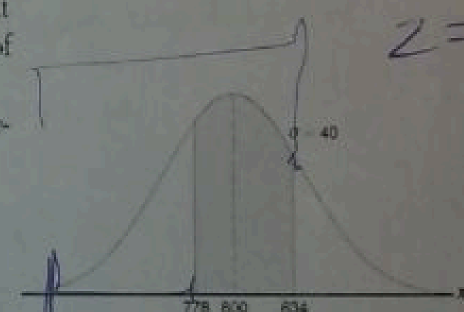


Figure 6.15: Area for Example 6.8.

Example 6.10: Gauges are used to reject all components for which a certain dimension is not within the specification $1.50 \pm d$. It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.2. Determine the value d such that the specifications "cover" 95% of the measurements.

→ label $\boxed{x = 1.5 \pm d}$

$P(X > 1.5+d) = 0.025 \Leftrightarrow P(X < 1.5+d) = 0.975$

$P(X < 1.5-d) = 0.025$

Solution: From Table A.3 we know that

احتمالية كبيرة

$$P(-1.96 < Z < 1.96) = 0.95.$$

$\mu = 1.5$
$\sigma = 0.2$

0.05

$$P\left(Z \le \frac{(1.5-d)-\mu}{\sigma}\right) = 0.025$$

$$P\left(Z \le \frac{(1.5-d)-1.5}{0.20}\right) = 0.025$$

$$P\left(Z \le \frac{-d}{0.20}\right) = 0.025$$

$$\frac{-d}{0.20} = -1.96$$

Therefore,

$$1.96 = \frac{(1.50 + d) - 1.50}{0.2},$$

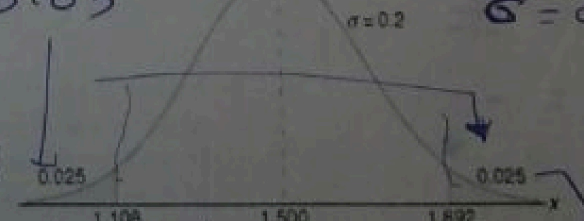
$x - \mu$

from which we obtain

$x = (1.96 * 0.2) + 1.5$

$\boxed{x = 1.5 \pm d}$ → $d = (0.2)(1.96) = 0.392.$

An illustration of the specifications is shown in Figure 6.17.



1.5 − d   d + 1.5

Figure 6.17: Specifications for Example 6.10.

لحساب قيمة Z
دبعرين بطول x

1 − 0.95

In an industrial process, the diameter of a ball bearing is an important measurement. The buyer sets specifications for the diameter to be 3.0 ± 0.01 cm. The implication is that no part falling outside these specifications will be accepted. It is known that in the process the diameter of a ball bearing has a normal distribution with mean $\mu = 3.0$ and standard deviation $\sigma = 0.005$. On average, how many manufactured ball bearings will be scrapped?

accepted range

$(2.99 - 3.01)$

$x \quad x$

Solution: The distribution of diameters is illustrated by Figure 6.16. The values corresponding to the specification limits are $x_1 = 2.99$ and $x_2 = 3.01$. The corresponding z values are

$$z_1 = \frac{2.99 - 3.0}{0.005} = -2.0 \text{ and } z_2 = \frac{3.01 - 3.0}{0.005} = +2.0.$$

$1 - P(-2.0 < Z < 2.0) = 1 - P(Z < 2.0) + P(Z < -2.0)$

$= P(Z > 2.0) + P(Z < -2.0) = 2 \cdot 0.0228 = 0.0456.$

Hence,

$$P(2.99 < X < 3.01) = P(-2.0 < Z < 2.0).$$

From Table A.3, $P(Z < -2.0) = 0.0228$. Due to symmetry of the normal distribution, we find that

$P(Z=2) - P(Z=-2)$

$= 0.9772 - 0.0228$

$= 0.9544$

$$P(Z < -2.0) + P(Z > 2.0) = 2(0.0228) = 0.0456.$$

As a result, it is anticipated that, on average, 4.56% of manufactured ball bearings will be scrapped. ∎

