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Probability and Statistics

Chapter 3

أنواع المتغيرات العشوائية

متصل
منفصل

Chapter 3

Random Variables and Probability Distributions

probability distribution

خروجي! نتاج من 3

Three electrical components are tested:

$$S = \{ \overset{1}{N}\overset{1}{N}\overset{1}{N}, \overset{1}{N}\overset{1}{N}\overset{2}{D}, \overset{1}{N}\overset{2}{D}\overset{2}{N}, \overset{1}{D}\overset{2}{N}\overset{2}{N}, \overset{2}{N}\overset{2}{D}\overset{3}{D}, \overset{2}{D}\overset{2}{N}\overset{3}{D}, \overset{2}{D}\overset{3}{D}\overset{3}{D} \}$$

If we are concerned with the number of defectives that occur?

0, 1, 2, or 3 ← متغير عشوائي: رقم حقيقي

$$E = \{ \overset{1}{D}\overset{2}{D}\overset{3}{N}, \overset{1}{D}\overset{2}{N}\overset{3}{D}, \overset{1}{N}\overset{2}{D}\overset{3}{D} \}$$

random variable ← 2 2 2

X → تعريف المتغير العشوائي
X → قيمة المتغير العشوائي
مثل الخزان الحما

sample space

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متغير عشوائي Z = (0, 1, 2)

Z → number of black

Definition 3.1: A random variable is a function that associates a real number with each element in the sample space.

Example 3.1: Two balls are drawn in succession without replacement from an urn containing 4 red balls and 3 black balls. The possible outcomes and the values y of the random variable Y , where Y is the number of red balls, are

Sample Space	y	
RR	2	احتمالات
RB	1	سواء
BR	1	موت
BB	0	

Example 3.2: A stockroom clerk returns three safety helmets at random to three steel mill employees who had previously checked them. If Smith, Jones, and Brown in that order receive one of the three hats, list the sample points for the possible orders of returning the helmets, and find the value m of the random variable M that represents the number of correct matches.

Solution: If S, J, and B stand for Smith's, Jones's, and Brown's helmets, respectively, then the possible arrangements in which the helmets may be returned and the number of correct matches are

Sample Space	m	
SJB	3	
SBJ	1	
BJS	1	
JSB	1	
JBS	0	
BSJ	0	

random variable

متغير عشوائي

Example 3.3: Consider the simple condition in which components are arriving from the production line and they are stipulated to be defective or not defective. Define the random variable X by

$$X = \begin{cases} 1, & \text{if the component is defective,} \\ 0, & \text{if the component is not defective.} \end{cases}$$

D D/N
 defective
 Value
 بالثاني
 في عدد
 صفتين

Example 3.4: Statisticians use sampling plans to either accept or reject batches or lots of material. Suppose one of these sampling plans involves sampling independently 10 items from a lot of 100 items in which 12 are defective.

12 defective
 D N

Let X be the random variable defined as the number of items found defective in the sample of 10. In this case, the random variable takes on the values 0, 1, 2, ..., 9, 10.

10 items sample

Example 3.5: Suppose a sampling plan involves sampling items from a process until a defective is observed. The evaluation of the process will depend on how many consecutive items are observed. In that regard, let X be a random variable defined by the number of items observed before a defective is found. With N a nondefective and D a defective, sample spaces are $S = \{D\}$ given $X = 1$, $S = \{ND\}$ given $X = 2$, $S = \{NND\}$ given $X = 3$, and so on.

مختصا
 خارج الثاني
 خرابان

defective $y = 1$
 item
 بجملة
 1 defective

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متغير عشوائي متقطع يتغير كمتغير
 probability
 بالتزجاء عند نقطة

x	0	1	2
$P(x)$			

Specific point
 $x=1$

اطساحة
 حول النقطة
 Count
 لكل اطةاحة تحت المعنى
 بالتالي اذا خطنا
 يكون عند نقطة الجواب
 يكون قريب الصفر
 بالتالي تكون infersity

Example 3.6: Interest centers around the proportion of people who respond to a certain mail order solicitation. Let X be that proportion. X is a random variable that takes on all values x for which $0 \leq x \leq 1$.

Definition 3.2: If a sample space contains a finite number of possibilities or an unending sequence with as many elements as there are whole numbers, it is called a discrete sample space.

Definition 3.3: If a sample space contains an infinite number of possibilities equal to the number of points on a line segment, it is called a continuous sample space.

على طول
 $a < x < b$

Discrete Probability Distributions

The probability that no employee gets back his right helmet

Sample Space

SJB
SBJ
BJS
JSB
JBS
BSJ

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Random Variable

m	0	1	3
$P(M=m)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
$f(x)$			

Definition 3.4:

The set of ordered pairs $(x, f(x))$ is a probability function, probability mass function, or probability distribution of the discrete random variable X if, for each possible outcome x ,

- $f(x) \geq 0$, Probability
- $\sum_x f(x) = 1$, Probability
- $P(X=x) = f(x)$.

$$P(X=0) = \frac{n}{N} = \frac{2}{6}$$

$$P(X=1) = \frac{3}{6} = \frac{1}{2}$$

$$P(X=3) = \frac{1}{6} \quad \text{or} \quad \frac{4}{6} - \frac{2}{6} - \frac{3}{6} = \frac{1}{6}$$

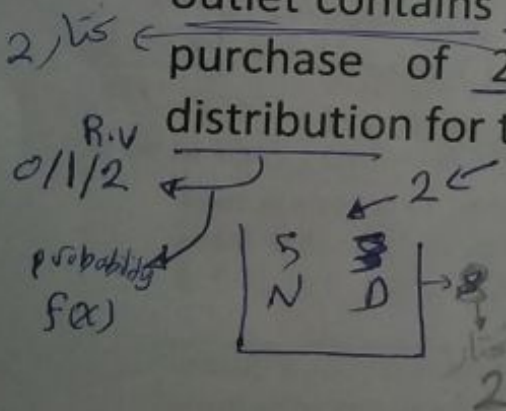
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احتمالية

$n=5$
 $r=2$

51
2131

Example 3.8 A shipment of 8 similar microcomputers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives?



$$f(0) = P(X=0) = \frac{\binom{3}{0} \binom{5}{2}}{\binom{8}{2}} = \frac{10}{28}$$

$$f(1) = P(X=1) = \frac{\binom{3}{1} \binom{5}{1}}{\binom{8}{2}} = \frac{15}{28}$$

$$f(2) = P(X=2) = \frac{\binom{3}{2} \binom{5}{0}}{\binom{8}{2}} = \frac{3}{28}$$

$$n = \binom{5}{2} = 10$$

$$N = \binom{8}{2} = 28$$

بهم طريقة يمكن
اختيار 2 من أصل 8

Thus the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{10}{28}$	$\frac{15}{28}$	$\frac{3}{28}$

$$1 - \frac{10}{28} - \frac{15}{28} = \frac{3}{28}$$

$$\binom{n}{n-1} = n \quad \binom{n}{0} = 1$$

$$\binom{n}{1} = n$$

Example 3.9: If a car agency sells 50% of its inventory of a certain foreign car equipped with side airbags, find a formula for the probability distribution of the number of cars with side airbags among the next 4 cars sold by the agency. The number of ways of selling 3 cars with side airbags

Solution: Since the probability of selling an automobile with side airbags is 0.5, the $2^4 = 16$ points in the sample space are equally likely to occur. Therefore, the denominator for all probabilities, and also for our function, is 16. To obtain the number of ways of selling 3 cars with side airbags, we need to consider the number of ways of partitioning 4 outcomes into two cells, with 3 cars with side airbags assigned to one cell and the model without side airbags assigned to the other. This can be done in $\binom{4}{3} = 4$ ways. In general, the event of selling x models with side airbags and $4 - x$ models without side airbags can occur in $\binom{4}{x}$ ways, where x can be 0, 1, 2, 3, or 4. Thus, the probability distribution $f(x) = P(X = x)$ is

$$f(x) = \frac{1}{16} \binom{4}{x}, \quad \text{for } x = 0, 1, 2, 3, 4.$$

There are many problems where we may wish to compute the probability that the observed value of a random variable X will be less than or equal to some real number x . Writing $F(x) = P(X \leq x)$ for every real number x , we define $F(x)$ to be the cumulative distribution function of the random variable X .

$$P(x=0) = \frac{n}{N} = \frac{\binom{4}{0}}{16} = \frac{1}{16}$$

$$P(x=1) = \frac{\binom{4}{1}}{16}$$

$$P(x=2) = \frac{\binom{4}{2}}{16}$$

$$P(x=3) = \frac{\binom{4}{3}}{16}$$

style air bag $\rightarrow N=16$

x	0	1	2	3	4
$f(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$\sum f(x) = 1$ or $\frac{16}{16} = 1$

$P = P(X \leq 3)$

$P(x > 4) = 0$

Definition 3.5: The cumulative distribution function $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for } -\infty < x < \infty.$$

For the random variable M , the number of correct matches in Example 3.2, we have

$$F(2) = P(M \leq 2) = f(0) + f(1) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The cumulative distribution function of M is

$$F(m) = \begin{cases} 0, & \text{for } m < 0, \\ \frac{1}{3}, & \text{for } 0 \leq m < 1, \\ \frac{5}{6}, & \text{for } 1 \leq m < 3, \\ 1, & \text{for } m \geq 3. \end{cases}$$

Sample Space	m
SJB	3
SBJ	1
BJS	1
JSB	1
JBS	0
BSJ	0

m	0	1	3
$P(M = m)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
$f(x)$			

Solution: Direct calculations of the probability distribution of Example 3.9 give $f(0) = 1/16$, $f(1) = 1/4$, $f(2) = 3/8$, $f(3) = 1/4$, and $f(4) = 1/16$. Therefore,

