Monday 17/11/2025

* Horizontal asymptote: the line $y = L$ is called horizontal asymptote to $f(x)$: if $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$

Ex) find a horizontal asymptote for:

① $f(x) = \dfrac{\sqrt{x^4 - 1}}{x+5}$

① $\lim_{x\to\infty} f(x) \dfrac{\sqrt{x^4-1}}{x+5} = \lim_{x\to\infty} \dfrac{\sqrt{x^4}}{x} = \lim_{x\to\infty} \dfrac{x^2}{x}$

$= \lim_{x\to\infty} x = \infty$ ← ما في Horizontal asymptote لانه طلع ∞ وليس عدد.

② $\lim_{x\to-\infty} \dfrac{\sqrt{x^4-1}}{x+5} = \lim_{x\to-\infty} \dfrac{\sqrt{x^4}}{x} = \lim_{x\to-\infty} \dfrac{x^2}{x}$

$= \lim_{x\to-\infty} x = -\infty$

$\therefore$ there is no horizontal asymptote.

② $f(x) = \dfrac{x-3}{3x-5}$

① $\lim_{x\to\infty} \dfrac{x-3}{3x-5} = \dfrac{x}{3x}$

$= \lim_{x\to\infty} \dfrac{1}{3} = \dfrac{1}{3}$

$\lim_{x\to-\infty} = \dfrac{1}{3}$

$\therefore y = \dfrac{1}{3}$ is horizontal asymptote

as $x \to \infty$

$x \to -\infty$

③ $f(x) = \dfrac{\sqrt{4x^2+1}}{2x-3}$

① $\lim_{x\to\infty} \dfrac{\sqrt{4x^2+1}}{2x-3} = \lim_{x\to\infty} \dfrac{\sqrt{4x^2}}{2x}$

$= \lim_{x\to\infty} \dfrac{2|x|}{2x} = \lim_{x\to\infty} \dfrac{2x}{2x} = 1$

② $\lim_{x\to-\infty} \dfrac{\sqrt{4x^2+1}}{2x-3} = \lim_{x\to-\infty} \dfrac{2|x|}{2x}$

$\lim_{x\to-\infty} \dfrac{2(-x)}{2x} = -1$

$\therefore$ there is two H.A

$y = 1$ as $x \to \infty$ and

$y = -1$ as $x \to -\infty$

Monday 17/11/2025

④ $f(x) = e^x$

① $\lim_{x\to\infty} e^x = \infty$

② $\lim_{x\to-\infty} e^x = 0$

$\therefore$ $y=0$ is H.A as $x\to-\infty$

⑤ $f(x) = 2+\tan^{-1}(2x)$

① $\lim_{f(x)\to\infty} f(x) = \lim_{x\to\infty} 2+\tan^{-1}(2x)$

$= 2+\frac{\pi}{2}$

② $\lim_{x\to-\infty} 2+\tan^{-1}(2x) = 2-\frac{\pi}{2}$

$\therefore$ $y = 2+\frac{\pi}{2}$

$y = 2-\frac{\pi}{2}$ are H.A

⑥ $f(x) = 4\tan^{-1}x - 1$

① $\lim_{x\to\infty} 4\tan^{-1}x - 1 = 4\left(\frac{\pi}{2}\right)-1 = 2\pi-1$

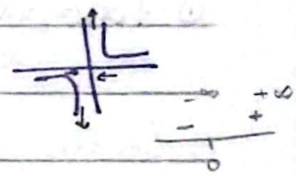
② $\lim_{x\to-\infty} 4\tan^{-1}x - 1 = 4\left(-\frac{\pi}{2}\right)-1 = -2\pi-1$

$y = 2\pi-1$ and $y = -2\pi-1$ are H.A

* vertical asymptote:
* the line $x=a$ is called vertical asymptote if

$$\lim_{\substack{x\to a \\ x\to a^+ \\ x\to a^-}} f(x) = \begin{cases} \infty \\ -\infty \\ \text{d.n.e} \end{cases}$$

ex) $f(x) = \frac{1}{x}$, $x=0$



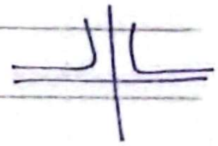
$x=0$ is V.A becose.

$\lim_{x\to0^+} \frac{1}{x} = +\infty$

$\lim_{x\to0^-} \frac{1}{x} = -\infty$

$\lim_{x\to0} = \text{d.n.e}$

Ex) $f(x) = \frac{1}{x^2}$



$\to x=0$ is V.A

$\lim_{x\to0^+} = \infty$

$x\to0^- = \infty$

$x\to0 = \infty$

Monday 17/11/2025

Ex) find the vertical asymptote of :-

① $f(x) = \dfrac{4}{x^2-16}$

$x^2-16=0 \Rightarrow x=4,-4$ are v. A

$\boxed{x=4 \quad , \quad x=-4}$

$\lim\limits_{x\to 4} \dfrac{4}{x^2-16} =$

$+ \quad -4 \quad - \quad 4 \quad +$

$x\to -4^+ = -\infty$	$x \to 4^+ = \infty$
$x\to -4^- = \infty$	$x \to 4^- = -\infty$
$x \to -4 = d.n.e$	$x \to 4 = d.n.e$

② $f(x) = \dfrac{x-3}{x^2-9} = \dfrac{\cancel{x-3}}{(\cancel{x-3})(x+3)} = \dfrac{1}{(x+3)}$

$\lim \dfrac{1}{x+3}$

$\therefore x=-3$ is v. A

$\lim\limits_{x\to -3^+} = \infty$

$- \quad -3 \quad + \quad 3 \quad +$

$x\to -3^- = -\infty$

$x \to -3 = d.n.e$

x كلها اي احسب v.A اشتم بتعويض احدهم الكتام بعد الاختصار

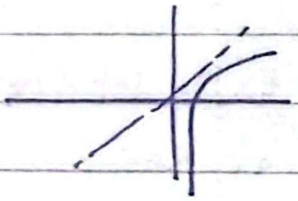
③ $f(x) = \dfrac{x-1}{|x|-1}$

$= \begin{cases} \dfrac{x-1}{x-1} , & x \geq 1 \\ \dfrac{x-1}{-x-1} , & x < 1 \end{cases}$

$-x-1=0$

$\boxed{x=-1}$ v A

④ $f(x) = \ln x$



$\lim\limits_{x\to 0^+} \ln x = -\infty$

$\therefore x=0$ v. A

$\lim (x-1)$

$\longrightarrow x-1=0$
$x=1$

$\lim\limits_{x\to 1^+} \ln(x-1) = -\infty$

Monday 17/11/2025

* Continuity الاقترانات المتصلة

→ Definition f(x) is Continuous (cont) at $x = a$ if:

① $f(a)$ defined

② $\lim_{x \to a} f(x)$ exist

③ $\lim_{x \to a} f(x) = f(a)$

Ex) let $f(x) = 3 + e^x + x^2$ is f cont at $x = 0$?

① $f(0) = 3 + e^0 + 0 = 4$

② $\lim_{x \to 0} f(x) = \lim_{x \to 0} 3 + e^x + x^2$

$= 3 + 1 + 0 = 4$

③ $f(0) = \lim_{x \to 0} f(x)$

$\therefore$ f(x) cont at $x = 0$.

Ex) let $f(x) = \begin{cases} \dfrac{x^2 - 9}{x - 3}, & x \neq 3 \\ 4, & x = 3 \end{cases}$

is f cont at $x = 3$ or not?

① $f(3) = 4$

② $\lim_{x \to 3} f(x) = \dfrac{x^2 - 9}{x - 3} = \dfrac{0}{0}$ نحلل

دائما lim تكون مش لا مساواة.

$= \lim_{x \to 3} \dfrac{\cancel{(x-3)}(x+3)}{\cancel{x-3}} = \lim_{x \to 3} (x+3) = 6$

$f(3) \neq \lim_{x \to 3} f(x)$

$\therefore$ f(x) not cont at $x = 3$

discont

* Remark: if f(x) not cont at $x = a$ it is called discontinuous at $x = a$ (dis cont)

Monday 17/11/2025

Ex) is $f(x) = \frac{|x|}{x}$ cont ?

① نعيد التعريف لأنه الاقتران مطلق

$$f(x) = \begin{cases} \frac{x}{x} & , x \geq 0 \\ \frac{-x}{x} & , x < 0 \end{cases} = \begin{cases} 1 & , x \geq 0 \\ -1 & , x < 0 \end{cases}$$

$\lim_{x \to 0^+} f(x) = 1$, $\lim_{x \to 0^-} f(x) = -1$

$\therefore \lim_{x \to 0} f(x)$ d.n.e

$\rightarrow f(x)$ dis cont at $x = 0$.