$P(Z>2) = P(Z<-2) = 0.0228$

المطلوب هذه المنطقتين

$1 - 0.9544$

$= 0.0456$

#

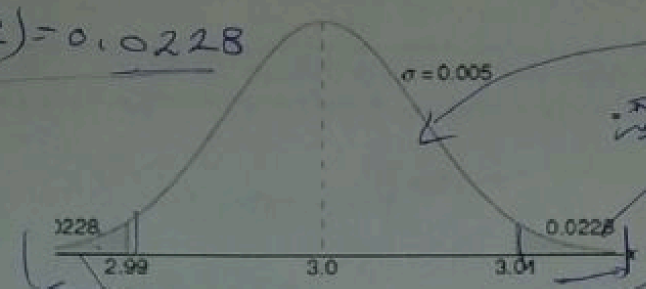


Figure 6.16: Area for Example 6.9.

مقربة أو غير مقربة عادي!!

Example 6.11: A certain machine makes electrical resistors having a mean resistance of 40 ohms and a standard deviation of 2 ohms. Assuming that the resistance follows a normal distribution and can be measured to any degree of accuracy, what percentage of resistors will have a resistance exceeding 43 ohms?

على الأقل 43

probability

Solution: A percentage is found by multiplying the relative frequency by 100%. Since the relative frequency for an interval is equal to the probability of a value falling in the interval, we must find the area to the right of $x = 43$ in Figure 6.18. This can be done by transforming $x = 43$ to the corresponding z value, obtaining the area to the left of z from Table A.3, and then subtracting this area from 1. We find

$\mu = 40$

$\sigma = 2$

$x = 43$

$$z = \frac{43 - 40}{2} = 1.5.$$

Therefore,

$$P(X > 43) = P(Z > 1.5) = 1 - P(Z < 1.5) = 1 - 0.9332 = 0.0668.$$

Hence, 6.68% of the resistors will have a resistance exceeding 43 ohms. ∎

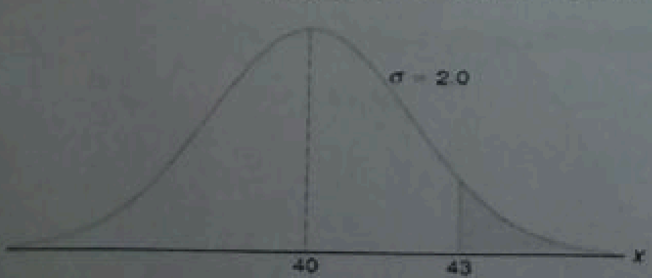


Figure 6.18: Area for Example 6.11.

Example 6.12: Find the percentage of resistances exceeding 43 ohms for Example 6.11 if resistance is measured to the nearest ohm.

42.5 ≃ 43

42.6

42.9

↓

43.4

Solution: This problem differs from that in Example 6.11 in that we now assign a measurement of 43 ohms to all resistors whose resistances are greater than 42.5 and less than 43.5. We are actually approximating a discrete distribution by means of a continuous normal distribution. The required area is the region shaded to the right of 43.5 in Figure 6.19. We now find that

$$z = \frac{43.5 - 40}{2} = 1.75.$$

Hence,

$$P(X > 43.5) = P(Z > 1.75) = 1 - P(Z < 1.75) = 1 - 0.9599 = 0.0401.$$

Therefore, 4.01% of the resistances exceed 43 ohms when measured to the nearest ohm. The difference 6.68% − 4.01% = 2.67% between this answer and that of Example 6.11 represents all those resistance values greater than 43 and less than 43.5 that are now being recorded as 43 ohms. ∎

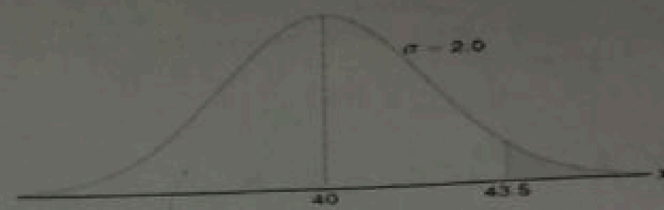


Figure 6.19: Area for Example 6.12.