$f(1) = 1/4$, $f(2) = 3/8$, $f(3) = 1/4$, and $f(4) = 1/16$. Therefore,

$$f(x) = \frac{1}{16} \binom{4}{x}, \quad \text{for } x = 0, 1, 2, 3, 4$$

$$F(2) = f(0) + f(1) + f(2) = \frac{11}{16}.$$

$$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1.$$

Hence,

$$f(2) = \frac{15}{16}$$

$$f(x=3) - f(3) = F(3) - F(2) = \frac{15}{16} - \frac{11}{16} = \frac{4}{16}$$

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Now

$$\underline{f(2)} = F(2) - F(1) = \frac{11}{16} - \frac{5}{16} = \frac{3}{8}$$

$$P(X=2) = f(2)$$

$$\hookrightarrow f(3) = \frac{15}{16}$$

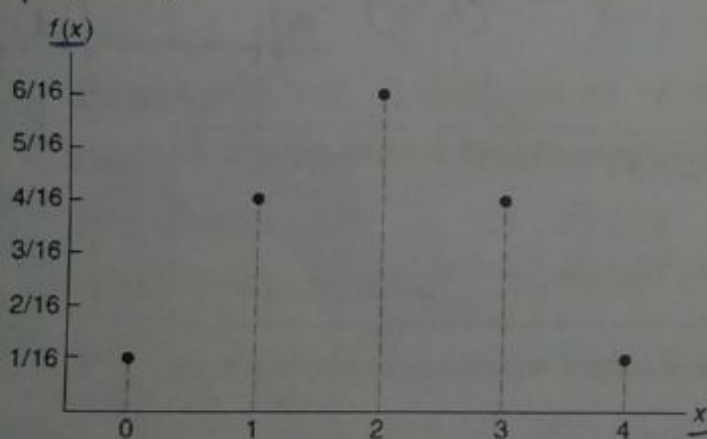
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Figure 3.1: Probability mass function plot.

x	f(x)
0	1/16
1	4/16
2	6/16
3	4/16
4	1/16

Figure 3.2: Probability histogram.

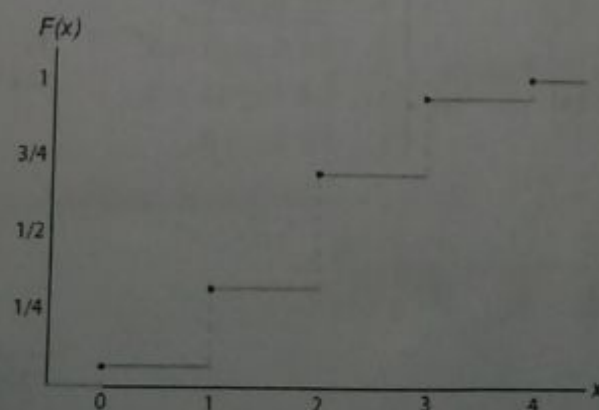


Figure 3.3: Discrete cumulative distribution function.

EX. 3.13: The probability distribution of A, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by:

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

Sumation of
probability

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ 0.41, & \text{for } 0 \leq x < 1, \\ 0.78, & \text{for } 1 \leq x < 2, \\ 0.94, & \text{for } 2 \leq x < 3, \\ 0.99, & \text{for } 3 \leq x < 4, \\ 1, & \text{for } x \geq 4. \end{cases}$$

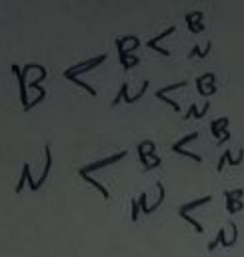
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x	0	1	2
$f(x)$	$\frac{n}{N}$		

$x = 0 / 1 / 2 \sim \text{مرتب } B$

2	2
N	B

- 3.2** An overseas shipment of 5 foreign automobiles contains 2 that have slight paint blemishes. If an agency receives (3) of these automobiles at random, list the elements of the sample space S using the letters B and N for blemished and nonblemished, respectively; then to each sample point assign a value x of the random variable X representing the number of automobiles purchased by the agency with paint blemishes.



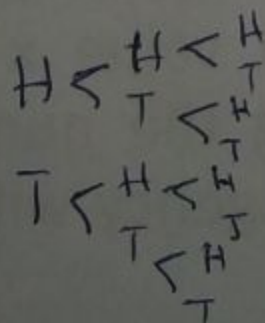
Sample Space	x
NNN	0
NNB	1
NBN	1
BNN	1
NBB	2
BNB	2
BBN	2
BBB	3

random
Variable

$$x = \{0, 1, 2, 3\}$$

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3.3 Let W be a random variable giving the number of heads minus the number of tails in three tosses of a coin. List the elements of the sample space S for the three tosses of the coin and to each sample point assign a value w of W .



$$w = HHH = 3 - 0$$

$$w = HHT = 2 - 1 = 1$$

Sample Space	w
HHHH	3
HHHT 2-1	1
HTHH 2-1	1
THHH 2-1	1
HTTT 1-2	-1
THTT 1-2	-1
TTHT 1-2	-1
TTTT 0-3	-3

$\rightarrow 8 = N$
جاري

فادخلها بالانشاء العشوائية بتفضل موجبة يا

$w \rightarrow$	-3	-1	1	3
$\leftarrow P(w)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$0 \leq P \leq 1$

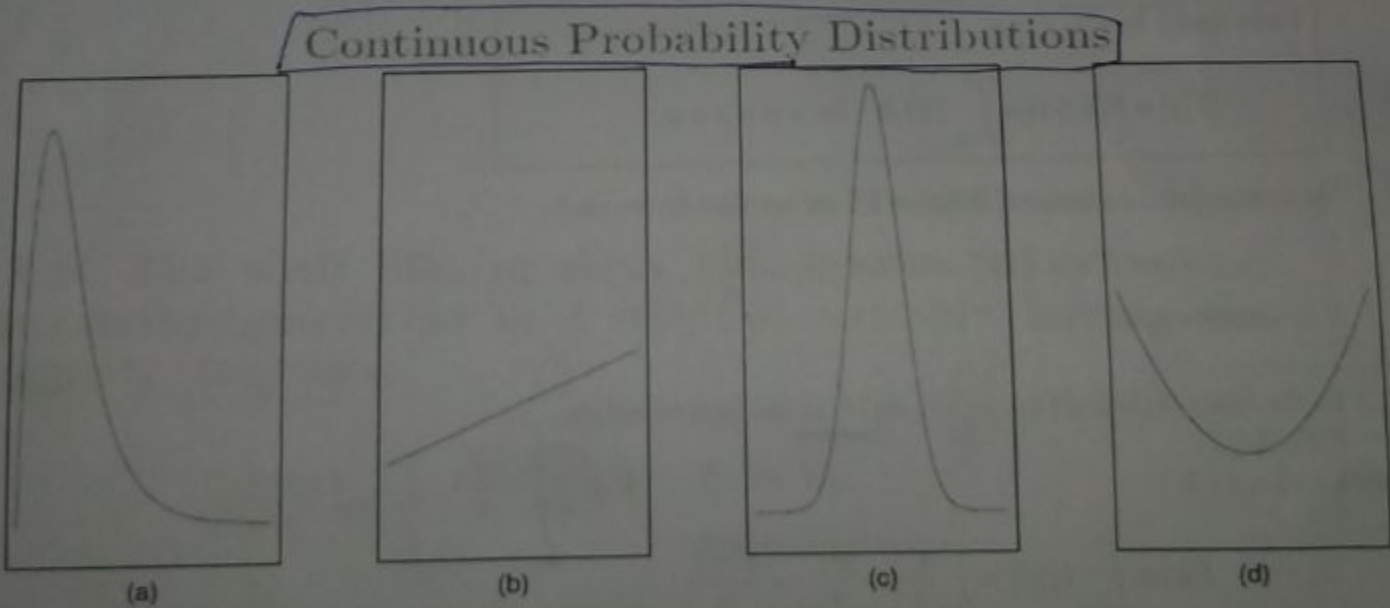
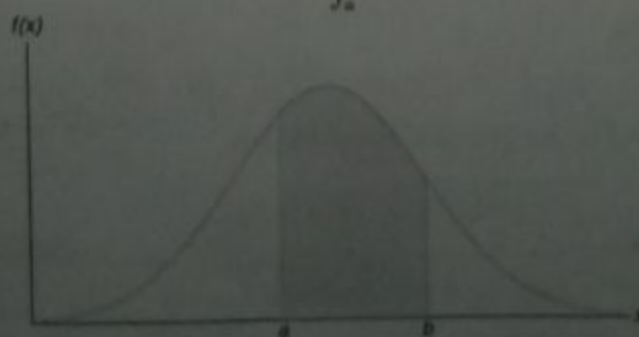


Figure 3.4: Typical density functions.

$$P(a < X < b) = \int_a^b f(x) dx.$$



a long line $\rightarrow$ prob

Figure 3.5: $P(a < X < b)$.

Definition 3.6:

The function $f(x)$ is a probability density function (pdf) for the continuous random variable X , defined over the set of real numbers, if

1. $f(x) \geq 0$, for all $x \in R$.

2. $\int_{-\infty}^{\infty} f(x) dx = 1$. $\leadsto \sum p = 1$ (discrete)

3. $P(a < X < b) = \int_a^b f(x) dx$.

Example 3.11: Suppose that the error in the reaction temperature, in $^{\circ}C$, for a controlled laboratory experiment is a continuous random variable X having the probability density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify that $f(x)$ is a density function.

(b) Find $P(0 < X \leq 1)$.

Solution: We use Definition 3.6.