Ex) find the values of a & b such that:

$$f(x) = \begin{cases} ax^2+1 & , x > 2 \\ -11 & , x = 2 \\ x^3+b & , x < 2 \end{cases}$$ is cont ?

since $f(x)$ cont $\Rightarrow f(x)$ cont at $x=2$

② $f(2) = \lim_{x \to 2^+ / x \to 2^-} f(x)$

① $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

$\boxed{4a+1 = 8+b}$

شرح →

② $\lim_{x \to 2^+} f(x) = f(2)$

$4a+1 = -11$

$4a = -12$

$a =$

$\boxed{a = -3}$

③ $\lim_{x \to 2^-} f(x) = f(2)$

$8 + b = -11$

$b = -11 - 8 = -19$

$\boxed{b = -19}$

* Defn : $f(x)$ is cont on the open interval (a,b) if $f(x)$ is cont for all $C \in (a,b)$

* حتى يكون الاقتران متصل على فترة مفتوحة لازم يكون متصل عند كل نقطة تنتمي لهذه الفترة.

(a,b) or $(-\infty, b)$ or (a, ∞) or $(-\infty, \infty)$

* Defn : $f(x)$ is cont every where iff $f(x)$ is cont on $(-\infty, \infty)$

Thm : polynomials are continuous every where.

Monday 17/11/2025

* Thrm polynomials are continuous every where.

Ex) let $f(x) = |x|$. show that $f(x)$ is cont every where

$$|x| = \begin{cases} x & , x \geq 0 \\ -x & , x > 0 \end{cases}$$

① $\lim\limits_{x \to 0} f(x) \overset{?}{=} f(0)$

$0 = 0$ $\therefore$ Cont at $x = 0$

and x poly $x \geq 0$

$-x$ poly $x < 0$

$\therefore f(x) = |x|$ cont every where

Ex) if $f(x) = \dfrac{x+3}{x^2 + ax + 1}$ find the values of a that makes $f(x)$ cont every where.

$x^2 + ax + 1 \neq 0$ $f(x) = ax^2 + bx + c$ الصورة العامة للمقدار التربيعي

لا يحلل

المميز < صفر

$b^2 - 4ac < 0$

$a^2 - (4 * 1 * 1) < 0$

$a^2 - 4 < 0$

$(a-2)(a+2) < 0$ + -2 - 2 +

$\therefore a \in (-2, 2)$

* Remark: if $f(x) = \dfrac{P(x)}{q(x)}$, where $p(x), q(x)$ are polynomials then $f(x)$ is cont every where except when $q(x) = 0$

Ex) $f(x) = \dfrac{x^3 + 3}{x^3 - x}$ cont every

where - $x^3 - x = 0$

$x(x^2 - 1) = 0$

$\boxed{x = 0}$ $\boxed{x = 1}$ $\boxed{x = -1}$

$\therefore$ cont every where $x \in \{0, \pm 1\}$

Monday 17/11/2025

* Defn:

① f(x) is cont at $x=c$ from the right iff

$$\lim_{x\to c^+} f(x) = f(c)$$

② f(x) is cont at $x=c$ from the left iff

$$\lim_{x\to c^-} f(x) = f(c)$$

③ f(x) is cont on $[a,b]$ iff

① f(x) is cont on (a,b)

② f(x) is cont at a from the right

③ f(x) is cont at b from the left

e.g) let $f(x)=\sqrt{9-x^2}$. show that f(x) is cont on $[-3,3]$

① let $c \in (-3,3)$ then

$$\lim_{x\to c} f(x) = \lim_{x\to c}\sqrt{9-x^2} = \sqrt{9-c^2}$$

$f(c) = \sqrt{9-c^2}$

$\therefore f(c) = \lim_{x\to c} f(x) \therefore$ f(x) cont on $(-3,3)$

② $\lim_{x\to -3^+} f(x) \overset{?}{=} f(-3)$

$0 = 0 \therefore$ f(x) cont at $x=-3$ from right

③ $\lim_{x\to 3^-} \overset{?}{=} f(3)$

$0 = 0 \therefore$ f(x) cont $x=3$ from left

$\therefore$ f(x) cont $[-3,3]$

* Thrm:

let f(x), g(x) be cont at $x=c$, then:

① $f+g$ cont at $x=c$

② $f-g$ cont at $x=c$

③ $f\cdot g$ cont at $x=c$

④ $\frac{f}{g}$ cont at $x=c$ if $g(c)\neq 0$

* Thrm:

① if g(x) is cont at $x=c$, and f(x) is cont at g(x), then $f\circ g$ is cont at $x=c$

② if g(x) is cont every where and f(x) is cont every where then $f\circ g$ is cont every where.

* Thrm:

if f(x) is one to one and cont on it's domain, the $f^{-1}(x)$ is cont on the range of f

Monday 17/11/2025

*Remark :

the following types of functions are cont at every number in their domains.-

1. polynomials.
2. rational functions.
3. root functions.
4. trigonometic functions.
5. inverse trigonometic functions.
6. exponential functions.
7. logarithmic functions.

e.g) discuss the continuty of :

① $f(x) = x^2 - 3x + 5$

cont on $\mathbb{R}$

② $f(x) = \frac{x+1}{x^2-4}$ cont on $\mathbb{R} - \{\pm 2\}$

$x^2 - 4 = 0$

$x^2 = 4$

$\boxed{x = \mp 2}$

③ $f(x) = \frac{2x}{x^2+1}$ cont on $\mathbb{R} - \{\cancel{x^2}\}$

$x^2 + 1 \neq 0$ cont on $\mathbb{R}$

$\cancel{④ \; f(x) = \frac{2x+1}{|x|+2}}$

$\cancel{|x|+2 \neq 0} \Rightarrow |x| \neq -2$

④ $f(x) = \frac{2x+1}{|x|+2}$ $\to \mathbb{R}$

$|x|+2 \neq 0$ $\quad |x| \neq -2$ $\to \mathbb{R}$

cont on $\mathbb{R}$

⑤ $f(x) = \frac{2x+1}{2+x}$ $\mathbb{R} - \{x+2=0\}$

cont on $\mathbb{R} - \{-2\}$

⑥ $f(x) = \frac{2x+1}{|x|-2}$ cont on $\mathbb{R} - \{2, -2\}$

$|x| = 2$

$x = 2$ or $x = -2$

⑦ $f(x) = \frac{\ln(2x-4)}{\sqrt{9-x^2}}$ $\to 2x-4 > 0$

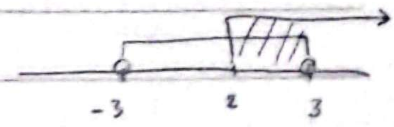
$x > 2$

$(2, \infty)$

$\sqrt{9-x^2} = 0$ $\quad\downarrow$ نصف دائرة

$9 - x^2 = 0$ $\quad [-3, 3]$

$\boxed{x = \pm 3}$



cont on $(2, 3)$

Monday 17/11/2025

⑧ $f(x) = |\cos x|$ $\quad g|x| \quad h: \cos x$

cont every where or $\mathbb{R}$

⑨ $f(x) = \sin\left(\frac{2\pi}{x-\pi}\right)$ $\quad \mathbb{R}-\{\pi\} \cap \mathbb{R}$

cont on $\mathbb{R}-\{\pi\}$

⑩ $f(x) = \log_{(x-1)}(x+1) = \frac{\log(x+1)}{\log(x-1)}$

البسط : $x+1>0 \Rightarrow x>-1 \quad (-1,\infty)$

المقام : $x-1>0 \Rightarrow x>1 - \{2\} \quad (1,\infty)$

$x-1=1 \Rightarrow \boxed{x=2}$

cont on $(1,\infty)-\{2\}$

e.g) the discont points of the function

$f(x) = \frac{x^2+2x-3}{(x-2)(x^2-1)}$ ← اصفار المقام

$x-2=0 \quad , \quad x^2-1=0$

$\boxed{x=2} \quad \boxed{x=\pm 1}$

$\therefore$ discont points $x \in \{\pm 1, 2\}$

e.g) $f(x) = \begin{cases} a(\tan^{-1}x+2), & x<0 \\ 2e^{bx}+1, & 0\le x\le 3 \\ \ln(x-2)+x^2, & x>3 \end{cases}$

find a, b such that $f(x)$ is cont?

$f(x)$ cont $\Rightarrow$ $f(x)$ cont at $x=0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$2e^0+1 = a(\tan^{-1}0+2)$

$3 = 2a$

$\boxed{a = \frac{3}{2}}$

$f(x)$ cont $\Rightarrow$ $f(x)$ cont at $x=3$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

$\cancel{\ln 1} + 9 = 2e^{3b}+1$

$9 = 2e^{3b}+1$

$2e^{3b} = 8 \Rightarrow e^{3b} = 4$

$3b = \ln 4$

$\boxed{b = \frac{1}{3}\ln 4}$ or $\boxed{b = \ln\sqrt[3]{4}}$

Monday 17/11/2025

* Thrm: if $f(x)$ is cont, then

$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

$(x \to c,\ x \to c^{+},\ x \to c^{-}$

$x \to \infty,\ x \to -\infty)$

e.g) $\lim_{x \to 0} e^{\sin x} = \cancel{e}$

$= e^{\lim_{x \to 0} \sin x} = e^{0} = 1$

e.g) $\lim_{x \to \infty} \ln\left(\frac{x^6 - 1}{x^6 + 1}\right)$

$= \ln \lim_{x \to \infty} \frac{x^6 - 1}{x^6 + 1} = \ln 1 = 0$

e.g) $\lim_{x \to 3} \sqrt{\frac{3x - 9}{2x^2 - 18}} = \sqrt{\lim_{x \to 3} \frac{3x - 9}{\underset{2(x^2-9)}{2x^2 - 18}}}$

حلل $\leftarrow \frac{0}{0}$

$\sqrt{\lim_{x \to 3} \frac{3\cancel{(x-3)}}{2\cancel{(x-3)}(x+3)}} = \sqrt{\lim_{x \to 3} \frac{3}{2(x+3)}}$

$= \sqrt{\lim \frac{3}{2 \cdot 6}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$

e.g) $\lim_{x \to \infty} \sin(\tan^{-1} x)$

$\sin\left(\lim_{x \to \infty} \tan^{-1} x\right) = \sin \frac{\pi}{2} = 1$

e.g) $\lim_{x \to 0^{+}} e^{-\frac{2}{x}} = e^{\lim_{x \to 0^{+}} -\frac{2}{x}}$ $\quad \frac{-2}{0} \quad \frac{-2}{0^{+}} \to -\infty$

$= e^{-\infty} = 0$

* limits involving trig functions

note that:

* $\lim_{x \to a} \sin x = \sin a$, $\lim_{x \to a} \cos x = \cos a$

since $\sin x, \cos x$ are cont every where.