$\boxed{\mu = 74}$ $\boxed{\sigma = 7}$

Example 6.13: The average grade for an exam is 74, and the standard deviation is 7. If 12% of the class is given As, and the grades are curved to follow a normal distribution, what is the lowest possible A and the highest possible B?

$P(A) = 0.12$

Solution: In this example, we begin with a known area of probability, find the z value, and then determine x from the formula $x = \sigma z + \mu$. An area of 0.12, corresponding to the fraction of students receiving As, is shaded in Figure 6.20. We require a z value that leaves 0.12 of the area to the right and, hence, an area of 0.88 to the left. From Table A.3, $P(Z < 1.18)$ has the closest value to 0.88, so the desired z value is 1.18. Hence,

$P(A < Z) = 0.12$

$$x = (7)(1.18) + 74 = 82.26.$$

المطلوب: أقل قيمة لـ A

وأعلى قيمة لـ B

Therefore, the lowest A is 83 and the highest B is 82. ∎

$Z = -1.18$

$\boxed{Z = 1.18}$

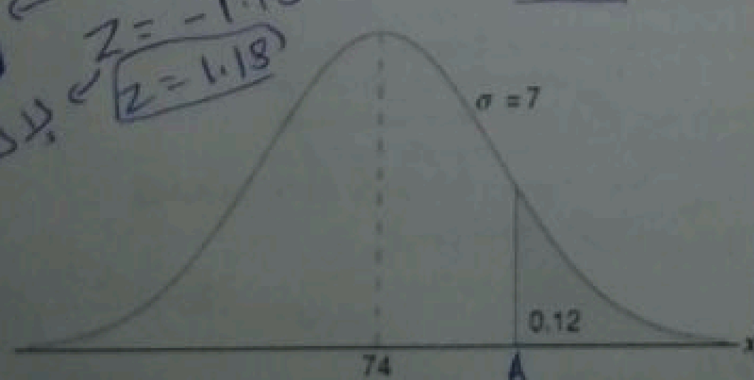


Figure 6.20: Area for Example 6.13.

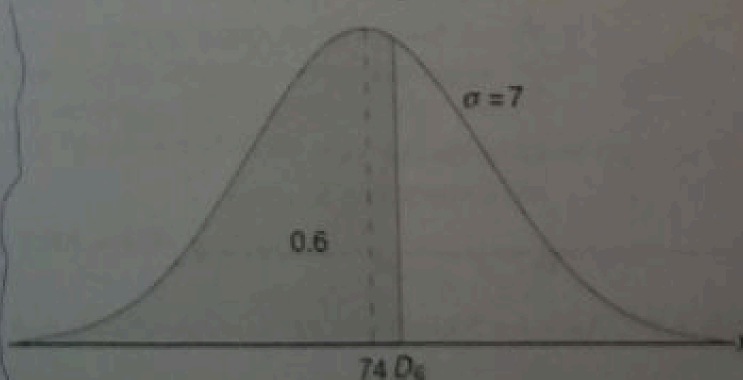
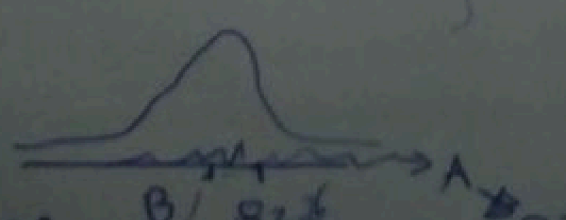


Figure 6.21: Area for Example 6.14.



and Exponential Distributions

6.2: The **gamma function** is defined by

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \text{for } \alpha > 0.$$

The following are a few simple properties of the gamma function.

