* Rules:-

1) $(A^T)^T = A$ 2) $(A+B)^T = A^T + B^T$

3) $[CA]^T = CA^T$ 4) $(AB)^T = B^T A^T$

symmetric Matrix

* if $A^T = A$

Skew-symmetric Matrix

iff $A^T = -A$

1) upper triangular Matrix:

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

2) Lower triangular Matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}$$

* diagonal Matrix D:-

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

* Scalar Matrix S:-

$$S = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

c = any scalar

* Unit Matrix or Identity Matrix

$I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$AI = IA = A$

Linear system of equation:
Gauss - elimination
Let linear system with m-equation and n-unkowns

$a_1 x_1 + \dots + a_{1n} x_n = b_1$

$a_2 x_1 + \dots + a_{2n} x_n = b_2$

$\vdots$

$a_{m1} x_1 + \dots + a_{mn} x_n = b_m$

1) Linear system $\Rightarrow x_1 x_2 \dots x_n$ of order 1.
2) For linear system with all b_j are zero $\Rightarrow$ called homogeneous linear system
3) For linear system with at least one (b_j) is not zero called non-homogeneous L. system

Linear system $AX=b$

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

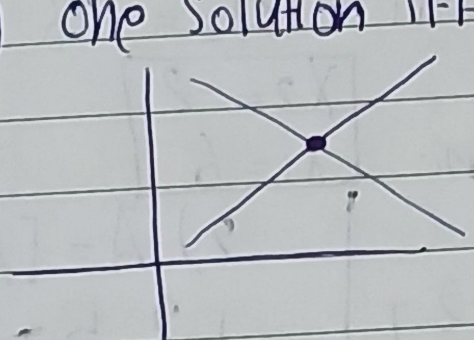
$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad b=\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

* Augmented Matrix:

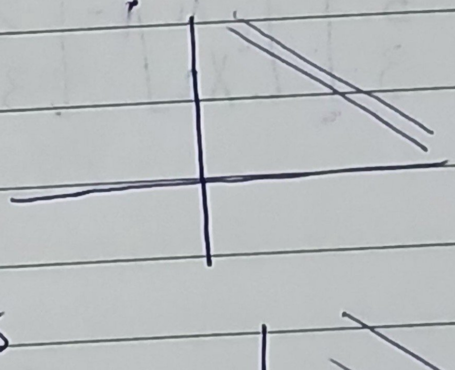
$$\hat{A}=[\,A : b\,]$$

for $AX=b$ has the Possible Cases:

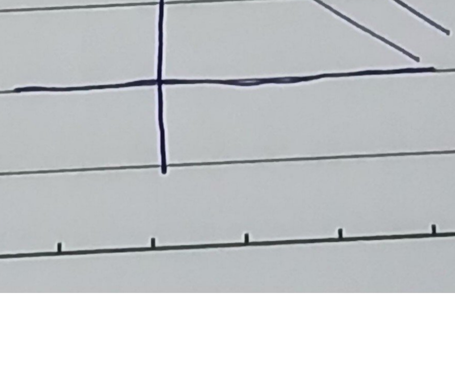
1) Precisely one solution iff lines intersect:-



2) Infintely many solutions if the lines coincide.



3) no-solution if the lines are Parallel



Gauss-elimination and Back substitution:-

$2x_1 + 5x_2 = 2$ $\qquad x_2 = -2 \rightsquigarrow -26/13$

det $\qquad +13x_2 = -26$ $\qquad x_1 = 6 \rightsquigarrow \frac{1}{2}(2-5x_2)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad 6+2$

$$\begin{bmatrix} 2 & 5 \\ 0 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

Ex: $2x_1 + 5x_2 = 2$ ..... eq(1)

$-4x_1 + 3x_2 = -30$ .... eq(2)

find x_1 x_2 ??

eq(2) + 2 eq(1)

$0x_1 + 13x_2 = -26$

$$\boxed{\begin{matrix} x_2 = -2 \\ x_1 = 6 \end{matrix}}$$

Using Gauss-elimination:-

1) $AX = b$

$$\begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -30 \end{bmatrix}$$

2) $\tilde{A} = [A \vdots b]$

$$\begin{bmatrix} 2 & 5 & \vdots & 2 \\ -4 & 3 & \vdots & -30 \end{bmatrix} \begin{matrix} \to R_1 \\ \to R_2 \end{matrix}$$

$$= \begin{bmatrix} 2 & 5 & \vdots & 2 \\ 0 & 13 & \vdots & -26 \end{bmatrix} \to R_1 + 2R_1$$

$2x_1 + 5x_2 = 2$ $\qquad x_2 = -2$

$+13x_2 = -26 \to$ $\qquad x_1 = 6$

EX:- with one Solution

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$
$$+10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 \quad = 80$$

Find x_1 x_2 x_3 ?
using G-E

1) $Ax = b$

$$= \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10. & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 90 \\ -80 \end{bmatrix}$$