* $\lim_{x \to \infty} \sin x = \text{d.n.e}$, $\lim_{x \to \infty} \cos x = \text{d.n.e}$

* $\lim_{x \to 0} \frac{\sin x}{x} = 1$

* $\lim_{x \to 0} \frac{\sin ax}{bx} = \frac{a}{b}$

* $\lim_{x \to 0} \frac{bx}{\sin ax} = \frac{b}{a}$

* $\lim_{x \to 0} \frac{\tan ax}{bx} = \frac{a}{b}$

* $\lim_{x \to 0} \frac{bx}{\tan ax} = \frac{b}{a}$

* $\lim_{x \to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$

Monday 17/11/2025

e.g) $\lim_{x\to 0} \frac{\sin 3x}{\tan 5x} = \frac{3}{5}$

e.g) $\lim_{x\to 0} x^2 \csc x = \lim \frac{x^2}{\sin x}$

$= \lim \frac{x}{\sin x} \cdot x$

$= 1 \cdot 0 = 0$

e.g) $\lim_{x\to 0} \frac{2x + \sin x}{\tan x + 10x}$

$= \lim_{x\to 0} \frac{\frac{2x}{x} + \frac{\sin x}{x}}{\frac{\tan x}{x} + \frac{10x}{x}} = \frac{2+1}{1+10} = \frac{3}{11}$

e.g) $\lim_{x\to \frac{\pi}{2}} \frac{\sin\left(x - \frac{\pi}{2}\right)}{\left(x - \frac{\pi}{2}\right)}$ $x - \frac{\pi}{2} = y$

$x \to \frac{\pi}{2}$
$y \to 0$

$\lim_{x\to 0} \frac{\sin y}{y} = 1$

e.g) $\lim_{x\to 0} \frac{1-\cos x}{x} * \frac{1+\cos x}{1+\cos x}$ نضرب بالمرافق

$= \lim_{x\to 0} \frac{1-\cos^2 x}{x(2)}$ $x=0$ $\cos x$

$= \frac{1}{2} \lim_{x\to 0} \frac{\sin^2 x}{x} = \frac{1}{2} \lim_{x\to 0} \frac{\sin x}{x} \cdot \sin x$

$= \frac{1}{2}(1)(0) = 0$

e.g) if $f(x) = \begin{cases} \frac{\sin(kx)}{x}, & x \le 0 \\ 3x + 2k^2, & x \ge 0 \end{cases}$

is cont find k?

$f(x)$ cont at $x=0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$\lim_{x\to 0^+} 3x + 2k^2 = \lim_{x\to 0^-} \frac{\sin(kx)}{x}$

$2k^2 = k$

$2k^2 - k = 0$

$k(2k-1) = 0$

$k=0$, $k = \frac{1}{2}$

e.g) if $\lim_{x\to 0} f(x) = 4$ then $\lim_{x\to 0} \frac{\tan 8x}{x\, f(4x)}$

$4x = y \Rightarrow x = \frac{y}{4}$: $x \to 0$, $y \to 0$

$4 \lim_{y\to 0} \frac{\tan\left(8 \cdot \frac{y}{4}\right)}{\frac{y}{4}\, f\left(y\right)}$

$= 4 \cdot \lim_{y\to 0} \frac{\tan 2y}{y \cdot f(y)} = 4(2) \frac{1}{4} = 2$

Monday 17/11/2025

e.g) if $\lim_{x\to 0} \frac{\sin(2ax)}{\sin(8x)} = \frac{14}{8}$ then $a = ??$

$\frac{2a}{8} = \frac{14}{8}$

$2a = 14 \Rightarrow \boxed{a = 7}$

* squeezing Thrm :

suppose that $g(x) \leq f(x) \leq h(x)$

for all x also suppose that $\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L$

then $\lim_{x\to c} f(x) = L$ $x\to c,\ x\to c^+,\ x\to c^-$

$x\to\infty,\ x\to-\infty$

e.g) $\lim_{x\to\infty} \frac{\sin x}{x} = \frac{\text{d.n.e}}{\infty}$ نلجئ للحصر

$-1 \leq \sin x \leq 1$

$\frac{-1}{x} \leq \frac{\sin x}{x} \leq \frac{1}{x}$

$\lim_{x\to\infty} \frac{-1}{x} = 0$ $\lim_{x\to\infty} \frac{1}{x} = 0$

$= 0$ by squeezing Thrm

e.g) $\lim_{x\to 0} x^4 \cos\left(\frac{2}{x}\right) = 0 \cdot \text{d.n.e}$ نلجئ للحصر

$-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$

$-x^4 \leq x^4\cos\left(\frac{2}{x}\right) \leq x^4$

$\lim_{x\to 0} -x^4 = 0$ $\lim_{x\to 0} x^4 = 0$

$= 0$ by squeezing Thrm.

e.g) $\lim_{x\to\infty} e^{-3x}\sin^2 x = 0 \cdot \text{d.n.e}$ نلجئ للحصر

$0 \leq \sin^2 x \leq 1$

$0 \leq e^{-3x}\sin^2 x \leq e^{-3x}$

$\lim_{x\to\infty} 0 = 0$ $\lim_{x\to\infty} e^{-3x} = 0$

$= 0$ by squeezing Thrm.

e.g) if $3x \leq f(x) \leq x^5+2$ for all $x \in (-2,2)$

find $\lim_{x\to 1} f(x) = ?$

$3x \leq f(x) \leq x^5+2$

$\lim_{x\to 1} 3x = 3$ $\lim_{x\to 1} x^5+2 = 3$

$\lim f(x) = 3$

Subject Monday 17/11/2025 Date No.

e.g) $\lim_{x\to\infty} \dfrac{2x+\sin 2x}{x+\cos x} = \lim_{x\to\infty} \dfrac{\frac{2x}{x} + \frac{\sin 2x}{x}}{\frac{x}{x} + \frac{\cos x}{x}}$

$= \dfrac{\cancel{\lim} 2 + \lim_{x\to\infty} \frac{\sin 2x}{x}}{1 + \lim_{x\to\infty} \frac{\cos x}{x}}$ $\to 0$

squeezing

$\lim_{x\to\infty} -\frac{1}{x} \leq \frac{\sin 2x}{x} \leq \lim \frac{1}{x}$

$= 0 \quad 0 \quad 0$

$\lim_{x\to\infty} \quad -\frac{1}{x} \leq \frac{\cos x}{x} \leq \frac{1}{x}$

$\downarrow 0 \quad \downarrow 0 \quad \downarrow 0$

$= 2$

* Differentiability : الاشتقاق

• Defn: for $f(x)$ we define the first derivative at $x=a$ by:

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

$$f'(a) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

• Rules of derivative:

• Remark: ① if $f(x) = c \Rightarrow f'(x) = 0$

② if $f(x) = x^n, n \in R \Rightarrow f'(x) = nx^{n-1}$

③ if $f(x), g(x)$ exist $\Rightarrow (f \pm g)'(x) = f'(x) \pm g'(x)$

e.g) find $f'(x)$

① $f(x) = x^3 - 6x^2 + 5$

$f'(x) = 3x^2 - 12x$

② $f(x) = 3x^5 - x^{\frac{1}{3}} + x^{-1}$

$f'(x) = 15x - \frac{1}{3}x^{-\frac{2}{3}} - x^{-2}$

③ $f(x) = \frac{1}{3}x^{\frac{1}{2}} - \underbrace{\frac{5}{x^4}}_{-5x^{-4}}$

$f'(x) = \frac{1}{6}x^{-\frac{1}{2}} + 20x^{-5}$

• Remark: if $f'(x), g'(x)$ exist, then,

① $(f \cdot g)'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

② $[(f(x))^n]' = n(f(x))^{n-1} \cdot f'(x)$

③ $\left(\frac{f}{g}\right)'(x) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)}$

e.g) find $\frac{dy}{dx}$

① $y = (x^2+3x)^4 \cdot (2x^2-5)$

$y' = (x^2+3x)^4(4x) + (2x^2-5)\, 4(x^2+3x)^3(2x+3)$

② $y = \left(\dfrac{x^2-1}{x+3}\right)^5$

$y' = 5\left(\dfrac{x^2-1}{x+3}\right)^4 \left(\dfrac{(x+3)(2x) - (x^2-1)(1)}{(x+3)^2}\right)$