(a) $\Gamma(n) = (n-1)(n-2)\cdots(1)\Gamma(1)$, for a positive integer n.

To see the proof, integrating by parts with $u = x^{\alpha-1}$ and $dv = e^{-x}\,dx$, we obtain

$$\Gamma(\alpha) = -e^{-x}\, x^{\alpha-1}\Big|_0^\infty + \int_0^\infty e^{-x}(\alpha-1)x^{\alpha-2}\, dx = (\alpha-1)\int_0^\infty x^{\alpha-2}e^{-x}\, dx,$$

for $\alpha > 1$, which yields the recursion formula

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1).$$

Gamma Distribution: The continuous random variable X has a **gamma distribution**, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

تحدد شكل المنحنى

where $\alpha > 0$ and $\beta > 0$.

scale parameter (avg)

$\Gamma(10) = (n-1)!$

$\boxed{\alpha = 1}$ is exponential function

specific number of events is the parameter α in the gamma density function

a (alpha) is known as the shape parameter, while b (beta) is referred to as the scale parameter

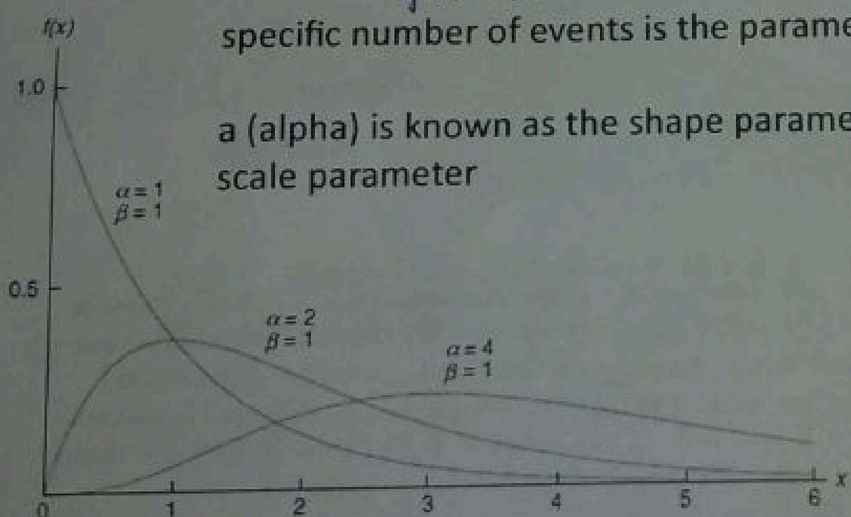


Figure 6.28: Gamma distributions.

اي اشي متعلق بال time

Exponential Distribution: The continuous random variable X has an **exponential distribution**, with parameter β, if its density function is given by

$$f(x;\beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta}, & x > 0, \\ 0, & \text{elsewhere}, \end{cases}$$

$\boxed{\alpha = 1}$

where $\beta > 0$.

Theorem 6.4: The mean and variance of the gamma distribution are

$$\mu = \alpha\beta \text{ and } \sigma^2 = \alpha\beta^2.$$

The proof of this theorem is found in Appendix A.26.

Corollary 6.1: The mean and variance of the exponential distribution are

$$\mu = \beta \text{ and } \sigma^2 = \beta^2.$$

!! مهم

Example 6.17: Suppose that a system contains a certain type of component whose time, in years, to failure is given by T. The random variable T is modeled nicely by the exponential distribution with mean time to failure $\beta = 5$. If 5 of these components are installed in different systems, what is the probability that at least 2 are still functioning at the end of 8 years?