(a) Obviously, $f(x) \geq 0$. To verify condition 2 in Definition 3.6, we have

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-1}^2 \frac{x^2}{3} dx = \frac{x^3}{9} \Big|_{-1}^2 = \frac{8}{9} - \left(-\frac{1}{9}\right) = \frac{8}{9} + \frac{1}{9} = 1.$$

(b) Using formula 3 in Definition 3.6, we obtain

$$P(0 < X \leq 1) = \int_0^1 \frac{x^2}{3} dx = \frac{x^3}{9} \Big|_0^1 = \frac{1}{9}.$$

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Definition 3.7: The cumulative distribution function $F(x)$ of a continuous random variable X with density function $f(x)$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(t) dt, \text{ for } -\infty < x < \infty.$$

As an immediate consequence of Definition 3.7, one can write the two results

$$P(a < X < b) = F(b) - F(a) \text{ and } f(x) = \frac{dF(x)}{dx},$$

if the derivative exists.

Example 3.12: For the density function of Example 3.11, find $F(x)$, and use it to evaluate $P(0 < X \leq 1)$.

Solution: For $-1 < x < 2$,

$$F(x) = \int_{-\infty}^x f(t) dt = \int_{-1}^x \frac{t^2}{3} dt = \frac{t^3}{9} \Big|_{-1}^x = \frac{x^3 + 1}{9}.$$

Therefore,

$$F(x) = \begin{cases} 0, & x < -1, \\ \frac{x^3 + 1}{9}, & -1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

The cumulative distribution function $F(x)$ is expressed in Figure 3.6. Now

$$P(0 < X \leq 1) = F(1) - F(0) = \frac{2}{9} - \frac{1}{9} = \frac{1}{9},$$

which agrees with the result obtained by using the density function in Example 3.11.

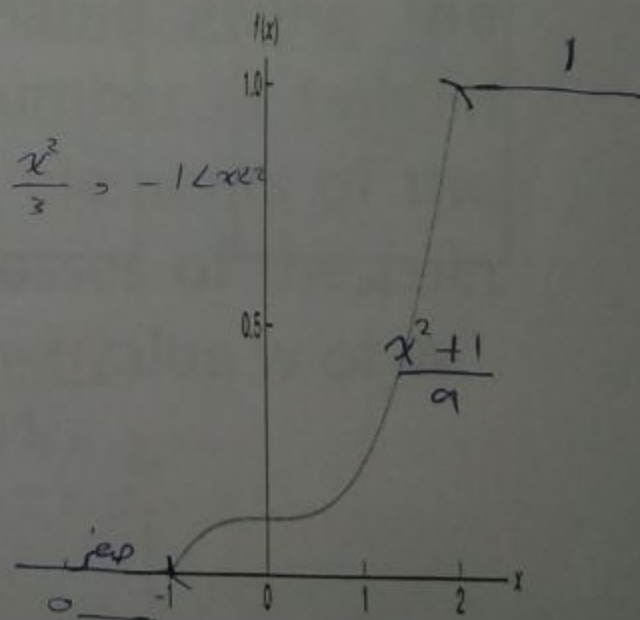


Figure 3.6: Continuous cumulative distribution function.

Example 3.13: The Department of Energy (DOE) puts projects out on bid and generally estimates what a reasonable bid should be. Call the estimate b . The DOE has determined that the density function of the winning (low) bid is

continuous
تابع مستمر

$$f(y) = \begin{cases} \frac{5}{8b}, & \frac{2}{5}b \leq y \leq 2b, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $F(y)$ and use it to determine the probability that the winning bid is less than the DOE's preliminary estimate b .

Solution: For $2b/5 \leq y \leq 2b$,

تستخدم y لأن الاقتران بدلالة y

$$F(y) = \int_{2b/5}^y \frac{5}{8b} dy = \frac{5t}{8b} \Big|_{2b/5}^y = \frac{5y}{8b} - \frac{1}{4}.$$

Thus,

مكونة من جزئين

$$F(y) = \begin{cases} 0, & y < \frac{2}{5}b, \\ \frac{5y}{8b} - \frac{1}{4}, & \frac{2}{5}b \leq y < 2b, \\ 1, & y \geq 2b. \end{cases}$$

To determine the probability that the winning bid is less than the preliminary bid estimate b , we have

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$$P(Y \leq b) = F(b) = \frac{5}{8} - \frac{1}{4} = \frac{3}{8}.$$

$$Y \leq b$$

3.6 The shelf life, in days, for bottles of a certain prescribed medicine is a random variable having the density function

$$f(x) = \begin{cases} \frac{20,000}{(x+100)^3}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the probability that a bottle of this medicine will have a shelf life of

(a) at least 200 days; $x \geq 200$

(b) anywhere from 80 to 120 days.

$$\int_{200}^{\infty} 20000 (x+100)^{-3} dx$$

$$\frac{20000}{-2} (x+100)^{-2} \Big|_{200}^{\infty}$$

$$(a) = 1/9$$

$$(b) = 0.1020$$

$$\int_{80}^{120} 20000 (x+100)^{-3} dx$$

3.7 The total number of hours, measured in units of 100 hours, that a family runs a vacuum cleaner over a period of one year is a continuous random variable X that has the density function

$$f(x) = \begin{cases} x, & 0 < x < 1, \\ 2-x, & 1 \leq x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

1 unit = 100 hours

$x = 1.2$

$x = 1.2$

Find the probability that over a period of one year, a family runs their vacuum cleaner

(a) less than 120 hours;

(b) between 50 and 100 hours,

$P(50 < X < 100)$

$$\int_0^{1.2} f(x) dx = \int_0^1 x dx + \int_1^{1.2} (2-x) dx$$

(a) 0.68

(b) = 0.375

$$\int_{0.5}^1 x dx \rightarrow \frac{1}{2} x^2 \Big|_{0.5}^1$$

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Definition 3.8:

The function $f(x, y)$ is a joint probability distribution or probability mass function of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y) ,

2. $\sum_x \sum_y f(x, y) = 1$,

3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum_A f(x, y)$.

Example 3.14: Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

(a) the joint probability function $f(x, y)$,

(b) $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

Solution: The possible pairs of values (x, y) are $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, and $(2, 0)$.

(a) Now, $f(0, 1)$, for example, represents the probability that a red and a green pens are selected. The total number of equally likely ways of selecting any 2 pens from the 8 is $\binom{8}{2} = 28$. The number of ways of selecting 1 red from 2 red pens and 1 green from 3 green pens is $\binom{2}{1} \binom{3}{1} = 6$. Hence, $f(0, 1) = 6/28 = 3/14$. Similar calculations yield the probabilities for the other cases, which are presented in Table 3.1. Note that the probabilities sum to 1. In Chapter

$$\frac{n}{n-1} = n$$

$$x \rightarrow 0/1/2$$

$P(x)$

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$$y \rightarrow 0/1/2$$

$x \backslash y$	0	1	2
0	$\frac{6}{28} = \frac{3}{14}$	$\frac{4}{28} = \frac{1}{7}$	$\frac{3}{28}$
1	$\frac{6}{28}$	$\frac{5}{28}$	0
2	$\frac{1}{28}$	0	0

$f(0,0)$

$P(X=0, Y=0)$

$\frac{n}{N}$

5, it will become clear that the joint probability distribution of Table 3.1 can be represented by the formula

$$f(x, y) = \frac{\binom{3}{x} \binom{2}{y} \binom{3}{2-x-y}}{\binom{8}{2}},$$

for $x = 0, 1, 2$; $y = 0, 1, 2$; and $0 \leq x + y \leq 2$.

(b) The probability that (X, Y) fall in the region A is

$$\begin{aligned} P[(X, Y) \in A] &= P(X + Y \leq 1) = f(0, 0) + f(0, 1) + f(1, 0) \\ &= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{9}{14}. \end{aligned}$$

Table 3.1: Joint Probability Distribution for Example 3.14

$f(x, y)$		x			Row
		0	1	2	Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$h(x) P(x+y \leq 4)$
 $\downarrow$
 $= 1$
 !!

$g(x)$

When X and Y are continuous random variables, the **joint density function** $f(x, y)$ is a surface lying above the xy plane, and $P[(X, Y) \in A]$, where A is any region in the xy plane, is equal to the volume of the right cylinder bounded by the base A and the surface.

Continuous

Definition 3.9: The function $f(x, y)$ is a **joint density function** of the **continuous random variables** X and Y if

1. $f(x, y) \geq 0$, for all (x, y) ,
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y) dx dy$, for any region A in the xy plane.

Example 3.15: A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y , respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

Joint

$$f(x, y) = \begin{cases} \frac{2}{5}(2x + 3y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Verify condition 2 of Definition 3.9.
- (b) Find $P[(X, Y) \in A]$, where $A = \{(x, y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

Solution: (a) The integration of $f(x, y)$ over the whole region is

$$\begin{aligned}\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy &= \int_0^1 \int_0^1 \frac{2}{5} (2x + 3y) dx dy \\ &= \int_0^1 \left(\frac{2x^2}{5} + \frac{6xy}{5} \right) \Big|_{x=0}^{x=1} dy \\ &= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5} \right) dy = \left(\frac{2y}{5} + \frac{3y^2}{5} \right) \Big|_0^1 = \frac{2}{5} + \frac{3}{5} = 1.\end{aligned}$$

(b) To calculate the probability, we use

الوقت حدد التفاضل

$$\begin{aligned}P[(X, Y) \in A] &= P\left(0 < X < \frac{1}{2}, \frac{1}{4} < Y < \frac{1}{2}\right) \\ &= \int_{1/4}^{1/2} \int_0^{1/2} \frac{2}{5} (2x + 3y) dx dy \\ &= \int_{1/4}^{1/2} \left(\frac{2x^2}{5} + \frac{6xy}{5} \right) \Big|_{x=0}^{x=1/2} dy = \int_{1/4}^{1/2} \left(\frac{1}{10} + \frac{3y}{5} \right) dy \\ &= \left(\frac{y}{10} + \frac{3y^2}{10} \right) \Big|_{1/4}^{1/2} \\ &= \frac{1}{10} \left[\left(\frac{1}{2} + \frac{3}{4} \right) - \left(\frac{1}{4} + \frac{3}{16} \right) \right] = \frac{13}{160}.\end{aligned}$$

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along along
Joint probability distribution ← محسب في كلاً من المحاور

Given the joint probability distribution $f(x, y)$ of the discrete random variables X and Y , the probability distribution $g(x)$ of X alone is obtained by summing $f(x, y)$ over the values of Y . Similarly, the probability distribution $h(y)$ of Y alone is obtained by summing $f(x, y)$ over the values of X . We define $g(x)$ and $h(y)$ to be the marginal distributions of X and Y , respectively. When X and Y are continuous random variables, summations are replaced by integrals. We can now make the following general definition.