Monday 17/11/2025

• note that

قابل للاشتقاق ← متصل

متصل ⇄ قابل / غير قابل

1. if $f'(a)$ exist then $f(x)$ cont at $x=a$

2. if $f(x)$ dis cont at $x=a$ then $f'(a)$ not exist

e.g) if $f(x) = \begin{cases} x^2\cos\left(\frac{2}{x}\right) & , x\neq 0 \\ 5 & , x=0 \end{cases}$

is $f'(x)$ exist? " differentiable"

$f(0) = 5$, $\lim_{x\to 0} x^2\cos\left(\frac{2}{x}\right)$

$$-1 \leq \cos\left(\frac{2}{x}\right) \leq 1$$

$$-x^2 \leq x^2\cos\left(\frac{2}{x}\right) \leq x^2$$

$\lim_{x\to 0} -x^2 = 0$ $\quad$ 0 $\quad$ $\lim_{x\to 0} x^2 = 0$

$\lim_{x\to 0} f(x) = 0 \neq f(0)$ ∴ dis cont at $x=0$

$f'(x)$ d.n.e

غير قابل للاشتقاق

eg) let $f(x) = \begin{cases} 5x-1 & , x>1 \\ x^2+9 & , x\leq 1 \end{cases}$

is $f'(x)$ exist, find $f'(x)$, $f'(2)$, $f'(0)$

$\lim_{x\to 1^+} f(x) = 4$, $\lim_{x\to 1^-} = 10$ ∴ dis cont at $x=1$

∴ $f'(x) = \begin{cases} 5 & , x>1 \\ 2x & , x<1 \\ \text{d.n.e} & x=1 \end{cases}$

$f'(2) = 5$, $f'(0) = 0$

e.g) let $f(x) = \begin{cases} x^3+2x & , x\leq 1 \\ 5x^2-2 & , x>1 \end{cases}$

find $f'(1)$, $f'(3)$

$\lim_{x\to 1^+} = 3$, $\lim_{x\to 1^-} = 3$

∴ cont at $x=1$

$f'(x) = \begin{cases} 3x^2+2 & , x<1 \\ 10x & , x>1 \\ \text{d.n.e} & , x=1 \end{cases}$

$f'(1^+) = 10$, $f'(1^-) = 5$ ∴ d.ne

$\boxed{f'(3) = 30}$

• Remark:

$\boxed{1}$ $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$

$\boxed{2}$ $\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{f'(x)}{2\sqrt{f(x)}}$

e.g) find $\frac{dy}{dx}$

① $y = \sqrt{x^2 - 3x + 1}$

$y' = \frac{2x - 3}{2\sqrt{x^2-3x+1}}$

② $y = \sqrt[5]{x - x^3 + 1} \Rightarrow y = (x - x^3 + 1)^{\frac{1}{5}}$

$y' = \frac{1}{5}\left(x - x^3 + 1\right)^{-\frac{4}{5}}\left(1 - 3x^2\right)$

• Remark. $f(x) \longrightarrow f'(x)$

$\sin(g(x)) \longrightarrow g'(x)\cos(g(x))$

$\cos(g(x)) \longrightarrow -g'(x)\sin(g(x))$

$\tan(g(x)) \longrightarrow g'(x)\sec^2(g(x))$

$\cot(g(x)) \longrightarrow -g'(x)\csc^2(g(x))$

$\sec(g(x)) \longrightarrow g'(x)\sec(g(x))\tan(g(x))$

$\csc(g(x)) \longrightarrow -g'(x)\csc(g(x))\cot(g(x))$

e.g) find $\frac{dy}{dx}$

① $y = \cos\left(2x^2 - 1\right) + \sin 5x$

$y' = -4x\sin\left(2x^2 - 1\right) + 5\cos 5x$

② $y = \sin^3\left(\cos\left(x^2+1\right)\right) = \left[\sin\left(\cos(x^2+1)\right)\right]^3$

$y' = 3\left[\sin\left(\cos(x^2+1)\right)\right]^2 \cdot -2x\sin(x^2+1) \cdot \cos\left(\cos(x^2+1)\right)$

③ $y = \sec^3\left(\tan^2(2x+5)\right)$

$\cancel{y' = 3\sec^2\left(\tan^2(2x+5)\right) \cdot 2\tan(2x+5) \cdot 2\sec^2(2x+5)}$

$3\sec^2\left(\tan^2(2x+5)\right) \cdot \sec\left(\tan^2(2x+5)\right) \cdot \tan\left(\tan^2(2x+5)\right) \cdot 2\tan(2x+5)\,\sec^2(2x+5) \cdot 2$

④ $y = x^2\sin^4(2x-5)$

$\cancel{y' = x^2\left(4\sin^3(2x-5)(2)\right) + 2x\sin^4(2x-5)}$

$y' = x^2\left(4\sin^3(2x-5)\cos(2x-5) \cdot 2\right) + 2x\sin^4(2x-5)$

⑤ $y = \sqrt{1 + \sin 2x}$

$y' = \frac{2\cos 2x}{2\sqrt{1+\sin 2x}} = \frac{\cos 2x}{\sqrt{1+\sin 2x}}$

Monday 17/11/2025

e.g) $f(x) = -\sin^2(x) + (\sin 2)x$ ~~$(\sin 2x)$~~

find $f'(\frac{\pi}{4}) = ?$

$f'(x) = -2\sin(x)\cos(x) + \sin 2$ ~~$2\sin(2x)$~~

$f'(\frac{\pi}{4}) = -\sin(2x) + \sin 2$

$= -\sin(2\frac{\pi}{4}) + \sin 2$

$= -1 + \sin 2$

e.g) if $f(x) = 3\tan x - 2\csc x$ find $f'(\frac{\pi}{3})$

$f'(x) = 3\sec^2 x + 2\csc x \cot x$

$f'(x) = \frac{3}{\cos^2 x} + 2\frac{1}{\cos x} \cdot \frac{1}{\tan x}$

$f'(\frac{\pi}{3}) = \frac{3}{(\frac{1}{2})^2} + 2\frac{1}{(\frac{1}{2})} \cdot \frac{1}{\sqrt{3}}$

$= 12 + 4 \cdot \frac{1}{\sqrt{3}} = 12 + \frac{4}{\sqrt{3}}$

• chain Rule : قاعدة السلسلة

① if $y = f(t)$, $t = g(x)$ then

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

② $(f\circ g)'(x) = f'(g(x)) \cdot g'(x)$

e.g) let $y = t^3 + 4t^{10}$, $t = \sin x^2$ find $\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx}$

$= (3t^2 + 40t^9) \cdot (2x\cos x^2)$

$= 3\sin^2 x^2 + 40\sin^9 x^2 \cdot 2x\cos x^2$

e.g) if $f(1) = 2$, $f'(3) = 5$, $f'(1) = 1$, $g(1) = 3$, $g'(1) = -1$ find

① $(f\circ g)'(1) = f'(g(1)) \cdot g'(1)$

$= f'(3) \cdot (-1)$

$= 5(-1) = -5$

② $(3f^2 + 4g^3)'(1) =$ ~~$6f'(1) + 12[g'(1)]^2$~~

~~$= 6 + 12 = 18$~~

$= 6f(1)f'(1) + 12g^2(1)g'(1)$

$= 12 + -108 = -96$

monday 17/11/2025

e.g) if $y = f(2x-3)$ such that $f'(1) = 2$

find $\left.\frac{dy}{dx}\right|_{x=2}$

$\frac{dy}{dx} = f'(2x-3)\cdot 2 \Big|_{x=2}$

$= f'(1)\cdot 2 = 4$

e.g) if $f(x^3-1) = \frac{3x^2}{x^2+1}$ find $f'(7)$

$f'(x^3-1)\cdot 3x^2 = \frac{(x^2+1)(6x)-(3x^2)(2x)}{(x^2+1)^2}$

$x^3-1 = 7$

$x^3 = 8$

$\boxed{x=2}$

$f'(7)\cdot \cancel{12} = \frac{\cancel{12}}{25}$

$f'(7) = \frac{1}{25}$

e.g) if $\frac{d}{dx}[f(2x+4)] = 4x+2$.

find $\frac{d}{dx}[f(x)]\Big|_{x=4}$ $f'(4)$

$2x+4 = 4$

$\boxed{x=0}$

$[f'(2x+4)\cdot 2] = 4x+2$

$f'(4)\cdot 2 = 4(0)+2$

$\boxed{f'(4) = 1}$

- Implicit Diff - الاشتقاق الضمني

For $f(x,y) = c$ we use implicit diff to find $\frac{dy}{dx}$

e.g) find $\frac{dy}{dx}$:

$\boxed{1}$ $y^3 + xy^2 = 5 - x^2 + y$

$3y^2y' + x(2yy') + y^2 = -2x + y'$

$3y^2y' + 2xyy' + y^2 = -2x + y'$

$3y^2y' + 2xyy' - y' = -y^2 - 2x$

$y'(3y^2 + 2xy - 1) = -y^2 - 2x$

$$\boxed{y' = \frac{-y^2 - 2x}{(3y^2 + 2xy - 1)}}$$

$\boxed{2}$ $\sin^2(xy) = x - y^2 + 3$

$2\sin(xy)\cdot xy' + y = 1 - 2yy'$

$2\sin(xy) + y - 1 = -xy' - 2yy'$

$y' = \frac{2\sin(xy) + y - 1}{(-x - 2y)}$

Monday 17/11/2025

[2] $\sin^2(xy) = x - y^2 + 3$

$2\sin(xy)\cos(xy)(xy' + y) = 1 - 2yy'$

$xy' 2\sin(xy)\cos(xy) + 2y\sin(xy)\cos(xy) = 1 - 2yy'$

$2y\sin(xy)\cos(xy) - 1 = -2y'x\sin(xy)\cos(xy) \boxed{-2yy'}$

$$y' = \frac{2y\sin(xy)\cos(xy) - 1}{-2x\sin(xy)\cos(xy) - 2y}$$

e.g) For $xy^2 + (x+y)^3 = 5x + 2y - 2$

find $\left.\frac{dy}{dx}\right|_{(1,-1)}$

$x(2yy') + y^2 + 3(x+y)^2(1+y') = 5 + 2y'$

$-2y' + 1 = 5 + 2y'$

$1 - 5 = 4y'$

$-4 = 4y'$

$\boxed{y' = -1}$

• Diff of log and exponential function

$f(x) \longrightarrow f'(x)$

$\log_a(g(x)) \longrightarrow \frac{g'(x)}{g(x)\ln a}$

$\ln(g(x)) \longrightarrow \frac{g'(x)}{g(x)}$

$a^{g(x)} \longrightarrow a^{g(x)} \cdot g'(x) \cdot \ln a$

$e^{g(x)} \longrightarrow e^{g(x)} \cdot g'(x)$

e.g)

① $y = \log_3(x^2+5) - x \cdot 4^{x^2}$

$$y' = \frac{2x}{(x^2+5)\ln 3} - x\left(4^{x^2}(2x)\ln 4\right) + 4^{x^2}$$

② $\ln(x^2+y^2) - 5y^3 = xe^y + 3$

$$\frac{2x + 2yy'}{(x^2+y^2)} - 15y^2y' = xe^yy' + e^y$$

$$\frac{2x + 2yy'}{(x^2+y^2)} = y'xe^y + e^y + 15y'y^2$$

$2x + 2yy' = (x^2+y^2)y'xe^y + (x^2+y^2)e^y + (x^2+y^2)15y'y^2$

يتبع

monday ~~Nov~~ 17/11/2025

$$2x - e^{y}(x^2+y^2) = y'xe^{y}(x^2+y^2) + 15y'y^2(x^2+y^2)$$

$$\boxed{y' = \frac{2x - e^{y}(x^2+y^2)}{xe^{y}(x^2+y^2)+15y^2(x^2+y^2)}}$$

③ $\frac{d}{dx}\left[\tan^3\sqrt{1+e^x}\right]$

$$y' = 3\tan^2\sqrt{1+e^x} \cdot \sec^2\sqrt{1+e^x} \cdot \frac{e^x}{2\sqrt{1+e^x}}$$

④ if $f(x) = \log_{(x+3)}(x+1)$ then $f'(0) = ??$

*الدكتورة حولتها ← ln
اسهل في الاشتقاق

$$f(x) = \frac{\ln(x+1)}{\ln(x+3)}$$

$$f'(x) = \frac{\ln(x+3)\left(\frac{1}{x+1}\right) - \ln(x+1)\left(\frac{1}{x+3}\right)}{\left(\ln(x+3)\right)^2}$$

$$f'(0) = \frac{\ln 3 - \cancel{\ln(1)}^{0}\left(\frac{1}{3}\right)}{(\ln(3))^{\cancel{2}}} = \frac{1}{\ln(3)}$$

⑥ if $f(x) = \cos(\log_2 x)$ the $f'(x) = ??$

$$f'(x) = -\sin(\log_2 x)\ \frac{1}{x\ln 2}$$

⑦ $\frac{d}{dx}\left[\ln\left(\frac{\sqrt{x+1}\ x^5}{(7+x)^3}\right)^3\right]$

$$y = 3\left(\ln\sqrt{x+1} + 5\ln x - 3\ln(7+x)\right)$$

$$y' = 3\left[\frac{1}{2\sqrt{x+1}} + \frac{5}{x} - \frac{3}{7+x}\right]$$

$$y' = \frac{3}{2\sqrt{x+1}} + \frac{15}{x} - \frac{9}{7+x}$$

⑧ $\frac{d}{dx}\left[\pi^{3x+1}\right] =$

$$y' = \pi^{3x+1} \cdot 3 \cdot \ln\pi$$

⑨ if $\ln y \overset{\ln}{=} x^{\sin x}$ find $\frac{dy}{dx}$

$$\ln y = \sin x \ln x$$

$$\frac{y'}{y} = \sin\frac{1}{x} + \ln x \cdot \cos x$$

$$y' = y\left[\frac{\sin x}{x} + \cos x \ln x\right]$$

$$y' = x^{\sin x}\left[\frac{\sin x}{x} + \cos x \ln x\right]$$

Monday 17/11/2025

e.g)

x	$f(x)$	$f'(x)$
1	3	0
2	1	4
3	5	6

x	$g(x)$	$g'(x)$
1	2	7
2	3	8
3	1	0

① if $h(x) = f(g(x))$ then $h'(1) =$

$$h'(x) = f'(g(x)) \cdot g'(x)$$

$$h'(1) = f'(g(1)) \cdot g'(1)$$

$$= f'(2)(7)$$

$$= (4)(7)$$

$$\boxed{h'(1) = 28}$$

② if $h(x) = \sqrt{3 + f(x)}$ the $h'(2) =$

$$h'(x) = \frac{f'(x)}{2\sqrt{3+f(x)}}$$

$$h'(2) = \frac{f'(2)}{2\sqrt{3+f(2)}} = \frac{4}{2\sqrt{3+1}}$$

$$h'(2) = \frac{4}{4}$$

$$\boxed{h'(2) = 1}$$

③ if $h(x) = g(x)\cos\left(\frac{\pi}{4}x\right)$ then $h'(2) =$

$$h'(x) = g(x)\left(-\sin\left(\frac{\pi}{4}x\right)\frac{\pi}{4}\right) + g'(x)\cos\left(\frac{\pi}{4}x\right)$$

$$h'(2) = g(2)\left(-\frac{\pi}{4}\sin\left(\frac{\pi}{2}\right)\right) + g'(2)\cos\frac{\pi}{2}$$

$$= (3)\left(-\frac{\pi}{4}\right) + \cancel{(8)(0)}$$

$$\boxed{h'(x) = -\frac{3\pi}{4}}$$

④ if $h(x) = \frac{4^x}{f(x)}$ then $h'(1) =$

$$h'(x) = \frac{f(x)\,4^x \cdot \ln 4 - 4^x f'(x)}{(f(x))^2}$$

$$h'(1) = \frac{f(1)(4)(\ln 4) - 4f'(1)}{(f(1))^2}$$

$$= \frac{12\ln 4 - 0}{9} = \frac{\cancel{12}^{4}\ln 4}{\cancel{9}_{3}}$$

$$\boxed{h'(1) = \frac{4\ln 4}{3}}$$

⑤ if $h(x) = x^{g(x)}$ then $h'(3) =$

$$\ln h(x) = g(x) \ln x$$

$$\frac{h'(x)}{h(x)} = g(x)\frac{1}{x} + \ln x \, g'(x)$$

$$\frac{h'(3)}{h(3)} = g(3)\left(\frac{1}{3}\right) + \ln 3 \, g'(3)$$

$$\frac{h'(3)}{3} = (1)\left(\frac{1}{3}\right) \cancel{+ \ln 3 \,(0)}$$

$$\boxed{h'(3) = 1}$$

$h(3) = 3^{g(3)}$

$h(3) = 3^{1}$

$h(3) = 3$

• Differentiation of inverse function :-

if $y = f(x)$ is one to one, the $f^{-1}(x)$ exist

and $\frac{d}{dx}\left[f^{-1}(x)\right] = \frac{1}{f'(f^{-1}(x))}$

e.g) if $f(x) = x^3 + 3x + 2$, find $(f^{-1})'(6)$

① $f^{-1}(6) = 1$

② $f'(x) = 3x^2 + 3$

③ $(f^{-1})'(6) = \frac{1}{f'(1)} = \frac{1}{6}$

e.g) if $f(x) = 3e^{2x-4} + x^3$, find $(f^{-1})'(11) = ??$

① $f^{-1}(11) = 2$

② $f'(x) = 6e^{2x-4} + 3x^2$

③ $(f^{-1})'(11) = \frac{1}{f'(2)} = \frac{1}{18}$

* • Diff of inverse trigonometric functions.