على الاقل 2 من اصل 5 تشتغل

Binomial

Solution: The probability that a given component is still functioning after 8 years is given by

Binomial

$$P(T > 8) = \frac{1}{5}\int_8^\infty e^{-t/5}\,dt = e^{-8/5} \approx 0.2.$$

A.1 table ال من

Let X represent the number of components functioning after 8 years. Then using the binomial distribution,

$$P(X \geq 2) = \sum_{x=2}^{5} b(x; 5, 0.2) = 1 - \sum_{x=0}^{1} b(x; 5, 0.2) = 1 - 0.7373 = 0.2627.$$

$f(1)$

$\beta = 5$

$\mu = \beta$

$$\int_8^\infty \frac{1}{5} e^{-t/5}\,dt = -\frac{5}{5} e^{-t/5}\Big|_8^\infty = 0.2$$

نعوض في β دون التغير x او t

gamma

modelling for time → gamma

ما بيجي تكامل gamma function

Example 6.18: Suppose that telephone calls arriving at a particular switchboard follow a Poisson process with an average of 5 calls coming per minute. What is the probability that up to a minute will elapse by the time 2 calls have come in to the switchboard?

$\frac{1}{\lambda t} = \beta = 5$ $(0,1)$

$\alpha = 2$ عدد الاحداث

خلال one minute يكون عندي 2 calls

Solution: The Poisson process applies, with time until 2 Poisson events following a gamma distribution with $\beta = 1/5$ and $\alpha = 2$. Denote by X the time in minutes that transpires before 2 calls come. The required probability is given by

$$P(X \leq 1) = \int_0^1 \frac{1}{\beta^2} x e^{-x/\beta}\,dx = 25\int_0^1 x e^{-5x}\,dx = 1 - e^{-5}(1+5) = 0.96.$$

While the origin of the gamma distribution deals in time (or space) until the occurrence of α Poisson events, there are many instances where a gamma distribution works very well even though there is no clear Poisson structure. This is particularly true for survival time problems in both engineering and biomedical applications.

* the mean of the exponential distribution is the parameter θ, the reciprocal of the parameter in the Poisson distribution

β

$\beta = \frac{1}{\beta}$ poisson

Ch. 3

Example 6.21: Consider Exercise 3.31 on page 94. Based on extensive testing, it is determined that the time Y in years before a major repair is required for a certain washing machine is characterized by the density function

$$f(y) = \begin{cases} \frac{1}{4}e^{-y/4}, & y > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution: Consider the cumulative distribution function $F(y)$ for the exponential distribution,

$$F(y) = \frac{1}{\beta}\int_0^y e^{-t/\beta}\, dt = 1 - e^{-y/\beta}.$$

Then

$$P(Y > 6) = 1 - F(6) = e^{-3/2} = 0.2231.$$

Thus, the probability that the washing machine will require major repair after year six is 0.223. Of course, it will require repair before year six with probability 0.777. Thus, one might conclude the machine is not really a bargain. The probability that a major repair is necessary in the first year is

$$P(Y < 1) = 1 - e^{-1/4} = 1 - 0.779 = 0.221.$$

3.31 Based on extensive testing, it is determined by the manufacturer of a washing machine that the time Y (in years) before a major repair is required is characterized by the probability density function

$$f(x) = \begin{cases} \frac{1}{4}e^{-y/4} & y \geq 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Critics would certainly consider the product a bargain if it is unlikely to require a major repair before the sixth year. Comment on this by determining $P(Y > 6)$.

(b) What is the probability that a major repair occurs in the first year? 20

(a) For $y \geq 0$, $F(y) = \frac{1}{4}\int_0^y e^{-t/4}\, dy = 1 - e^{y/4}$. So, $P(Y > 6) = e^{-6/4} = 0.2231$. This probability certainly cannot be considered as "unlikely."

(b) $P(Y \leq 1) = 1 - e^{-1/4} = 0.2212$, which is not so small either.