Definition 3.10: The marginal distributions of X alone and of Y alone are

$$g(x) = \sum_y f(x, y) \quad \text{and} \quad h(y) = \sum_x f(x, y) \quad \text{Conf}$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x, y) dx$$

for the continuous case.

The term *marginal* is used here because, in the discrete case, the values of $g(x)$ and $h(y)$ are just the marginal totals of the respective columns and rows when the values of $f(x, y)$ are displayed in a rectangular table.

one dimension → لا اهتمام

Example 3.16: Show that the column and row totals of Table 3.1 give the marginal distribution of X alone and of Y alone.

Solution: For the random variable X , we see that

$$g(0) = f(0,0) + f(0,1) + f(0,2) = \frac{3}{28} + \frac{3}{14} + \frac{1}{28} = \frac{5}{14},$$

$$g(1) = f(1,0) + f(1,1) + f(1,2) = \frac{9}{28} + \frac{3}{14} + 0 = \frac{15}{28},$$

and

$$g(2) = f(2,0) + f(2,1) + f(2,2) = \frac{3}{28} + 0 + 0 = \frac{3}{28},$$

which are just the column totals of Table 3.1. In a similar manner we could show that the values of $h(y)$ are given by the row totals. In tabular form, these marginal distributions may be written as follows:

x	0	1	2
$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

Table 3.1: Joint Probability Distribution for Example 3.14

$f(x,y)$	x	y	Row Totals
	0	1	2
0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$
1	$\frac{3}{14}$	$\frac{3}{14}$	0
2	$\frac{1}{28}$	0	0
Column Totals	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

$$\rightarrow h(0)$$

$$\rightarrow h(1)$$

$$\rightarrow h(2)$$

$$\leftarrow h(y)$$

$$g(x) \rightarrow g(0) \quad g(1) \quad g(2)$$

الفكرة إنهم جازوا Joint
فصلناهم فصل y عن x
أول x عن y

table 1 بي 1

دعنا نكتب $g(x)$ و $h(y)$

table 2 بي 1

Summation 1

marginal distribution

Example 3.17: Find $g(x)$ and $h(y)$ for the joint density function of Example 3.15.

Solution: By definition,

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \leq x \leq 1, 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

$$g(x) = \int_{-\infty}^{\infty} f(x,y) dy = \int_0^1 \frac{2}{5}(2x+3y) dy = \left(\frac{4xy}{5} + \frac{6y^2}{10} \right) \Big|_{y=0}^{y=1} = \frac{4x+3}{5},$$

for $0 \leq x \leq 1$, and $g(x) = 0$ elsewhere. Similarly,

$$h(y) = \int_{-\infty}^{\infty} f(x,y) dx = \int_0^1 \frac{2}{5}(2x+3y) dx = \frac{2(1+3y)}{5},$$

for $0 \leq y \leq 1$, and $h(y) = 0$ elsewhere.

The fact that the marginal distributions $g(x)$ and $h(y)$ are indeed the probability distributions of the individual variables X and Y alone can be verified by showing that the conditions of Definition 3.4 or Definition 3.6 are satisfied. For example, in the continuous case

$$\int_{-\infty}^{\infty} g(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dy dx = 1,$$

and

$$P(a < X < b) = P(a < X < b, -\infty < Y < \infty) \\ = \int_a^b \int_{-\infty}^{\infty} f(x,y) dy dx = \int_a^b g(x) dx.$$

In Section 3.1, we stated that the value x of the random variable X represents an event that is a subset of the sample space. If we use the definition of conditional probability as stated in Chapter 2,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) > 0,$$

$f(x,y) \rightarrow$ احتمال مشترك
احتمال x وحده
احتمال y وحده

احتمال مشترك
احتمال x وحده
احتمال y وحده

where A and B are now the events defined by $X = x$ and $Y = y$, respectively,

احتمال لحدث مشترك
احتمال الذي نتحدث
Joint Function

$$P(Y = y | X = x) = \frac{P(X = x, Y = y)}{P(X = x)} = \frac{f(x, y)}{g(x)}, \text{ provided } g(x) > 0,$$

where X and Y are discrete random variables.

It is not difficult to show that the function $f(x, y)/g(x)$, which is strictly a function of y with x fixed, satisfies all the conditions of a probability distribution. This is also true when $f(x, y)$ and $g(x)$ are the joint density and marginal distribution, respectively, of continuous random variables. As a result, it is extremely important that we make use of the special type of distribution of the form $f(x, y)/g(x)$ in order to be able to effectively compute conditional probabilities. This type of distribution is called a conditional probability distribution; the formal definition follows.

Definition 3.11: Let X and Y be two random variables, discrete or continuous. The **conditional distribution** of the random variable Y given that $X = x$ is

دالة الاحتمال
Joint Function

$$f(y|x) = \frac{f(x, y)}{g(x)}, \text{ provided } g(x) > 0.$$

Similarly, the conditional distribution of X given that $Y = y$ is

$$f(x|y) = \frac{f(x, y)}{h(y)}, \text{ provided } h(y) > 0.$$

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If we wish to find the probability that the discrete random variable X falls between a and b when it is known that the discrete variable $Y = y$, we evaluate

$$P(a < X < b | Y = y) = \sum_{a < x < b} f(x|y),$$

where the summation extends over all values of X between a and b . When X and Y are continuous, we evaluate

$$P(a < X < b | Y = y) = \int_a^b f(x|y) dx,$$

Function
نفسه على
التابع

Example 3.18: Referring to Example 3.14, find the conditional distribution of X , given that $Y = 1$, and use it to determine $P(X = 0 | Y = 1)$.

Solution: We need to find $f(x|y)$, where $y = 1$. First, we find that

$$h(1) = \sum_{x=0}^2 f(x, 1) = \frac{3}{14} + \frac{3}{14} + 0 = \frac{3}{7}.$$

Now

$$f(x|1) = \frac{f(x, 1)}{h(1)} = \left(\frac{7}{3}\right) f(x, 1), \quad x = 0, 1, 2.$$

Therefore,

$$f(0|1) = \left(\frac{7}{3}\right) f(0, 1) = \left(\frac{7}{3}\right) \left(\frac{3}{14}\right) = \frac{1}{2}, \quad f(1|1) = \left(\frac{7}{3}\right) f(1, 1) = \left(\frac{7}{3}\right) \left(\frac{3}{14}\right) = \frac{1}{2},$$

$$f(2|1) = \left(\frac{7}{3}\right) f(2, 1) = \left(\frac{7}{3}\right) (0) = 0,$$

and the conditional distribution of X , given that $Y = 1$, is

x	0	1	2
$f(x 1)$	$\frac{1}{2}$	$\frac{1}{2}$	0

Finally,

$$P(X = 0 | Y = 1) = f(0|1) = \frac{1}{2}.$$

Therefore, if it is known that 1 of the 2 pen refills selected is red, we have a probability equal to $1/2$ that the other refill is not blue.

اللي هو نابت
 $f(x|y) = \frac{f(x,y)}{h(y)}$

Table 3.1: Joint Probability Distribution for Example 3.14

	$f(x, y)$	y			Row Totals
		0	1	2	
x	0	$\frac{3}{14}$	$\frac{3}{14}$	$\frac{3}{14}$	$\frac{9}{14}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{6}{14}$
	2	$\frac{1}{14}$	0	0	$\frac{1}{14}$
Column Totals		$\frac{6}{14}$	$\frac{6}{14}$	$\frac{3}{14}$	1

dis

$$\frac{3}{14} \times \frac{7}{3} = \frac{1}{2}$$

$$\frac{3}{14} \times \frac{7}{3}$$

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Example 3.19: The joint density for the random variables (X, Y) , where X is the unit temperature change and Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f(x, y) = \begin{cases} 10xy^2, & 0 < x < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Find the marginal densities $g(x)$, $h(y)$, and the conditional density $f(y|x)$.
 (b) Find the probability that the spectrum shifts more than half of the total observations, given that the temperature is increased by 0.25 unit.