مهم

1. $\frac{d}{dx}\left[\sin^{-1}(g(x))\right] = \frac{g'(x)}{\sqrt{1 - g^2(x)}}$

2. $\frac{d}{dx}\left[\cos^{-1}(g(x))\right] = \frac{-g'(x)}{\sqrt{1 - g^2(x)}}$

3. $\frac{d}{dx}\left[\tan^{-1}(g(x))\right] = \frac{g'(x)}{1 + g^2(x)}$

4. $\frac{d}{dx}\left[\cot^{-1}(g(x))\right] = -\frac{g'(x)}{1 + g^2(x)}$

5. $\frac{d}{dx}\left[\sec^{-1}(g(x))\right] = \frac{g'(x)}{|g(x)| \cdot \sqrt{g^2(x) - 1}}$

6. $\frac{d}{dx}\left[\csc^{-1}(g(x))\right] = \frac{-g'(x)}{|g(x)| \sqrt{g^2(x) - 1}}$

Monday 17/11/2025

e.g) find $\frac{dy}{dx}$

$\boxed{1}$ $y = \tan^{-1} x + \sin^{-1} x$

$$y' = \frac{1}{1+x^2} + \frac{1}{\sqrt{1-x^2}}$$

$\boxed{2}$ $y = \sin^{-1}(2x) + 3\cos^{-1} x^2$

$$y' = \frac{2}{\sqrt{1-4x^2}} - 3\,\frac{2x}{\sqrt{1-x^4}}$$

$$y' = \frac{2}{\sqrt{1-4x^2}} - \frac{6x}{\sqrt{1-x^4}}$$

$\boxed{3}$ $y = \sec^{-1}(\ln x)$

$$y' = \frac{\frac{1}{x}}{|\ln x|\sqrt{(\ln x)^2 - 1}}$$

$\boxed{4}$ $y = \pi^{\cot^{-1}(5x)}$

$$y' = \pi^{\cot^{-1}(5x)} \cdot \frac{-5}{25x^2+1} \ln \pi$$

$\boxed{5}$ $y = \sin^{-1}(\cos(5x)) + \tan^{-1}(e^{3x+7})$

$$y' = \frac{-5\sin 5x}{\sqrt{1-\cos^2(5x)}} + \frac{3e^{3x+4}}{(e^{3x+7})^2+1}$$

$\boxed{6}$ $x^3 + x\tan^{-1} y = e^y$

$$3x^2 + x\left(\frac{y'}{1+y^2}\right) + \tan^{-1} y = y'e^y$$

$$3x^2 + \tan^{-1} y = y'e^y - \frac{xy'}{1+y^2}$$

$$y' = \frac{3x^2 + \tan^{-1} y}{e^y - \frac{x}{1+y^2}}$$

e.g) if $f(x) = e^{2\tan^{-1} x}$ then $f'(0) = ??$

$$f'(x) = e^{2\tan^{-1} x} \cdot 2\left(\frac{1}{1+x^2}\right)$$

$$f'(0) = e^0\, 2(1)$$

$$f'(0) = 2$$

Monday 17/11/2025

- slope and tangent line

- Remark : for any Curve $y = f(x)$

[1] the slope of the tangent line at $x_0 = f'(x_0)$

[2] the equation of the tangent line at $x_0 =$

$$y - y_0 = m(x - x_0)$$

$m = \text{slope} = f'(x_0)$, $y_0 = f(x_0)$

e.g) Let $f(x) = x^3 - 3x^2 + 1$

[1] find the slope at $(1,-1)$

slope $= f'(x) = 3x^2 - 6x$ ← $\boxed{x=1}$

$f'(1) = -3$

[2] find the equation of tangent line at $(1,-1)$

① $m = f'(1) = -3$

② $y - y_0 = m(x - x_0)$

$y + 1 = -3(x - 1)$

$y + 1 = -3x + 3$

$\boxed{y = -3x + 2}$

e.g) let $f(x) = \dfrac{3\cos 2x}{1 + \sin x}$ find the equation of tangent line at $x = 0$
$y = 3$

① $m = f'(x) = \dfrac{(1+\sin x)(-3(2)(\sin 2x) - (3\cos 2x)(\cos x)}{(1+\sin x)^2}$

$= f'(0) = \dfrac{\cancel{(1+0)(0)} - 3}{(1+0)^2}$

$f'(0) = -3$

$\therefore\ y - 3 = -3(x - 0)$

$\boxed{y = -3 + 3}$

e.g) find the slope of the tangent line to the curve : $2x^2y + y^2 + \cos\left(\frac{\pi x}{2}\right) = 3$ at $(1,1)$

$2x^2y' + 4xy + 2yy' + \left(\frac{\pi}{2}\right)\left(-\sin\left(\frac{\pi x}{2}\right)\right) = 0$

$2y' + 4 + 2y' - \frac{\pi}{2} = 0$

$4y' + 4 - \frac{\pi}{2} = 0$

$\frac{4y'}{4} = \frac{-4 + \frac{\pi}{2}}{4}$

$\boxed{y' = -1 + \frac{\pi}{8}}$ ← slope

Monday 17/11/2025

e.g) find the equation of tangent line for $f(x) = x e^x + \ln(x^2+1) - 1$ at $x=0$

$y = -1$

① $f'(x) = x e^x + e^x + \frac{2x}{x^2+1}$ ← $x=0$

$= 1 + 0$

$\boxed{f'(0) = 1}$ ← slope

② $y + 1 = 1(x - 0)$

$\boxed{y = x - 1}$

e.g) find the equation of tangent line for: $x^2y^2 - 2x = 4 - 4y$ at $(2,-2)$

① $x^2(2yy') + y^2(2x) - 2 = -4y'$

$4(-4)y' + (4)(4) - 2 = -4y'$

$-16y' + 14 = -4y'$

$14 = 12y'$

$\boxed{y' = \frac{14}{12}}$

② $y + 2 = \frac{14}{12}(x - 2)$

$y + 2 = \frac{14}{12}x - \frac{7}{3}$

$\boxed{y = \frac{14}{12}x - \frac{7}{3} - 2}$

e.g) find the equation of tangent line of $y = f^{-1}(x)$ at $x=3$ where $f(x) = x^3 - 5$

$f'(x) = 3x^2$

① $x_0 = 3$

② $y_0 = f^{-1}(3) = 2$

③ $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))} = \frac{1}{f'(2)} = \frac{1}{12}$

$\therefore y - 2 = \frac{1}{12}(x - 3)$

• Remark:

1 if the tangent is horizontal then slope $= 0$

2 if $f(x) \parallel g(x)$, then $m_f = m_g$ ⇒ paralle

3 if $f(x) \perp g(x)$, then $m_f \cdot m_g = -1$

e.g) find x on $f(x) = \frac{x^2+1}{x-1}$ where $f(x)$ paralle (يوازي) to $y = x$ if exist.

① $f'(x) = \frac{(x-1)(2x) - (x^2+1)}{(x-1)^2} = 1$

② $(x-1)(2x) - (x^2+1) = (x-1)^2$

$2x^2 - 2x - x^2 - 1 = x^2 - 2x + 1$

$\cancel{x^2} - \cancel{2x} - 1 = \cancel{x^2} - \cancel{2x} + 1$

$-1 \neq 1$ False

No solu ⟹ No x

Monday 17/11/2025

e.g) let $f(x) = \ln(x - 4x^2)$ the $f(x)$ has a horizontal tangent line at $x =$

$m = 0$
الخاصية

$f'(x) = 0$

$0 = \frac{1 - 8x}{x - 4x^2}$

$1 - 8x = 0$

$\boxed{x = \frac{1}{8}}$

---

e.g) the points of the tangent lines to the curve $x^2 + y^2 = 4$ are horizontal are:

$y' = 0$

$2x + \cancel{2y}(y') = 0$

$2x = 0 \Rightarrow \boxed{x = 0} \rightarrow \boxed{y = \pm 2}$

$\therefore (0, 2), (0, -2)$ horizontal line

---

e.g) find the slope of the normal ~~tangent~~ line for $f(x) = x e^{-x}$ at $x = 0$?