Chi-Squared Distribution The continuous random variable X has a **chi-squared distribution**, with v **degrees of freedom**, if its density function is given by

α

تكامل بالأجزاء

$$f(x;v) = \begin{cases} \frac{1}{2^{v/2}\Gamma(v/2)} x^{v/2-1} e^{-x/2}, & x > 0, \\ 0, & \text{elsewhere}. \end{cases}$$

Theorem 6.5: The mean and variance of the chi-squared distribution are

$$\boxed{\mu = v \text{ and } \sigma^2 = 2v.}$$

Definition 6.3: A **beta function** is defined by

α

$$B(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ for } \alpha,\beta > 0,$$

where $\Gamma(\alpha)$ is the gamma function.

Beta Distribution The continuous random variable X has a **beta distribution** with parameters $\alpha > 0$ and $\beta > 0$ if its density function is given by

$$f(x) = \begin{cases} \frac{1}{B(\alpha,\beta)} x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1, \\ 0, & \text{elsewhere}. \end{cases}$$

Note that the uniform distribution on $(0,1)$ is a beta distribution with parameters $\alpha = 1$ and $\beta = 1$.

$\Gamma(8) = (n-1)! = 7!$

Theorem 6.6: The mean and variance of a beta distribution with parameters α and β are

$$\mu = \frac{\alpha}{\alpha+\beta} \text{ and } \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$

respectively.

For the uniform distribution on $(0,1)$, the mean and variance are

$$\mu = \frac{1}{1+1} = \frac{1}{2} \text{ and } \sigma^2 = \frac{(1)(1)}{(1+1)^2(1+1+1)} = \frac{1}{12},$$

4×3

The continuous random variable X has a **lognormal distribution** if the random variable $Y = \ln(X)$ has a normal distribution with mean μ and standard deviation σ. The resulting density function of X is

$$f(x;\mu,\sigma) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2\sigma^2}[\ln(x)-\mu]^2}, & x \ge 0, \\ 0, & x < 0. \end{cases}$$

α

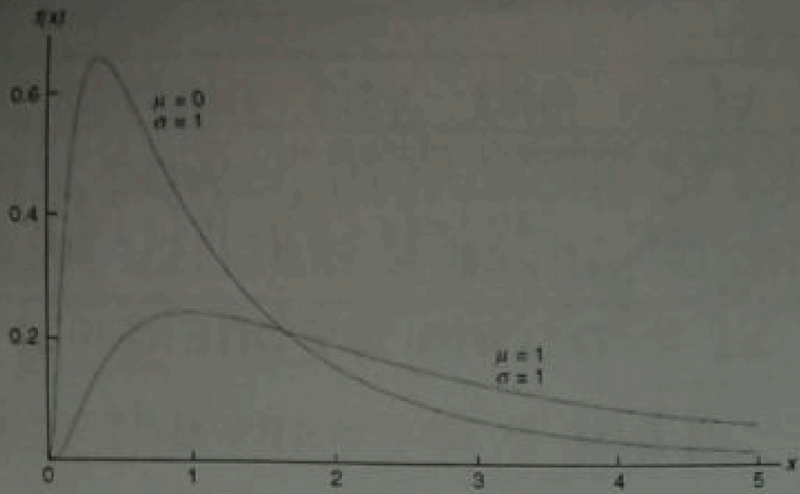


Figure 6.29: Lognormal distributions.

The mean and variance of the lognormal distribution are

$$\mu = e^{\mu+\sigma^2/2} \text{ and } \sigma^2 = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1).$$

ln ← log (الطبيعي) normal distribution

Example 6.22: Concentrations of pollutants produced by chemical plants historically are known to exhibit behavior that resembles a lognormal distribution. This is important when one considers issues regarding compliance with government regulations. Suppose it is assumed that the concentration of a certain pollutant, in parts per million, has a lognormal distribution with parameters $\mu = 3.2$ and $\sigma = 1$. What is the probability that the concentration exceeds 8 parts per million?