Solution: (a) By definition,

x alone $\rightarrow g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_x^1 10xy^2 dy$

$$= \frac{10}{3} xy^3 \Big|_{y=x}^{y=1} = \frac{10}{3} x(1 - x^3), \quad 0 < x < 1.$$

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^y 10xy^2 dx = 5x^2 y^2 \Big|_{x=0}^{x=y} = 5y^4, \quad 0 < y < 1.$$

Now

$$f(y|x) = \frac{f(x, y)}{g(x)} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)} = \frac{3y^2}{1-x^3}, \quad 0 < x < y < 1.$$

(b) Therefore,

$$P\left(Y > \frac{1}{2} \mid X = 0.25\right) = \int_{1/2}^1 f(y | x = 0.25) dy = \int_{1/2}^1 \frac{3y^2}{1 - 0.25^3} dy = \frac{8}{9}.$$

في الفونكشن

Example 3.20: Given the joint density function

$$f(x, y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2, \quad 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

find $g(x)$, $h(y)$, $f(x|y)$, and evaluate $P(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3})$. ✓

Solution: By definition of the marginal density, for $0 < x < 2$,

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_0^1 \frac{x(1+3y^2)}{4} dy = \left(\frac{xy}{4} + \frac{xy^3}{4} \right) \Big|_{y=0}^{y=1} = \frac{x}{2},$$

and for $0 < y < 1$,

$$h(y) = \int_{-\infty}^{\infty} f(x, y) dx = \int_0^2 \frac{x(1+3y^2)}{4} dx = \left(\frac{x^2}{8} + \frac{3x^2 y^2}{8} \right) \Big|_{x=0}^{x=2} = \frac{1+3y^2}{2}.$$

Therefore, using the conditional density definition, for $0 < x < 2$,

$$f(x|y) = \frac{f(x, y)}{h(y)} = \frac{x(1+3y^2)/4}{(1+3y^2)/2} = \frac{x}{2},$$

and

$$\Rightarrow P\left(\frac{1}{4} < X < \frac{1}{2} \mid Y = \frac{1}{3}\right) = \int_{1/4}^{1/2} \frac{x}{2} dx = \frac{3}{64}.$$

$$P(A) = P(B)$$

Statistical Independence

If $f(x|y)$ does not depend on y , as is the case for Example 3.20, then $f(x|y) = g(x)$ and $f(x, y) = g(x)h(y)$. The proof follows by substituting

$$f(x, y) = f(x|y)h(y)$$

into the marginal distribution of X . That is,

$$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{-\infty}^{\infty} f(x|y)h(y) dy.$$

If $f(x|y)$ does not depend on y , we may write

$$g(x) = f(x|y) \int_{-\infty}^{\infty} h(y) dy.$$

Now

$$\int_{-\infty}^{\infty} h(y) dy = 1,$$

since $h(y)$ is the probability density function of Y . Therefore,

$$g(x) = f(x|y) \quad \text{and then} \quad f(x, y) = g(x)h(y).$$

$$A \cap B = P(A) * P(B)$$

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Definition 3.12: Let X and Y be two random variables, discrete or continuous, with joint probability distribution $f(x, y)$ and marginal distributions $g(x)$ and $h(y)$, respectively. The random variables X and Y are said to be **statistically independent** if and only if

$$f(x, y) = g(x)h(y) \quad \text{Independent}$$

for all (x, y) within their range.

Example 3.21: Show that the random variables of Example 3.14 are not statistically independent.

Proof: Let us consider the point $(0, 1)$. From Table 3.1 we find the three probabilities $f(0, 1)$, $g(0)$, and $h(1)$ to be

$$f(0, 1) = \frac{3}{14}$$

$$g(0) = \sum_{y=0}^2 f(0, y) = \frac{3}{28} + \frac{3}{14} + \frac{1}{28} = \frac{5}{14}$$

$$h(1) = \sum_{x=0}^2 f(x, 1) = \frac{3}{14} + \frac{3}{14} + 0 = \frac{3}{7}$$

Clearly,

$$\frac{3}{14} \neq \frac{5}{14} * \frac{3}{7}$$

and therefore X and Y are not statistically independent. $\blacksquare$

All the preceding definitions concerning two random variables can be generalized to the case of n random variables. Let $f(x_1, x_2, \dots, x_n)$ be the joint probability function of the random variables $X_1, X_2, \dots, X_n$. The marginal distribution of X_1 , for example, is

$$g(x_1) = \sum_{x_2} \dots \sum_{x_n} f(x_1, x_2, \dots, x_n)$$

Table 3.1: Joint Probability Distribution for Example 3.14

$f(x,y)$		x			Row
		0	1	2	Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$\frac{3}{14} \neq \frac{15}{98}$$

for the discrete case, and

$$g(x_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_n) dx_2 dx_3 \cdots dx_n$$

for the continuous case. We can now obtain **joint marginal distributions** such as $g(x_1, x_2)$, where

$$g(x_1, x_2) = \begin{cases} \sum_{x_3} \cdots \sum_{x_n} f(x_1, x_2, \dots, x_n) & \text{(discrete case),} \\ \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, x_2, \dots, x_n) dx_3 dx_4 \cdots dx_n & \text{(continuous case).} \end{cases}$$

We could consider numerous conditional distributions. For example, the **joint conditional distribution** of X_1, X_2 , and X_3 , given that $X_4 = x_4, X_5 = x_5, \dots, X_n = x_n$, is written

$$f(x_1, x_2, x_3 | x_4, x_5, \dots, x_n) = \frac{f(x_1, x_2, \dots, x_n)}{g(x_4, x_5, \dots, x_n)}$$

where $g(x_4, x_5, \dots, x_n)$ is the joint marginal distribution of the random variables $X_4, X_5, \dots, X_n$.

A generalization of Definition 3.12 leads to the following definition for the mutual statistical independence of the variables $X_1, X_2, \dots, X_n$.

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مستقلة احصائي (statistical independence)

Definition 3.13: Let $X_1, X_2, \dots, X_n$ be n random variables, discrete or continuous, with joint probability distribution $f(x_1, x_2, \dots, x_n)$ and marginal distribution $f_1(x_1), f_2(x_2), \dots, f_n(x_n)$, respectively. The random variables $X_1, X_2, \dots, X_n$ are said to be mutually **statistically independent** if and only if

$$f(x_1, x_2, \dots, x_n) = \underline{f_1(x_1)} \underline{f_2(x_2)} \cdots \underline{f_n(x_n)}$$

for all $(x_1, x_2, \dots, x_n)$ within their range.

Example 3.22: Suppose that the shelf life, in years, of a certain perishable food product packaged in cardboard containers is a random variable whose probability density function is given by

$$f(x) = \begin{cases} e^{-x}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

Let X_1, X_2 , and X_3 represent the shelf lives for three of these containers selected independently and find $P(X_1 < 2, 1 < X_2 < 3, X_3 > 2)$.

Solution: Since the containers were selected independently, we can assume that the random variables X_1, X_2 , and X_3 are statistically independent, having the joint probability density

$$f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3) = e^{-x_1}e^{-x_2}e^{-x_3} = e^{-x_1-x_2-x_3},$$

for $x_1 > 0, x_2 > 0, x_3 > 0$, and $f(x_1, x_2, x_3) = 0$ elsewhere. Hence

$$P(X_1 < 2, 1 < X_2 < 3, X_3 > 2) = \int_2^{\infty} \int_1^3 \int_0^2 e^{-x_1-x_2-x_3} dx_1 dx_2 dx_3 = (1 - e^{-2})(e^{-1} - e^{-3})e^{-2} = 0.0372.$$

38

من فترة السؤال الاحصائية

• **Ex. 3.47**

- The amount of kerosene, in thousands of liters, in a tank at the beginning of any day is a random amount Y from which a random amount X is sold during that day. Suppose that the tank is not resupplied during the day so that $x < y$, and assume that the joint density is

$$f_{X,Y}(x,y) = \begin{cases} 2, & 0 < x \leq y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

- (a) Determine if X and Y are independent.
- (b) Find $P(1/4 < X < 1/2 \mid Y = 3/4)$.

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(a)

$$g(x) = 2 \int_x^1 dy = 2(1 - x) \text{ for } 0 < x < 1;$$

$$h(y) = 2 \int_0^y dx = 2y, \text{ for } 0 < y < 1.$$

Since $f(x,y) \neq g(x)h(y)$, X and Y are not independent.

(b)

$$f(x|y) = f(x,y)/h(y) = 1/y, \text{ for } 0 < x < y.$$

$$\text{Therefore, } P(1/4 < X < 1/2 \mid Y = 3/4) = \frac{4}{3} \int_{1/4}^{1/2} dx = \frac{1}{3}.$$

Mathematical Expectation

$$S = \{TT / TH / HT / HH\}$$

16 مرة
تجارب

number of head $\leftarrow X = \{0, 1, 2\}$

avg / 6

4.1 Mean of a Random Variable

In Chapter 1, we discussed the sample mean, which is the arithmetic mean of the data. Now consider the following. If two coins are tossed 16 times and X is the number of heads that occur per toss, then the values of X are 0, 1, and 2. Suppose that the experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively. The average number of heads per toss of the two coins is then

$$\text{mean} = \frac{(0)(4) + (1)(7) + (2)(5)}{16} = 1.06$$

This is an average value of the data and yet it is not a possible outcome of $\{0, 1, 2\}$. Hence, an average is not necessarily a possible outcome for the experiment. For instance, a salesman's average monthly income is not likely to be equal to any of his monthly paychecks.

Let us now restructure our computation for the average number of heads so as to have the following equivalent form:

$$\text{mean} = (0) \left(\frac{4}{16} \right) + (1) \left(\frac{7}{16} \right) + (2) \left(\frac{5}{16} \right) = 1.06$$

16 مرة
H 0.5

4 → 0 H

7 → 1 H

5 → 2 H

Assuming that 1 fair coin was tossed twice, we find that the sample space for our experiment is

$$S = \{HH, HT, TH, TT\}$$

Since the 4 sample points are all equally likely, it follows that

$$P(X=0) = P(TT) = \frac{1}{4}, \quad P(X=1) = P(TH) + P(HT) = \frac{1}{2}, \quad P(X=2) = P(HH) = \frac{1}{4}$$

and

$$P(X=2) = P(HH) = \frac{1}{4}$$

where a typical element, say TH , indicates that the first toss resulted in a tail followed by a head on the second toss. Now, these probabilities are just the relative frequencies for the given events in the long run. Therefore,

expected mean value $\mu = E(X) = (0) \left(\frac{1}{4}\right) + (1) \left(\frac{1}{2}\right) + (2) \left(\frac{1}{4}\right) = 1$

Probability Variable

x	0	1	2

Definition 4.1: Let X be a random variable with probability distribution $f(x)$. The mean, or expected value, of X is

$$\mu = E(X) = \sum x f(x)$$

mean $\leftarrow$ expected value

If X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx$$

if X is continuous

Example 4.1: A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Solution: Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x} \binom{3}{3-x}}{\binom{7}{3}}, \quad x = 0, 1, 2, 3$$

Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, and $f(3) = 4/35$. Therefore,

$$\mu = E(X) = (0) \left(\frac{1}{35}\right) + (1) \left(\frac{12}{35}\right) + (2) \left(\frac{18}{35}\right) + (3) \left(\frac{4}{35}\right) = \frac{12}{7} = 1.7$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

بالاحتمال
بيني نحسب
عن نقطة
واحدة

7/3

Example 4.2: A salesperson for a medical device company has two appointments on a given day. At the first appointment, he believes that he has a 70% chance to make the deal, from which he can earn \$1000 commission if successful. On the other hand, he thinks he only has a 40% chance to make the deal at the second appointment, from which, if successful, he can make \$1500. What is his expected commission based on his own probability belief? Assume that the appointment results are independent of each other.