العمودي

$\frac{-1}{\text{الخاصية}}$

(1) $f'(x) = -x e^{-x} + e^{-x} \leftarrow \boxed{x = 0}$

$f'(0) = 1$

$\therefore$ slope of the normal $= \frac{-1}{f'(0)} = \frac{-1}{1} = -1$

Monday 24/11/2025

* اسئلة مراجعة لمادة الكالكس *

① if $\lim \dfrac{\sin 7x}{\sin(2bx)} = \dfrac{7}{8}$, then $b = ??$

$\dfrac{7}{2b} = \dfrac{7}{8} \Rightarrow 2b = 8 \Rightarrow \boxed{b=4}$

② let $y = -3\tan x \csc x$, $\dfrac{dy}{dx}$ at $x = \dfrac{\pi}{4}$

$y = -3\dfrac{\cancel{\sin x}}{\cos x} \cdot \dfrac{1}{\cancel{\sin x}} \Rightarrow y = \dfrac{-3}{\cos x}$

$y' = \dfrac{-3(-\sin x)}{\cos^2 x} \quad \swarrow x = \dfrac{\pi}{4}$

$y' = \dfrac{\frac{3}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^2} = \dfrac{6}{\sqrt{2}} = 3\sqrt{2}$

$3 \cdot ② \; \sqrt{2}\,\cancel{\sqrt{2}}$

③ $\lim\limits_{x \to -\infty} \dfrac{\sqrt{16x^2+4} - 5}{-x+5}$

$= \lim\limits_{x \to -\infty} \dfrac{\sqrt{16x^2}}{-x} = \lim\limits_{x \to -\infty} \dfrac{4|x|}{-x} =$

لانها تؤول الى $-\infty$

$\lim\limits_{x \to -\infty} \dfrac{4(\cancel{-x})}{\cancel{-x}} = 4$

④ Let $f(4)=4$, $f'(4)=4$, $f''(4)=3$

then $\left(\dfrac{f'}{f}\right)'(4) = ??$

$= \dfrac{f(4)f''(4) - f'(4)f'(4)}{[f(4)]^2} = \dfrac{-4}{16} = \dfrac{-1}{4}$

⑤ $f(x) = \sin^2 x + (\sin 2)\,x$ the $f'(\frac{\pi}{4}) =$

$f'(x) = 2\sin x\cos x + \sin 2 \quad \leftarrow x = \dfrac{\pi}{4}$

$f'(\frac{\pi}{4}) = \cancel{2}\left(\dfrac{1}{\cancel{\sqrt{2}}}\right)\left(\dfrac{1}{\sqrt{2}}\right) + \sin 2$

$f'(\frac{\pi}{4}) = 1 + \sin 2$

⑥ let $\dfrac{d}{dx}[f(2x+4)] = 4x+4$

then $\dfrac{d}{dx}(f(x))\Big|_{x=4}$. $?? = f'(4)$ يعني بده

$\boxed{x=0}$

$f'(2x+4) \cdot 2 = 4x+4$

$f'(4) \cdot 2 = 4$

$\boxed{f'(4) = 2}$

⑦ if $f(x) = \cos(\log_2 x)$, $f'(x) = ??$

$f'(x) = -\sin(\log_2 x) \cdot \dfrac{1}{x\ln 2}$

⑧ $f(x) = \cos^{-1}(\tan 2x)$, $f'(x) = ??$

$f'(x) = \dfrac{-2\sec^2 2x}{\sqrt{1-(\tan 2x)^2}}$

9 $e^{2\tan^{-1}x}$, $f'(0) = ??$

$$f'(x) = e^{2\tan^{-1}x} \cdot \frac{2}{1+x^2}$$

$$f'(0) = e^{2\tan^{-1}(0)} \cdot \frac{2}{1+0} = 1 \cdot 2 = 2$$

10 slope of the tangent line :

$$2x^2y + y^2 + \cos\left(\frac{\pi x}{2}\right) = 3$$

at $(1,1)$

$$2x^2y' + 4xy + 2yy' - \frac{\pi}{2}\sin\left(\frac{\pi x}{2}\right) = 0 \leftarrow (1,1)$$

$$2y' + 4 + 2y' - \frac{\pi}{2} = 0$$

$$\frac{4y'}{4} = \frac{-4 + \frac{\pi}{2}}{4}$$

$$\boxed{y' = \frac{\pi}{8} - 1}$$

11 The points at which The tangent line to the curve $x^2+y^2=9$ on horizontal :

$y'=0$

$$2x + 2y\cancel{y'} = 0$$

$\boxed{x=0}$

$y^2 = 9$

$\boxed{y = \pm 3}$

points

$(0, -3)$

$(0, +3)$

$\frac{\infty}{\infty}$ | $\frac{0}{0}$ لوبيتال

12 $$\lim_{x\to 1} \frac{1}{x-1} + \frac{1}{x^2-3x+2}$$

$$\lim_{x\to 1} \frac{1}{x-1} \cdot \frac{(x-2)}{(x-2)} \quad \frac{1}{(x-1)(x-2)}$$

$$= \lim_{x\to 1} \frac{(x-2)}{(x-1)(x-2)} + \frac{1}{(x-1)(x-2)}$$

$$= \lim_{x\to 1} \frac{\cancel{x-1}}{(\cancel{x-1})(x-2)} = \lim_{x\to 1} \frac{1}{(x-2)}$$

$$= \frac{1}{-1} = -1$$

13 $$\lim_{x\to 0^+} \tan^{-1}(\ln x)$$

$$\tan^{-1}\left(\lim_{x\to 0^+} \ln x\right)$$

$$= \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

14 $$\lim_{x\to 0} \frac{\sqrt{ax+b} - 2}{x} = 1$$, find a,b

$\frac{0}{0} \Rightarrow \sqrt{b} - 2 = 0 \Rightarrow \sqrt{b} = 2 \Rightarrow \boxed{b=4}$

$$\lim_{x\to 0} \frac{\sqrt{ax+4} - 2}{x} * \frac{\sqrt{ax+4} + 2}{\sqrt{ax+4} + 2} = 1$$

$x=0$

$$\frac{(ax+4) - 4}{x\,(4)} = 1$$

$$\frac{a\cancel{x}}{4\cancel{x}} = 1 \Rightarrow \boxed{a=4}$$

Subject Monday 24/11/2025 Date No.

- L'H Rule

- Main Idea: for $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}, \frac{\infty}{\infty}, \frac{-\infty}{-\infty}, \frac{\pm\infty}{\pm\infty}, \frac{\infty}{-\infty}$

$$= \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

e.g) $\lim\limits_{x\to\infty} \frac{x}{e^x} = \frac{\infty}{\infty}$ L'H نستخدم

$$= \frac{1}{e^x} = \frac{1}{\infty} = 0$$

e.g) $\lim\limits_{x\to\infty} \frac{\ln(3x)}{\ln(2x)} = \frac{\infty}{\infty}$,

L'H $\lim\limits_{x\to\infty} \frac{\frac{\cancel{3}}{\cancel{3}x}}{\frac{\cancel{2}}{\cancel{2}x}} = \lim\limits_{x\to\infty} 1 = 1$

e.g) $\lim\limits_{x\to 0} \frac{1-\cos 4x}{2x^2} = \frac{0}{0}$

L'H $= \lim\limits_{x\to 0} \frac{\cancel{4}\sin 4x}{\cancel{4}x} = \frac{0}{0}$

$$= \lim_{x\to 0} \frac{4\cos 4x}{1} = 4$$

e.g) $\lim\limits_{x\to 0} \frac{5x}{\tan^{-1}x} = \frac{0}{0}$,

L'H $\lim\limits_{x\to 0} \frac{5}{\frac{1}{1+x^2}} = \lim\limits_{x\to 0} 5+5x^2 = 5$

e.g) $\lim\limits_{x\to 0^+} \frac{1-\ln x}{e^{\frac{1}{x}}} = 1 - {-\infty} = \frac{\infty}{\infty}$

L'H $\lim\limits_{x\to 0^+} \frac{-\frac{1}{x}}{e^{\frac{1}{x}} \cdot \frac{-1}{x^2}} = \lim\limits_{x\to 0^+} \frac{\cancel{-}1}{\cancel{x}} \cdot \frac{\cancel{-}x^{\cancel{2}}}{e^{\frac{1}{x}}}$

$$= \lim_{x\to 0^+} \frac{x}{e^{\frac{1}{x}}} = \frac{0}{\infty} = 0$$

e.g) $\lim\limits_{x\to 0^+} \frac{\ln(\sin x)}{\ln(\tan x)} = \frac{-\infty}{-\infty}$

L'H $\lim\limits_{x\to 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}} = \lim\limits_{x\to 0^+} \frac{\cos x}{\cancel{\sin x}} \cdot \frac{\cancel{\tan x}}{\sec^2 x}$

$$= \lim_{x\to 0^+} \cos^2 x = 1$$

L'H Rule $\frac{0}{0}, \frac{\pm\infty}{\pm\infty}$

Not L'H Rule:

* case 1 :- $0 \cdot \infty$

$\lim\limits_{x\to a} f(x)\cdot g(x) = 0\cdot\infty = \lim\limits_{x\to a} \frac{f(x)}{1/g(x)}$ $\frac{0}{1/\infty} = \frac{0}{0}$

or $= \lim\limits_{x\to a} \frac{g(x)}{\frac{1}{f(x)}} = \frac{\infty}{\frac{1}{0}} = \frac{\infty}{\infty}$ L'H

Monday 24/11/2025

e.g) $\lim_{x\to\infty} x e^{-x} = \infty \cdot 0$ مش من أشكال لوبيتال

الاسهل اني اقلب e^{-x}

$$\lim_{x\to\infty} \frac{x}{e^{x}} = \frac{\infty}{\infty}$$

$$L'H \quad \lim_{x\to\infty} \frac{1}{e^{x}} = \frac{1}{\infty} = 0$$

---

e.g) $\lim_{x\to\infty} x \sin\left(\frac{\pi}{x}\right) = \infty \cdot 0$ نقلب واحد منهم حتى تحول المسألة لـ لوبيتال.