2) $\hat{A} = [A \vdots b]$

$$= \begin{bmatrix} 1 & -1 & 1 & \vdots & 0 \\ -1 & 1 & -1 & \vdots & 0 \\ 0 & 10 & 25 & \vdots & 90 \\ 20 & 10 & 0 & \vdots & 80 \end{bmatrix} \begin{matrix} \rightarrow \text{Pivot} \\ \rightarrow R_2 + R_1 \\ \\ \rightarrow R_4 + 20R_1 \end{matrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 & \vdots & 0 \\ 0 & 10 & 25 & \vdots & 90 \\ 0 & 30 & -20 & \vdots & 80 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix} \begin{matrix} \cancel{\text{Pivot}} \\ \rightarrow \text{Pivot} \\ \\ \\ \end{matrix}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow R_3 - 3R_2$$

$1X_1 - 1X_2 + 1X_3 = 0$ $\quad X_3 = 2$

$10X_2 + 25X_3 = 90$ $\quad X_2 = 4$ $\quad X = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

$-95X_3 = -190$ $\quad X_1 = 2$

Elementary Row operations for matrix:-

1) Interchange of two Rows.
2) addition of constant Multiple of one row to another row
3) Multiplication of a row by a nonzero constant C.

has no-effect on the solution

EX:-

1) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$

2) $\begin{bmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$

1 + 2 has same solution

1) one determined linear system:
if it has more equation than unknowns
2) determined linear system:
if the number of equation equal number of unknowns.
3) under determined linear system
if has fewer equation then unknowns
4) consistent L.S $\Rightarrow$ if it has at least one solution.
5) In consistent if it has no-solution

Ex: with infinitly many solution:

$$3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$$
$$0.6x_1 + 1.5x_2 + 1.5x_3 - 5.4x_4 = 2.7$$
$$1.2x_1 - 0.3x_2 - 0.3x_3 + 2.4x_4 = 2.1$$

find x_1 x_2 x_3 x_4??
using G-E

1) $AX = b$

$$\begin{bmatrix} 3 & 2 & 2 & -5 \\ 0.6 & 1.5 & 1.5 & -5.4 \\ 1.2 & -0.3 & -0.3 & 2.4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 2.7 \\ 2.1 \end{bmatrix}$$

2) $\tilde{A} = [A \vdots b]$

$$\left[\begin{array}{cccc|c} 3 & 2 & 2 & -5 & 8 \\ 0.6 & 1.5 & 1.5 & -5.4 & 2.7 \\ 1.2 & -0.3 & -0.3 & 2.4 & 2.1 \end{array}\right] \rightarrow \text{pivot}$$

$$\text{Pivot} \rightarrow \left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & -1.1 & -1.1 & 4.4 & -1.1 \end{array}\right] \begin{array}{l} \\ \rightarrow R_2 - 0.2R_1 \\ \rightarrow R_3 - 0.4R_1 \end{array}$$

$$\left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ \\ \rightarrow R_3 - R_2 \end{array}$$

$$3X_1 + 2X_2 + 2X_3 - 5X_4 = 8$$

$$1.1X_2 + 1.1X_3 - 4.4X_4 = 1.1$$

$$X_2 + X_3 - 4X_4 = 1$$

$$X_2 + X_3 = 1 + 4X_4$$

$$3X_1 + 2[1 + 4X_4] - 5X_4 = 8$$

$$3X_1 + 2 + 8X_4 - 5X_4 = 8$$

$$3X_1 + 3X_4 = 6$$

$$X_1 = 2 - X_4$$

$$X_2 = 1 + 4X_4 - X_3$$

X_3 = arbitrary

X_4 = arbitery

$$X = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

infinitly many solution

EX: no- Solution Find x_1 x_2 x_3?

$3x_1 + 2x_2 + x_3 = 3$

$2x_1 + x_2 + x_3 = 0$

$6x_1 + 2x_2 + 4x_3 = 6$

1) $Ax = b$

$$\begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix}$$

2) $\tilde{A} = [A \vdots b]$

$$\left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 2 & 1 & 1 & 0 \\ 6 & 2 & 4 & 6 \end{array}\right] \rightarrow \text{Pivot}$$

$$= \left[\begin{array}{ccc:c} -3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ 0 & -2 & 2 & 0 \end{array}\right] \begin{array}{l} \\ \rightarrow R_2 - 2/3 R_1 \\ \rightarrow R_3 - 2R \end{array}$$

$$= \left[\begin{array}{ccc:c} 3 & 2 & 1 & 3 \\ 0 & -1/3 & 1/3 & -2 \\ \boxed{0 \quad 0 \quad 0} & & & 2 \end{array}\right] \rightarrow R_3 - 6R_2$$

$3x_1 + 2x_2 + x_3 = 3$

$-1/3 x_2 + 1/3 x_3 = -2$

$0 = 12$

False statment

has no solution

* Raw - echelon form and information from it:

1) linear system $Ax = b$

2) $\tilde{A} = [A \vdots b]$

3) after doing G-E $\rightarrow [R : f]$

$$[R : f] = \left[\begin{array}{cccc|c} V_{11} & V_{12} & \cdots & V_{1n} & f_1 \\ & V_{22} & \cdots & V_{2n} & f_2 \\ & & \ddots V_{rr} & \rightarrow V_{rn} & f_r \\ & & & & f_{r+1} \\ & & & & \vdots \\ & & & & f_m \end{array}\right]$$

1) $V_{11} \neq 0$

2) $r \leq m$

3) all element in the tringle + reclangle is zero

1) no-solution:-

if $r < m$ and at least one number $f_{r+1} \ldots f_m$ is not zero

$r_{11} = 3 \neq 0$

$r_{rn} = V_{23} = 1/3$

$r = 2$

$n = 3$

$f_m = f_3 = 12$

$m = 3$

no-solution

one solution:

if $r = n$ all number $f_{r+1} \ldots f_m$ are zero

$$[R : f] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

infinitly Mony solution
if f_m and all numbers
$f_{r+1} \dots f_m$ are zero

$$[A : f] = \left[\begin{array}{cccc:c} 3 & 2 & 2 & -5 & 8 \\ 0 & 1.1 & 1.1 & -4.4 & 1.1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$r = 3 \neq 0$ $\quad r = 2$

$r_{2n} = \cancel{r}_{2r} = -4.4$ & $n = 4$

$f_3 = f_m = 0$ $\quad m = 3$

* linear Independence Rank of matrix:-

Def: let $a_1\ a_2 \dots a_m$ are $\overset{a_{in}}{\text{one}}$ ~~matri~~ vectors
$c_1, c_2 \dots c_m$ are m-scalar

then $a_1 c_1 + a_2 c_2 \dots + a_m c_m = 0 \rightarrow$ eq(1)

1) if eq(1) hold for all C's are zero then system [linearly Indep]

2) if eq(1) hold for at least one C's are not zero then system is [linearly dep]

def: $a_1 = [3\ 0\ 2\ 2]$
$a_2 = [-6\ 42\ 24\ 54]$
$a_3 = [21\ -21\ 0\ -15]$

are $a_1\ a_2\ a_3$ linearly dep??

$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0$

$c_1 [3 \; 0 \; 2 \; 2] + c_2 [-6 \; 42 \; 24 \; 54]$
$+ c_3 [21 \; -21 \; 0 \; -15] = 0$

$3c_1 - 6c_2 + 21c_3 = 0$

$0c_1 + 42c_2 - 21c_3 = 0$

$2c_1 + 24c_2 + 0c_3 = 0$

$2c_1 + 54c_2 - 15c_3 = 0$

Using G-E $\quad c_1 = 6 \quad c_2 = -1/2 \quad c_3 = -1$

$a_1 \; a_2 \; a_3$ are linearly dep

Rank A:
is the maximum number of linearly indep
Row vect of A.

Ex: let $A = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$ find Rank A?

Using G-E:

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \begin{matrix} \to \text{Pivot} \\ \to R_2 + 2R_1 \\ \to R_3 - 7R_1 \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to R_3 + 1/2 R_2$$

Rank of $A = 2$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Rank} = 1$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{Rank} = 3$$

Row- equivelant Matrix:-

1) Matrix A_1 is Row equivelent to Matrix A_2 if A_1 Can be obtained from A_2 by elementary Row opention

2) Row -equivelent Matrix have Same Rank.

3)

Therem:-

Con Sider P-vectors that each have n- Component then theses Vectors are linearly Indep.

if matrix formed with these vectors as Row vector has Rank = P, however theses vector are linearly dep if Matrix has Rank less than P.

Theorem: Rank in terms of column vector
The Rank of matrix A equals the Maximum number of linearly Indep colum vector of A.

Theorem:- linearly dep of vectors consider P vectors each have n-component if $n < P$ then these vectors are linearly dep

EX: let $P_1 = [1\ 2]$ are P_s linerly dep?
$P_2 = [3\ 4]$ $P = 5$ $n < P$
$P_3 = [5\ 6]$ $n = 2$ $2 < 5$
$P_4 = [7\ 8]$ $P_s \Rightarrow$ linearly dep
$P_5 = [9\ 10]$

(*) $\text{Rank}(A) = \text{Rank}(A^T)$

(*) Theorem: solution of linear system Existence uniquness.

let $Ax = b$ has m-equation
n-unknowns

~~let~~ $\tilde{A} = [A : b]$

1) Existence: linear system has a solution then it is consistent if A
$\text{Rank}\, A = \text{Rank}(\tilde{A})$

2) Uniqueness: linear system has one-solution iff $Rank(A) = Rank(\tilde{A}) = n$

3) Infinitly many solution iff:

$Rank(A) < n$

$Rank(\tilde{A}) < n$

4) no-solution:

linear system is inconsistant iff $Rank(A) \neq Rank(\tilde{A})$