مقر
 $X = \{0, 1000, 1500, 2500\}$
 1000
 2500

* Solution: First, we know that the salesperson, for the two appointments, can have 4 possible commission totals: \$0, \$1000, \$1500, and \$2500. We then need to calculate their associated probabilities. By independence, we obtain

مقر متوقع
 صاري

$$f(\$0) = (1 - 0.7)(1 - 0.4) = 0.18, \quad f(\$2500) = (0.7)(0.4) = 0.28, \\ f(\$1000) = (0.7)(1 - 0.4) = 0.42, \quad \text{and} \quad f(\$1500) = (1 - 0.7)(0.4) = 0.12.$$

Therefore, the expected commission for the salesperson is

إذا جلب حسارة
 بفرضه يساوي

$$E(X) = (\$0)(0.18) + (\$1000)(0.42) + (\$1500)(0.12) + (\$2500)(0.28) \\ = \$1300.$$

dis

Continuous

Example 4.3: Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

$$f(x) = \begin{cases} \frac{20,000}{x^3}, & x > 100, \\ 0, & \text{elsewhere.} \end{cases}$$

$E(X)$

Find the expected life of this type of device.

* Solution: Using Definition 4.1, we have

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20,000}{x^3} dx = \int_{100}^{\infty} \frac{20,000}{x^2} dx = 200.$$

Therefore, we can expect this type of device to last, on average, 200 hours.

Now let us consider a new random variable $g(X)$, which depends on X ; that is, each value of $g(X)$ is determined by the value of X . For instance, $g(X)$ might be X^2 or $3X - 1$, and whenever X assumes the value 2, $g(X)$ assumes the value $g(2)$. In particular, if X is a discrete random variable with probability distribution $f(x)$, for $x = -1, 0, 1, 2$, and $g(X) = X^2$, then

$$\begin{aligned} \rightarrow P[g(X) = 0] &= P(X = 0) = f(0), \\ \rightarrow P[g(X) = 1] &= P(X = -1) + P(X = 1) = f(-1) + f(1), \\ \rightarrow P[g(X) = 4] &= P(X = 2) = f(2), \end{aligned}$$

and so the probability distribution of $g(X)$ may be written

$g(x)$	0	1	4
$P[g(X) = g(x)]$	$f(0)$	$f(-1) + f(1)$	$f(2)$

By the definition of the expected value of a random variable, we obtain

$$\begin{aligned} \mu_{g(X)} = E[g(X)] &= 0f(0) + 1[f(-1) + f(1)] + 4f(2) \\ &= (-1)^2 f(-1) + (0)^2 f(0) + (1)^2 f(1) + (2)^2 f(2) = \sum_x g(x)f(x). \end{aligned}$$

احتمال الاقتران

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Theorem 4.1: Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_x g(x)f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x) dx$$

if X is continuous.

Example 4.4: Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

Solution: By Theorem 4.1, the attendant can expect to receive

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$$\begin{aligned} E[g(X)] &= E(2X - 1) = \sum_{x=1}^9 (2x - 1)f(x) \\ &= (7) \left(\frac{1}{12}\right) + (9) \left(\frac{1}{12}\right) + (11) \left(\frac{1}{4}\right) + (13) \left(\frac{1}{4}\right) \\ &\quad + (15) \left(\frac{1}{6}\right) + (17) \left(\frac{1}{6}\right) = \underline{\underline{\$12.67}} \end{aligned}$$

Example 4.5: Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

Solution: By Theorem 4.1, we have

$$E(4X + 3) = \int_{-1}^2 \frac{(4x+3)x^2}{3} dx = \frac{1}{3} \int_{-1}^2 (4x^3 + 3x^2) dx = 8.$$

We shall now extend our concept of mathematical expectation to the case of two random variables X and Y with joint probability distribution $f(x, y)$.

Definition 4.2: Let X and Y be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$\mu_{g(X,Y)} = E[g(X, Y)] = \sum_x \sum_y g(x, y)f(x, y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)f(x, y) dx dy$$

if X and Y are continuous.

Example 4.6: Let X and Y be the random variables with joint probability distribution indicated in Table 3.1 on page 96. Find the expected value of $g(X, Y) = XY$. The table is reprinted here for convenience.

		x			Row
		0	1	2	Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution: By Definition 4.2, we write

$$\begin{aligned}
 E(XY) &= \sum_{x=0}^2 \sum_{y=0}^2 xyf(x, y) \\
 &= (0)(0)f(0, 0) + (0)(1)f(0, 1) \\
 &\quad + (1)(0)f(1, 0) + (1)(1)f(1, 1) + (2)(0)f(2, 0) \\
 &= f(1, 1) = \frac{3}{14}.
 \end{aligned}$$

Example 4.7: Find $E(Y/X)$ for the density function

$$f(x, y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2, 0 < y < 1 \\ 0, & \text{elsewhere.} \end{cases}$$

بثلاثة بالنسبة لـ x بحرين y

Solution: We have

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^2 \frac{y(1+3y^2)}{4} dx dy = \int_0^1 \frac{y+3y^3}{2} dy = \frac{5}{8}.$$

Note that if $g(X, Y) = X$ in Definition 4.2, we have

margin

$$E(X) = \begin{cases} \sum_x \sum_y xf(x, y) = \sum_x \underline{xg(x)} & \text{(discrete case),} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x, y) dy dx = \int_{-\infty}^{\infty} \underline{xg(x)} dx & \text{(continuous case),} \end{cases}$$

where $g(x)$ is the marginal distribution of X . Therefore, in calculating $E(X)$ over a two-dimensional space, one may use either the joint probability distribution of X and Y or the marginal distribution of X . Similarly, we define

$$E(Y) = \begin{cases} \sum_y \sum_x yf(x, y) = \sum_y \underline{yh(y)} & \text{(discrete case),} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x, y) dx dy = \int_{-\infty}^{\infty} \underline{yh(y)} dy & \text{(continuous case),} \end{cases}$$

where $h(y)$ is the marginal distribution of the random variable Y .

Exercises

4.23 Suppose that X and Y have the following joint probability function:

		x	
		2	4
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

- (a) Find the expected value of $g(X,Y) = XY^2$.
 (b) Find μ_X and μ_Y .

Answer 4.23:

$$(a): E[g(X,Y)] = E(XY^2) = \sum_x \sum_y xy^2 f(x,y)$$

		x		
		2	4	
y	1	0.10	0.15	0.25
	3	0.20	0.30	0.5
	5	0.10	0.15	0.25
		$\frac{2}{5}$	$\frac{3}{5}$	

$$= 2^2 \cdot 1^2 \cdot 0.1 + 2^2 \cdot 3^2 \cdot 0.2 + 2^2 \cdot 5^2 \cdot 0.1 + 4^2 \cdot 1^2 \cdot 0.15 + 4^2 \cdot 3^2 \cdot 0.3 + 4^2 \cdot 5^2 \cdot 0.15$$

$$= 35.2$$

(b)

$$\mu_X = E(X) = \frac{2(0.1+0.2+0.1) + 4(0.15+0.3+0.15)}{0.25 + 0.5 + 0.25} = \frac{3.20}{1} = 3.20 \rightarrow \sum x g(x)$$

$$\mu_Y = E(Y) = \frac{1(0.1+0.15) + 3(0.2+0.3) + 5(0.1+0.15)}{0.25 + 0.5 + 0.25} = \frac{3}{1} = 3 \rightarrow \sum y h(y)$$

4.2 Variance and Covariance of Random Variables

The mean, or expected value, of a random variable X is of special importance in statistics because it describes where the probability distribution is centered. By itself, however, the mean does not give an adequate description of the shape of the distribution. We also need to characterize the variability in the distribution. In Figure 4.1, we have the histograms of two discrete probability distributions that have the same mean, $\mu = 2$, but differ considerably in variability, or the dispersion of their observations about the mean.

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في مقاييس التشتت

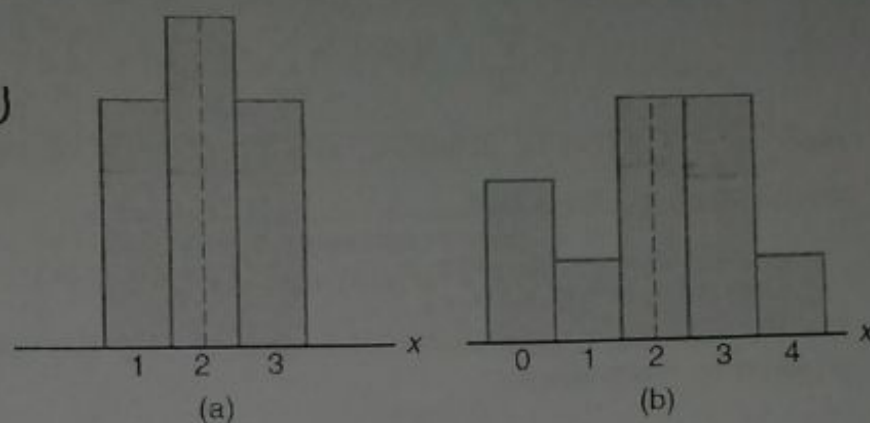


Figure 4.1: Distributions with equal means and unequal dispersions.