تجاوز الـ8

Solution: Let the random variable X be pollutant concentration. Then

$$P(X > 8) = 1 - P(X \le 8).$$

Since $\ln(X)$ has a normal distribution with mean $\mu = 3.2$ and standard deviation $\sigma = 1$,

$$P(X \le 8) = \Phi\left[\frac{\ln(8) - 3.2}{1}\right] = \Phi(-1.12) = 0.1314.$$

$= x - \mu$, σ , Z

$\to 1 - 0.1314 = 0.8686$

Example 6.23: The life, in thousands of miles, of a certain type of electronic control for locomotives has an approximately lognormal distribution with $\mu = 5.149$ and $\sigma = 0.737$. Find the 5th percentile of the life of such an electronic control.

Solution: From Table A.3, we know that $P(Z < -1.645) = 0.05$. Denote by X the life of such an electronic control. Since $\ln(X)$ has a normal distribution with mean $\mu = 5.149$ and $\sigma = 0.737$, the 5th percentile of X can be calculated as

$$\ln(x) = 5.149 + (0.737)(-1.645) = 3.937.$$

Hence, $x = 51.265$. This means that only 5% of the controls will have lifetimes less than 51,265 miles.

$P = \frac{5}{100}$

Weibull Distribution

The continuous random variable X has a **Weibull distribution**, with parameters α and β, if its density function is given by

$$f(x;\alpha,\beta)=\begin{cases}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}, & x>0,\\ 0, & \text{elsewhere,}\end{cases}$$

where $\alpha > 0$ and $\beta > 0$.

Theorem 6.8: The mean and variance of the Weibull distribution are

$$\mu=\alpha^{-1/\beta}\Gamma\left(1+\frac{1}{\beta}\right) \quad\text{and}\quad \sigma^2=\alpha^{-2/\beta}\left\{\Gamma\left(1+\frac{2}{\beta}\right)-\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2\right\}.$$

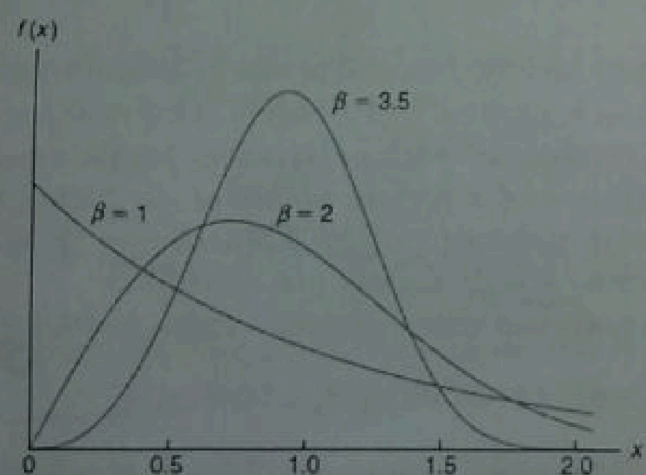


Figure 6.30: Weibull distributions ($\alpha = 1$).

cdf for Weibull Distribution

The cumulative distribution function for the Weibull distribution is given by

$$F(x)=1-e^{-\alpha x^{\beta}}, \qquad \text{for } x\geq 0,$$

for $\alpha > 0$ and $\beta > 0$.

Example 6.24: The length of life X, in hours, of an item in a machine shop has a Weibull distribution with $\alpha = 0.01$ and $\beta = 2$. What is the probability that it fails before eight hours of usage?

Solution: $P(X < 8) = F(8) = 1 - e^{-(0.01)8^{2}} = 1 - 0.527 = 0.473.$

$x = 8$

$P(x>8) = 1-(1-e^{-\alpha x^{\beta}})$

Ex:

In a certain city, the daily consumption of electric power, in millions of kilowatt-hours, is a random variable X having a gamma distribution with $\mu = 6$ and variance $\sigma^2 = 12$.

(a) Find the values of α and β.