$$\lim_{x\to\infty} \frac{\sin\left(\frac{\pi}{x}\right)}{\left(\frac{1}{x}\right)} = \frac{0}{0}$$

$$L'H \quad \lim_{x\to\infty} \frac{-\frac{\pi}{x^2}\cos\left(\frac{\pi}{x}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\cancel{-}\pi\cos\left(\frac{\pi}{x}\right)}{\cancel{x^2}} \cdot \cancel{-x^2}$$

$$= \pi$$

---

e.g) $\lim_{x\to\frac{\pi}{4}^+} (1-\tan x)\sec 2x = 0 \cdot \infty$; $\frac{1}{\cos\frac{\pi}{2}} = \frac{1}{0}$

$$\lim_{x\to\frac{\pi}{4}^+} \frac{(1-\tan x)}{\frac{1}{\sec 2x}} = \lim_{x\to\frac{\pi}{4}^+} \frac{(1-\tan x)}{\cos 2x} = \frac{0}{0}$$

$$= L'H \lim_{x\to\frac{\pi}{4}^+} \frac{-\sec^2 x}{-2\sin 2x} = \lim_{x\to\frac{\pi}{4}^+} \frac{1}{2} \cdot \frac{1}{\cos^2 x \cdot \sin 2x}$$

$\left(\frac{1}{2}\right)\cdot(1)$

$$= 1$$

L'H Rule

• case (2) $\infty - \infty$ / $-\infty + \infty$

e.g) $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{\sin x} = \infty - \infty$ → نوحد المقامات

$$\lim_{x\to 0^+} \frac{\sin x}{x\sin x} - \frac{x}{x\sin x}$$

$$= \lim_{x\to 0^+} \frac{\sin x - x}{x\sin x} = \frac{0}{0}$$

$$L'H \quad \lim_{x\to 0^+} \frac{\cos x - 1}{x\cos x + \sin x} = \frac{0}{0}$$

$$L'H \quad \lim_{x\to 0^+} \frac{-\sin x}{-x\sin x + \cos x + \cos x}$$

$$\lim_{x\to 0^+} \frac{-\sin x}{-x\sin x + 2\cos x} = \frac{0}{0+1+1} = \frac{0}{2} = 0$$

---

e.g) $\lim_{x\to 0^+} \frac{1}{x} - \frac{1}{e^{x}-1} = \infty - \infty$ نوحد مقامات

$$\lim_{x\to 0^+} \frac{e^{x}-1-x}{x(e^{x}-1)} = \frac{0}{0}$$

$$L'H \lim_{x\to 0^+} \frac{e^{x}-1}{xe^{x}+e^{x}-1} = \frac{0}{0}$$

$$L'H \lim_{x\to 0^+} \frac{e^{x}}{xe^{x}+e^{x}+e^{x}} = \frac{1}{0+1+1} = \frac{1}{2}$$

Monday 24/11/2025

e.g) $\lim_{x\to0^+} \frac{1}{x} - \frac{1}{x^2} = \infty - \infty$

نوحد مقامات

$\lim_{x\to0^+} \frac{x-1}{x^2} = \frac{-1}{0^+} = -\infty$

e.g) $\lim_{x\to0^+} \frac{1}{\ln(x+1)} - \frac{1}{x} = \infty - \infty$

نوحد مقامات

$\lim_{x\to0^+} \frac{x-\ln(x+1)}{x\ln(x+1)} = \frac{0}{0}$

L'H $\lim_{x\to0^+} \frac{1-\frac{1}{x+1}}{x\left(\frac{1}{x+1}\right)+\ln(x+1)} = \frac{\frac{x+1-1}{x+1}}{\frac{x+(x+1)\ln(x+1)}{(x+1)}}$

$= \lim_{x\to0^+} \frac{x}{\cancel{x+1}} \cdot \frac{\cancel{x+1}}{x+(x+1)\ln(x+1)}$

$= \lim_{x\to0^+} \frac{x}{x+(x+1)\ln(x+1)} = \frac{0}{0}$

L'H $\lim_{x\to0^+} \frac{1}{1+\cancel{(x+1)}\frac{1}{\cancel{(x+1)}} + \ln(x+1)}$

$= \lim_{x\to0^+} \frac{1}{1+1+\ln(x+1)} = \frac{1}{1+1+0} = \frac{1}{2}$

e.g) $\lim_{x\to\infty} 2x - \ln(x^2+1) \quad \infty-\infty$

$\lim_{x\to\infty} \ln e^{2x} - \ln(x^2+1)$

$= \lim_{x\to\infty} \ln\left(\frac{e^{2x}}{x^2+1}\right)$

$= \ln\left(\lim_{x\to\infty} \frac{e^{2x}}{x^2+1} = \frac{\infty}{\infty}\right) = \cancel{\ln\frac{\infty}{\infty}}$

$= \ln\left(\text{L'H} \lim_{x\to\infty} \frac{\cancel{2}e^{2x}}{\cancel{2}x}\right)$

$= \ln\left(\lim_{x\to\infty} \frac{e^{2x}}{x} = \frac{\infty}{\infty}\right)$

$= \ln\left(\text{L'H} \lim_{x\to\infty} \underbrace{2e^{2x}}_{\infty}\right) = \ln\infty =$

- case 3: $0^0 / \infty^0 / 1^\infty$

$\lim_{x\to a} (f(x))^{g(x)}$

① $y = (f(x))^{g(x)}$

② $\ln y = \ln(f(x))$

③ $\ln y = g(x)\ln(f(x))$

④ $\lim_{x\to a} \ln y = \lim_{x\to a} g(x)\ln(f(x))$ $\infty\cdot0$

e.g) $\lim\limits_{x\to0^+}(1+\sin x)^{\frac{1}{x}} = 1^\infty$

$y = (1+\sin x)^{\frac{1}{x}}$

lim
ln

$\ln y = \frac{1}{x}\ln(1+\sin x)$

$\lim\limits_{x\to0^+}\ln y = \lim\limits_{x\to0^+}\frac{\ln(1+\sin(x))}{x}$ ← $\infty\cdot0$

$\ln y \overset{L'H}{=} \lim\limits_{x\to0^+}\frac{\frac{\cos x}{1+\sin x}}{1}$

$\ln y = \lim\limits_{x\to0^+}\frac{\cos x}{1+\sin x}$ $\frac{1}{1}$

$e^{\ln y} = e^{1}$

$\boxed{y=e^1} \Leftarrow \lim\limits_{x\to0^+}(1+\sin x)^{\frac{1}{x}}$

e.g) $\lim\limits_{x\to0^+}(e^{2x}-1)^{x}$ 0^0

$y = (e^{2x}-1)^{x}$

$\ln y = x\ln(e^{2x}-1)$

$\lim\limits_{x\to0^+}\ln y = \lim\limits_{x\to0^+} x\ln(e^{2x}-1)$ ← $\ln 0$ $0\cdot-\infty$

$\ln y = \lim\limits_{x\to0^+}\frac{\ln(e^{2x}-1)}{\frac{1}{x}}$

$\ln y \overset{L'H}{=} \lim\limits_{x\to0^+}\frac{\frac{2e^{2x}}{e^{2x}-1}}{\frac{-1}{x^2}}$

$\ln y = \lim\limits_{x\to0^+}\frac{2e^{2x}}{e^{2x}-1}\cdot -x^2$ $\infty\cdot0$

$\ln y = \lim\limits_{x\to0^+}\frac{-2x^2e^{2x}}{e^{2x}-1}$

$\ln y \overset{L'H}{=} \lim\limits_{x\to0^+}\frac{-4x^2e^{2x}-4xe^{2x}}{2e^{2x}} \to \frac{0+0}{2}=0$

$e^{\ln y} = e^{0} \Rightarrow \boxed{y=1}$

e.g) $\lim\limits_{x\to\infty} x^{\frac{-1}{\ln x}}$ ∞^0

$y = x^{\frac{-1}{\ln x}}$

$\ln y = \frac{-1}{\ln x}\ln x$

$\lim\limits_{x\to\infty}\ln y = \lim\limits_{x\to\infty}\frac{-1}{\cancel{\ln x}}\cancel{\ln x}$

$e^{\ln y} = e^{-1}$

$\boxed{y=\frac{1}{e}}$

Subject Monday 24/11/2025 Date

• Rule : $\lim_{x \to \infty} \left(1+\frac{n}{x}\right)^{x} = e^{n}$ 1^{∞}

e.g) $\lim_{x \to \infty} \left(1+\frac{3}{x}\right)^{x} = e^{3}$

e.g) $\lim_{x \to \infty} \left(1-\frac{2}{x}\right)^{4x} = \left(\lim_{x \to \infty} \left(1-\frac{2}{x}\right)^{x}\right)^{4}$

$= \left(e^{-2}\right)^{4} = e^{-8}$

e.g) $\lim_{x \to \infty} \left(\frac{x+1}{x-1}\right)^{x} = \lim_{x \to \infty} \left(\frac{\cancel{x}\left(1+\frac{1}{x}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)}\right)^{x}$

$= \lim \left(\frac{1+\frac{1}{x}}{1-\frac{1}{x}}\right)^{x} = \frac{e^{1}}{e^{-1}} = e^{2}$