Definition 4.3.

Let X be a random variable with probability distribution $f(x)$ and mean μ . The variance of X is

$$\sigma^2 = E[(X - \mu)^2] = \sum_x (x - \mu)^2 f(x), \quad \text{if } X \text{ is discrete, and}$$

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, \quad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ , is called the **standard deviation** of X .

$$\sum (x - \mu)^2 f(x)$$

Example 4.8: Let the random variable X represent the number of automobiles that are used for official business purposes on any given workday. The probability distribution for company A [Figure 4.1(a)] is

x	1	2	3
$f(x)$	0.3	0.4	0.3

and that for company B [Figure 4.1(b)] is

x	0	1	2	3	4
$f(x)$	0.2	0.1	0.3	0.3	0.1

Show that the variance of the probability distribution for company B is greater than that for company A.

Solution: For company A, we find that

$$\mu_A = E(X) = (1)(0.3) + (2)(0.4) + (3)(0.3) = 2.0,$$

and then

$$\sigma_A^2 = \sum_{x=1}^3 (x - 2)^2 f(x) = (1 - 2)^2(0.3) + (2 - 2)^2(0.4) + (3 - 2)^2(0.3) = 0.6.$$

For company B, we have

$$\mu_B = E(X) = (0)(0.2) + (1)(0.1) + (2)(0.3) + (3)(0.3) + (4)(0.1) = 2.0,$$

and then

$$\begin{aligned} \sigma_B^2 &= \sum_{x=0}^4 (x - 2)^2 f(x) \\ &= (0 - 2)^2(0.2) + (1 - 2)^2(0.1) + (2 - 2)^2(0.3) \\ &\quad + (3 - 2)^2(0.3) + (4 - 2)^2(0.1) = 1.6. \end{aligned}$$

EQS = Probability
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Random
Variable

calculations, is stated in the following theorem.

Theorem 4.2: The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2.$$

Proof: For the discrete case, we can write

$$\begin{aligned}\sigma^2 &= \sum_x (x - \mu)^2 f(x) = \sum_x (x^2 - 2\mu x + \mu^2) f(x) \\ &= \sum_x x^2 f(x) - 2\mu \sum_x x f(x) + \mu^2 \sum_x f(x).\end{aligned}$$

Since $\mu = \sum_x x f(x)$ by definition, and $\sum_x f(x) = 1$ for any discrete probability distribution, it follows that

$$\sigma^2 = \sum_x x^2 f(x) - \mu^2 = E(X^2) - \mu^2.$$

For the continuous case the proof is step by step the same, with summations replaced by integrations. $\blacksquare$

Example 4.9: Let the random variable X represent the number of defective parts for a machine when 3 parts are sampled from a production line and tested. The following is the probability distribution of X .

x	0	1	2	3
$f(x)$	0.51	0.38	0.10	0.01

Using Theorem 4.2, calculate σ^2 .

Solution: First, we compute

$$\mu = (0)(0.51) + (1)(0.38) + (2)(0.10) + (3)(0.01) = 0.61.$$

Now,

$$E(X^2) = (0)(0.51) + (1)(0.38) + (4)(0.10) + (9)(0.01) = 0.87.$$

Therefore,

$$\sigma^2 = 0.87 - (0.61)^2 = 0.4979. \quad \blacksquare$$

Example 4.10: The weekly demand for a drinking-water product, in thousands of liters, from a local chain of efficiency stores is a continuous random variable X having the probability density

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the mean and variance of X .

$$\mu = E(X) = 2 \int_1^2 x(x-1) dx = \frac{5}{3}$$

and

$$E(X^2) = 2 \int_1^2 x^2(x-1) dx = \frac{17}{6}$$

Therefore,

$$\sigma^2 = \frac{17}{6} - \left(\frac{5}{3}\right)^2 = \frac{1}{18}.$$

Definition 4.3:

Let X be a random variable with probability distribution $f(x)$ and mean μ . The variance of X is

$$\sigma^2 = E[(X - \mu)^2] = \sum_x (x - \mu)^2 f(x), \quad \text{if } X \text{ is discrete, and}$$

$$\sigma^2 = E[(X - \mu)^2] = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) dx, \quad \text{if } X \text{ is continuous.}$$

The positive square root of the variance, σ , is called the standard deviation of X .

The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2$$

Theorem 4.3:

Let X be a random variable with probability distribution $f(x)$. The variance of the random variable $g(X)$ is

$$\sigma_{g(X)}^2 = E\{[g(X) - \mu_{g(X)}]^2\} = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

if X is discrete, and

$$\sigma_{g(X)}^2 = E\{[g(X) - \mu_{g(X)}]^2\} = \int_{-\infty}^{\infty} [g(x) - \mu_{g(X)}]^2 f(x) dx$$

if X is continuous.

Proof: Since $g(X)$ is itself a random variable with mean $\mu_{g(X)}$ as defined in Theorem 4.1, it follows from Definition 4.3 that

$$\sigma_{g(X)}^2 = E\{[g(X) - \mu_{g(X)}]^2\}.$$

Now, applying Theorem 4.1 again to the random variable $[g(X) - \mu_{g(X)}]^2$ completes the proof. ■

Example 4.11: Calculate the variance of $g(X) = 2X + 3$ where X is a random variable with probability distribution

x	0	1	2	3
$f(x)$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{2}$	$\frac{1}{8}$

$$\sigma_{g(X)}^2 = \sum_x [g(x) - \mu_{g(X)}]^2 f(x)$$

$$\mu_{2X+3} = E(2X + 3) = \sum_{x=0}^3 (2x + 3)f(x) = 6.$$

Now, using Theorem 4.3, we have

$$\begin{aligned} \sigma_{2X+3}^2 &= E\{[(2X + 3) - \mu_{2X+3}]^2\} = E[(2X + 3 - 6)^2] \\ &= E(4X^2 - 12X + 9) = \sum_{x=0}^3 (4x^2 - 12x + 9)f(x) = 4. \end{aligned}$$

Example 4.12: Let X be a random variable having the density function given in Example 4.5 on page 115. Find the variance of the random variable $g(X) = 4X + 3$.

Solution: In Example 4.5, we found that $\mu_{4X+3} = 8$. Now, using Theorem 4.3,

$$\begin{aligned} \sigma_{4X+3}^2 &= E\{[(4X + 3) - 8]^2\} = E[(4X - 5)^2] \\ &= \int_{-1}^2 (4x - 5)^2 \frac{x^2}{3} dx = \frac{1}{3} \int_{-1}^2 (16x^4 - 40x^3 + 25x^2) dx = \frac{51}{5}. \end{aligned}$$

Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

If $g(X, Y) = (X - \mu_X)(Y - \mu_Y)$, where $\mu_X = E(X)$ and $\mu_Y = E(Y)$, Definition 4.2 yields an expected value called the **covariance** of X and Y , which we denote by σ_{XY} or $\text{Cov}(X, Y)$.

Definition 4.4: Let X and Y be random variables with joint probability distribution $f(x, y)$. The covariance of X and Y is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \sum_x \sum_y (x - \mu_X)(y - \mu_Y)f(x, y)$$

if X and Y are discrete, and

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_X)(y - \mu_Y)f(x, y) dx dy$$

if X and Y are continuous.

Theorem 4.4: The covariance of two random variables X and Y with means μ_X and μ_Y , respectively, is given by

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y.$$

Proof: For the discrete case, we can write

$$\begin{aligned}\sigma_{XY} &= \sum_x \sum_y (x - \mu_X)(y - \mu_Y)f(x, y) \\ &= \sum_x \sum_y xyf(x, y) - \mu_X \sum_x \sum_y yf(x, y) \\ &\quad - \mu_Y \sum_x \sum_y xf(x, y) + \mu_X \mu_Y \sum_x \sum_y f(x, y).\end{aligned}$$

Since

$$\mu_X = \sum_x xf(x, y), \quad \mu_Y = \sum_y yf(x, y), \quad \text{and} \quad \sum_x \sum_y f(x, y) = 1$$

for any joint discrete distribution, it follows that

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y - \mu_Y \mu_X + \mu_X \mu_Y = E(XY) - \mu_X \mu_Y.$$

For the continuous case, the proof is identical with summations replaced by integrals. $\blacksquare$

Example 4.13: Example 3.14 on page 95 describes a situation involving the number of blue refills X and the number of red refills Y . Two refills for a ballpoint pen are selected at random from a certain box, and the following is the joint probability distribution:

		x			
		0	1	2	$h(y)$
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$g(x)$		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$\begin{aligned}E(XY) &= \sum_{x=0}^2 \sum_{y=0}^2 xyf(x, y) \\ &= (0)(0)f(0, 0) + (0)(1)f(0, 1) \\ &\quad + (1)(0)f(1, 0) + (1)(1)f(1, 1) + (2)(0)f(2, 0) \\ &= f(1, 1) = \frac{3}{14}.\end{aligned}$$

Find the covariance of X and Y .

Solution: From Example 4.6, we see that $E(XY) = 3/14$. Now

$$\mu_X = \sum_{x=0}^2 xg(x) = (0)\left(\frac{5}{14}\right) + (1)\left(\frac{15}{28}\right) + (2)\left(\frac{3}{28}\right) = \frac{3}{4},$$

and

$$\mu_Y = \sum_{y=0}^2 yh(y) = (0)\left(\frac{15}{28}\right) + (1)\left(\frac{3}{7}\right) + (2)\left(\frac{1}{28}\right) = \frac{1}{2}.$$

Therefore,

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{3}{14} - \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = -\frac{9}{56}.$$

المسألة يد ل على إجابة حل وحدة بتقالي لكلا (الملاقة) بيت خدوي

Example 1.1.1: The fraction X of male runners and the fraction Y of female runners who compete in marathon races are described by the joint density function

$$f(x, y) = \begin{cases} 8xy, & 0 \leq y \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the covariance of X and Y .