(b) Find the probability that on any given day the daily power consumption will exceed 12 million kilowatthours. $P(x > 12)$

$\beta = \left(\frac{6}{\alpha}\right)$

- Answer:

(a) $\mu = \alpha\beta = 6$ $\boxed{\alpha\beta = 6}$ ①

$\sigma^2 = \alpha\beta^2 = 12$ $\boxed{\alpha\beta^2 = 12}$ ②

So, $\beta = 2$ and then $\alpha = 3$. $\alpha\left(\frac{36}{\alpha^2}\right) = 12$ $\boxed{\beta = \frac{6}{3} = 2}$

$\boxed{\alpha = \frac{36}{12} = 3}$

(b) $P(X > 12) = \frac{1}{16}\int_{12}^{\infty} x^2 e^{-x/2}\, dx$

$$P(X > 12) = \frac{1}{16}\left[-2x^2e^{-x/2} - 8xe^{-x/2} - 16e^{-x/2}\right]\Big|_{12}^{\infty} = 25e^{-6} = 0.0620$$

معهم !!

- The length of time for one individual to be served at a cafeteria is a random variable having an exponential distribution with a mean of 4 minutes. What is the probability that a person is served in less than 3 minutes on at least 4 of the next 6 days? Binomial

$\boxed{\beta = 4}$

$$P(X < 3) = \frac{1}{4}\int_0^3 e^{-x/4}\, dx = -e^{-x/4}\Big|_0^3 = 1 - e^{-3/4} = 0.5276.$$

$$P(Y \geq 4) = \sum_{x=4}^{6} b(y; 6, 1-e^{-3/4}) = \binom{6}{4}(0.5276)^4(0.4724)^2 + \binom{6}{5}(0.5276)^5(0.4724) + \binom{6}{6}(0.5276)^6 = 0.3968.$$

35

Probability $P(x \leq 3) = \int_0^3 \frac{1}{4} e^{-x/4}\, dx = -\frac{4}{4} e^{-x/4}\Big|_0^3 = -e^{3/4} + 1$

$= 0.5276$

$P(x \geq 4) = 1 - P(x < 4)$

$= 1 - 0.6563 = 0.3437$

تقريباً !!

End of Chapter 6

finally !!

Ch. 6

$\boxed{\text{uniform distribution}}$

$$f(x; A, B) = \begin{cases} \frac{1}{B-A} & A \le x \le B \\ 0 \end{cases}$$

$$\boxed{\mu = \frac{A+B}{2}} \qquad \boxed{\sigma^2 = \frac{(B-A)^2}{12}}$$

$\boxed{\text{Normal distribution}}$ A.3

$$n(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(x-\mu)^2} \qquad -\infty < x < \infty$$

$$Z = \frac{x-\mu}{\sigma}$$

$\boxed{\text{Gamma and exponental distribution}}$

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{\beta} x \, e^{\frac{-x}{\beta}} & , x > 0 \\ 0 \end{cases}$$

$$\boxed{\begin{array}{c} \mu = \alpha\beta \\ \sigma^2 = \alpha\beta^2 \end{array}}$$

expo

$$f(x; \beta) = \begin{cases} \frac{1}{\beta} e^{-x/\beta} & , x > 0 \end{cases}$$

$$\boxed{\begin{array}{c} \mu = \beta \\ \sigma^2 = \beta^2 \end{array}}$$

$\boxed{\text{chi squared}}$ $\mu = v \qquad \sigma^2 = 2v$

$\boxed{\text{Beta distribution}}$

$$F(x) = \begin{cases} \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} \\ 0 < x < 1 \end{cases}$$

$$\boxed{\mu = \frac{\alpha}{\alpha+\beta}} \qquad \sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

$\boxed{\text{lognormal distribution.}}$ $\mu = e^{\mu+\sigma^2} \qquad \sigma^2 = e^{2\mu+\sigma^2}(e^{\sigma^2}-1)$

$\boxed{\text{Weibull distribution}}$

$$F(x) = 1 - e^{-\alpha x^{\beta}}$$