Solution: We first compute the marginal density functions. They are

أخذنا x لوحدها $\leftarrow g(x) = \begin{cases} 4x^3, & 0 \leq x \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$

and

$$h(y) = \begin{cases} 4y(1 - y^2), & 0 \leq y \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

From these marginal density functions, we compute

$$\mu_x = E(X) = \int_0^1 4x^4 dx = \frac{4}{5} \text{ and } \mu_y = \int_0^1 4y^2(1 - y^2) dy = \frac{8}{15}.$$

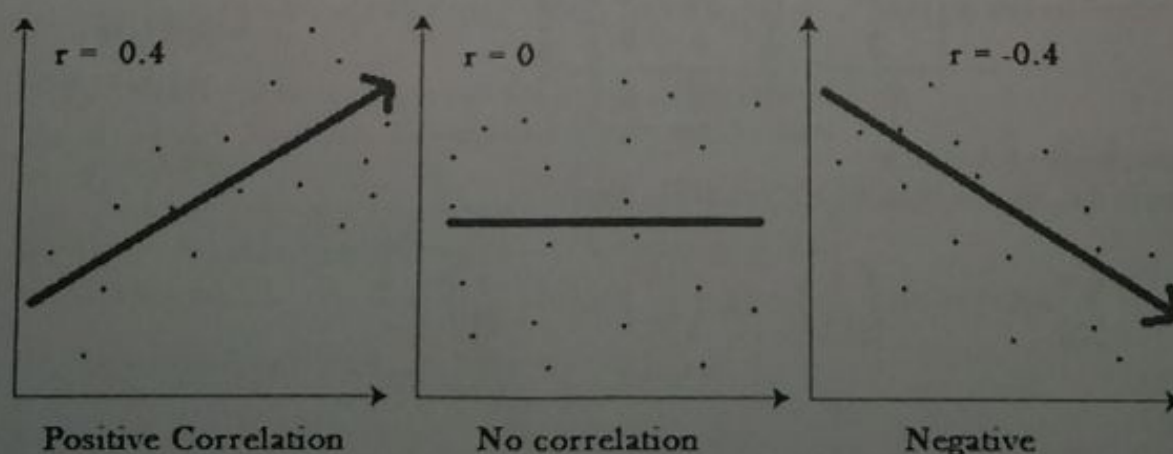
From the joint density function given above, we have

$$E(XY) = \int_0^1 \int_y^1 8x^2y^2 dx dy = \frac{4}{9}.$$

Then

$$\sigma_{xy} = E(XY) - \mu_x \mu_y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

Correlation Coefficient



Definition 4.5: Let X and Y be random variables with covariance σ_{XY} and standard deviations σ_X and σ_Y , respectively. The correlation coefficient of X and Y is

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}.$$

Example 4.13: Find the correlation coefficient between X and Y in Example 4.13

Solution: Since

$$E(X^2) = (0^2) \left(\frac{5}{14} \right) + (1^2) \left(\frac{15}{28} \right) + (2^2) \left(\frac{3}{28} \right) = \frac{27}{28}$$

and

$$E(Y^2) = (0^2) \left(\frac{15}{28} \right) + (1^2) \left(\frac{3}{7} \right) + (2^2) \left(\frac{1}{28} \right) = \frac{1}{2}$$

$$\mu_X = \sum_{x=0}^2 xg(x) = (0) \left(\frac{5}{14} \right) + (1) \left(\frac{15}{28} \right) + (2) \left(\frac{3}{28} \right) = \frac{3}{4}$$

$$\sigma_X^2 = \frac{27}{28} - \left(\frac{3}{4} \right)^2 = \frac{45}{112}$$

$$\mu_Y = \sum_{y=0}^2 yh(y) = (0) \left(\frac{15}{28} \right) + (1) \left(\frac{3}{7} \right) + (2) \left(\frac{1}{28} \right) = \frac{1}{2}$$

$$\sigma_Y^2 = \frac{1}{2} - \left(\frac{1}{2} \right)^2 = \frac{9}{28}$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{3}{14} - \left(\frac{3}{4} \right) \left(\frac{1}{2} \right) = -\frac{9}{56}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{-9/56}{\sqrt{(45/112)(9/28)}} = -\frac{1}{\sqrt{5}}$$

The variance of a random variable X is

$$\sigma^2 = E(X^2) - \mu^2.$$

		x			$h(y)$
		0	1	2	
y	0	$\frac{5}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
$g(x)$		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

$$E(XY) = \sum_{x=0}^2 \sum_{y=0}^2 xyf(x, y)$$

$$= (0)(0)f(0,0) + (0)(1)f(0,1)$$

$$+ (1)(0)f(1,0) + (1)(1)f(1,1) + 2(0)f(2,0)$$

$$= f(1,1) = \frac{3}{14}$$

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probability

Example 4.16: Find the correlation coefficient of X and Y in Example 4.14.

Solution: Because

$$E(X^2) = \int_0^1 4x^5 dx = \frac{2}{3} \text{ and } E(Y^2) = \int_0^1 4y^3(1-y^2) dy = 1 - \frac{2}{3} = \frac{1}{3},$$

we conclude that

$$\sigma_X^2 = \frac{2}{3} - \left(\frac{4}{5}\right)^2 = \frac{2}{75} \text{ and } \sigma_Y^2 = \frac{1}{3} - \left(\frac{8}{15}\right)^2 = \frac{11}{225}.$$

$$\sigma_{XY} = E(XY) - \mu_X \mu_Y = \frac{4}{9} - \left(\frac{4}{5}\right)\left(\frac{8}{15}\right) = \frac{4}{225}.$$

$$\rho_{XY} = \frac{4/225}{\sqrt{(2/75)(11/225)}} = \frac{4}{\sqrt{66}}.$$

$$f(x, y) = \begin{cases} 24xy & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\sigma^2 = E(X^2) - \mu^2$$

$$g(x) = \begin{cases} 4x^5 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_X = E(X) = \int_0^1 4x^4 dx = \frac{4}{5}$$

$$h(y) = \begin{cases} 4y^3(1-y^2) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_Y = \int_0^1 4y^3(1-y^2) dy = \frac{8}{15}$$

$$E(XY) = \int_0^1 \int_0^1 24x^2 y^2 dx dy = \frac{4}{9}$$

Theorem 4.7: The expected value of the sum or difference of two or more functions of the random variables X and Y is the sum or difference of the expected values of the functions. That is,

دالة الجواب $E[g(X, Y) \pm h(X, Y)] = E[g(X, Y)] \pm E[h(X, Y)].$

Proof: By Definition 4.2,

$$\begin{aligned} E[g(X, Y) \pm h(X, Y)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [g(x, y) \pm h(x, y)] f(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f(x, y) dx dy \pm \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) f(x, y) dx dy \\ &= E[g(X, Y)] \pm E[h(X, Y)]. \end{aligned}$$

Corollary 4.3: Setting $g(X, Y) = g(X)$ and $h(X, Y) = h(Y)$, we see that

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)].$$

$$E[X \pm Y] = E[X] \pm E[Y]$$

If X represents the daily production of some item from machine A and Y the daily production of the same kind of item from machine B , then $X + Y$ represents the total number of items produced daily by both machines. Corollary 4.4 states that the average daily production for both machines is equal to the sum of the average daily production of each machine.

Theorem 4.8: Let X and Y be two independent random variables. Then

$$E(XY) = E(X)E(Y).$$

Proof: By Definition 4.2,

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) \, dx \, dy.$$

Independent $\Rightarrow$ $f(x, y) = g(x)h(y)$

Since X and Y are independent, we may write

$$f(x, y) = g(x)h(y),$$

where $g(x)$ and $h(y)$ are the marginal distributions of X and Y , respectively. Hence,

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyg(x)h(y) \, dx \, dy = \int_{-\infty}^{\infty} xg(x) \, dx \int_{-\infty}^{\infty} yh(y) \, dy \\ &= E(X)E(Y). \end{aligned}$$

Corollary 4.5: Let X and Y be two independent random variables. Then $\sigma_{XY} = 0$.

Proof: The proof can be carried out by using Theorems 4.4 and 4.8.

$$\text{Cov}(X, Y) := \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$

$$\begin{aligned} \text{Cov}(X, Y) &= \mathbb{E}[(X - \mu_X)(Y - \mu_Y)] \\ &= \mathbb{E}[(X - \mu_X)]\mathbb{E}[(Y - \mu_Y)] \\ &= (\mathbb{E}[X] - \mu_X)(\mathbb{E}[Y] - \mu_Y) \\ &= 0. \end{aligned}$$

Example 4.21: It is known that the ratio of gallium to arsenide does not affect the functioning of gallium-arsenide wafers, which are the main components of microchips. Let X denote the ratio of gallium to arsenide and Y denote the functional wafers retrieved during a 1-hour period. X and Y are independent random variables with the joint density function

$$f(x, y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2, \ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that $E(XY) = E(X)E(Y)$, as Theorem 4.8 suggests.

Solution: By definition,

$$E(XY) = \int_0^1 \int_0^2 \frac{x^2 y (1 + 3y^2)}{4} dx dy = \frac{5}{6}, \quad E(X) = \frac{4}{3}, \quad \text{and} \quad E(Y) = \frac{5}{8}.$$

Hence,

$$E(X)E(Y) = \left(\frac{4}{3}\right) \left(\frac{5}{8}\right) = \frac{5}{6} = E(XY).$$

We conclude this section by proving one theorem and presenting several corollaries that are useful for calculating variances or standard deviations. ■